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Intersection-complexes. I. Combinatory Theory.

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The theory of the intersection-complex $C_h . C_k$, due originally to Veblen, Weyl, Alexander and Lefschetz, has recently been extended by Flexner* to cover intersections on a "topological n-manifold" (locally Cartesian n-space) in the case $h + k = n$. In his theory the whole of Lefschetz's work for ordinary simplicial manifolds is presupposed, including the rather difficult approximations. The object of the present paper is to provide a combinatory theory of the general intersection, $(h + k \geqslant n)$, which would serve equally well as a basis for Flexner's work, and avoids the wasteful process of using two independent approximations.

The idea of defining the products inductively by means of the "looping coefficients," $(h + k - n = -1)$, is not new; it is suggested as a possibility, for example, in Lefschetz's *Topology*, p. 216; but the definitions do not appear to have been worked out in detail.

I am much indebted to Mr L. C. Young, who read the manuscript and substantially shortened several of the proofs.

§ 1. *Normal Intersection.*

1. The theory is combinatory in the strictest sense, i.e. a complex C_N is determined by specifying which groups of its vertices determine an N-simplex.

We shall be concerned with relations between *chains* on a complex, and trivial exceptions to many theorems are avoided by adopting Alexander's formal definition†. The vertices, denoted by small Greek letters, $\alpha, \beta, \ldots$, are regarded as "marks" from which symbolical polynomials with integer coefficients can be formed, and addition and multiplication of these polynomials obey all the usual rules (associative, distributive and commutative laws, and division of equations by integer factors), save that

$$\alpha\beta = -\beta\alpha,$$

whence $\alpha\alpha = 0$.

An *n-simplex*, A_n, is the product of $n + 1$ different vertices; an *n-chain*, C_n, is a form of degree $n + 1$ in the marks, i.e. a sum, $\sum_i x_i A_r^i$, of n-simplexes multiplied by integers. Evidently‡

$$A_h A_k = (-)^{(h+1)(k+1)} A_k A_h.$$

* W. W. Flexner, *Annals of Math.* (2), 32 (1931), 393—406.

† *Annals of Math.* 31 (1930), p. 294.

‡ The product $C_h C_k$, an $(h + k + 1)$-chain, is not, of course, to be confused with the intersection $C_h . C_k$, to be defined later.

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A k-component ($0 \leqslant k \leqslant n$) of C_n is a k-simplex whose vertices all belong to one term of C_n, and it is agreed that every non-zero complex has the unique -1-component, 1.

It is convenient to introduce also the *quotients* A_n/A_k and C_n/A_k. If A_k is a component of A_n, and $A_n = \epsilon A_k A_{n-k-1}$, where $\epsilon = \pm 1$, then $A_n/A_k = \epsilon A_{n-k-1}$; if A_k is not a component of A_n, $A_n/A_k = 0$. If $C_n = \sum_i x_i A_n^i$, $C_n/A_k = \sum_i x_i (A_n^i/A_k)$. Thus geometrically A_n/A_k is the component of A_n opposite A_k, C_n/A_k the chain whose join to A_k is the "A_k-cluster" in C_n. In particular if $C_n = \Sigma x_i A_n^i$, then $C_n/A_n^i = x_i$, the "multiplicity" of A_n^i.

The *boundary*, $\widehat{A_n}$, of A_n is by definition $\Sigma A_n/\alpha_i$ summed over the vertices of A_n; and

$$\widehat{\Sigma x_i A_n^i} = \Sigma x_i \widehat{A_n^i}.$$

Clearly the boundary of $\widehat{C_n}$ is 0. If $\widehat{C_n} = 0$, C_n is an *n-cycle.*

It is to be noticed that $\widehat{C_0}$ is not in general zero, but a -1-chain, i.e. a number. These numerical boundaries play a fundamental part in the intersection theory. The boundary of all -1-chains is 0.

I 1. $(C_h/A_k)/A_r = C_h/A_r A_k$.

(Follows from the easy special case $C_h = A_h$.)

I 2. $\widehat{C_k} = \sum_i C_k/\alpha_i$, *summed over all vertices of* C_k.

This states merely that $\sum_j x_j (\sum_i A_k^j/\alpha_i) = \sum_i (\sum_j x_j A_k^j)/\alpha_i$.

