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OPTIMAL ALLOCATION POLICIES FOR RELEVATIONS IN COHERENT SYSTEMS

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Abstract

The relevation model is a fundamental tool in reliability engineering for assessing the effectiveness of redundancy allocation in coherent systems. In this study, we address the problem of allocation of relevations for one or two nodes in a coherent system with independent components to enhance system reliability. We establish results concerning the usual stochastic and hazard rate orders for coherent systems. Moreover, we illustrate our findings with a range of examples and counterexamples. In addition, we conduct a simulation-based study and a real data analysis to further illustrate the application of our results. Lastly, we study the case of the minimal repair policy in detail.

Keywords: Coherent system; minimal repair; relevation; stochastic orders

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1. Introduction

Many real-world systems are both expensive and highly complex. As a result, system failures can lead to significant consequences. Therefore, a key challenge in reliability theory is determining how to extend the system's lifetime. This issue is commonly addressed by incorporating one or more redundant components (spares) into the system. Redundancy allocation is an effective strategy for improving the lifetime of a system. Usually, two common methods of redundancy allocation are utilized to enhance the reliability of the system: active redundancy and standby redundancy. Active redundancy allocation involves assigning redundancy to a component so that both the original and redundant counterpart operate simultaneously. In contrast, standby redundancy allocation connects the redundancy such that it only activates when the original component fails. A significant amount of research has been done on these two allocations strategies.

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The redundancy allocation methods discussed above may not always be feasible due to the possible increase in overall load and various economic considerations in certain scenarios [10]. As an alternative, relevation allocation may be a more effective option than traditional active and standby redundancies. The concept of relevation was initially introduced by Krakowski [20]. The relevation model is commonly used in reliability engineering to describe the effectiveness of redundancy allocation in coherent systems. Its detailed mathematical definition is as follows: consider two non-negative independent random variables T and S having survival functions $\overline{F}_T$ and $\overline{F}_S$, then the survival function of the relevation operation between T and S (denoted as $T\#S$) is given as [20]

$$\overline{F}_{T\#S}(t) = \overline{F}_T(t) + \overline{F}_S(t) \int_0^t \frac{1}{\overline{F}_S(x)} \, \mathrm{d}F_T(x), \quad \text{for } t > 0. \tag{1}$$

To understand this, let us consider two items with independent non-negative random lifetimes T and S. Once the item with lifetime T fails, it is replaced by another item with lifetime S such that it has the same age as that of the failed item. Then, the survival for the time until the second failure is represented by (1) [11, 34]. Due to its practical significance, relevation is applied across a range of real-world situations. We give two examples for relevation allocation.

Example 1. In a car, a machine part fails after several months of operation. Rather than replacing the entire machine, a new part of the same age, made with upgraded materials, is installed. Though similar in function to the original, this new part follows a different lifetime distribution than the old one. Consequently, the machine's overall distribution changes after the new part is installed. Thus, the lifetime of the machine can be represented by relevation transformation.

Example 2. In a heating, ventilation, and air conditioning (HVAC) system, a component such as a fan motor fails after prolonged use. Rather than replacing the entire HVAC unit, a new fan motor of the same age but made from more heat-resistant materials is installed. While similar in function to the original, this new part follows a different lifetime distribution from the old one. As a result, the distribution of the HVAC system changes after the new fan motor is installed.

Before describing the relevant works in the literature, let us first review some fundamental concepts and definitions related to coherent systems. A system is considered coherent if all components are relevant, and whenever a failed component is replaced by a good one, the system lifetime does not decrease [12]. Series, parallel, and k-out-of-n systems are examples of coherent systems. Consider a coherent system consisting of n independent components with structure function τ. Assume the components are labeled by the numbers $1, 2, \ldots, n$, and for each component i, its lifetime is denoted by T_i, $i = 1, \ldots, n$. A path set $\mathcal{P}$ is defined as a collection of components whose functioning can guarantee the functioning of the system, and a cut set $\mathcal{C}$ is defined as a collection of components whose failures cause the failure of the system. Specifically, a path set $\mathcal{P}$ is called a minimal path set if any of its proper subsets is not a path set, and a cut set $\mathcal{C}$ is called a minimal cut set if no proper subset is a cut set. According to Barlow and Proschan [5], the system lifetime can be expressed as

$$\tau(\boldsymbol{T}) = \tau\,(T_1, \ldots, T_n) = \max_{\mathcal{P} \in \mathcal{A}} \min_{j \in \mathcal{P}} T_j = \min_{C \in \mathcal{S}} \max_{j \in \mathcal{C}} T_j.$$

Suppose $\mathcal{A}$ denotes the class of all minimal path sets, and $\mathcal{S}$ denotes the class of all minimal cut sets. In addition, let $\mathcal{A}_i = \{\mathcal{I} \in \mathcal{A}, i \in \mathcal{I}\}$ be the collection of minimal path sets in which each minimal path set contains component i, and $\mathcal{S}_i = \{\mathcal{J} \in \mathcal{S}, i \in \mathcal{J}\}$ be the collection of minimal cut sets in which each minimal cut set contains component i. Then, for any two components i and j, their structural relationship can be classified into the following six types [30, 34]:

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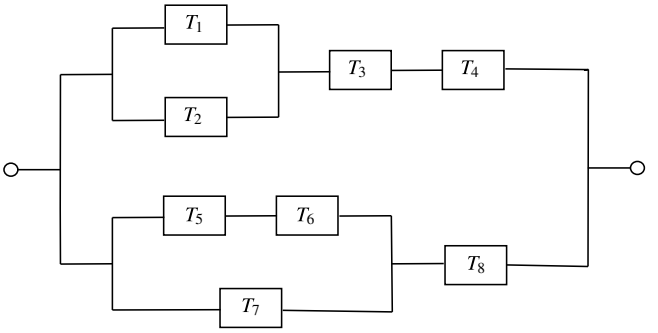


FIGURE 1. Schematic diagram of an eight-component system.

(i) $\mathcal{A}_i = \mathcal{A}_j$, i.e. components i and j are included in same minimal path sets;

(ii) $\mathcal{S}_i = \mathcal{S}_j$, i.e. components i and j are included in same minimal cut sets;

(iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$, i.e. the minimal path sets including component i is a subset of those including component j;

(iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$, i.e. the minimal cut sets including component i is a subset of those including component j;

(v) $\mathcal{A}_i$ and $\mathcal{A}_j$ do not contain each other;

(vi) $\mathcal{S}_i$ and $\mathcal{S}_j$ do not contain each other.

Zhang and Zhang [34, p. 5] mentioned, 'It is worth noting that some of the above six settings might occur simultaneously'. In light of this, an example of a system structure that satisfies cases (i)–(vi) is as follows: consider an eight-component coherent system with component lifetimes $T_1, T_2, \ldots, T_8$, as depicted in Figure 1. It is easy to verify that $\mathcal{A}_1 = \{\{1, 3, 4\}\}$, $\mathcal{A}_2 = \{\{2, 3, 4\}\}$, $\mathcal{A}_3 = \{\{1, 3, 4\}, \{2, 3, 4\}\}$, $\mathcal{A}_4 = \{\{1, 3, 4\}, \{2, 3, 4\}\}$, $\mathcal{A}_5 = \{\{5, 6, 8\}\}$, $\mathcal{A}_6 = \{\{5, 6, 8\}\}$, $\mathcal{A}_7 = \{\{7, 8\}\}$, $\mathcal{A}_8 = \{\{7, 8\}\}$, $\{5, 6, 8\}\}$, $\mathcal{S}_1 = \{\{1, 2, 5, 7\}, \{1, 2, 6, 7\}, \{1, 2, 8\}\}$, $\mathcal{S}_2 = \{\{1, 2, 5, 7\}, \{1, 2, 6, 7\}, \{1, 2, 8\}\}$, $\mathcal{S}_3 = \{\{3, 5, 7\}, \{3, 6, 7\}, \{3, 8\}\}$, $\mathcal{S}_4 = \{\{4, 5, 7\}, \{4, 6, 7\}, \{4, 8\}\}$, $\mathcal{S}_5 = \{\{1, 2, 5, 7\}, \{3, 5, 7\}, \{4, 5, 7\}\}$, $\mathcal{S}_6 = \{\{1, 2, 6, 7\}, \{3, 6, 7\}, \{4, 6, 7\}\}$, and $\mathcal{S}_7 = \{\{1, 2, 5, 7\}, \{1, 2, 6, 7\}, \{3, 5, 7\}, \{3, 6, 7\}, \{4, 5, 7\}, \{4, 6, 7\}\}$, $\mathcal{S}_8 = \{\{3, 8\}, \{4, 8\}\}$. Here, $\mathcal{A}_1 = \{\{1, 3, 4\}\}$ denotes the minimal path sets containing component 1, $\mathcal{S}_1 = \{\{1, 2, 5, 7\}, \{1, 2, 6, 7\}, \{1, 2, 8\}\}$ denotes the minimal cut sets containing component 1, and so on. Observe that $\mathcal{A}_5 = \mathcal{A}_6$, $\mathcal{S}_1 = \mathcal{S}_2$, $\mathcal{A}_1 \subseteq \mathcal{A}_3$, $\mathcal{S}_5 \subseteq \mathcal{S}_7$, whereas $\mathcal{A}_1$ and $\mathcal{A}_2$ do not contain each other (i.e. $\mathcal{A}_1 \nsubseteq \mathcal{A}_2$ and $\mathcal{A}_2 \nsubseteq \mathcal{A}_1$), and neither do $\mathcal{S}_1$ and $\mathcal{S}_2$ (i.e. $\mathcal{S}_2 \nsubseteq \mathcal{S}_3$ and $\mathcal{S}_3 \nsubseteq \mathcal{S}_2$).

