East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.260416.270217a

Vol. 7, No. 2, pp. 396-416
May 2017

Semilocal Convergence Analysis for MMN-HSS Methods under Hölder Conditions

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Received 26 April 2016; Accepted (in revised version) 27 February 2017.

Abstract. Multi-step modified Newton-HSS (MMN-HSS) methods, which are variants of inexact Newton methods, have been shown to be competitive for solving large sparse systems of nonlinear equations with positive definite Jacobian matrices. Previously, we established these MMN-HSS methods under Lipschitz conditions, and we now present a semilocal convergence theorem assuming the nonlinear operator satisfies milder Hölder continuity conditions. Some numerical examples demonstrate our theoretical analysis.

AMS subject classifications: 65F10, 65F50, 65H10

Key words: MMN-HSS method, large sparse systems of nonlinear equation, Hölder conditions, positive-definite Jacobian matrices, semilocal convergence.

1. Introduction

Systems of nonlinear equations arise in many practical applications, such as discretizations of nonlinear differential and integral equations, numerical optimization, see [3, 21, 27, 29] and references therein. We consider the general form

$$F(x) = 0\,, \tag{1.1}$$

where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a nonlinear continuously differentiable operator mapping from an open convex subset $\mathbb{D}$ of the n-dimensional complex linear space $\mathbb{C}^n$ into $\mathbb{C}^n$, and $F = (F_1, \cdots, F_n)^T$ with $F_i = F_i(x)$, $i = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n)^T$ is such that the Jacobian matrix $F'(x)$ is large, sparse, non-Hermitian and positive definite.

Inexact Newton methods [17, 18, 22, 25] are especially useful for solving such systems, and the general algorithm can be summarised briefly as follows.

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Algorithm 1. IN (inexact Newton method)
1. Let x_0 be given.
2. For $k = 0, 1, \cdots$ until convergence do:
   2.1. For a given $\eta_k \in [0, 1)$, find s_k that satisfies

$$\left\|F(x_k) + F'(x_k)s_k\right\| \leq \eta_k \left\|F(x_k)\right\|. \tag{1.2}$$

   2.2. Set

$$x_{k+1} = x_k + s_k.$$

Here $\eta_k \in [0, 1)$, commonly called the *forcing term*, is used to control the level of accuracy.

These methods are obviously variants of the classical Newton method where

$$F'(x_k)\, s_k = -F(x_k), \quad k \geq 0 \tag{1.3}$$

is solved approximately at each iteration. Local convergence analysis for the inexact Newton methods represented above likewise shows that the sequence $\{x_k\}$ converges if x_0 is sufficiently close to a solution x_* of the nonlinear system (1.1) and η_k is suitably uniformly bounded [17]. When the dimension n of the problem is large, linear iteration methods such as splitting methods and modern Krylov subspace methods [31] are usually applied to solve Eq. (1.3). Inexact Newton methods are inner-outer iteration methods, where the *inner iteration* invokes a linear method and a nonlinear *outer iteration* method generates the sequences $\{x_k\}$. The Newton-CG and Newton-GMRES methods, using CG and GMRES methods as the respective inner iterations, have often been used and studied [1, 13, 14]. For non-Hermitian positive-definite linear systems, Bai *et al.* [10] proposed the Hermitian and skew-Hermitian splitting (HSS) method, which converges unconditionally to the exact solution of the system of linear equations with the same upper bound for the convergence rate as the CG method when optimal parameters are used. Because of its effectiveness and robustness in successfully solving Stokes problems and distributed control problems for example [4, 5, 12], the HSS method has also been studied extensively [7–9, 11].

Using the HSS method as the inner iteration, Bai & Guo [6] proposed the Newton-HSS method for solving the system of nonlinear equations with non-Hermitian positive-definite Jacobian matrices.

Algorithm 2. NHSS (Newton-HSS method)
1. Given an initial guess x_0, positive constants α and tol, and a positive integer sequence $\{l_k\}_{k=0}^{\infty}$:
2. For $k = 0, 1, \cdots$ until $\|F(x_k)\| \leq tol\|F(x_0)\|$ do:
   2.1. Set $d_{k,0} := 0$.
   2.2. For $l = 0, 1, \cdots, l_k - 1$, apply algorithm HSS:

$$\begin{cases} \big(\alpha I + H(x_k)\big) d_{k,l+\frac{1}{2}} = \big(\alpha I - S(x_k)\big) d_{k,l} - F(x_k), \\ \big(\alpha I + S(x_k)\big) d_{k,l+1} = \big(\alpha I - H(x_k)\big) d_{k,l+\frac{1}{2}} - F(x_k), \end{cases}$$

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and obtain d_{k,l_k} such that

$$\left\|F(x_k)+F'(x_k)d_{k,l_k}\right\| \le \eta_k \left\|F(x_k)\right\|, \quad \eta_k \in [0,1),$$

where $H(x_k)=(F'(x_k)+F'(x_k)^*)/2$ and $S(x_k)=(F'(x_k)-F'(x_k)^*)/2$.

2.3. Set

$$x_{k+1}=x_k+d_{k,l_k}.$$

Here $F'(x)^*$ denotes the conjugate transpose of $F'(x)$. The numerical results for two-dimensional nonlinear convection-diffusion equations have shown that the Newton-HSS method requires fewer iteration steps and less CPU time than the Newton-USOR, the Newton-GMRES and the Newton-CG methods.

Recently, the HSS method has been used to solve modified Newton equations [16, 33], and Wu & Chen [15, 30] proposed a modified Newton-HSS method as follows.

Algorithm 3. MN-HSS (modified Newton-HSS method)

1. Given an initial guess x_0, positive constants α and tol, and two positive integer sequences $\{l_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$:

2. For $k=0,1,\cdots$ until $\|F(x_k)\| \le tol\ \|F(x_0)\|$ do:

2.1. Set $d_{k,0}=h_{k,0}:=0$.

2.2. For $l=0,1,\cdots,l_k-1$, apply Algorithm HSS:

$$\begin{cases}\left(\alpha I+H(x_k)\right)d_{k,l+\frac{1}{2}}=\left(\alpha I-S(x_k)\right)d_{k,l}-F(x_k),\\ \left(\alpha I+S(x_k)\right)d_{k,l+1}=\left(\alpha I-H(x_k)\right)d_{k,l+\frac{1}{2}}-F(x_k),\end{cases}$$

and obtain d_{k,l_k} such that

$$\left\|F(x_k)+F'(x_k)d_{k,l_k}\right\| \le \eta_k \left\|F(x_k)\right\|, \quad \eta_k \in [0,1), \tag{1.4}$$

where $H(x_k)=(F'(x_k)+F'(x_k)^*)/2$ and $S(x_k)=(F'(x_k)-F'(x_k)^*)/2$.

2.3. Set

$$y_k=x_k+d_{k,l_k}.$$

2.4. Compute $F(y_k)$.

2.5. For $m=0,1,\cdots,m_k-1$, apply Algorithm HSS:

$$\begin{cases}\left(\alpha I+H(x_k)\right)h_{k,m+\frac{1}{2}}=\left(\alpha I-S(x_k)\right)h_{k,m}-F(y_k),\\ \left(\alpha I+S(x_k)\right)h_{k,m+1}=\left(\alpha I-H(x_k)\right)h_{k,m+\frac{1}{2}}-F(y_k),\end{cases} \tag{1.5}$$

and obtain h_{k,m_k} such that

$$\left\|F(y_k)+F'(x_k)h_{k,m_k}\right\| \le \widetilde{\eta}_k \left\|F(y_k)\right\|, \quad \widetilde{\eta}_k \in [0,1). \tag{1.6}$$

2.6. Set

$$x_{k+1}=y_k+h_{k,m_k}.$$

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We earlier proposed the following multi-step modified Newton-HSS method [26], which includes the Newton-HSS method and the modified Newton-HSS method as special cases for the parameter values $m = 1$ and $m = 2$, respectively.