I 3. $\widehat{C_k/A_h} = (-)^{h+1} \widehat{C_k}/A_h$,

$$\widehat{C_k/A_h} = \sum_i (C_k A_h)/\alpha_i$$

$$= (-)^{h+1} (\sum_i C_k/\alpha_i)/A_h, \text{ by I 1.}$$

Hence

I 4. *If* A_h *does not belong to* C_k, C_k/A_h *is an unbounded chain.*

I 5. $\widehat{C_h C_k} = \widehat{C_h} C_k + (-)^{h+1} C_h \widehat{C_k}$.

For

$$\widehat{C_h C_k} = \sum_i (C_h C_k)/\alpha_i = \sum_i (C_h/\alpha_i)\, C_k + (-)^{h+1} \sum_i C_h (C_k/\alpha_i).$$

2. A chain C_h is *on* a chain C_n if $C_h/A_h = 0$ whenever $C_n/A_h = 0$. If, for all $j < n$, every j-cycle on $C_n \sim 0$ on C_n, (i.e. bounds a $(j+1)$-chain on C_n), C_n is *simply-connected.*

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The combinatory theory of true "n-spheres," "n-elements," and "n-manifolds" being rather complicated* we use instead the more general "combinatory n-spheres" etc. introduced by Alexander. They are defined inductively as follows:

Definition of Π_h. A Π_{-1} is either of the -1-chains ± 1; $\Pi_0 = \alpha$, or $\alpha - \beta$; and if $h > 0$, Π_h is a simply-connected h-chain, $\neq 0$, such that Π_h/α is a Π_{h-1} for all vertices α of Π_h.

Definition of M_h. If $h = -1$, or 0, an M_h is a Π_h. If $h > 0$, M_h is a connected h-chain such that M_h/α is a Π_{h-1} for all α of M_h.

The inductive proofs of the following properties follow very readily from I 1—5:

I 6. *M_h is a simple chain* (i.e. $M_h/A_h = \pm 1$ or 0 for all A_h).

I 7. *An M_h is $(h-1)$-connected* (i.e. every two h-simplexes are connected by a series in which neighbours have a common $(h-1)$-component).

I 8. *If $l \geqslant 0$, M_h/A_l is a Π_{h-l-1} or 0.*

I 9. *$A_h \Pi_k$ is a Π_{h+k+1}, $\widehat{A_h}\Pi_k$ is a Π_{h+k}.*

I 10. *If C_n is on M_n, and $\widehat{C_n}$ (if it exists) is on $\widehat{M_n}$, the numbers C_n/A_n and M_n/A_n have a constant ratio for all A_n of M_n.*

The number whose existence follows from I 10 is called the *degree* of C_n on M_n. By I 6 it is an integer.

3. *Subdivision.* ${}^1A_n$ denotes the n-chain formed by "starring" all components of A_n†. By definition if $C_n = \Sigma x_i A_n^i$,

$$ {}^1C_n = \Sigma x_i ({}^1A_n^i), \quad {}^pC_n = {}^1({}^{p-1}C_n), \quad {}^1C_{-1} = C_{-1}. $$

Evidently $\widehat{{}^1C_n} = {}^1(\widehat{C_n})$, and the following properties are easily proved.

I 11. *If A_k is a component of C_k, and A_k^1 and A_k^2 any k-components of ${}^1A_k$, then*

$$ {}^1C_k/A_k^1 = {}^1C_k/A_k^2, $$

and is isomorphic‡ with ${}^1(C_h/A_k)$: symbolically, ${}^1(C_h/A_k)$ is isomorphic with ${}^1C_h/{}^1A_k$.

I 12. *If C_n is simply connected so is ${}^1C_n$.*

I 13. *${}^1\Pi_h$ is a Π_h, ${}^1M_h$ is an M_h.*

* Cf. Alexander, *op. cit.*; Newman, *Journal Lond. Math. Soc.* 6 (1931), 186—192.