An intriguing decision problem for system design engineers is determining how to allocate a given number of redundant components to a coherent system to maximize the system's overall lifetime. In this regard, a few studies have been presented that investigate the optimal allocation policy for one or two revelations to a coherent system. Belzunce *et al.* [10] studied the optimal allocation policy for one relevation performed to a series (parallel) system and then generalized it to a coherent system by assuming that the components where relevation is performed share similar minimal path (minimal cut) sets (i.e. for the cases: (i) $\mathcal{A}_i = \mathcal{A}_j$ and (ii) $\mathcal{S}_i = \mathcal{S}_j$, where $\mathcal{A}_i$, $\mathcal{A}_j$, $\mathcal{S}_i$, and $\mathcal{S}_j$ are as defined previously). For this purpose, the authors considered a series system with two components having lifetimes T_1 and T_2. Suppose two relevations

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with lifetimes S_1 and S_2 are available, but due to the budget constraint, we can perform only one relevation at a time. Then, the authors demonstrated that if $T_1 \leq_{hr} T_2$ and $S_1 \geq_{hr} S_2$, then $U_1 \geq_{st} U_2$, where $U_1 = \min\{T_1 \# S_1, T_2\}$ with $U_2 = \min\{T_1, T_2 \# S_2\}$. They further extended this result to the coherent system with n components and showed that for components i and j in common path sets, the relevation should be allocated to the component with the higher hazard rate. In contrast, for components i and j in common cut sets, the relevation should be allocated to the component with the higher reversed hazard rate. Later, Wu *et al.* [30] complemented the results of [10] by considering one relevation allocation problem to a coherent system for cases (iii)–(vi), and compared system lifetimes in the sense of usual stochastic order. Further, under the scenario where there is no budget issue and both the relevations can be performed at the same time (i.e. two possible allocations policies: either S_1 is attached with T_i and S_2 with T_j, or S_2 is attached with T_i and S_1 with T_j), Yan *et al.* [31] studied the allocation of two relevations to a coherent system simultaneously. They extended the work of [10] by applying two simultaneous relevations in a coherent system, under the condition that the components where relevation is allocated share similar minimal path sets, i.e. for case (i) $\mathcal{A}_i = \mathcal{A}_j$. Recently, Zhang and Zhang [34] have studied the optimal allocation policy for two relevations to a coherent system for cases (iii)–(vi) in terms of minimal path sets and cut sets. They made the stochastic comparison of coherent systems in the sense of the usual stochastic order.

From the aforementioned discussion, we make the following observations:

(a) very little work has been done so far in the literature on relevation allocation to coherent systems;

(b) most of the works have established the usual stochastic order between system lifetimes;

(c) although the allocation of two relevations (performed one at a time) to a coherent system has garnered the attention of researchers for cases (i)–(vi) [10, 30], the same does not hold true for the situation when two relevations are allocated (both at the same time) to a coherent system [31, 34].

In this study, we aim to address these gaps in the literature. With regards to observation (a), we establish results on the stochastic comparison of system lifetimes after allocating two relevations (one at a time and both simultaneously) to a coherent system. Next, to address observation (b), we stochastically compare system lifetimes with respect to hazard rate order after allocations of two relevations (performed one at a time) (see Theorems 1 and 2). By doing so, we extend the results of [10] and also answer the call for future research on hazard rate order. Finally, with regards to observation (c), we note that stochastic comparison results were established in [34] in the case of two relevations (performed simultaneously) on the series system. Thus, in this study, we address the gap on two relevation allocations (performed simultaneously) to a series system (by adding some more sufficient conditions in the literature) and a coherent system for case (i) $\mathcal{A}_i = \mathcal{A}_j$ (see Theorems 3 and 4). In addition, by building on the result for two relevation allocation to a parallel system (performed simultaneously), we provide stochastic comparison results in the case of two relevations (performed simultaneously) on the coherent system for case (ii) $\mathcal{S}_i = \mathcal{S}_j$, and also provide additional sets of sufficient conditions for cases (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$, and (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$ (see Theorems 4–7 and 8). By obtaining new findings on two relevations allocations (performed one at a time or both at the same time), we not only extend the results of [10, 30], but also generalize the results of [34]. In particular, for cases (i) $\mathcal{A}_i = \mathcal{A}_j$ and (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$, we provided additional sets of sufficient conditions to those obtained in [34], and we also addressed the conjecture in Theorem 2 of Zhang and Zhang [34]

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by employing our Theorem 6 and establishing the usual stochastic order for two relevations allocation (performed both at the same time) to a coherent system for case (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$.

As an application of our main results, we study the optimal allocation policy for two minimal repairs (performed one at a time) in a series system and a coherent system for case (i) $\mathcal{A}_i = \mathcal{A}_j$, and make the stochastic comparison in sense of hazard rate order (see Theorems 9 and 10). For the case of multiple minimal repairs, the majority of the results focused either on identical components or on the same number of minimal repairs performed to series and coherent systems (see, for example, [3, 24, 25]). With this in mind, we concentrate on determining the optimal allocation policy for multiple minimal repairs (which may vary in number) on a single node (or component) within coherent systems composed of nonidentical components. To fill these gaps for multiple minimal repairs, we provide the results for coherent systems in cases (i) $\mathcal{A}_i = \mathcal{A}_j$ and (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$ by employing the result for a series system (see Theorems 11, 12, and 13). In summary, the key contributions of this paper are as follows.

- This work explores the optimal allocation policy for two relevation performed (one at a time and both at a time) to a series or parallel system and then later generalized to a general coherent system for cases (i)–(iv) in the sense of the usual stochastic order and the hazard rate order.
- We study the optimal allocation policy for two minimal repairs (performed one at a time) and multiple minimal repairs to a series system and a coherent system for cases (i) $\mathcal{A}_i = \mathcal{A}_j$ and (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$ in the sense of the usual stochastic order and the hazard rate order.

The rest of the paper is structured as follows. The next section introduces several definitions and lemmas that will be instrumental in proving our main results. In Section 3, we provide our main results. Specifically, we examine the allocation problems of relevations for one (or two) node (or nodes) in a coherent system with independent components to enhance the system's reliability. We have established several results concerning the usual stochastic order and hazard rate order for a coherent system. Next, Section 4 presents an application of our results through a simulation-based study and an analysis of real data. In addition, we study the case of the minimal repair policy in detail (see Subsection 4.3). Finally, in Section 5 we conclude our study.

2. Preliminaries and auxiliary results

Let X and Y be non-negative independent random variables with distribution functions F_X and F_Y, density functions f_X and f_Y, and survival functions $\overline{F}_X$ and $\overline{F}_Y$, respectively. Throughout the paper, all random variables under consideration are assumed to be mutually independent. Moreover, 'increasing' refers to 'nondecreasing', and 'decreasing' refers to 'nonincreasing'. In addition, $X \overset{d}{=} Y$ denotes that X and Y are identically distributed.

We first present the definitions of some commonly used stochastic orders, which are employed in this study. Stochastic orders are well-known to be useful for comparing the random lifetimes of two systems.

Definition 1. *A random variable X is said to be smaller than random variable Y in the:*

(i) *usual stochastic order (denoted by $X \leq_{st} Y$) if $\overline{F}_X(t) \leq \overline{F}_Y(t)$, for all $t \in [0, \infty)$;*

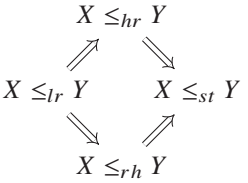
(ii) *hazard rate order (denoted by $X \leq_{hr} Y$) if $\overline{F}_Y(t)/\overline{F}_X(t)$ is increasing in $t \in [0, \infty)$;*

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(iii) *reversed hazard rate order (denoted by* $X \leq_{rh} Y$*) if* $F_Y(t)/F_X(t)$ *is increasing in* $t \in [0, \infty)$*;*

(iv) *likelihood ratio order (denoted by* $X \leq_{lr} Y$*) if* $f_Y(t)/f_X(t)$ *is increasing in* $t \in [0, \infty)$*.*

For a more detailed discussion on stochastic orders, readers can consult [7, 23, 29]. The following diagram depicts the chain of implications among the aforementioned stochastic orders.



The article also requires the idea of totally positive (reverse regular) functions of order 2 ($TP_2(RR_2)$). In the domain of statistical reliability theory, totally positive (reverse regular) functions of order 2 ($TP_2(RR_2)$) naturally emerge when analyzing the reliability of coherent systems. Karlin made substantial contributions to this field, laying its foundations within probability and statistical theory. In 1968, he authored a seminal book on the subject, which spurred numerous advancements. Karlin [19] defined a $TP_2(RR_2)$ function as follows.

Definition 2. *Let* S_1 *and* S_2 *be two subsets of the real line* $\mathbb{R}$*. A non-negative function* $k : S_1 \times S_2 \to [0, \infty)$ *is said to be* TP_2 (RR_2) *if, for* $x_1 \leq x_2, y_1 \leq y_2 (x_i \in S_1; y_i \in S_2,\ i = 1, 2)$*,*

$$k(x_1, y_1)\, k(x_2, y_2) \geq (\leq) k(x_1, y_2)\, k(x_2, y_1)\,.$$

The following lemma is the improvement of Lemma 3(iii) of [13], in which we show that condition $g_1(t) \geq g_2(t)$ is not needed to prove $(g(t) + g_1(t))/(g(t) + g_2(t))$ is increasing in $t > 0$.