Algorithm 4. MMN-HSS (multi-step modified Newton-HSS method)
1. Given an initial guess x_0, positive constants α and tol.
2. For $k = 0, 1, \cdots$ until $\|F(x_k)\| \le tol\ \|F(x_0)\|$ do:
   2.1. Let $x_k^{(0)} = x_k$.
   2.2. For $i = 1, 2, \cdots, m$,
      2.2.1. Set $d_{k,0}^{(i)} := 0$.
      2.2.2. For $l = 0, 1, \cdots, l_k^{(i)} - 1$, apply Algorithm HSS:

$$\begin{cases} \big(\alpha I + H(x_k)\big) d_{k,l+\frac{1}{2}}^{(i)} = \big(\alpha I - S(x_k)\big) d_{k,l}^{(i)} - F(x_k^{(i-1)}), \\ \big(\alpha I + S(x_k)\big) d_{k,l+1}^{(i)} = \big(\alpha I - H(x_k)\big) d_{k,l+\frac{1}{2}}^{(i)} - F(x_k^{(i-1)}), \end{cases}$$

      and obtain $d_{k,l_k^{(i)}}^{(i)}$ such that

$$\big\|F(x_k^{(i-1)}) + F'(x_k) d_{k,l_k^{(i)}}^{(i)}\big\| \le \eta_k^{(i)} \big\|F(x_k^{(i-1)})\big\|, \quad \eta_k^{(i)} \in [0, 1), \tag{1.7}$$

      where $H(x_k) = (F'(x_k) + F'(x_k)^*)/2$ and $S(x_k) = (F'(x_k) - F'(x_k)^*)/2$.
      2.2.3. Set

$$x_k^{(i)} = x_k^{(i-1)} + d_{k,l_k^{(i)}}^{(i)}.$$

   2.3. Set $x_{k+1} = x_k^{(m)}$.

After some calculations, we obtain an equivalent expression for the MMN-HSS method:

$$\begin{cases} x_k^{(0)} = x_k, \\ x_k^{(i)} = x_k^{(i-1)} - \big(I - T(\alpha; x_k)^{l_k^{(i)}}\big) F'(x_k)^{-1} F(x_k^{(i-1)}), \quad 1 \le i \le m, \\ x_{k+1} = x_k^{(m)}, \end{cases} \tag{1.8}$$

where

$$T(\alpha; x) = \big(\alpha I + S(x)\big)^{-1} \big(\alpha I - H(x)\big) \big(\alpha I + H(x)\big)^{-1} \big(\alpha I - S(x)\big).$$

We previously considered local convergence of the MMN-HSS method when the derivative satisfies Lipschitz conditions, and proved a semilocal convergence theorem [26]. In some cases however, the derivative does not satisfy Lipschitz conditions — e.g. when we choose initial points that satisfy only Lipschitz conditions, the generalised MMN-HSS method cannot be used to solve problems involving convection-diffusion equations [6, 20]. We note that Chen *et al.* [15] discussed convergence of the MN-HSS method under milder Hölder conditions, and in Section 2 we prove a new semilocal convergence theorem for the MMN-HSS method assuming the derivative satisfies the Hölder continuity conditions.

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Two numerical examples are given to illustrate its effectiveness and robustness under the Hölder continuity conditions in Section 3, and some brief concluding remarks are made in Section 4.

2. Semilocal Convergence Theorem Under the Hölder Conditions

We first discuss the Hölder continuity conditions and several lemmas, then present and prove the semilocal convergence theorem when the derivative satisfies the Hölder conditions. Thus if the initial point x_0 satisfies some conditions, the existence of the solution of the nonlinear system (1.1) can be ascertained directly from the iteration process.

Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$ of a point $x_0 \in \mathbb{D}$ at which $F'(x)$ is continuous and positive definite. Suppose $F'(x) = H(x) + S(x)$, where $H(x) = (F'(x) + F'(x)^*)/2$ and $S(x) = (F'(x) - F'(x)^*)/2$ are the Hermitian and skew-Hermitian parts of the Jacobian matrix $F'(x)$, respectively. Let $\mathbb{N}(x_0, r)$ denote an open ball centred at x_0 with radius $r > 0$.

Assumption 1. Let $x_0 \in \mathbb{C}^n$ and assume the following conditions hold:
(A1)(Bounded Condition) there exist positive constants β, γ and δ such that

$$\max\left\{\|H(x_0)\|, \|S(x_0)\|\right\} \le \beta\,, \quad \left\|F'(x_0)^{-1}\right\| \le \gamma\,, \quad \left\|F(x_0)\right\| \le \delta\,; \tag{2.1}$$

(A2)(Hölder Condition) there exist nonnegative constants K_h and K_s such that, for any $x \in \mathbb{N}(x_0, r)$,

$$\left\|H(x) - H(x_0)\right\| \le K_h \|x - x_0\|^p, \tag{2.2}$$

$$\left\|S(x) - S(x_0)\right\| \le K_s \|x - x_0\|^p, \tag{2.3}$$

with the exponent $p \in (0, 1]$.

Lemma 2.1. *Under Assumption* 1, *if* $r \in (0, (\frac{1}{\gamma K})^{1/p})$ *for all* $x \in \mathbb{N}(x_0, r)$ *then* $F'(x)^{-1}$ *exists and we have the following inequalities:*

$$\left\|F'(x) - F'(x_0)\right\| \;\le\; K\|x - x_0\|^p, \tag{2.4}$$

$$\left\|F'(x)^{-1}\right\| \le \frac{\gamma}{1 - \gamma K \|x - x_0\|^p}, \tag{2.5}$$

where $K := K_h + K_s$.

Proof. From the inequalities in the Hölder Condition, we have

$$\begin{aligned}\left\|F'(x) - F'(x_0)\right\| &= \left\|H(x) + S(x) - H(x_0) - S(x_0)\right\| \le \left\|H(x) - H(x_0)\right\| + \left\|S(x_0) - S(x)\right\| \\ &\le (K_h + K_s)\|x - x_0\|^p = K\|x - x_0\|^p\,,\end{aligned}$$

hence

$$\left\|F'(x_0)^{-1}(F'(x_0) - F'(x))\right\| \le \left\|F'(x_0)^{-1}\right\|\left\|F'(x_0) - F'(x)\right\| \le K\gamma\|x - x_0\|^p < 1\,.$$

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From the Banach Lemma (e.g. see Theorem V.4.3 in Ref. [24]), clearly $F'(x)^{-1}$ exists and we have

$$\left\|F'(x)^{-1}\right\| \leq \frac{\left\|F'(x_0)^{-1}\right\|}{1-\left\|F'(x_0)^{-1}(F'(x_0)-F'(x))\right\|} \leq \frac{\gamma}{1-\gamma K\|x-x_0\|^p},$$

which completes the proof. □

Define the functions

$$\varphi(t) = \frac{a}{1+p}t^{1+p} - bt + c\,, \tag{2.6}$$

$$\psi(t) = dt^p - 1\,, \quad t \in [0, d^{-\frac{1}{p}}], \tag{2.7}$$

where the positive constants a, b, c and d satisfy the inequalities

$$a > bd\,, \qquad 0 < b \leq 1-\eta\,, \qquad d > 0\,, \qquad \frac{c}{b}\frac{1+p}{p} < \left(\frac{b}{a}\right)^{\frac{1}{p}}; \tag{2.8}$$

and also the sequences $\{t_k\}, \{s_k^{(1)}\}, \cdots, \{s_k^{(m-1)}\}$ by the following schemes:

$$\begin{cases} t_0 = 0\,, \\ s_k^{(1)} = t_k - \dfrac{\varphi(t_k)}{\psi(t_k)}, \\ s_k^{(2)} = s_k^{(1)} - \dfrac{\varphi(s_k^{(1)})}{\psi(t_k)}, \\ \quad \cdots\cdots \\ t_{k+1} = s_k^{(m-1)} - \dfrac{\varphi(s_k^{(m-1)})}{\psi(t_k)}, \quad k = 0, 1, \cdots. \end{cases} \tag{2.9}$$

Letting $s_k^{(0)} = t_k$ and $t_{k+1} = s_k^{(m)}$, we can re-express these schemes as

$$s_k^{(i+1)} = s_k^{(i)} - \frac{\varphi(s_k^{(i)})}{\psi(s_k^{(0)})}, \quad i = 0, 1, \cdots, m-1\,, \quad k = 0, 1, \cdots.$$

Some properties of function $\varphi(t)$ and sequences $\{t_k\}$, $\{s_k^{(1)}\}$, $\cdots$, $\{s_k^{(m-1)}\}$ are summarised in the following lemmas.