† I.e. to each component of A_n corresponds a vertex of ${}^1A_n$, and to each rising sequence of components, $A_0, A_1, \ldots, A_n$, of $A_n (A_i \subset A_{i+1})$, corresponds an n-component, $\beta^0\beta^1 \ldots \beta^n$, of ${}^1A_n$. The indicatrix is fixed thus: Each A_i contains one vertex, α^i, not in A_{i-1}. If

$$ \alpha^0\alpha^1 \ldots \alpha^n = \epsilon A_n, \quad (\epsilon = \pm 1) $$

then $\epsilon\beta^0\beta^1 \ldots \beta^n$ is the corresponding component of ${}^1A_n$.

‡ Two chains are isomorphic if they are the same functions of the same or different "marks."

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Two chains, C_h and C_k on M_n, have *normal intersection on* M_n if they have no common $(h+k-n+1)$-component.

I 14. *If C_h and C_k are chains on M_n, and $\widehat{C_h}$ and C_k have normal intersection on M_n, there exists (for some integers) a chain $C^*{}_h$ on ${}^sM_n$ homologous to ${}^sC_h$ on ${}^sM_n$ and having normal intersection with ${}^sC_k$.*

In order to prove this inductively we generalise it as follows:

I 15. *If C_h and C_k are chains on M_n, r an integer $> h+k-n$, and $\widehat{C_h}$ and C_k have no common r-component, there is an integer s and an h-chain $C^*{}_h$ homologous to ${}^sC_h$ on ${}^sM_n$ such that $C^*{}_h$ and ${}^sC_k$ have no common r-component.*

The theorem is trivial if $h=n$, since then $r>k$; in particular it is trivial if $n=-1$. Assume then that $h<n$ and that I 15 $(n-1)$ is true.

Suppose that $p\geqslant 0$ is the largest integer such that C_h, C_k have a common A_{r+p}; let α be the centre of ${}^1A_{r+p}$. Since $\widehat{C_h}$ does not contain A_{r+p}, ${}^1C_h/\alpha$ is a cycle on ${}^1M_n/\alpha$ and therefore bounds an h-chain Δ_h on it. ${}^1C_h/\alpha$ and ${}^1C_k/\alpha$ have no common $(r+p)$-component, or its join to α would be a common $(r+p+1)$-component. Thus the chains Δ_h and ${}^1C_k/\alpha$ satisfy the conditions of I 15 $(n-1)$, with $r+p$ in place of r. Moreover, since ${}^1C_k$ meets ${}^1M_n/\alpha$ in ${}^1C_k/\alpha$ any chain on ${}^t({}^1M_n/\alpha)$ has the same components in common with ${}^{t+1}C_k$ as with ${}^t({}^1C_k/\alpha)$. Hence, by I 15 $(n-1)$, there exists on ${}^t({}^1M_n/\alpha)$ a chain K_h^1, homologous to ${}^t\Delta_h$ and having no $(r+p)$-component in common with ${}^{t+1}C_k$; and

$$\begin{aligned}{}^t[\alpha({}^1C_h/\alpha)]-K_h^1 &= {}^t\widehat{\alpha\Delta_h}+{}^t\Delta_h-K_h^1\\ &\sim {}^t\widehat{\alpha\Delta_h}\sim 0.\end{aligned}$$

It follows that

$${}^{t+1}C_h\sim {}^{t+1}C_h-\sum_j {}^t[\alpha^j({}^1C_h/\alpha^j)]+\sum_j K_h^j,$$

which has no $(r+p)$-component in common with ${}^{t+1}C_k$.

The $(r+p-1)$-, ..., r-components common to the two chains may now be removed in the same way.

The following generalisation of I 14 is proved similarly. The set $C_{h_1}, C_{h_2}, \ldots, C_{h_q}$, are said to have *normal intersection on* M_n if no selection of them,

$$C_{a_1}, C_{a_2}, \ldots, C_{a_p},$$

have a common $(\Sigma a_i - pn+n+1)$-component.