Lemma 1. *Suppose real functions* g_1*,* g_2*, and* g *such that* $g_1(t)$*,* $g_2(t) \geq 0$*, and* $g(t) \geq 0$ *for all* $t > 0$*. If both* $g_1(t)/g_2(t)$ *and* $g(t)/(g_2(t) - g_1(t))$ *are increasing in* t*, then* $(g(t) + g_1(t))/(g(t) + g_2(t))$ *is also increasing in* $t > 0$*.*

Proof. The proof is given in the Appendix. □

The following lemma is due to Misra and Naqvi [22], which is very useful for proving our main results.

Lemma 2. *Let* $h_i(x, \theta) : [0, \infty) \times [0, \infty) \to \mathbb{R}$*,* $i = 1, 2$*, be a function and* $g_i(\theta)$*,* $i = 1, 2$*, be the Lebesgue probability distribution function of the random variable* Θ_i*,* $i = 1, 2$*. Let*

$$K(x) = \frac{\int_0^\infty h_2(x, \theta)\, \mathrm{d}G_2(\theta)}{\int_0^\infty h_1(x, \theta),\ \mathrm{d}G_1(\theta)}.$$

Now, we define the following two pairs of conditions for the positive function h_i*,* $i = 1, 2$*: (i)* $h_1(x, \theta)$ *or* $h_2(x, \theta)$ *is* TP_2 *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$*; (ii)* $h_2(x, \theta)/h_1(x, \theta)$ *increasing in* $x \in (0, \infty)$ *and* $\theta \in (0, \infty)$*; (iii)* $h_1(x, \theta)$ *or* $h_2(x, \theta)$ *is* RR_2 *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$*; and (iv)* $h_2(x, \theta)/h_1(x, \theta)$ *decreasing in* $x \in (0, \infty)$ *and increasing in* $\theta \in (0, \infty)$*.*

Suppose that conditions (i) and (ii) ((iii) and (iv)) hold. Further, if any of the following three conditions hold:

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(a) $\Theta_1 \leq_{lr} \Theta_2$;

(b) $\Theta_1 \leq_{hr} \Theta_2$ *and* $h_1(x, \theta)$ *or* $h_2(x, \theta)$ *is increasing in* $\theta \in (0, \infty)$;

(c) $\Theta_1 \leq_{rh} \Theta_2$ *and* $h_1(x, \theta)$ *or* $h_2(x, \theta)$ *is decreasing in* $\theta \in (0, \infty)$.

Then the function $K(x)$ *is increasing (decreasing) in* $x \in (0, \infty)$ (*or, equivalently,* $\int_0^\infty h_i(x, \theta) g_i(\theta)\, \mathrm{d}\theta$ *is* $TP_2(RR_2)$ *in* $(i, x) \in \{1, 2\} \times [0, \infty)$).

3. Main results

In this section, we consider two 2-component series (parallel) systems and establish optimal allocations of two relevations to these two series (parallel) systems. Moreover, we also study the optimal allocation policy for a general coherent system in the sense of various stochastic orderings, for cases (i)–(iv).

3.1. Allocation of two relevations in cases: (i) $\mathcal{A}_i = \mathcal{A}_j$ and (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$

In this subsection, we discuss the optimal allocation policies of one and two relevations for a series system and later generalize them to a coherent system in the sense of usual stochastic order and hazard rate order, for the cases: (i) $\mathcal{A}_i = \mathcal{A}_j$ and (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$.

Suppose that a series system has two components with lifetimes T_1 and T_2, and two relevations are available having lifetimes S_1 and S_2. However, due to time and budget issues, suppose one can perform only one relevation at a time to the series system. That is, we are interested in the comparison between $U_1 = \min\{T_1 \# S_1, T_2\}$ and $U_2 = \min\{T_1, T_2 \# S_2\}$, where the distribution function, survival function, and density function of a random variable U_1 (U_2) are F_{U_1} (F_{U_2}), $\overline{F}_{U_1}$ ($\overline{F}_{U_2}$), and f_{U_1} (f_{U_2}), respectively. Suppose that the component lifetimes and the available relevation lifetimes are mutually independent. Belzunce *et al.* [10] proved that if $T_1 \leq_{hr} T_2$ and $S_1 \geq_{hr} S_2$ then $U_1 \geq_{st} U_2$. One natural question arises that under what sufficient conditions $U_1 \geq_{st} U_2$ can be strengthened to $U_1 \geq_{hr} U_2$? To answer this question, we provide the following result on the optimal allocation policy of two relevations (performed one at a time) for hazard rate order in the series system.

Theorem 1. *Let* $U_1 = \min\{T_1 \# S_1, T_2\}$ *and* $U_2 = \min\{T_1, T_2 \# S_2\}$. *Suppose* $S_1 \geq_{hr} T_1$ *or* $S_2 \geq_{hr} T_2$. *In addition, if* $T_1 \leq_{hr} (\geq_{hr}) T_2$ *and* $S_1 \geq_{hr} (\leq_{hr}) S_2$ *then* $U_1 \geq_{hr} (\leq_{hr}) U_2$.

Proof. First, we prove that $U_1 \geq_{hr} U_2$ under the conditions '$S_1 \geq_{hr} T_1$', '$T_1 \leq_{hr} T_2$', and '$S_1 \geq_{hr} S_2$'. To prove $U_1 \geq_{hr} U_2$, it suffices to show that $\dfrac{\overline{F}_{U_1}(t)}{\overline{F}_{U_2}(t)}$ is increasing in $t > 0$. We have

$$\overline{F}_{U_1}(t) = \left(\overline{F}_{T_1}(t) + \overline{F}_{S_1}(t) \int_0^t \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\, \mathrm{d}u\right) \overline{F}_{T_2}(t), \quad t > 0, \tag{2}$$

and

$$\overline{F}_{U_2}(t) = \left(\overline{F}_{T_2}(t) + \overline{F}_{S_2}(t) \int_0^t \frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)}\, \mathrm{d}u\right) \overline{F}_{T_1}(t), \quad t > 0. \tag{3}$$

From (2) and (3), it can be seen that

$$\frac{\overline{F}_{U_1}(t)}{\overline{F}_{U_2}(t)} = \frac{\overline{F}_{T_1}(t)\,\overline{F}_{T_2}(t) + \overline{F}_{T_2}(t)\,\overline{F}_{S_1}(t) \displaystyle\int_0^t \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\, \mathrm{d}u}{\overline{F}_{T_1}(t)\,\overline{F}_{T_2}(t) + \overline{F}_{T_1}(t)\,\overline{F}_{S_2}(t) \displaystyle\int_0^t \frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)}\, \mathrm{d}u} = \frac{g(t) + g_1(t)}{g(t) + g_2(t)}, \quad t > 0, \tag{4}$$

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where

$$g(t) = \overline{F}_{T_1}(t)\overline{F}_{T_2}(t),\ g_1(t) = \overline{F}_{T_2}(t)\overline{F}_{S_1}(t)\int_0^t \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\,\mathrm{d}u,$$

and

$$g_2(t) = \overline{F}_{T_1}(t)\overline{F}_{S_2}(t)\int_0^t \frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)}\,\mathrm{d}u, \quad t > 0.$$

To prove that $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$ is increasing in $t > 0$, it suffices to show that

$$\frac{g(t) + g_1(t)}{g(t) + g_2(t)} \quad \text{is increasing in } t > 0. \tag{5}$$

Now, we employ Lemma 1 to show that (5) holds true. It is evident that $g_1(t)$, $g_2(t) \geq 0$, and $g(t) \geq 0$, for all $t > 0$. Further, we have

$$\begin{aligned}\frac{g_1(t)}{g_2(t)} &= \frac{\overline{F}_{T_2}(t)}{\overline{F}_{T_1}(t)} \cdot \frac{\overline{F}_{S_1}(t)\int_0^t \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\,\mathrm{d}u}{\overline{F}_{S_2}(t)\int_0^t \frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)}\,\mathrm{d}u} \\ &= \frac{\overline{F}_{T_2}(t)}{\overline{F}_{T_1}(t)} \cdot \frac{\int_0^t \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)} f_{T_1}(u)\,\mathrm{d}u}{\int_0^t \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)} f_{T_2}(u)\,\mathrm{d}u} \\ &= \phi_1(t)\,\phi_2(t), \end{aligned} \tag{6}$$

where $\phi_1(t) = \overline{F}_{T_2}(t)/\overline{F}_{T_1}(t)$ and

$$\phi_2(t) = \frac{\int_0^t \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)} f_{T_1}(u)\,\mathrm{d}u}{\int_0^t \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)} f_{T_2}(u)\,\mathrm{d}u}.$$

If $T_1 \leq_{hr} T_2$ then $\phi_1(t)$ is increasing in $t > 0$. To prove $\phi_2(t)$ is increasing, it is equivalent to prove that $1/\phi_2(t)$ is decreasing for $t > 0$. Now,

$$\frac{1}{\phi_2(t)} = \frac{\int_0^t \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)} f_{T_2}(u)\,\mathrm{d}u}{\int_0^t \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)} f_{T_1}(u)\,\mathrm{d}u} = \frac{\int_0^t h_2(t,u) g_2(u)\,\mathrm{d}u}{\int_0^t h_1(t,u) g_1(u)\,\mathrm{d}u},$$

where, $h_i(t,u) = \overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ and $g_i(u) = f_{T_i}(u)$, for $i = 1, 2$. On employing Lemma 2 and using the conditions '$T_1 \leq_{hr} T_2$' and '$S_2 \leq_{hr} S_1$', we get $1/\phi_2(t)$ is decreasing in $t > 0$. Thus, from (6), we find that

$$\frac{g_1(t)}{g_2(t)} \quad \text{is increasing in } t > 0. \tag{7}$$

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Now, we have to show that $g(t)/(g_2(t) - g_1(t))$ is increasing in $t > 0$. We have

$$\frac{g(t)}{g_2(t) - g_1(t)} = \frac{\overline{F}_{T_1}(t)\overline{F}_{T_2}(t)}{\overline{F}_{T_1}(t)\overline{F}_{S_2}(t)\int_0^t \frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)}\,\mathrm{d}u - \overline{F}_{T_2}(t)\overline{F}_{S_1}(t)\int_0^t \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\,\mathrm{d}u}$$

$$= \frac{1}{\int_0^t \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)}\frac{f_{T_2}(u)}{\overline{F}_{T_2}(t)}\,\mathrm{d}u - \int_0^t \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)}\frac{f_{T_1}(u)}{\overline{F}_{T_1}(t)}\,\mathrm{d}u}, \quad t > 0. \tag{8}$$