Lemma 2.2. *If $\varphi(t)$ is decreasing in $[0, (a/b)^{1/p})$ but increasing in $[(a/b)^{1/p}, +\infty)$, and if*

$$\frac{c}{b}\frac{1+p}{p} < \left(\frac{b}{a}\right)^{\frac{1}{p}}$$

and $\varphi(t) = 0$ has two roots t_* and t_{**} in $(0, +\infty)$, then the following inequality holds:

$$0 < t_* < \frac{c}{b}\frac{1+p}{p} < t_{**}\,.$$

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Proof. The derivative of the function $\varphi(t)$ is $\varphi'(t) = at^p - b$. Obviously, $(b/a)^{1/p}$ is the unique zero of the function $\varphi'(t)$ in $(0, +\infty)$, and since

$$\frac{c}{b}\frac{1+p}{p} < \left(\frac{b}{a}\right)^{\frac{1}{p}},$$

we have

$$\begin{aligned}\varphi\left(\left(\frac{b}{a}\right)^{\frac{1}{p}}\right) &= \frac{a}{1+p}\left(\left(\frac{b}{a}\right)^{\frac{1+p}{p}}\right) - b\left(\frac{b}{a}\right)^{\frac{1}{p}} + c \\ &= -\frac{p}{1+p}b\left(\frac{b}{a}\right)^{\frac{1}{p}} + c < 0\,.\end{aligned}$$

From $\varphi(0) = c > 0$ and $\varphi(+\infty) = +\infty$, we therefore know that $\varphi(t) = 0$ has two roots t_*, t_{**} and

$$\begin{aligned}\varphi\left(\frac{1+p}{p}\frac{c}{b}\right) &= \frac{a}{1+p}\left(\frac{1+p}{p}\right)^{1+p}\left(\frac{c}{b}\right)^{1+p} - \frac{1+p}{p}\frac{c}{b}b + c \\ &= \frac{(1+p)^p}{p^{1+p}}a\left(\frac{c}{b}\right)^{1+p} - \frac{c}{p} \\ &= \frac{c}{b}\left(\left(\frac{1+p}{p}\right)^p\frac{a}{b}\left(\frac{c}{b}\right)^p - 1\right) \\ &< \frac{c}{b}\left(\left(\frac{1+p}{p}\right)^p\left(\frac{p}{1+p}\right)^p - 1\right) = 0\,.\end{aligned}$$

The above inequalities imply that

$$0 < t_* < \frac{1+p}{p}\frac{c}{b} < t_{**}\,,$$

to complete the proof. □

Lemma 2.3. *Suppose the sequences* $\{t_k\}$, $\{s_k^{(1)}\}$, $\cdots$, $\{s_k^{(m-1)}\}$ *are generated by the formulas* (2.9). *Let* t_* *be the smaller nonnegative solution of equation* $\varphi(t) = 0$. *Then the sequences* $\{t_k\}$, $\{s_k^{(1)}\}$, $\cdots$, $\{s_k^{(m-1)}\}$ *converge to* t_* *and satisfy the inequalities*

$$0 < t_k < s_k^{(1)} < s_k^{(2)} < \cdots < s_k^{(m-1)} < t_{k+1} < t_*\,.$$

Proof. Setting

$$U_1(x) = x - \frac{\varphi(x)}{\psi(x)} \quad \text{and} \quad U_{i+1}(x) = U_i(x) - \frac{\varphi(U_i(x))}{\psi(x)}, \quad i = 1, 2, \cdots, m-1\,,$$

we prove

$$0 < U_1(x) < U_2(x) < \cdots < U_m(x) < U_m(t_*) = t_*,$$

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for $x \in [0, t_*)$ by induction. When $i = 1$, we have

$$U_1'(x) = \frac{(\psi(x)\varphi'(x))\psi(x) + \varphi(x)\psi'(x)}{(\psi(x))^2}.$$

Because $\varphi(x) > 0$, $\psi(x) < \varphi'(x) < 0$, $\psi'(x) > 0$ and $\psi'(x)$ increases for $x \in [0, t_*)$, obviously $U_1(x)$ is strictly increasing on $[0, t_*)$. So $x < U_1(x) < U_1(t_*) = t_*$ for $x \in [0, t_*)$. Since

$$U_2'(x) = \frac{(\psi(x) - \varphi'(U_1(x)))U_1'(x)\psi(x) + \varphi(U_1(x))\psi'(x)}{\psi(x)^2} > 0$$

and $U_1'(x) > 0$, $\psi(x) < 0$, $\psi(U_1(x)) > 0$, $\psi'(x) > 0$, $\psi(x) < \varphi'(x) < \varphi'(U_1(x))$ when $x \in [0, t_*)$, we have that $U_2(x)$ also strictly increases on $[0, t_*)$. So

$$0 < U_1(x) < U_2(x) < U_2(t_*) = t_*\,.$$

Supposing that

$$0 < U_1(x) < U_2(x) < \cdots < U_{m-1}(x) < U_{m-1}(t_*) = t_*\,,$$

when $i = m$ we have

$$U_m'(x) = \frac{(\psi(x) - \varphi'(U_{m-1}(x)))U_{m-1}'(x)\psi(x) + \varphi(U_{m-1}(x))\psi'(x)}{\psi(x)^2} > 0.$$

Since

$$U_{m-1}'(x) > 0\,, \quad \psi(x) < 0\,, \quad \psi\big(U_{m-1}(x)\big) > 0\,, \psi'(x) > 0\,, \quad \psi(x) < \varphi'(x) < \varphi'\big(U_{m-1}(x)\big)$$

when $x \in [0, t_*)$, $U_m(x)$ also increases strictly on $[0, t_*)$ and

$$0 < U_{m-1}(x) < U_m(x) < U_m(t_*) = t_*\,,$$

and consequently

$$0 < U_1(x) < U_2(x) < \cdots < U_m(x) < U_m(t_*) = t_*$$

when $x \in [0, t_*)$. Moreover,

$$s_k^{(i)} = U_i(t_k), \quad i = 1, 2, \cdots, m-1, \quad t_{k+1} = U_m(t_k),$$

so we readily obtain the inequalities

$$0 < t_k < s_k^{(1)} < s_k^{(2)} < \cdots < s_k^{(m-1)} < t_{k+1} < t_*$$

to complete the proof. □

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Theorem 2.1. *Under the assumptions of Lemmas* 2.1 *and* 2.2*, define* $r := \min(r_1, r_2)$ *where*

$$r_1 = \left(\frac{\alpha+\beta}{K}\left(\sqrt{1+\frac{2\tau\alpha\theta}{\gamma(2+\tau\theta)(\alpha+\beta)^2}}-1\right)\right)^{\frac{1}{p}},$$
$$r_2 = \frac{1+p}{p}\frac{c}{b}.$$

Constants b *and* c *satisfy the relevant inequalities in* (2.8)*; and further define* $u = \min_{1\le i\le m}\{l_*^{(i)}\}$, $l_*^{(i)} = \liminf_{k\to\infty} l_k^{(i)}$ *satisfying* $u > \lfloor \ln\eta/\ln((\tau+1)\theta)\rfloor$, $\eta = \max_k\{\max_{1\le i\le m}\eta_k^{(i)}\}$, *where the symbol* $\lfloor\cdot\rfloor$ *is used to denote the smallest integer no less than the corresponding real number,* $\tau \in (0, 1/\theta - 1)$ *and* $\theta \equiv \theta(\alpha; x_0) = \|T(\alpha; x_0)\| < 1$. *If*

$$\left(\frac{(1+\eta)\gamma}{1-\eta}\right)^{\frac{1+p}{p}} K^{\frac{1}{p}}\delta < \frac{p}{1+p},$$

then the iteration sequence $\{x_k\}_{k=0}^{\infty}$ *generated by the MMN-HSS method is well-defined and converges to* x_* *that satisfies* $F(x_*) = 0$.