I 16. *If $\widehat{C_{h_0}}, C_{h_1}, \ldots, C_{h_q}$ on M_n have normal intersection there exists a chain $C^*{}_{h_0}$ on ${}^sM_n$, homologous to ${}^sC_{h_0}$ such that $C^*{}_{h_0}$, ${}^sC_{h_1}, \ldots, {}^sC_{h_q}$, have normal intersection.*

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§ 2. *Intersection-complexes.*

4. Two chains, C_h and C_k, on an M_n, will be called an "admissible pair" on M_n if all the pairs (C_h, C_k), $(\widehat{C_h}, C_k)$, and $(C_h, \widehat{C_k})$ have normal intersection on M_n, i.e. if

A (i). *C_h and C_k have no common $(l+1)$-component $(l = h+k-n)$;*

A (ii). *neither of the pairs $\widehat{C_h}$ and C_k, C_h and $\widehat{C_k}$ has a common l-component;*

A (ii a). *when $l = -1$, M_n is a Π_n.*

Notice that, since every non-zero complex has a -1-component, it follows from A (ii) that, when $l = -1$, $\widehat{C_h} = 0$ or $C_k = 0$, and $\widehat{C_k} = 0$ or $C_h = 0$; i.e. *either* one of C_h and C_k is zero, *or* both C_h and C_k are unbounded.

I 17. *If C_h and C_k are an admissible pair on M_n, C_h/A_j and C_k/A_j are admissible on M_n/A_j.*

A (i). If C_h/A_j and C_k/A_j had a common $(l-j)$-component the join to A_j would be an $(l+1)$-component of C_h and C_k.

A (ii). Similar to A (i), using $\widehat{C_h/A_j} = (-)^{h+1}\,\widehat{C_h}/A_j$.

A (ii a) is satisfied in all cases save the trivial $j = -1$.

The symbol $(C_h \,.\, C_k)_{M_n}$ is defined inductively as an l-chain, when C_h and C_k are an admissible pair on M_n, by the following rules †:

$$\textit{Case I.}\quad l \geqslant 0\colon \frac{(C_h \,.\, C_k)_{M_n}}{A_l} = \left(\frac{C_h}{A_l} \,.\, \frac{C_k}{A_l}\right)_{M_n/A_l}.$$

The dimension number, $n-l-1$, of M_n/A_l being less than n, the multiplicity of A_l in $(C_h \,.\, C_k)_{M_n}$ is determined as a -1-dimensional intersection on an M_n of lower dimension number.

Case II. $l = -1$ (and therefore $M_n = \Pi_n$):

(*a*) If $C_h = 0$, $(C_h \,.\, C_{n-h-1})_{\Pi_n} = 0$.

(*b*) If $C_h \neq 0$ and $h \geqslant 0$, condition A (ii) gives $\widehat{C_h} = \widehat{C_k} = 0$. By I 14 there is on ${}^t\Pi_n$ a chain C^*_{n-h} bounded by ${}^tC_{n-h-1}$, and forming with ${}^tC_n$ an admissible pair on ${}^t\Pi_n$. We define

$$(C_h \,.\, C_{n-h-1})_{\Pi_n} = \widehat{({}^tC_h \,.\, C^*_{n-h})}\,{}^t\Pi_n.$$

† Lefschetz gives the two symbols $C_h \,.\, C_{n-h}$ and $(C_h \,.\, C_{n-h})$ distinct meanings. Although this notation has been followed by other writers the inconvenience of not being able to use brackets in their ordinary punctuating sense is so great that I have ventured to give up the distinction. Moreover, in the present paper no special definition is necessary, since Lefschetz's $(C_h \,.\, C_{n-h})$ appears naturally as $\widehat{C_h \,.\, C_{n-h}}$.

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(*c*) If $C_h \neq 0$ and $h = -1$, the chains are C_{-1} and C_n. Since C_n is unbounded it has a definite degree, d, on Π_n. We define

$$(C_{-1} \cdot C_n)_{\Pi_n} = dC_{-1}.$$

Cases (*a*) and (*c*) of II cover between them all possibilities when $n = -1$, and so start the induction.

The case $h + k - n < -1$ is really excluded by A (i, ii), but we agree to set $(C_h \cdot C_k)_{M_n} = 0$ in these cases.