Now, we can see that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_0^t \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)}\frac{f_{T_2}(u)}{\overline{F}_{T_2}(t)}\,\mathrm{d}u - \int_0^t \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)}\frac{f_{T_1}(u)}{\overline{F}_{T_1}(t)}\,\mathrm{d}u\right)\\
&= \left(\frac{f_{T_2}(t)}{\overline{F}_{T_2}(t)} - \frac{f_{T_1}(t)}{\overline{F}_{T_1}(t)}\right) + \int_0^t \frac{\partial}{\partial t}\left(\frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)}\frac{f_{T_2}(u)}{\overline{F}_{T_2}(t)} - \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)}\frac{f_{T_1}(u)}{\overline{F}_{T_1}(t)}\right)\mathrm{d}u\\
&= \left(\frac{f_{T_2}(t)}{\overline{F}_{T_2}(t)} - \frac{f_{T_1}(t)}{\overline{F}_{T_1}(t)}\right) + \int_0^t \frac{\partial}{\partial t}\left\{\left(\frac{\overline{F}_{S_1}(t)}{\overline{F}_{T_1}(t)}\right)\left(\frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_1}(t)}\frac{\overline{F}_{T_1}(t)}{\overline{F}_{T_2}(t)}\frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)} - \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\right)\right\}\mathrm{d}u\\
&\le \int_0^t \frac{\partial}{\partial t}\left\{\left(\frac{\overline{F}_{S_1}(t)}{\overline{F}_{T_1}(t)}\right)\left(\frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_1}(t)}\frac{\overline{F}_{T_1}(t)}{\overline{F}_{T_2}(t)}\frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)} - \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\right)\right\}\mathrm{d}u\\
&\le 0, \quad \text{for all } t > 0,
\end{aligned} \tag{9}$$

where the second last inequality follows from the condition '$T_1 \le_{hr} T_2$', and, the last inequality follows from the conditions '$T_1 \le_{hr} T_2$', '$S_1 \ge_{hr} S_2$', and '$S_1 \ge_{hr} T_1$', since these conditions imply that $\overline{F}_{S_1}(t)/\overline{F}_{T_1}(t)$ is increasing, $\overline{F}_{S_2}(t)/\overline{F}_{S_1}(t)$ and $\overline{F}_{T_1}(t)/\overline{F}_{T_2}(t)$ are decreasing in $t > 0$, and

$$\frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_1}(t)}\frac{\overline{F}_{T_1}(t)}{\overline{F}_{T_2}(t)}\frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)} - \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)} \le 0 \quad \text{for all } t > 0.$$

Therefore, we get that

$$\frac{g(t)}{g_2(t) - g_1(t)} \quad \text{is increasing in } t > 0. \tag{10}$$

Now, by using Lemma 1, and from (7) and (10), we get that (4) is increasing (i.e. $(g(t) + g_1(t))/(g(t) + g_2(t))$ is increasing in $t > 0$), thereby implying that $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$ is increasing in $t > 0$, i.e. $U_1 \ge_{hr} U_2$. Thus, $U_1 \ge_{hr} U_2$ under the conditions '$S_1 \ge_{hr} T_1$', '$T_1 \le_{hr} T_2$', and '$S_1 \ge_{hr} S_2$'. Similarly, we can prove $U_1 \ge_{hr} U_2$ under the conditions '$S_2 \ge_{hr} T_2$', '$T_1 \le_{hr} T_2$', and '$S_1 \ge_{hr} S_2$'.

In a similar vein, we can prove that $U_1 \le_{hr} U_2$ under the assumptions '$S_1 \ge_{hr} T_1$ (or $S_2 \ge_{hr} T_2$)', '$T_1 \ge_{hr} T_2$', and '$S_1 \le_{hr} S_2$' hold true. Hence, the result is proven. □

The following example illustrates the application of Theorem 1. Recall that T is said to be Weibull random variable with shape parameter $\alpha(> 0)$ and scale parameter $\lambda(> 0)$, denoted by $T \sim W(\alpha, \lambda)$, if its survival function is given by $\overline{F}_T(t; \alpha, \lambda) = \mathrm{e}^{-(\lambda t)^\alpha}$, $t > 0$.

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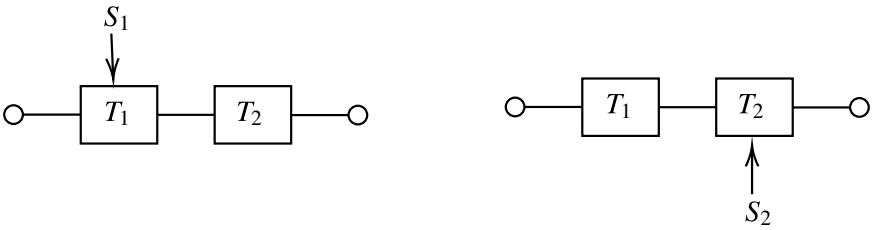


FIGURE 2. Schematic diagrams of systems with lifetimes U_1 (left) and U_2 (right) corresponding to Theorem 1.

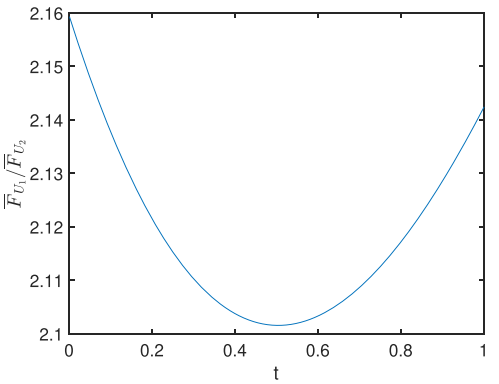


FIGURE 3. Plot of the ratio of survival functions of U_1 and U_2 (i.e. $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$).

Example 3. Suppose that the random variables $T_i \sim W(\alpha, \lambda_i)$ and $S_i \sim W(\alpha, \beta_i)$, where $\alpha > 0$, $\lambda_i > 0$, and $\beta_i > 0$, $i = 1, 2$. If $\lambda_1 \geq \lambda_2$, $\beta_1 \leq \beta_2$, and $\beta_1 \leq \lambda_1$ then the conditions $T_1 \leq_{hr} T_2$, $S_1 \geq_{hr} S_2$, and $S_1 \geq_{hr} T_1$ hold true, respectively. Consequently, the conditions of Theorem 1 are satisfied.

Note that Belzunce *et al.* [10] established $U_1 \geq_{st} U_2$ under the conditions $T_1 \leq_{hr} T_2$ and $S_1 \geq_{hr} S_2$. As we can see from Theorem 1 that when we move from a weaker stochastic ordering (i.e. usual stochastic order) to a stronger stochastic ordering (i.e. hazard rate order), an extra condition '$S_1 \geq_{hr} T_1$ or $S_2 \geq_{hr} T_2$' is needed for the result to hold true. The following counterexample shows that if we relax the condition '$S_1 \geq_{hr} T_1$' in Theorem 1, the result $U_1 \not\geq_{hr} U_2$ no longer holds true.

Counterexample 1. *Suppose that the random variables* $T_i \sim W(\alpha, \lambda_i)$ *and* $S_i \sim W(\alpha, \beta_i)$, *where* $\alpha > 0$, $\lambda_i > 0$, *and* $\beta_i > 0$, $i = 1, 2$. *Consider the resulting systems in Figure 2. If* $\lambda_1 = 4.03$, $\lambda_2 = 2.02$, $\beta_1 = 6$, $\beta_2 = 8.95$, *and* $\alpha = 1$, *this implies that the conditions* $T_1 \leq_{hr} T_2$, $S_1 \geq_{hr} S_2$ *hold, but the condition* $S_1 \geq_{hr} T_1$ *does not holds. Then, from Figure 3 it can be seen that* $U_1 \not\geq_{hr} U_2$.

With regards to relevation allocation to a coherent system, Belzunce *et al.* [10] investigated the optimal allocation policies for two relevation (performed one at a time) to a coherent system for case (i) $\mathcal{A}_i = \mathcal{A}_j$ in the sense of usual stochastic order. They demonstrated that for components i and j in common path sets, the relevation should be applied to the component with

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the higher hazard rate. Further, they called for future research on comparing system lifetimes in terms of the hazard rate order after allocating a single relevation to a coherent system. To address this, we provide the following result, which employs Theorem 1 above on series systems. For this purpose, consider an n-component coherent system with components' lifetimes $T_1, T_2, \ldots, T_n$, having structure function ϕ, and minimal path sets $P_1, P_2, \ldots, P_k$. The system lifetime can be expressed as

$$\tau(\boldsymbol{T}) = \tau\,(T_1, \ldots, T_n) = \max_{\mathcal{P}\in\mathcal{A}} \min_{j\in\mathcal{P}} T_j.$$

In addition, let $\boldsymbol{T}^{(i,j)} = T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{j-1}, T_{j+1}, \ldots, T_n$, and denote the lifetime of the two coherent systems by $W_1 \sim \tau\left(T_i\#S_1, T_j, \boldsymbol{T}^{(i,j)}\right)$ and $W_2 \sim \tau\left(T_i, T_j\#S_2, \boldsymbol{T}^{(i,j)}\right)$ after performing one relevation at a time.