Proof. First, we note the norm of the iterative matrix $T(\alpha; x)$ $(x \in B(x_0, r))$ for the linear solver satisfies $\|T(\alpha; x)\| \le (\tau+1)\theta < 1$, as in the proof of Theorem 3.2 in Ref. [23]. Define $a = (1+\eta)K\gamma$, $b \le 1-\eta$, $c = (1+\eta)\gamma\delta$ and $d = K\gamma$, and we proceed to prove the following formulas by induction:

$$\begin{cases} \|x_k - x_0\| \le t_k - t_0, \\ \left\|F(x_k)\right\| \le \dfrac{1}{(1+\eta)\gamma}\varphi(t_k), \\ \|x_k^{(1)} - x_k\| \le s_k^{(1)} - t_k, \\ \left\|F(x_k^{(i)})\right\| \le \dfrac{1}{(1+\eta)\gamma}\varphi(s_k^{(i)}), \\ \|x_k^{(i+1)} - x_k^{(i)}\| \le s_k^{(i+1)} - s_k^{(i)}, \quad i = 1, 2, \cdots, m-2, \\ \left\|F(x_k^{(m-1)})\right\| \le \dfrac{1}{(1+\eta)\gamma}\varphi(s_k^{(m-1)}), \\ \|x_{k+1} - x_k^{(m-1)}\| \le t_{k+1} - s_k^{(m-1)}. \end{cases} \tag{2.10}$$

For $k = 0$ we have

$$\|x_0 - x_0\| = 0 \le t_0 - t_0,$$
$$\left\|F(x_0)\right\| \le \delta \le \frac{\gamma(1+\eta)\delta}{\gamma(1+\eta)} = \frac{c}{\gamma(1+\eta)} = \frac{\varphi(t_0)}{\gamma(1+\eta)},$$
$$\|x_0^{(1)} - x_0\| \le \left\|I - T(\alpha; x_0)^{l_0^{(1)}}\right\| \cdot \left\|F'(x_0)^{-1}\right\| \cdot \left\|F(x_0)\right\| \le (1+\theta^{l_0^{(1)}})\gamma\delta < (1+\eta)\gamma\delta = s_0^{(1)}.$$

Suppose the following inequalities hold for any positive integers i less than $m-1$:

$$\begin{cases} \left\|F(x_0^{(i)})\right\| \le \dfrac{1}{(1+\eta)\gamma}\varphi(s_0^{(i)}), \\ \|x_0^{(i+1)} - x_0^{(i)}\| \le s_0^{(i+1)} - s_0^{(i)}, \quad i = 1, 2, \cdots, m-2. \end{cases} \tag{2.11}$$

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We now prove the inequalities (2.11) hold for $m-1$. From inequalities (2.4) and (2.5)

$$\begin{aligned}\left\|F(x_0^{(m-1)})\right\| \le& \left\|F(x_0^{(m-1)})-F(x_0^{(m-2)})-F'(x_0)(x_0^{(m-1)}-x_0^{(m-2)})\right\| \\ &+\left\|F(x_0^{(m-2)})+F'(x_0)(x_0^{(m-1)}-x_0^{(m-2)})\right\| \\ \le& \left\|F(x_0^{(m-1)})-F(x_0^{(m-2)})-F'(x_0^{(m-2)})(x_0^{(m-1)}-x_0^{(m-2)})\right\| \\ &+\left\|F'(x_0^{(m-2)})-F'(x_0)\right\|\|x_0^{(m-1)}-x_0^{(m-2)}\|+\eta\left\|F(x_0^{(m-2)})\right\| \\ \le& \frac{K}{1+p}\|x_0^{(m-1)}-x_0^{(m-2)}\|^{1+p}+K\|x_0^{(m-2)}-x_0\|^p\|x_0^{(m-1)}-x_0^{(m-2)}\| \\ &+\eta\left\|F(x_0^{(m-2)})\right\|,\end{aligned}$$

imply

$$\left\|F(x_0^{(m-1)})\right\| \le \frac{K}{1+p}(s_0^{(m-1)}-s_0^{(m-2)})^{1+p}+K(s_0^{(m-2)})^p(s_0^{(m-1)}-s_0^{(m-2)})+\frac{\eta}{(1+\eta)\gamma}\varphi(s_0^{(m-2)}),$$

since

$$\begin{aligned}&\|x_0^{(m-1)}-x_0^{(m-2)}\| \le s_0^{(m-1)}-s_0^{(m-2)}, \\ &\|x_0^{(m-2)}-x_0\| \le \|x_0^{(m-2)}-x_0^{(m-3)}\|+\cdots+\|x_0^{(1)}-x_0\| \\ &\qquad\qquad \le (s_0^{(m-2)}-s_0^{(m-3)})+\cdots+(s_0^{(1)}-t_0) \le s_0^{(m-2)}-t_0=s_0^{(m-2)}, \\ &\left\|F(x_0^{(m-2)})\right\| \le \frac{1}{(1+\eta)\gamma}\varphi(s_0^{(m-2)}).\end{aligned}$$

From inequality $\frac{K}{1+p}(t_k-t_{k-1})^p \le \frac{K}{1+p}t_k^{1+p}-\frac{K}{1+p}t_{k-1}^{1+p}-Kt_{k-1}^p(t_k-t_{k-1})$ in Ref. [32],

$$\begin{aligned}\left\|F(x_0^{(m-1)})\right\| \le& \frac{K}{1+p}(s_0^{(m-1)})^{1+p}-\frac{K}{1+p}(s_0^{(m-2)})^{1+p}-K(s_0^{(m-2)})^p(s_0^{(m-1)}-s_0^{(m-2)}) \\ &+K(s_0^{(m-2)})^p(s_0^{(m-1)}-s_0^{(m-2)})+\frac{\eta}{(1+\eta)\gamma}\varphi(s_0^{(m-2)}) \\ =& \frac{K}{1+p}(s_0^{(m-1)})^{1+p}-\frac{K}{1+p}(s_0^{(m-2)})^{1+p}+\frac{\eta}{(1+\eta)\gamma}\varphi(s_0^{(m-2)}) \\ =& \frac{1}{(1+\eta)\gamma}\left[\frac{a}{1+p}(s_0^{(m-1)})^p-\frac{a}{1+p}(s_0^{(m-2)})^p\right]+\frac{\eta}{(1+\eta)\gamma}\varphi(s_0^{(m-2)}) \\ =& \frac{1}{(1+\eta)\gamma}\left[\varphi(s_0^{(m-1)})+bs_0^{(m-1)}-c-\varphi(s_0^{(m-2)})-bs_0^{(m-1)}+c\right] \\ &+\frac{\eta}{(1+\eta)\gamma}\varphi(s_0^{(m-2)}) \\ =& \frac{1}{(1+\eta)\gamma}\varphi(s_0^{(m-1)})+\frac{b+\eta-1}{(1+\eta)\gamma}\varphi(s_0^{(m-2)}) \\ \le& \frac{1}{(1+\eta)\gamma}\varphi(s_0^{(m-1)}),\end{aligned}$$

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so we have

$$\begin{aligned}\|x_1-x_0^{(m-1)}\| \le& \left\|I-T(\alpha;x_0)^{l_0^{(m)}}\right\|\left\|F'(x_0)^{-1}\right\|\left\|F(x_0^{(m-1)})\right\| \\ \le& \left(1+\left((\tau+1)\theta\right)^{l_0^{(m)}}\right)\gamma\frac{1}{(1+\eta)\gamma}\varphi(s_0^{(m-1)}) \\ \le& (1+\eta)\gamma\frac{1}{(1+\eta)\gamma}\varphi(s_0^{(m-1)}) \\ =& \varphi(s_0^{(m-1)}) \\ =& (t_1-s_0^{(m-1)})\big(-\psi(t_0)\big) \\ =& t_1-s_0^{(m-1)}\end{aligned}$$

and that the formulas (2.10) are satisfied for $k=0$.