It will be noticed that there is an apparent ambiguity in Case II, since t and C^*_{n-h} may be variously determined; and since I is referred back to II the doubt extends to all values of l. It will be shewn, however, that the ambiguity is not a real one: the intersection-complex is unique. This is clearly true if $n = -1$. We assume:

$H[n]$. *The definition determines a unique l-chain $(C_h \cdot C_k)_{M_m}$ for all admissible pairs when either*

$$m \leqslant n-1, \quad h+k-m = -1; \quad \text{or} \quad m \leqslant n, \quad h+k-m \neq -1.$$

This combined range of m, h and k we denote by $[n]$. From $H[n]$ there follow:

I 18 $[n]$. *If a and b are integers and all the intersections exist,*

$$(aC_h + bC_h') \cdot C_k = a(C_h \cdot C_k) + b(C_h' \cdot C_k),$$

$$C_h \cdot (aC_k + bC_k') = a(C_h \cdot C_k) + b(C_h \cdot C_k').$$

(By induction.)

I 19 $[n]$. *If C_h and C_k are an admissible pair on M_m,*

$$\frac{(C_h \cdot C_k)_{M_m}}{A_j} = \left(\frac{C_h}{A_j} \cdot \frac{C_k}{A_j}\right)_{M_m/A_j}.$$

If $j = -1$, or $j > l$ $(= h + k - m)$ the theorem is trivial.
If $0 \leqslant j \leqslant l$, let $A_{l-j-1} A_j = A_l$. Then, by I 1 and Case I,

$$\frac{(C_h \cdot C_k)_{M_m}}{A_j} \Big/ A_{l-j-1} = \frac{(C_h \cdot C_k)_{M_m}}{A_l} = \left(\frac{C_h}{A_l} \cdot \frac{C_k}{A_l}\right)_{M_m/A_l},$$

and

$$\left(\frac{C_h}{A_j} \cdot \frac{C_k}{A_j}\right)_{\frac{M_m}{A_j}} \Big/ A_{l-j-1} = \left(\frac{C_h}{A_l} \cdot \frac{C_k}{A_l}\right)_{M_m/A_l},$$

by Case I and then I 1.

The multiplicities of the two given $(l-j-1)$-chains being thus equal on all $(l-j-1)$-components, they are identical.

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I 20 [n]. *If* $\widehat{C_h} = 0$, $(\widehat{C_h \,.\, C_{k+1}})_{M_m} = (C_h \,.\, \widehat{C_{k+1}})_{M_m}$.

If $l\,(=h+k-m) < -1$, both sides $= 0$.

If $l = -1\,(m \leqslant n-1)$ the definition, I, states that the left-hand side is one of the values of the right-hand, and hypothesis $H[n]$ that it is the only one.

If $l > 0\,(m \leqslant n)$

$$(\widehat{C_h \,.\, C_k})_{M_m}/A_{l-1} = (-)^l(\widehat{C_h \,.\, C_k})_{M_m}/A_{l-1}$$

$$= (-)^l\left(\widehat{\frac{C_h}{A_{l-1}} \,.\, \frac{C_k}{A_{l-1}}}\right)_{\frac{M_m}{A_{l-1}}}.$$

Now M_m/A_{l-1} is a Π_{m-l}, C_h/A_{l-1} is unbounded, and $m-l < m$. Hence by the case $l = -1$, $(m \leqslant n-1)$,

$$(\widehat{C_h \,.\, C_k})_{M_m}/A_{l-1} = (-)^l\left(\frac{C_h}{A_{l-1}} \,.\, \widehat{C_k/A_{l-1}}\right)_{M_m/A_{l-1}}$$

$$= (C_h \,.\, \widehat{C_k})_{M_m}/A_{l-1}.$$

I 20 a [n]. *If* $\widehat{C_h} = 0$ *and* $C_k \sim 0$ *then* $C_h \,.\, C_k \sim 0$.

(C_k is the $\widehat{C_{k+1}}$ of I 20.)

5. *Subdivision.* The following theorem is asserted for *all* l, provided that $m \leqslant n$. (When $l = -1$ it asserts that any value of the left-hand side is a value of the right-hand side.)

I 21. $(m \leqslant n)$ ${}^1\{(C_h \,.\, C_k)_{M_m}\} = ({}^1C_h \,.\, {}^1C_k)_{{}^1M_m}$.