Theorem 2. *Let* $W_1 \sim \tau\left(T_i\#S_1, T_j, \boldsymbol{T}^{(i,j)}\right)$ *and* $W_2 \sim \tau\left(T_i, T_j\#S_2, \boldsymbol{T}^{(i,j)}\right)$. *Consider two locations* $i, j \in \{1, \ldots, n\}$ *such that* $i < j$ *and* $\{i, j\} \subseteq P_r$, *for* $r = i_1, \ldots, i_r$, *where* $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$, *i.e.* $\mathcal{A}_i = \mathcal{A}_j$. *Suppose*

$$\frac{\mathrm{P}\left(\max\limits_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l} \{T_m\} > t\right)}{\mathrm{P}\left(\max\limits_{s\in\{i_1,\ldots,i_r\}} \min\limits_{m\in P_s, m\neq i,j} \{T_m\} > t,\ \max\limits_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l} \{T_m\} \le t\right)}$$

is decreasing in $t > 0$. *If* $T_i \le_{hr} (\ge_{hr})\, T_j$, $S_1 \ge_{hr} (\le_{hr})\, S_2$, *and* $S_1 \ge_{hr} T_i$ *(or* $S_2 \ge_{hr} T_j$*) then* $W_1 \ge_{hr} (\le_{hr})\, W_2$.

Proof. To prove the result, let us consider two series systems having lifetime $U_1 = \min\{T_i\#S_1,\ T_j\}$ and $U_2 = \min\left\{T_i, T_j\#S_2\right\}$. First, we prove that $W_1 \ge_{hr} W_2$ under the assumptions $T_i \le_{hr} T_j$, $S_1 \ge_{hr} S_2$, and $S_1 \ge_{hr} T_i$ (or $S_2 \ge_{hr} T_j$). Then, for some minimal path set P_r such that $\{i, j\} \subseteq P_r$, the lifetimes of systems W_1 and W_2 can be expressed as

$$W_1 = \max\left\{\min\left\{\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j} \{T_m\},\ T_i\#S_1,\ T_j\right\},\ \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l} \{T_m\}\right\},$$

and

$$W_2 = \max\left\{\min\left\{\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j} \{T_m\},\ T_i,\ T_j\#S_2\right\},\ \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l} \{T_m\}\right\}.$$

Thus, we get

$$\begin{aligned}\overline{F}_{W_1}(t) = 1 - \mathrm{P}\Bigg(&\min\left\{\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j} \{T_m\},\ T_i\#S_1,\ T_j\right\} \le t,\\ &\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l} \{T_m\} \le t\Bigg)\\ = \mathrm{P}\,(U_1 > t)\,&\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j} \{T_m\} > t,\ \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l} \{T_m\} \le t\right)\\ &+ \mathrm{P}\left(\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l} \{T_m\} > t\right), \quad t > 0,\end{aligned}$$

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and

$$\overline{F}_{W_2}(t) = \mathrm{P}\,(U_2 > t)\,\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right)$$
$$+\,\mathrm{P}\left(\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} > t\right), \quad t > 0.$$

Now,

$$\frac{\overline{F}_{W_1}(t)}{\overline{F}_{W_2}(t)} = \frac{g(t)+g_1(t)}{g(t)+g_2(t)}, \quad t > 0,$$

where

$$g(t) = \mathrm{P}\left(\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} > t\right),$$

$$g_1(t) = \mathrm{P}\,(U_1 > t)\,\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right)$$

and

$$g_2(t) = \mathrm{P}\,(U_2 > t)\,\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right), \quad t > 0.$$

Now, from Theorem 1, we get that $g_1(t)/g_2(t)$ is increasing in $t > 0$, since '$T_i \le_{hr} T_j$' and '$S_2 \le_{hr} S_1$'. Further, we have

$$\frac{g(t)}{g_2(t)-g_1(t)} = \frac{\mathrm{P}\left(\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} > t\right)}{\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right)}$$
$$\times \frac{1}{\mathrm{P}\,(U_2 > t) - \mathrm{P}\,(U_1 > t)}, \quad t > 0.$$

Now, from the condition of Theorem 2, we get that

$$\frac{\mathrm{P}\left(\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} > t\right)}{\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right)}$$

is decreasing in $t > 0$ and from the proof of Theorem 1, we get $1/(\mathrm{P}(U_2 > t) - \mathrm{P}(U_1 > t))$ is increasing in $t > 0$ under the assumptions $T_i \le_{hr} T_j$, $S_1 \ge_{hr} S_2$, and $S_1 \ge_{hr} T_i$ (or $S_2 \ge_{hr} T_j$). By combining these two facts, we get $g(t)/(g_2(t) - g_1(t))$ is increasing in $t > 0$. Hence, $W_1 \ge_{hr} W_2$.

Similarly, we can prove that $W_1 \le_{hr} W_2$ under the assumptions $T_i \ge_{hr} T_j$, $S_1 \le_{hr} S_2$, and rest of the conditions are same. Hence, the result is proved. □

To illustrate the result of Theorem 2, we provide an example below.

Example 4. Consider a coherent system made up of four components C_1, C_2, C_3, and C_4 (illustrated in Figure 4). Let the first two components C_1 and C_2 have lifetimes T_1 and T_2,

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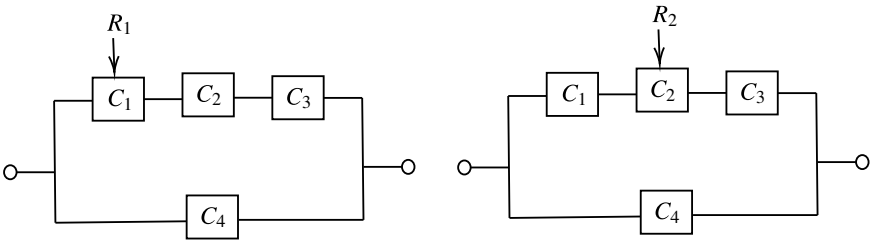


FIGURE 4. Schematic diagrams of systems with lifetimes W_1 (left) and W_2 (right) corresponding to Example 2.

respectively, with $T_i \sim W(\alpha, \lambda_i)$, where $\alpha > 0$, $\lambda_i > 0$, $i = 1, 2$, and the other two components C_3 and C_4 have lifetimes T_3 and T_4, respectively, such that $T_1 \overset{d}{=} T_4$ and $T_2 \overset{d}{=} T_3$. Further, suppose two relevations R_1 and R_2 having lifetimes $S_1 \sim W(\alpha, \beta_1)$ and $S_2 \sim W(\alpha, \beta_2)$, where $\alpha > 0$, $\beta_i > 0$, $i = 1, 2$, are available for allocation. Assume that relevation R_1 is performed to component C_1 and R_2 is performed to component C_2, then the resultant systems have lifetimes W_1 and W_2, respectively.

Note that the given coherent system has minimal path sets $P_1 = \{1, 2, 3\}$ and $P_2 = \{4\}$. Thus, we can say that $\mathcal{A}_1 = \mathcal{A}_2$ (i.e. components 1 and 2 are included in the same minimal path sets). Further, it is easy to see that when $\lambda_2 \leq \lambda_1$, then $T_1 \leq_{hr} T_2$. In addition, if $\beta_1 \leq \beta_2$ and $\beta_1 \leq \lambda_1$, then $S_2 \leq_{hr} S_1$ and $T_1 \leq_{hr} S_1$ hold true. Further, it can be seen that

$$\frac{\mathrm{P}\left(\max\limits_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l} \{T_m\} > t\right)}{\mathrm{P}\left(\max\limits_{s\in\{i_1,\ldots,i_r\}} \min\limits_{m\in P_s, m\neq i,j} \{T_m\} > t, \max\limits_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l} \{T_m\} \leq t\right)} = \frac{\mathrm{P}(T_4 > t)}{\mathrm{P}(T_3 > t)\,\mathrm{P}(T_4 \leq t)}$$

is decreasing in $t > 0$, if $\lambda_2 \leq \lambda_1$ (i.e. $T_4 \leq_{hr} T_3$). Thus, the conditions of Theorem 2 are satisfied, i.e. $W_1 \geq_{hr} W_2$.

Up to now, we considered the optimal allocation policy for two relevations (performed one at a time) to series and general coherent systems. Moving forward, suppose there is no budget constraint, and we can afford to allocate two relevations, with lifetimes S_1 and S_2, simultaneously to a system. It is, therefore, of interest to determine the optimal allocation policy for two relevation allocations simultaneously to a system. In this regard, Zhang and Zhang [34, Lemma 1], proved that if $T_1 \leq_{lr} T_2$ and $S_1 \geq_{hr} S_2$ then $V_1' \geq_{st} V_2'$, where $V_1' = \min\{T_1\#S_1, T_2\#S_2\}$ and $V_2' = \min\{T_1\#S_2, T_2\#S_1\}$. As they gave only one set of sufficient conditions for $V_1' \geq_{st} V_2'$ to hold true, we provide additional sufficient conditions, in the following theorem, for the same result to hold true.