Now supposing the formulas (2.10) hold for all nonnegative integers less than k, we proceed to prove they hold for k. From the hypothesis,

$$\begin{aligned}\|x_k-x_0\| \le& \|x_k-x_{k-1}^{(m-1)}\|+\|x_{k-1}^{(m-1)}-x_{k-1}^{(m-2)}\|+\cdots+\|x_{k-1}^{(1)}-x_{k-1}^{(0)}\|+\|x_{k-1}-x_0\| \\ \le& (t_k-s_{k-1}^{(m-1)})+(s_{k-1}^{(m-1)}-s_{k-1}^{(m-2)})+\cdots+(s_{k-1}^{(1)}-t_{k-1})+(t_{k-1}-t_0) \\ =& t_k-t_0<t_*<r\,,\end{aligned}$$

and since $x_{k-1},x_{k-1}^{(1)},\cdots,x_{k-1}^{(m-1)}\in\mathbb{N}(x_0,r)$ we also have

$$\begin{aligned}\left\|F(x_k)\right\| \le& \left\|F(x_k)-F(x_{k-1}^{(m-1)})-F'(x_{k-1})(x_k-x_{k-1}^{(m-1)})\right\| \\ &+\left\|F(x_{k-1}^{(m-1)})+F'(x_{k-1})(x_k-x_{k-1}^{(m-1)})\right\| \\ \le& \left\|F(x_k)-F(x_{k-1}^{(m-1)})-F'(x_{k-1}^{(m-1)})(x_k-x_{k-1}^{(m-1)})\right\| \\ &+\left\|F'(x_{k-1}^{(m-1)})-F'(x_{k-1})\right\|\|x_k-x_{k-1}^{(m-1)}\|+\eta\left\|F(x_{k-1}^{(m-1)})\right\| \\ \le& \frac{K}{1+p}\|x_k-x_{k-1}^{(m-1)}\|^{1+p}+K\|x_{k-1}^{(m-1)}-x_{k-1}\|^p\|x_k-x_{k-1}^{(m-1)}\|+\eta\left\|F(x_{k-1}^{(m-1)})\right\|.\end{aligned}$$

Now since

$$\begin{aligned}&\|x_k-x_{k-1}^{(m-1)}\| \le t_k-s_{k-1}^{(m-1)}, \\ &\|x_{k-1}^{(m-1)}-x_{k-1}\| \le \|x_{k-1}^{(m-1)}-x_{k-1}^{(m-2)}\|+\cdots+\|x_{k-1}^{(1)}-x_{k-1}\| \\ &\qquad\qquad \le (s_{k-1}^{(m-1)}-s_{k-1}^{(m-2)})+\cdots+(s_{k-1}^{(1)}-t_{k-1}) \\ &\qquad\qquad \le s_{k-1}^{(m-1)}-t_{k-1}\,, \\ &\left\|F(x_{k-1}^{(m-1)})\right\| \le \frac{1}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)}),\end{aligned}$$

we have

$$\left\|F(x_k)\right\| \le \frac{K}{1+p}(t_k-s_{k-1}^{(m-1)})^{1+p}+K(s_{k-1}^{(m-1)}-t_{k-1})^p(t_k-s_{k-1}^{(m-1)})+\frac{\eta}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)});$$

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and since $0 < t_{k-1} < s_{k-1}^{(m-1)} < t_*$ we obtain $(s_{k-1}^{(m-1)} - t_{k-1})^p < (s_{k-1}^{(m-1)})^p$ such that

$$\begin{aligned}
\left\|F(x_k)\right\| \le & \frac{K}{1+p}(t_k - s_{k-1}^{(m-1)})^{1+p} + K(s_{k-1}^{(m-1)})^p(t_k - s_{k-1}^{(m-1)}) + \frac{\eta}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)}) \\
\le & \frac{K}{1+p}t_k^{1+p} - \frac{K}{1+p}(s_{k-1}^{(m-1)})^{1+p} - K(s_{k-1}^{(m-1)})^p(t_k - s_{k-1}^{(m-1)}) \\
& + K(s_{k-1}^{(m-1)})^p(t_k - s_{k-1}^{(m-1)}) + \frac{\eta}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)}) \\
= & \frac{K}{1+p}t_k^{1+p} - \frac{K}{1+p}(s_{k-1}^{(m-1)})^{1+p} + \frac{\eta}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)}) \\
= & \frac{1}{(1+\eta)\gamma}\left[\frac{a}{1+p}t_k^{1+p} - \frac{a}{1+p}(s_{k-1}^{(m-1)})^{1+p}\right] + \frac{\eta}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)}) \\
= & \frac{1}{(1+\eta)\gamma}\left[\varphi(t_k) + bt_k - c - \varphi(s_{k-1}^{(m-1)}) - bs_{k-1}^{(m-1)} + c\right] + \frac{\eta}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)}) \\
= & \frac{1}{(1+\eta)\gamma}\varphi(t_k) + \frac{b+\eta-1}{(1+\eta)\gamma}\varphi(s_{k-1}^{(m-1)}) \\
\le & \frac{1}{(1+\eta)\gamma}\varphi(t_k)
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_k^{(1)} - x_k\| \le & \left\|I - T(\alpha; x_k)^{l_k^{(1)}}\right\|\left\|F'(x_k)^{-1}\right\|\left\|F(x_k)\right\| \\
\le & (1+\theta^{l_k^{(1)}})\frac{\gamma}{1-\gamma K t_k^p}\frac{1}{(1+\eta)\gamma}\varphi(t_k) \le (1+\eta)\frac{\gamma}{1-\gamma K t_k^p}\frac{1}{(1+\eta)\gamma}\varphi(t_k) \\
= & -\frac{\varphi(t_k)}{\psi(t_k)} = s_k^{(1)} - t_k\,.
\end{aligned}$$

Now we suppose the inequalities

$$\begin{cases}
\left\|F(x_k^{(i)})\right\| \le \dfrac{1}{(1+\eta)\gamma}\varphi(s_k^{(i)}), \\
\|x_k^{(i+1)} - x_k^{(i)}\| \le s_k^{(i+1)} - s_k^{(i)}, \quad i = 1, 2, \cdots, m-2
\end{cases} \tag{2.12}$$

hold for any positive integers less than $m-1$, and proceed to prove that the inequalities (2.12) hold for $m-1$. From the inequalities (2.4) and (2.5), we obtain

$$\begin{aligned}
\left\|F(x_k^{(m-1)})\right\| \le & \left\|F(x_k^{(m-1)}) - F(x_k^{(m-2)}) - F'(x_k)(x_k^{(m-1)} - x_k^{(m-2)})\right\| \\
& + \left\|F(x_k^{(m-2)}) + F'(x_k)(x_k^{(m-1)} - x_k^{(m-2)})\right\| \\
\le & \left\|F(x_k^{(m-1)}) - F(x_k^{(m-2)}) - F'(x_k^{(m-2)})(x_k^{(m-1)} - x_k^{(m-2)})\right\| \\
& + \left\|F'(x_k^{(m-2)}) - F'(x_k)\right\|\|x_k^{(m-1)} - x_k^{(m-2)}\| + \eta\left\|F(x_k^{(m-2)})\right\| \\
\le & \frac{K}{1+p}\|x_k^{(m-1)} - x_k^{(m-2)}\|^{1+p} + K\|x_k^{(m-2)} - x_k\|^p\|x_k^{(m-1)} - x_k^{(m-2)}\| \\
& + \eta\left\|F(x_k^{(m-2)})\right\|;
\end{aligned}$$

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and since

$$\begin{aligned}
&\|x_k^{(m-1)} - x_k^{(m-2)}\| \le s_k^{(m-1)} - s_k^{(m-2)},\\
&\|x_k^{(m-2)} - x_k\| \le \|x_k^{(m-2)} - x_k^{(m-3)}\| + \cdots + \|x_k^{(1)} - x_k\|\\
&\qquad\qquad \le (s_k^{(m-2)} - s_k^{(m-3)}) + \cdots + (s_k^{(1)} - t_k) \le s_k^{(m-2)} - t_k,\\
&\big\|F(x_k^{(m-2)})\big\| \le \frac{1}{(1+\eta)\gamma}\varphi(s_k^{(m-2)}),\\
&\big\|F(x_k^{(m-1)})\big\| \le \frac{K}{1+p}(s_k^{(m-1)} - s_k^{(m-2)})^{1+p} + K(s_k^{(m-2)} - t_k)^p(s_k^{(m-1)} - s_k^{(m-2)})\\
&\qquad\qquad + \frac{\eta}{(1+\eta)\gamma}\varphi(s_k^{(m-2)}).
\end{aligned}$$