(*a*) $l \geqslant 0$. Evidently only l-components, of either side, that belong to some ${}^1A_l$ have multiplicity $\neq 0$. Let $A^*{}_l$ be any l-component of ${}^1A_l$. Then evidently

$${}^1\{(C_h \,.\, C_k)_{M_m}\}/A^*{}_l = (C_h \,.\, C_k)_{M_m}/A_l.$$

Again, by definition,

$$({}^1C_h \,.\, {}^1C_k)_{{}^1M_m}/A^*{}_l = \left(\frac{{}^1C_h}{A^*{}_l} \,.\, \frac{{}^1C_k}{A^*{}_l}\right)_{{}^1M_m/A^*{}_l}.$$

Now ${}^1C_h/A^*{}_l$, ${}^1C_k/A^*{}_l$, and ${}^1M_m/A^*{}_l$, are by I 11 isomorphic with ${}^1(C_h/A_l)$, ${}^1(C_k/A_l)$, and ${}^1(M_m/A_l)$, and the correspondence between the vertices in the three congruence relations is the same. Hence

$$({}^1C_h \,.\, {}^1C_k)_{{}^1M_m}/A^*{}_l = \{{}^1(C_h/A_l) \,.\, {}^1(C_k/A_l)\}_{{}^1(M_m/A_l)}$$

$$= {}^1\left[\left(\frac{C_h}{A_l} \,.\, \frac{C_k}{A_l}\right)_{M_m/A_l}\right]$$

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by an inductive hypothesis, since $l \geqslant 0$,

$$= (C_h \,.\, C_k)_{M_m}/A_l,$$

since -1-complexes are not affected by subdivision.

(*b*) $l = -1$. Only the case $C_h \neq 0$, $h \neq -1$ requires attention.

We need only shew that if, given C_h and C_{m-h-1}, s is such that C^*_{m-h} exists on ${}^sM_m$ with the properties given in the definition, and if t is any positive integer, there exists *at least* one C^0_{m-h} on ${}^{s+t}M_m$ such that

$$({}^{s+t}C_h \,.\, C^0_{m-h})^{s+t}{}_{M_m} = ({}^sC_h \,.\, C^*_{m-h})^s{}_{M_m}.$$

But by case (*a*) it is sufficient to take $C^0_{m-h} = {}^tC^*_{m-h}$, for 0-spreads are unaffected by subdivisions.

I 22. ($m \leqslant n$, *all* $k \leqslant m$.) *If* C_m *is an* m*-cycle on* M_m *and* d *is the degree of* C_m *then* $(C_k \,.\, C_m)_{M_m} = dC_k$.

If $k = -1$ we have the definition, II (*b*).

If $k \geqslant 0$

$$(C_k \,.\, C_m)_{M_m}/A_k = \left(\frac{C_k}{A_k} \,.\, \frac{C_m}{A_k}\right)_{M_m/A_k}$$

Now the degree of C_m on M_m = degree of C_m/A_k on M_m/A_k. Hence, by case $k = -1$,

$$(C_k \,.\, C_m)_{M_m}/A_k = dC_k/A_k.$$

The uniqueness theorem can now be proved.

(It is agreed that from this point intersections without a suffix are to be taken on M_n (or Π_n) itself:

$$C_h \,.\, C_k = (C_h \,.\, C_k)_{M_n}.)$$

I 23. *The definitions determine* $C_h \,.\, C_k$ *uniquely.*

The assumption $H[n]$ is to be extended to the cases $m = n$, $l = -1$, and $m = n+1$, $l \neq -1$. Of these the second obviously follows from the first. We suppose then that $m = n$, $l = -1$, and $M_n = \Pi_n$. By I 21 it is sufficient to consider only one value of s.

(*a*) $h > 0$. Let C^1_{n-h} and C^2_{n-h} be chains on ${}^s\Pi_n$ both satisfying the conditions of II (*b*). Then $C^1_{n-h} - C^2_{n-h}$, being an unbounded $(n-h)$-chain on ${}^s\Pi_n$, ~ 0. Hence

$${}^sC_h \,.\, (C^1_{n-h} - C^2_{n-h}) \sim 0, \qquad (\text{I } 20a,\ k = n-h),$$

$$\overbrace{{}^sC_h \,.\, C^1_{n-h}} - \overbrace{{}^sC_h \,.\, C^2_{n-h}} = 0.$$

(*b*) $h = 0$. With the same notation

$$C_0 \,.\, (C^1_n - C^2_n) = dC_0, \qquad (\text{I } 22)$$

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where d is the degree of $C_n^1 - C_n^2$. Hence

$$\widehat{C_0 . C_n^1} - \widehat{C_0 . C_n^2} = d\widehat{C_0} = 0.$$

(*c*) $h = -1$. Obvious.