Theorem 3. *Let T_1, T_2, S_1, and S_2 be non-negative independent random variables, and let $V_1' = \min\{T_1\#S_1, T_2\#S_2\}$ and $V_2' = \min\{T_1\#S_2, T_2\#S_1\}$. If any of the following sets of conditions are satisfied:*

(a) *$T_1 \leq_{lr} (\geq_{lr}) T_2$ and $S_1 \geq_{hr} S_2$;*

(b) *$T_1 \geq_{lr} (\leq_{lr}) T_2$ and $S_1 \leq_{hr} S_2$;*

then $V_1' \geq_{st} (\leq_{st}) V_2'$.

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Proof. The proof of the theorem is analogous to [34, Lemma 1], hence it is omitted. □

If we assume that S follows an exponential distribution in the relevation survival formula (i.e. (1)), then the standby redundancy allocation becomes a special case of the relevation allocation. In light of this, one can observe that the first set of conditions, i.e. $T_1 \leq_{lr} (\geq_{lr}) T_2$ and $S_1 \geq_{hr} S_2$, is same as that in [21, Theorem 3.1], and the second set of conditions, i.e. $T_1 \geq_{lr} (\leq_{lr}) T_2$ and $S_1 \leq_{hr} S_2$, is same as that in [21, Theorem 3.2]. This highlights that in this case, if we shift from standby redundancy allocation to relevation allocation, the conditions remain the same.

The following counterexample shows that if we relax the likelihood ratio condition $T_1 \leq_{lr} (\geq_{lr}) T_2$ to a weaker stochastic order in Theorem 3, then the result no longer holds.

Counterexample 2. *Let T_1 and T_2 be independent random variables with distribution functions*

$$F_{T_1}(t) = \begin{cases} 1 - \exp\left(-at - \dfrac{bt^3}{3}\right), & \text{if } t \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F_{T_2}(t) = \begin{cases} \dfrac{1}{2}\left[1 + erf\left(\dfrac{\log(t)}{\sqrt{2}}\right)\right], & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b > 0$. From Figure 5(a), it can be seen that $F_{T_2} \leq F_{T_1}$, for $a = 0.8$, $b = 3$, implying that $T_1 \leq_{st} T_2$. It is also easy to see from Figure 5(b) (and (c)) that F_{T_2}/F_{T_1} (and f_{T_2}/f_{T_1}) is not monotonically increasing or decreasing for $a = 0.8$ and $b = 3$ (i.e. neither $T_1 \leq_{rh} T_2$ nor $T_1 \leq_{lr} T_2$).

Now, suppose that S_1 and S_2 are independent random variables following Gamma and Weibull distribution with density functions

$$f_{S_1}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha_1)}\, t^{\alpha_1 - 1}\, e^{-t}, & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_{S_2}(t) = \begin{cases} \alpha_2 t^{\alpha_2 - 1} e^{-t^{\alpha_2}}, & \text{if } t \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

respectively, where, $\alpha_1 > 0$, $\alpha_2 > 0$. One can easily prove that $S_1 \geq_{hr} S_2$ if $\alpha_1 = 2$ and $\alpha_2 = 1$. However, it is evident from Figure 5(d) that the result of Theorem 3 does not hold, i.e. if $T_1 \leq_{st} T_2$ and $S_1 \geq_{hr} S_2$, it does not imply that $V_1' \leq_{st} V_2'$. Hence, if we relax the condition $T_1 \leq_{lr} (\geq_{lr}) T_2$ to $T_1 \leq_{st} (\geq_{st}) T_2$ in Theorem 3, then the result (i.e. $V_1' \leq_{st} (\geq_{st}) V_2'$) no longer holds even if the other condition (i.e. $S_1 \geq_{hr} S_2$) remains the same.

With regards to the allocation of two relevations to a coherent system simultaneously, Yan *et al.* [31] provided the result for case (i) $\mathcal{A}_i = \mathcal{A}_j$. However, Zhang and Zhang [34] pointed out

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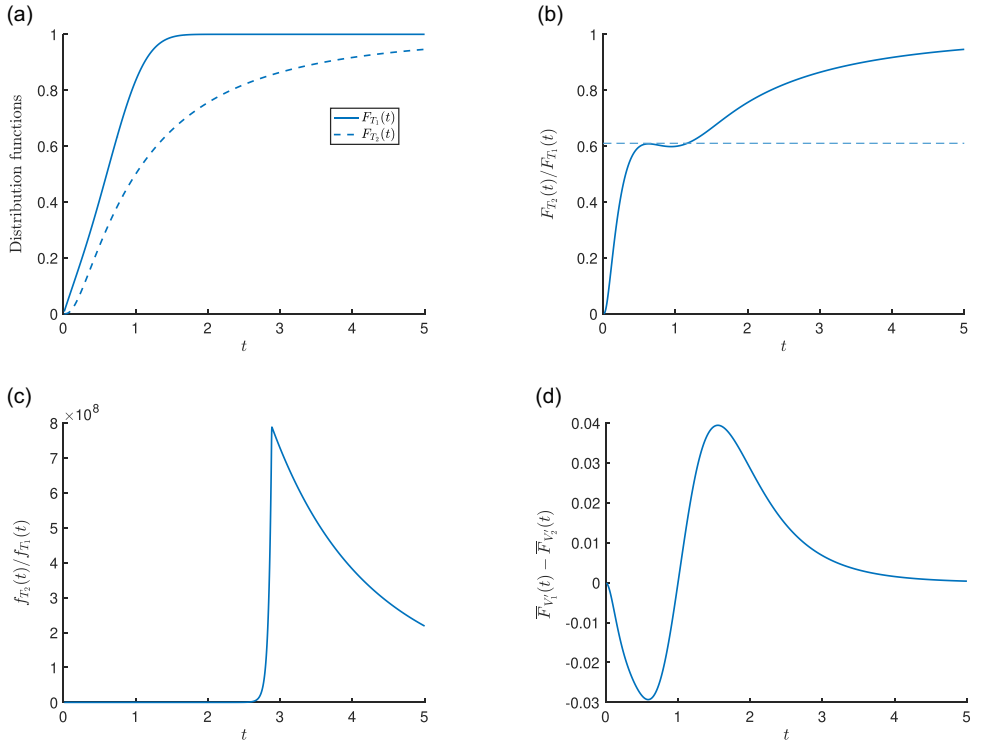


FIGURE 5. (a) Plot of the distribution functions of T_1 and T_2 (i.e. $F_{T_1}(t)$ and $F_{T_2}(t)$). (b) Plot of the ratio of the distribution functions of T_2 and T_1 (i.e. $F_{T_2}(t)/F_{T_1}(t)$). (c) Plot of the ratio of the density functions of T_2 and T_1 (i.e. $f_{T_2}(t)/f_{T_1}(t)$). (d) Plot of the difference of the survival functions of V_1' and V_2' (i.e. $\overline{F}_{V_1'}(t) - \overline{F}_{V_2'}(t)$).

an error in Theorem 1 of [31] for a two-component series system. Consequently, Theorem 4 in [31] for two relevations allocation to a coherent system for case (i) $\mathcal{A}_i = \mathcal{A}_j$ is also incorrect. In the following, we present the corrected version for two relevations allocation to a coherent system in case (i) $\mathcal{A}_i = \mathcal{A}_j$. For this purpose, we consider an n-component coherent system with components lifetimes $T_1, T_2, \ldots, T_n$, having structure function ϕ, and minimal path sets $P_1, P_2, \ldots, P_k$, where k is the total number of minimal path sets. The system lifetime can be expressed as

$$\tau(\boldsymbol{T}) = \tau\,(T_1, \ldots, T_n) = \min_{\mathcal{P}\in\mathcal{A}} \max_{j\in\mathcal{P}} T_j.$$

In addition, let $\boldsymbol{T}^{(i,j)} = T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{j-1}, T_{j+1}, \ldots, T_n$ and we denote the lifetime of the two coherent systems by $Z_1 \sim \tau\left(T_i\#S_1, T_j\#S_2, \boldsymbol{T}^{(i,j)}\right)$ and $Z_2 \sim \tau\left(T_i\#S_2, T_j\#S_1, \boldsymbol{T}^{(i,j)}\right)$ after performing two relevations.

Theorem 4. *Let* $Z_1 \sim \tau\left(T_i\#S_1, T_j\#S_2, \boldsymbol{T}^{(i,j)}\right)$ *and* $Z_2 \sim \tau\left(T_i\#S_2, T_j\#S_1, \boldsymbol{T}^{(i,j)}\right)$. *Consider two locations* $i, j \in \{1, \ldots, n\}$ *such that* $i < j$ *and* $\{i, j\} \subseteq P_r$, *for* $r = i_1, \ldots, i_r$, *where* $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$, *i.e.* $\mathcal{A}_i = \mathcal{A}_j$. *If any of the following sets of conditions are satisfied:*

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(a) $T_i \leq_{lr} (\geq_{lr}) T_j$ *and* $S_1 \geq_{hr} S_2$*;*

(b) $T_i \geq_{lr} (\leq_{lr}) T_j$ *and* $S_1 \leq_{hr} S_2$*;*

then $Z_1 \geq_{st} (\leq_{st}) Z_2$.