Because $0 < t_k < s_k^{(m-2)} < t_*$ and we also have $(s_k^{(m-2)} - t_k)^p < (s_k^{(m-2)})^p$, hence

$$\begin{aligned}
\big\|F(x_k^{(m-1)})\big\| \le& \frac{K}{1+p}(s_k^{(m-1)} - s_k^{(m-2)})^{1+p} + K(s_k^{(m-2)})^p(s_k^{(m-1)} - s_k^{(m-2)}) + \frac{\eta}{(1+\eta)\gamma}\varphi(s_k^{(m-2)})\\
\le& \frac{K}{1+p}(s_k^{(m-1)})^{1+p} - \frac{K}{1+p}(s_k^{(m-2)})^{1+p} - K(s_k^{(m-2)})^p(s_k^{(m-1)} - s_k^{(m-2)})\\
&+ K(s_k^{(m-2)})^p(s_k^{(m-1)} - s_k^{(m-2)}) + \frac{\eta}{(1+\eta)\gamma}\varphi(s_k^{(m-2)})\\
=& \frac{K}{1+p}(s_k^{(m-1)})^{1+p} - \frac{K}{1+p}(s_k^{(m-2)})^{1+p} + \frac{\eta}{(1+\eta)\gamma}\varphi(s_k^{(m-2)})\\
=& \frac{1}{(1+\eta)\gamma}\left[\frac{a}{1+p}(s_k^{(m-1)})^p - \frac{a}{1+p}(s_k^{(m-2)})^p\right] + \frac{\eta}{(1+\eta)\gamma}\varphi(s_k^{(m-2)})\\
=& \frac{1}{(1+\eta)\gamma}\left[\varphi(s_k^{(m-1)}) + bs_k^{(m-1)} - c - \varphi(s_k^{(m-2)}) - bs_k^{(m-1)} + c\right]\\
&+ \frac{\eta}{(1+\eta)\gamma}\varphi(s_k^{(m-2)})\\
=& \frac{1}{(1+\eta)\gamma}\varphi(s_k^{(m-1)}) + \frac{b+\eta-1}{(1+\eta)\gamma}\varphi(s_k^{(m-2)})\\
\le& \frac{1}{(1+\eta)\gamma}\varphi(s_k^{(m-1)}).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\|x_{k+1} - x_k^{(m-1)}\| \le& \big\|I - T(\alpha; x_k)^{l_k^{(m)}}\big\|\big\|F'(x_k)^{-1}\big\|\big\|F(x_k^{(m-1)})\big\|\\
\le& (1+\theta^{l_k^{(m)}})\frac{\gamma}{1-\gamma K t_k^p}\frac{1}{(1+\eta)\gamma}\varphi(s_k^{(m-1)})\\
\le& (1+\eta)\frac{\gamma}{1-\gamma K t_k^p}\frac{1}{(1+\eta)\gamma}\varphi(s_k^{(m-1)})\\
=& -\frac{\varphi(s_k^{(m-1)})}{\psi(t_k)} = t_{k+1} - s_k^{(m-1)},
\end{aligned}$$

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so the formulas (2.10) are true for all k. Since sequences $\{t_k\}, \{s_k^{(1)}\}, \cdots, \{s_k^{(m-1)}\}$ converge to t_* and

$$\begin{aligned}\|x_{k+1}-x_0\| &\le \|x_{k+1}-x_k^{(m-1)}\| + \|x_k^{(m-1)}-x_k^{(m-2)}\| + \cdots + \|x_k^{(1)}-x_k^{(0)}\| + \|x_k-x_0\| \\ &\le (t_{k+1}-s_k^{(m-1)}) + (s_k^{(m-1)}-s_k^{(m-2)}) + \cdots + (s_k^{(1)}-t_k) + (t_k-t_0) \\ &= t_{k+1}-t_0 < t_* < r\,,\end{aligned}$$

such that the sequence $\{x_k\}$ also converges, say to x_*. Finally, since $\|T(\alpha; x_*)\| < 1$ we have $F(x_*) = 0$ from the iteration (1.8). □

3. Numerical Tests

We considered two examples to illustrate the effectiveness and robustness of the MMN-HSS approach.

Example 3.1 (Two-dimensional nonlinear convection-diffusion equation [6, 20])**.**

$$\begin{cases} -(u_{xx}+u_{yy}) + q_1 u_x + q_2 u_y = u^{3/2}, & (x,y) \in \Omega\,, \\ u(x,y) = 0\,, & (x,y) \in \partial\Omega\,., \end{cases}$$

where the region $\Omega = (0,1)\times(0,1)$ has the boundary $\partial\Omega$ and q_1, q_2 are positive constants used to measure the magnitudes of the convective terms.

Applying the centred finite difference scheme on the equidistant discretisation grid with stepsize $h = 1/(N+1)$, where N is a prescribed positive integer, we obtain the system of nonlinear equations (1.1) in the form:

$$F(x) = Mx + h^2\phi(x) = 0\,,$$

with $M = T_x \otimes I + I \otimes T_y$, $\phi(x) = (x_1^{3/2}, x_2^{3/2}, \cdots, x_n^{3/2})^T$ and the tridiagonal matrices

$$T_x = \text{tridiag}(-1-Re_1, 2, -1+Re_1) \quad \text{and} \quad T_y = \text{tridiag}(-1-Re_2, 2, -1+Re_2)\,.$$

In the above form, $Re_j = q_j h/2$ for $j = 1, 2$, $Re = \max\{Re_1, Re_2\}$ is the mesh Reynolds number, $\otimes$ denotes the Kronecker product, and $n = N \times N$. It is easy to see that

$$F'(x) = M + \frac{3}{2}h^2 \text{diag}\left(x_1^{\frac{1}{2}}, x_2^{\frac{1}{2}}, \cdots, x_n^{\frac{1}{2}},\right),$$

and if the initial point x_0 is fixed the parameters δ, β and γ can be estimated from the Assumption 1. Moreover, we obtain the inequality

$$\|F'(x) - F'(y)\| \le \frac{3}{2}h^2\|x-y\|^{\frac{1}{2}} \quad \text{for any } x, y \in \mathbb{N}(x_0, r)\,, \tag{3.1}$$

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Table 1: $\eta = 0.1$, $N = 30$.

q	Method	α	C	Error estimate	CPU time(s)	Outer IT	Inner IT
600	MN-HSS	9.6774	-0.3328	3.2833×10^{-5}	0.9587	3	59
	3MN-HSS	9.6774	-0.3328	3.2698×10^{-5}	0.7632	2	59
	4MN-HSS	9.6774	-0.3328	2.6070×10^{-7}	0.9506	2	73
800	MN-HSS	12.9032	-0.3330	4.0043×10^{-5}	1.0470	3	70
	3MN-HSS	12.9032	-0.3330	3.9899×10^{-5}	0.8567	2	70
	4MN-HSS	12.9032	-0.3330	2.4020×10^{-7}	1.0969	2	88
1000	MN-HSS	16.1290	-0.3331	8.5361×10^{-5}	1.1307	3	78
	3MN-HSS	16.1290	-0.3331	8.5173×10^{-5}	0.9398	2	78
	4MN-HSS	16.1290	-0.3331	6.8376×10^{-7}	1.1432	2	98

where $\|\cdot\|$ denotes the 2-norm; and from inequality (3.1) the nonlinear system does not satisfy Lipschitz conditions, but the Hölder conditions are satisfied with $K = (3/2)h^2$ and $p = 1/2$.

In our computation, we adopt the positive constants $q_1 = q$, $q_2 = 1/h$ and $\alpha = qh/2$, where q is a given constant. The parameters θ and τ can be estimated from the definition of $\| T(\alpha; x_0) \|$ and the estimation of $\| T(\alpha; x) \|$, respectively; and the initial point x_0 is chosen to be **e**, the n-vector with all entries 1. With a 30×30 uniform grid discretisation and $\eta = 0.1$, the results for the modified Newton-HSS (MN-HSS), the three-step modified Newton-HSS (3MN-HSS) and the four-step modified Newton-HSS (4MN-HSS) methods are shown in Table 1 when $q = 600$, $q = 800$ and $q = 1000$, respectively. The results for a 40×40 uniform grid and $\eta = 0.1$ are likewise shown in Table 2, and for a 30×30 uniform grid and $\eta = 0.2$ in Table 3. (The results for a 40×40 uniform grid and $\eta = 0.2$ are similar to the results for a 30×30 uniform grid and $\eta = 0.2$, so not displayed.) In all of these calcultions,

$$C = \left(\frac{(1+\eta)\gamma}{1-\eta} \right)^{\frac{1+p}{p}} K^{\frac{1}{p}} \delta - \frac{p}{1+p} < 0$$

means the function satisfies inequality (2.1), so the MMN-HSS methods are appropriate to solve the nonlinear equations numerically. From Tables 1-3, we observe that:

- in general, the MMN-HSS methods including 3MN-HSS and 4MN-HSS are better than the MN-HSS method (cf. the CPU times);
- the 3MN-HSS method is better when η is small (cf. Tables 1 and 2); and
- the 4MN-HSS method is better when η is large and q is small (cf. Table 3).

Example 3.2. As the 3MN-HSS methods have been proved to be better in Example 3.1, six types of $\eta_k^{(i)}$ further test the algorithm. In particular, norm values between 0 to 10^{-14} with step 10^{-1} are chosen to compute the corresponding total numbers of inner iterations, outer iterations and CPU times — and if three norm values need the same numbers of outer iterations, the corresponding parts of the curve should be parallel to the vertical axis.