This also completes the proofs of I 18–22 for all m.

I 24. *If C_h and C_k are an admissible pair on M_n, and also (condition* A iii) *$\widehat{C_h}$ and $\widehat{C_k}$ have normal intersection on M_n (par.* 3), *then*

$$\widehat{C_h . C_k} = C_h . \widehat{C_k} + (-)^{(n-h)(n-k)} C_k . \widehat{C_h}.$$

I 25. *If C_h and C_k are an admissible pair,*

$$C_h . C_k = (-)^{(n-h)(n-k)} C_k . C_h.$$

These two theorems will be proved by a combined induction. (Notice that if $l = 0$ A (iii) requires $\widehat{C_h}$ or $\widehat{C_k} = 0$.)

If $n = -1$ both theorems are evidently true. Assume, then, that they are true for n smaller than the given one.

Proof of I 24 (*n*) *if* $l \neq 0$. If $l = -1$ both sides are 0; assume then that $l > 0$.

$$\widehat{C_h . C_k}/A_{l-1} = (-)^l \widehat{(C_h . C_k)/A_{l-1}}$$

$$= (-)^l \left(\widehat{\frac{C_h}{A_{l-1}} . \frac{C_k}{A_{l-1}}}\right)_{M_n/A_{l-1}}$$

$$= (-)^l \left[\frac{C_h}{A_{l-1}} . \widehat{C_k/A_{l-1}} + (-)^{(n-h)(n-k)} \frac{C_k}{A_{l-1}} . \widehat{C_h/A_{l-1}}\right]_{\frac{M_n}{A_{l-1}}}$$

by the inductive hypothesis, since $n - l < n$,

$$= [C_h . \widehat{C_k} + (-)^{(n-h)(n-k)} C_k . \widehat{C_h}]/A_{l-1}.$$

Proof of I 25 (*n*).

(*a*) Let $l \neq -1$,

$$\frac{C_h . C_k}{A_l} = \left(\frac{C_h}{A_l} . \frac{C_k}{A_l}\right)_{M_m/A_l} = (-)^{(n-h)(n-k)} \left(\frac{C_k}{A_l} . \frac{C_h}{A_l}\right)_{M_m/A_l}, \text{ etc.}$$

(*b*) Let $l = -1$, $h \neq -1$ or n. Let

$$C_h . C_{n-h-1} = {}^s\widehat{C_h . C^*_{n-h}},$$

$$C_{n-h-1} . C_h = {}^s C_{n-h-1} . C^*_{h+1}.$$

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The boundaries of C^*_{n-h} and C^*_{h+1} are C_{n-h-1} and C_h, and have no common vertex. Hence we may suppose that C^*_{n-h} and C^*_{h+1} are an admissible pair of complexes on Π_n. They also satisfy A (iii) of I 24. Hence, by the case of I 24 already proved,

$$\widehat{C^*_{n-h} \cdot C^*_{h+1}} = C^*_{n-h} \cdot {}^sC_h + (-)^{h(n-h-1)}\, C^*_{h+1} \cdot {}^sC_{n-h-1}.$$

The boundary of the spread on the right is therefore zero, giving

$$0 = \widehat{C^*_{n-h} \cdot {}^sC_h} + (-)^{h(n-h-1)}\, \widehat{C^*_{h+1} \cdot {}^sC_{n-h-1}}$$

$$= (-)^{h(n-h)} \left[{}^s\widehat{C_h \cdot C^*_{n-h}} - (-)^{(h+1)(n-h)}\, {}^s\widehat{C_{n-h} \cdot C^*_{h+1}} \right],$$

by case (a), just dealt with; and this is the required result.