Proof. To prove the result, let us consider two series systems having $V_1' = \min\left\{T_i\#S_1, T_j\#S_2\right\}$ and $V_2' = \min\left\{T_i\#S_2, T_j\#S_1\right\}$. Now, for some minimal path set P_r such that $\{i, j\} \subseteq P_r$, the lifetimes of systems Z_1 and Z_2 can be expressed as

$$Z_1 = \max\left\{\min\left\{\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j}\{T_m\}, T_i\#S_1, T_j\#S_2\right\}, \max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\}\right\},$$

and

$$Z_2 = \max\left\{\min\left\{\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j}\{T_m\}, T_i\#S_2, T_j\#S_1\right\}, \max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\}\right\}.$$

Thus, we get

$$\begin{aligned}\overline{F}_{Z_1}(t) &= 1 - \mathrm{P}\Bigg(\min\left\{\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j}\{T_m\}, T_i\#S_1, T_j\#S_2\right\} \leq t, \\ &\qquad \max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\} \leq t\Bigg) \\ &= \mathrm{P}\left(V_1' > t\right)\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\} \leq t\right) \\ &\quad + \mathrm{P}\left(\max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\} > t\right),\end{aligned}$$

and

$$\begin{aligned}\overline{F}_{Z_2}(t) &= \mathrm{P}\left(V_2' > t\right)\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\} \leq t\right) \\ &\quad + \mathrm{P}\left(\max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\} > t\right), \quad t > 0.\end{aligned}$$

Now,

$$\overline{F}_{Z_1}(t) - \overline{F}_{Z_2}(t) = \left(\mathrm{P}\left(V_1' > t\right) - \mathrm{P}\left(V_2' > t\right)\right) \times \delta(t), \tag{11}$$

where

$$\delta(t) = \mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l}\{T_m\} \leq t\right).$$

Thus, from (11), it can be seen that to prove $Z_1 \geq_{st} (\leq_{st}) Z_2$, it is equivalent to prove $V_1' \geq_{st} (\leq_{st}) V_2'$. Hence, from Theorem 3, the result is proved. □

Up to now, we focused on the first case, i.e. $\mathcal{A}_i = \mathcal{A}_j$. We now turn our attention to case (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$. In the situation where only one relevation out of the two can be allocated to a coherent system, Wu *et al.* [30, Theorem 2.1] showed that $W_1 \leq_{st} W_2$ under the conditions

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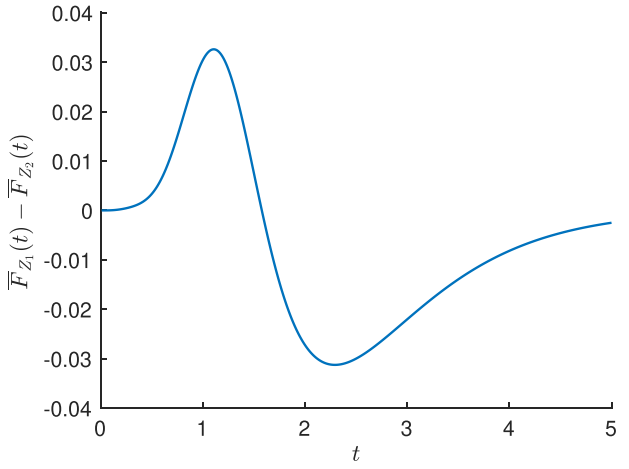


FIGURE 6. Plot of the difference of survival functions of Z_1 and Z_2 (i.e. $\overline{F}_{Z_1}(t) - \overline{F}_{Z_2}(t)$).

$T_i \geq_{hr} T_j$ and $S_1 \leq_{hr} S_2$. Further, when the two relevations can be allocated simultaneously to a coherent system, Zhang and Zhang [34, Theorem 1] proved $Z_1 \geq_{st} Z_2$ when $T_i \geq_{lr} T_j$ and $S_1 \leq_{hr} S_2$. Now, we provide additional set of sufficient conditions for $Z_1 \geq_{st} (\leq_{st}) Z_2$ in case (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$.

Theorem 5. *Let* $Z_1 \sim \tau\left(T_i\#S_1, T_j\#S_2, \boldsymbol{T}^{(i,j)}\right)$ *and* $Z_2 \sim \tau\left(T_i\#S_2, T_j\#S_1, \boldsymbol{T}^{(i,j)}\right)$. *Suppose* $\mathcal{A}_i \subseteq \mathcal{A}_j$. *If* $T_i \geq_{lr} (\leq_{lr}) T_j$ *and* $S_1 \leq_{hr} (\geq_{hr}) S_2$ *then* $Z_1 \geq_{st} (\leq_{st}) Z_2$.

Proof. Denote $V_1' = \min\left\{T_i\#S_1, T_j\#S_2\right\}$ and $V_2' = \min\left\{T_i\#S_2, T_j\#S_1\right\}$. In view of [34, Theorem 1], to prove $Z_1 \geq_{st} (\leq_{st}) Z_2$, it suffices to show that $V_1' \geq_{st} (\leq_{st}) V_2'$ and $\mathrm{P}\left(T_j\#S_2 > t\right) \geq (\leq) \mathrm{P}\left(T_j\#S_1 > t\right)$. Now, since T_j is non-negative, $S_1 \leq_{hr} S_2$ implies that $\mathrm{P}\left(T_j\#S_2 > t\right) \geq \mathrm{P}\left(T_j\#S_1 > t\right)$. Similarly, $S_1 \geq_{hr} S_2$ implies that $\mathrm{P}\left(T_j\#S_2 > t\right) \leq \mathrm{P}\left(T_j\#S_1 > t\right)$. Thus, from Theorem 3, we can see that if $T_i \geq_{lr} (\leq_{lr}) T_j$ and $S_1 \leq_{hr} (\geq_{hr}) S_2$ then $V_1' \geq_{st} (\leq_{st}) V_2'$ which further implies that $Z_1 \geq_{st} (\leq_{st}) Z_2$. □

The following counterexample shows that if we relax the likelihood ratio condition $T_1 \leq_{lr} (\geq_{lr}) T_2$ to a weaker stochastic order in Theorem 5, then the result no longer holds.

Counterexample 3. *Consider a coherent system made up of three components having lifetimes* T_1, T_2, *and* T_3. *Suppose, the lifetime of a coherent system is* $\tau(\boldsymbol{T}) = \min\{\max\{T_1, T_3\}, T_2\}$. *Thus, the minimal path sets are* $\{1, 2\}$ *and* $\{2, 3\}$, *and consequently* $\mathcal{A}_1 \subseteq \mathcal{A}_2$. *Suppose that either relevation* S_1 *is performed with* T_1 *and* S_2 *is performed with* T_2 *or* S_1 *is performed with* T_2 *and* S_2 *is performed with* T_1 *which gives that the resultant system has lifetime* Z_1 *or* Z_2, *i.e.* $Z_1 = \min\{\max\{T_1\#S_1, T_3\}, T_2\#S_2\}$ *and* $Z_2 = \min\{\max\{T_1\#S_2, T_3\}, T_2\#S_1\}$. *Further, in the setup of Counterexample 2, it can be seen that* $T_1 \leq_{st} T_2$ *holds for* $a = 0.8$, $b = 3$ *(see Figure 5(a)), but* $T_1 \leq_{rh} T_2$ *(and* $T_1 \leq_{lr} T_2$*) do not hold (see Figure 5(b) and (c)). In addition, we can see that* $S_1 \geq_{hr} S_2$ *is true for* $\alpha_1 = 2$ *and* $\alpha_2 = 1$. *However, it is evident from Figure 6 that the result of Theorem 5 does not hold, i.e. if* $T_1 \leq_{st} T_2$ *and* $S_1 \geq_{hr} S_2$ *hold, it does not imply that* $Z_1 \leq_{st} Z_2$. *Hence, if we relax the condition,* $T_1 \leq_{lr} (\geq_{lr}) T_2$ *to* $T_1 \leq_{st} (\geq_{st}) T_2$ *in Theorem 5, then the result (i.e.* $Z_1 \geq_{st} (\leq_{st}) Z_2$*) no longer holds even if the other condition (i.e.* $S_1 \geq_{hr} S_2$*) remains the same.*

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It is interesting to explore the problem of allocating two relevations simultaneously to a coherent system in the sense of hazard rate order. This continues to be an open problem requiring further exploration.

3.2. Allocation of two relevations in cases: (ii) $\mathcal{S}_i = \mathcal{S}_j$ and (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$

In this subsection, we discuss the optimal allocation policies of two relevations (performed simultaneously) for a parallel system and later generalize them to a coherent system for the cases: (ii) $\mathcal{S}_i = \mathcal{S}_j$ and (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$.

Belzunce *et al.* [10] studied the optimal allocation policy for two relevations (performed one at a time) in a two-component series system and later generalized to a coherent system in case (ii) $\mathcal{S}_i = \mathcal{S}_j$. They made the stochastic comparison in the sense of the usual stochastic order and demonstrated that for components i and j in common cut sets, the relevation should be done on the component with the higher reversed hazard rate. Later, Wu *et al.* [30] established the stochastic comparison for two relevations (performed one at a time) in the sense of the usual stochastic order for case (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$. Recently, Zhang and Zhang [34] studied the optimal allocation policy for two relevations (performed both at a time) in the sense of a usual stochastic order for case (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$. In their study on the allocation of two relevations to a coherent system, they proved that $\tau\left(T_i\#S_1, T_j\#S_2, \boldsymbol{T}^{(i,j)}\right) \leq_{st} \tau\left(T_i\#S_2, T_j\#S_1, \boldsymbol{T}^{(i,j)}\right)$ if $\mathcal{S}_i \subseteq \mathcal{S}_j$, $S_1 \geq_{hr} S_2$, and $F_{T_i\#S_1}(t)F_{T_j\#S_2}(t) \geq F_{T_i\#S_2}(t)F_{T_j\#S_1}(t)$, for all $t > 0$ (see [34, Theorem 2]). The authors further pointed out that it is of interest to find sufficient conditions for the assumption $F_{T_i\#S_1}(t)F_{T_j\#S_2}(t) \geq F_{T_i\#S_2}(t)F_{T_j\#S_1}(t)$ to hold true. They conjectured that it might require $T_i \leq_{hr} T_j$ or $T_i \leq_{rh} T_j$. To inquire into this conjecture, we first provide the following theorem for two-component parallel system and then investigate the case for a coherent system.