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Table 2: $\eta = 0.1,\ N = 40.$

q	Method	α	C	Error estimate	CPU time(s)	Outer IT	Inner IT
600	MN-HSS	7.3171	-0.3326	2.5112×10^{-5}	2.8233	3	58
	3MN-HSS	7.3171	-0.3326	2.5004×10^{-5}	2.2799	2	58
	4MN-HSS	7.3171	-0.3326	1.7292×10^{-7}	2.7408	2	71
800	MN-HSS	9.7561	-0.3329	5.2707×10^{-5}	2.9619	3	66
	3MN-HSS	9.7561	-0.3329	5.2665×10^{-5}	2.4051	2	66
	4MN-HSS	9.7561	-0.3329	3.0649×10^{-7}	2.9963	2	82
1000	MN-HSS	12.1951	-0.3330	5.9827×10^{-5}	3.4201	3	75
	3MN-HSS	12.1951	-0.3330	5.9714×10^{-5}	2.7357	2	75
	4MN-HSS	12.1951	-0.3330	3.9597×10^{-7}	3.2716	2	93

Table 3: $\eta = 0.2,\ N = 30.$

q	Method	α	C	Error estimate	CPU time(s)	Outer IT	Inner IT
600	MN-HSS	9.6774	-0.3323	6.4863×10^{-5}	1.3500	4	57
	3MN-HSS	9.6774	-0.3323	1.1890×10^{-5}	1.1312	3	62
	4MN-HSS	9.6774	-0.3323	6.4636×10^{-5}	0.9039	2	57
800	MN-HSS	12.9032	-0.3327	9.2682×10^{-5}	1.3114	4	67
	3MN-HSS	12.9032	-0.3327	1.7194×10^{-5}	1.2174	3	73
	4MN-HSS	12.9032	-0.3327	9.2241×10^{-5}	0.9377	2	67
1000	MN-HSS	16.1290	-0.3329	4.8373×10^{-6}	1.7206	5	90
	3MN-HSS	16.1290	-0.3329	2.6711×10^{-5}	1.3209	3	83
	4MN-HSS	16.1290	-0.3329	1.6832×10^{-7}	1.5940	3	104

Choice 1: For $\bar{\rho} = (\sqrt{5}+1)/2$ and any $\eta_0 \in [0, 2/3)$, let

$$\eta_k^{(i)} = \frac{\| \| F(x_k^{(i)}) \| - \| F(x_k^{(i)}) + F'(x_k^{(0)}) d_k^{(i+1)} \| \|}{\| F(x_k^{(i-1)}) \|}, \quad i = 1, 2, \cdots, m, \quad k = 1, 2, \cdots.$$

Choice 1 safeguards: $\eta_k^{(i)} = \max\{\eta_k^{(i)}, (\eta_k^{(i-1)})^{\bar{\rho}}\}$ wherever $(\eta_k^{(i-1)})^{\bar{\rho}} > 0.1$.

Choice 2: Given $\lambda \in [0, 1)$ and $\rho \in (1, 2]$, select any $\eta_0 \in [0, 2/3)$ and

$$\eta_k^{(i)} = \lambda \left(\frac{\| F(x_k^{(i)}) \|}{\| F(x_k^{(i-1)}) \|} \right)^{\rho}, \quad i = 1, 2, \cdots, m, \quad k = 1, 2, \cdots.$$

Choice 2 safeguards: $\eta_k^{(i)} = \max\{\eta_k^{(i)}, \lambda(\eta_k^{(i-1)})^{\bar{\rho}}\}$ wherever $\lambda(\eta_k^{(i-1)})^{\bar{\rho}} > 0.1$.

In both cases, after applying the relevant safeguard mentioned above another safeguard is necessary [28]:

$$\eta_k \leftarrow \min\{\eta_k, \eta_{max}\}.$$

Further, when $\eta_k \|F(x_k)\|$ is small enough, a larger η_k can be used for the outer iteration, and Pernice & Walker [28] imposed the following final safeguard:

$$\eta_k \leftarrow 0.8\varepsilon / \left\|F(x_k)\right\| \quad \text{whenever} \quad \eta_k \le 2\varepsilon / \left\|F(x_k)\right\|.$$

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In all, six types of forcing term will be considered:

- Choice 1 denotes Choice 1 with Choice 1 safeguard, additional safeguard and final safeguard;
- Choice 2 denotes Choice 2 with Choice 2 safeguard, additional safeguard and final safeguard;
- Choice 3 denotes $\eta_k = 0.1$ for all k;
- Choice 4 denotes $\eta_k = 0.0001$ for all k;
- Choice 5 denotes $\eta_k^{(i)} = \frac{|\|F(x_k^{(i)})\| - \|F(x_k^{(i)}) + F'(x_k^{(0)})d_k^{(i+1)}\||}{\|F(x_k^{(i)})\|}$, $i = 1, 2, \cdots, m$, $k = 1, 2, \cdots$, with Choice 1 safeguard, additional safeguard and final safeguard;
- Choice 6 denotes the new forcing term in Ref. [2].

Note: Choice 5 is a new option represented in Ref. [23].

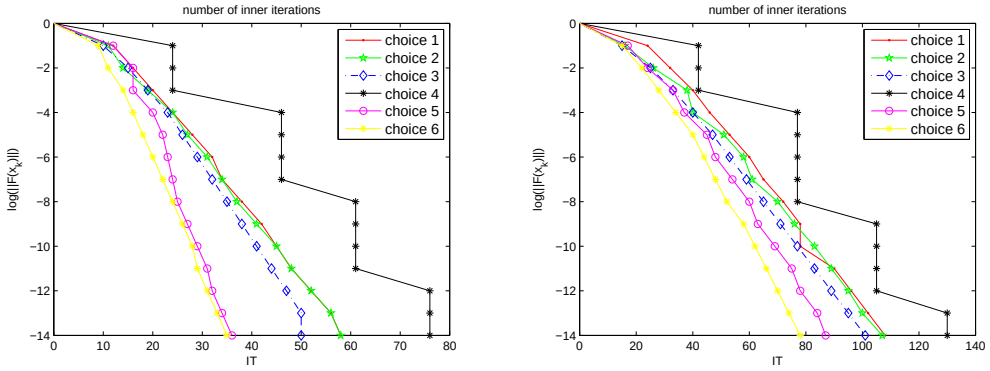
We set $\eta_0 = 0.1$ and $\eta_{max} = 0.6$ for safeguards in Choices 1, 2 and 5; and $\lambda = 1$ and $\rho = (1 + \sqrt{5})/2$ in Choice 2, as they are more effective [19]. Discretize on a 30×30 uniform grid, so the dimension in the implementing the 3MN-HSS method is $n = 900$. The results are illustrated in the following six figures — each showing $\log \|F(x_k)\|$ on the vertical axis versus the total number of inner iterations (denoted as "IT"), the total numbers of outer iterations (denoted as "OT") and the total CPU times on the horizontal axis. In every figure, results for two values of q (200 and 600) are shown, and we use $x_0 = 0.1\mathbf{e}$ in the first three figures and $x_0 = \mathbf{e}$ in the last three figures. We choose the two points because the computed solution is near 0 — i.e. far away from the accurate solution.

In Fig. 1 we observe that fewer iterations (IT) are needed for Choices 1, 2, 3, 5 and 6 and the poorest performer is Choice 4. However, Choice 6 requires the largest number of outer iterations (cf. Fig. 2), and the CPU times generally have the same behaviour as the number of outer iterations (cf. Fig. 3). When the starting point is $\mathbf{e}$, more inner iterations are needed than with $x_0 = 0.1\mathbf{e}$. The numbers of outer iterations remain the main influence on the CPU time (cf. Figs. 4, 5 and 6). Although Choice 3 performs relatively worse, we see that the 3MN-HSS method is generally better than the other methods in Example 3.1 (the results shown in Tables 1-3), so we consider that the 3MN-HSS methods have superior performance when suitable parameters are used.