(c) Let $h = -1$, $k = n$. By I 22

$$C_{-1} \cdot C_n = dC_{-1} \qquad (d = \text{degree of } C_n)$$

$$C_n \cdot C_{-1} = \widehat{C_n \cdot C^*_0} = \widehat{C^*_0 \cdot C_n} \qquad \text{(Case } a\text{)}$$

$$= \widehat{dC^*_0} = dC_{-1}.$$

Proof of I 24 (n) *if* $l = 0$. Condition A (iii) requires $\widehat{C_h} = 0$ or $\widehat{C_k} = 0$. This case is therefore, by I 25 (n), nothing but Case II (b) of the definition.

I 26. *If* A (iii) *is not satisfied in* I 24, *it is still true that* $\widehat{C_h \cdot C_k}$ *covers only common* A_{l-1}*'s of* $\widehat{C_h}$ *and* C_k *or of* C_h *and* $\widehat{C_k}$.

(If A (iii) is not satisfied the right-hand of I 24 does not exist. The proof of I 26 is similar to that of I 24, but is simplified by the fact that I 25 is now established.)

I 27. *If* $\widehat{C_h} = 0$ *and* $C_k \sim 0$ *then* $C_h \cdot C_k \sim C_k \cdot C_h \sim 0$. (From I 20 a and I 25.)

I 28. *If* (α), C_h, C_j, C_k, *and* (β), *all sets* $\widehat{C_r}$, C_s, C_t *have normal intersection on* M_n, *where* r, s, t *are* h, j, k *in any order, then*

$$C_h \cdot (C_j \cdot C_k) = (C_h \cdot C_j) \cdot C_k.$$

In view of I 25, 26, conditions (α) and (β) ensure the existence of both sides of the equation.

The proof is inductive through n.

Let $\lambda = h + j + k - 2n$. If $\lambda \neq -1$ the usual induction shews, with the help of I 19, that

$$\frac{C_h \cdot (C_j \cdot C_k)}{A_\lambda} = \frac{(C_h \cdot C_j) \cdot C_k}{A_\lambda}.$$

(Cf. proof of I 25.)

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Suppose then that $\lambda = -1$. If the trivial case where one of C_h, C_j, C_k is zero is excluded, condition (β) requires that all three chains be unbounded, and by I 22 the case $n = h$ is also obvious. Assume then that $h \neq n$.

Let $\widehat{C^1_{h+1}} = {}^s C_h$ where C^1_{h+1} forms an admissible pair with ${}^s(C_j \,.\, C_k)$. We drop the index ^{s} everywhere, as being irrelevant.

$$C_h \,.\, (C_j \,.\, C_k) = (-)^{(n-h)\,(h+1)}\,(C_j \,.\, C_k) \,.\, C_h$$

$$= (-)^{(n-h)\,(h+1)}\,\overbrace{(C_j \,.\, C_k) \,.\, C^1_{h+1}}.$$

It follows easily from I 16 and I 26 that there exists a chain C^2_{h+1}, homologous to C^1_{h+1}, such that C^2_{h+1}, C_j, C_k, satisfy (α) and (β). Hence, by the case $\lambda \neq -1$ and I 24, 25, 27,

$$(1) \qquad (-)^{(n-h)\,(h+1)}\,(C_j \,.\, C_k) \,.\, C^1_{h+1} \sim (-)^{\mu}\,(C^2_{h+1} \,.\, C_k) \,.\, C_j,$$

where $\mu = h + 1 + (n-k)(n-j)$.

Again $(C_h \,.\, C_j) \,.\, C_k = \overbrace{(C_h \,.\, C_j) \,.\, C^3_{k+1}}$

$$(2) \qquad = (-)^{(n-k-1)\,(k+1)}\,\overbrace{(C^4_{k+1} \,.\, C_h) \,.\, C_j}.$$

From the relation $\lambda = -1$ between h, j, k, and n, it follows that

$$\mu + (n-k-1)(k+1) \equiv (n-k-1)(n-h-1) + 1, \text{ mod. } 2,$$

whence by (1) and (2), with I 24,

$$C_h \,.\, (C_j \,.\, C_k) - (C_h \,.\, C_j) \,.\, C_k = (-)^{\mu}\,\widehat{K_0},$$

where $$K_0 = \overbrace{(C^2_{h+1} \,.\, C^4_{k+1})} \,.\, C_j.$$

Since $\widehat{C_j} = 0$ it follows that $\widehat{K_0} = 0$, which proves the theorem.

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