4. Conclusions

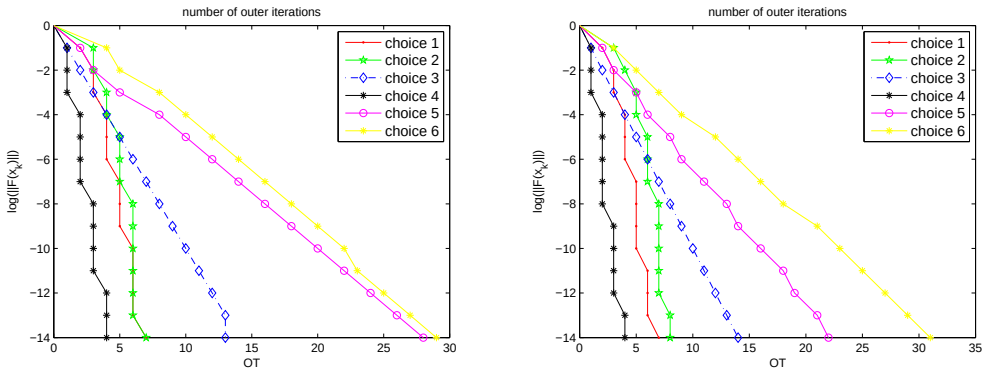
In some cases, the derivatives in systems of nonlinear equations do not satisfy the Lipschitz conditions, so in this article we propose and prove a semilocal convergence theorem under Hölder continuity conditions. Furthermore, numerical experiments indicate that the three-step MN-HSS method outperforms the MN-HSS method under the Hölder conditions, as found previously under Lipschitz conditions, and the three-step MN-HSS method is tested for six different types of forcing term. Local and global convergence analyses under the Hölder conditions are yet to be considered.

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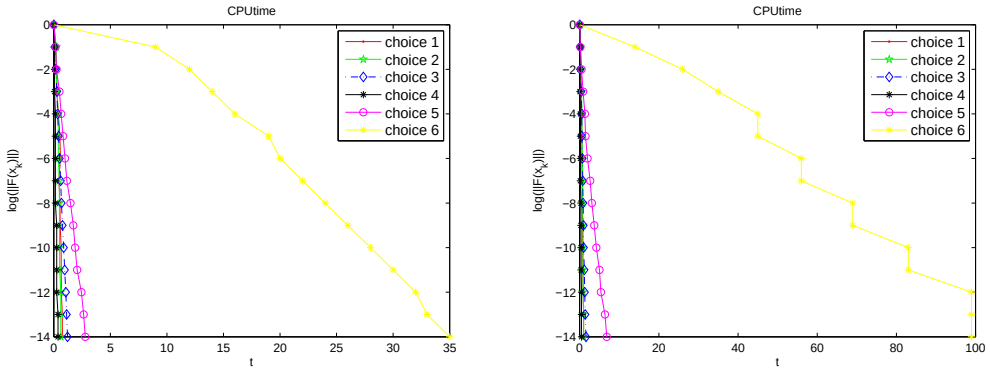
(a) $N = 30$, $q = 200$ (b) $N = 30$, $q = 600$

Figure 1: The total numbers of inner iterations versus the norms of the nonlinear function values when $q = 200$ and $q = 600$, respectively, with $x_0 = 0.1\mathbf{e}$.



(a) $N = 30$, $q = 200$ (b) $N = 30$, $q = 600$

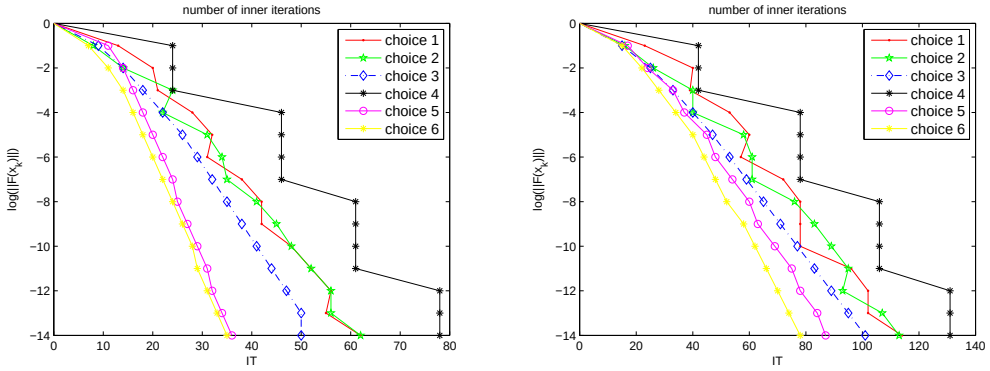
Figure 2: The total numbers of outer iterations versus the norms of the nonlinear function values when $q = 200$ and $q = 600$, respectively, with $x_0 = 0.1\mathbf{e}$.



(a) $N = 30$, $q = 200$ (b) $N = 30$, $q = 600$

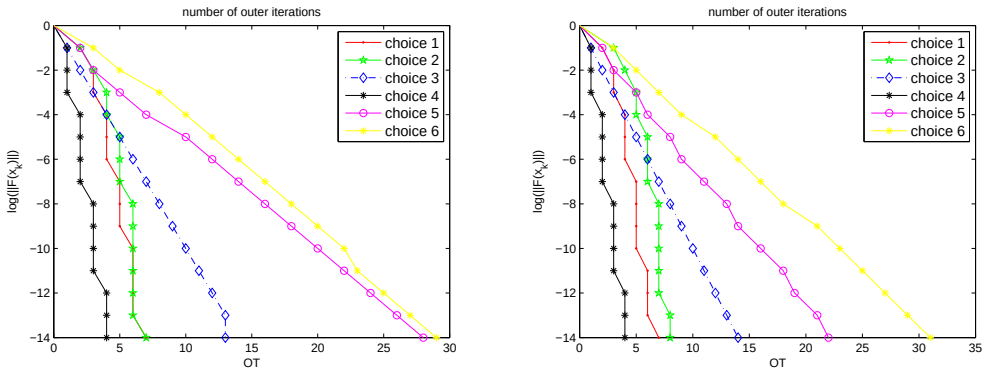
Figure 3: CPU times versus the norms of the nonlinear function values when $q = 200$ and $q = 600$, respectively, with $x_0 = 0.1\mathbf{e}$.

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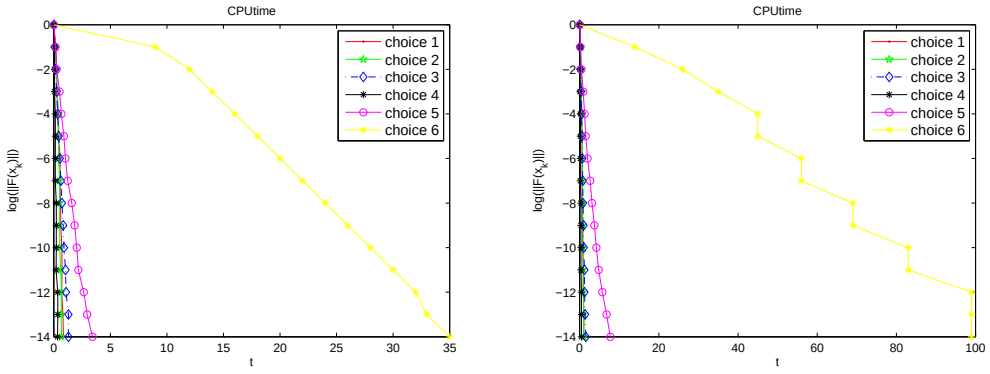
(a) $N = 30$, $q = 200$ (b) $N = 30$, $q = 600$

Figure 4: The total numbers of inner iterations versus the norms of the nonlinear function values when $q = 200$ and $q = 600$, respectively, with $x_0 = \mathbf{e}$.



(a) $N = 30$, $q = 200$ (b) $N = 30$, $q = 600$

Figure 5: The total numbers of outer iterations versus the norms of the nonlinear function values when $q = 200$ and $q = 600$, respectively, with $x_0 = \mathbf{e}$.



(a) $N = 30$, $q = 200$ (b) $N = 30$, $q = 600$

Figure 6: CPU times versus the norms of the nonlinear function values when $q = 200$ and $q = 600$, respectively, with $x_0 = \mathbf{e}$.

https://doi.org/10.4208/eajam.260416.270217a Published online by Cambridge University Press

Acknowledgments

The authors are very much indebted to the referees for their valuable comments and suggestions which greatly improved the original version of this paper. This research is partly supported by the National Natural Science Foundation of China (No. 11371145, No. 11471122), and the Science and Technology Commission of Shanghai Municipality (Grant No. 13dz2260400).

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