Theorem 6. *Let T_1, T_2, S_1, and S_2 be non-negative independent random variables, and let $V_1 = \max\{T_1\#S_1, T_2\#S_2\}$ and $V_2 = \max\{T_1\#S_2, T_2\#S_1\}$. If any of the following sets of conditions are satisfied:*

(a) $1 - (\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ *is RR_2 in* $(i, u) \in \{1, 2\} \times (0, t]$ *and* $T_1 \leq_{rh} (\geq_{rh}) T_2$*;*

(b) $1 - (\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ *is TP_2 in* $(i, u) \in \{1, 2\} \times (0, t]$ *and* $T_1 \geq_{rh} (\leq_{rh}) T_2$*;*
then

$$V_1 \geq_{st} (\leq_{st}) V_2.$$

Proof.

(a) First, we prove that $V_1 \geq_{st} V_2$ under the assumptions '$1 - (\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ is RR_2 in $(i, u) \in \{1, 2\} \times (0, t]$' and '$T_1 \leq_{rh} T_2$'. The distribution functions of V_1 and V_2 can be expressed as

$$\begin{aligned} F_{V_1}(t) &= \left(F_{T_1}(t) - \overline{F}_{S_1}(t)\int_0^t \frac{f_{T_1}(u)}{\overline{F}_{S_1}(u)}\,\mathrm{d}u\right)\left(F_{T_2}(t) - \overline{F}_{S_2}(t)\int_0^t \frac{f_{T_2}(u)}{\overline{F}_{S_2}(u)}\,\mathrm{d}u\right) \\ &= \int_0^t \left(1 - \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)}\right) f_{T_1}(u)\,\mathrm{d}u \int_0^t \left(1 - \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)}\right) f_{T_2}(u)\,\mathrm{d}u, \\ &= \int_0^t h(1, u)g_1(u)\,\mathrm{d}u \int_0^t h(2, u)g_2(u)\,\mathrm{d}u, \quad \text{for all } t > 0, \end{aligned}$$

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and

$$F_{V_2}(t) = \int_0^t \left(1 - \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u)}\right) f_{T_1}(u)\,\mathrm{d}u \int_0^t \left(1 - \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u)}\right) f_{T_2}(u)\,\mathrm{d}u,$$

$$= \int_0^t h(2, u) g_1(u)\,\mathrm{d}u \int_0^t h(1, u) g_2(u)\,\mathrm{d}u, \quad \text{for all } t > 0,$$

where, for fixed $t > 0$, $h(x, u) = 1 - \overline{F}_{S_x}(t)/\overline{F}_{S_x}(u)$ and $g_i(u) = f_{T_i}(u)$, for $i = 1, 2$, and $x = \{1, 2\}$. To prove $V_1 \geq_{st} V_2$, it suffices to prove that

$$F_{V_2}(t) \geq F_{V_1}(t), \quad \text{for all } t > 0.$$

Now, by using the Lemma 2, from conditions '$1 - (\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ is RR_2 in $(i, u) \in \{1, 2\} \times (0, t]$' and '$T_1 \leq_{rh} T_2$', we get

$$F_{V_1}(t) - F_{V_2}(t) = \int_0^t h(1, u) g_1(u) du \int_0^t h(2, u) g_2(u)\,\mathrm{d}u$$

$$- \int_0^t h(2, u) g_1(u)\,\mathrm{d}u \int_0^t h(1, u) g_2(u)\,\mathrm{d}u$$

$$\leq 0, \quad \text{for all } t > 0.$$

Hence, it can be seen that $V_1 \geq_{st} V_2$. In a similar vein, one can prove that $V_1 \leq_{st} V_2$ under the assumptions '$1 - (\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ is RR_2 in $(i, u) \in \{1, 2\} \times (0, t]$' and '$T_1 \geq_{rh} T_2$'. Thus, the result is proved.

(b) Similarly, we can prove that $V_1 \geq_{st} (\leq_{st}) V_2$ under the assumptions '$1 - (\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ is TP_2 in $(i, u) \in \{1, 2\} \times (0, t]$' and '$T_1 \geq_{rh} (\leq_{rh}) T_2$'. □

In the above result, it is easy to establish reversed hazard rate ordering between T_1 and T_2. To investigate the other sufficient condition '$1 - (\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ is RR_2 (TP_2) in $(i, u) \in \{1, 2\} \times (0, t]$', we illustrate in the following example that it is satisfied by some common family of distributions, such as Weibull and Pareto.

Example 5.

(a) Weibull. Let $S_1 \sim W(2, 3)$ and $S_2 \sim W(2, 2)$. Here, $\overline{F}_{S_1}(t)/\overline{F}_{S_1}(u) = \mathrm{e}^{-3^2(t^2-u^2)}$ and $\overline{F}_{S_2}(t)/\overline{F}_{S_2}(u) = \mathrm{e}^{-2^2(t^2-u^2)}$. Now, '$1 - \overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 (TP_2) in $(i, u) \in \{1, 2\} \times (0, t]$' if and only if $(1 - \mathrm{e}^{-2^2}(t^2 - u^2))/(1 - \mathrm{e}^{-3^2}(t^2 - u^2))$ is decreasing (increasing) in $0 \leq u \leq t$, which easily follows from the fact that $S_1 \leq_{rh} (\geq_{rh}) S_2$.

(b) Pareto. Recall that T is said to be a Pareto random variable with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$, denoted by $T \sim Pareto(\alpha, \lambda)$, if its survival function is given by $\overline{F}_T(t; \alpha, \lambda) = (\lambda/t)^\alpha$, $t > \lambda$. Let $S_1 \sim Pareto(2, 1)$, and $S_2 \sim Pareto(1, 1)$. Thus, for all $0 < u_1 \leq u_2 \leq t$, we have

$$\left[1 - \frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u_1)}\right]\left[1 - \frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u_2)}\right] = (t + u_1)(t - u_1)(t - u_2) \cdot t^{-3}$$

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and

$$\left[1-\frac{\overline{F}_{S_1}(t)}{\overline{F}_{S_1}(u_2)}\right]\left[1-\frac{\overline{F}_{S_2}(t)}{\overline{F}_{S_2}(u_1)}\right]=(t+u_2)(t-u_1)(t-u_2)\cdot t^{-3}.$$

Therefore, it is easy to see that condition '$1-\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 in $(i,u)\in\{1,2\}\times(1,t]$' holds true. Similarly, suppose $S_1\sim Pareto(1,1)$, and $S_2\sim Pareto(2,1)$, then '$1-\frac{\overline{F}_{S_i}(t)}{\overline{F}_{S_i}(u)}$ is TP_2 in $(i,u)\in\{1,2\}\times(0,t]$'.

Now, we are ready to present the results for the allocation of two relevations simultaneously to a coherent system for case (ii) $\mathcal{S}_i=\mathcal{S}_j$ and case (iv) $\mathcal{S}_i\subseteq\mathcal{S}_j$. It is to be noted that for case (ii) $\mathcal{S}_i=\mathcal{S}_j$, although a result was provided in [10] on optimal allocation policy for two relevations (performed one at a time), no result exists in the literature for two relevations (performed at the same time) allocation to a coherent system. To fill this gap in the literature, we provide the following result, which employs Theorem 6 and uses the notation from Subsection 3.1.

Theorem 7. *Let* $Z_1\sim\tau\left(T_i\#S_1,T_j\#S_2,\boldsymbol{T}^{(i,j)}\right)$ *and* $Z_2\sim\tau\left(T_i\#S_2,T_j\#S_1,\boldsymbol{T}^{(i,j)}\right)$. *Consider two locations* $i,j\in\{1,\ldots,n\}$ *such that* $i<j$ *and* $\{i,j\}\subseteq C_r$, *for* $r=i_1,\ldots,i_r$, *where* $\{i_1,\ldots,i_r\}\subseteq\{1,\ldots,m\}$, *i.e.* $\mathcal{S}_i=\mathcal{S}_j$. *If any of the following sets of conditions are satisfied:*

(i) $1-(\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ *is* RR_2 *in* $(i,u)\in\{1,2\}\times(0,t]$ *and* $T_i\le_{rh}(\ge_{rh})\,T_j$;

(ii) $1-(\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ *is* TP_2 *in* $(i,u)\in\{1,2\}\times(0,t]$ *and* $T_i\ge_{rh}(\le_{rh})\,T_j$;

then

$$Z_1\ge_{st}(\le_{st})\,Z_2.$$

Proof. The idea of the proof is borrowed from [10, Theorem 2]. First, we prove (i). To prove $Z_1\ge_{st}Z_2$ under the assumptions '$1-\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 in $(i,u)\in\{1,2\}\times(0,t]$' and '$T_i\le_{rh}T_j$', let us consider two parallel systems having lifetime $V_1=\max\left\{T_i\#S_1,T_j\#S_2\right\}$ and $V_2=\max\left\{T_i\#S_2,T_j\#S_1\right\}$. Then, for some minimal cut set C_r such that $\{i,j\}\subseteq C_r$, the survival functions of systems Z_1 and Z_2 can be expressed as

$$\begin{aligned}\overline{F}_{Z_1}(t)&=\mathrm{P}\Bigg(\min_{1\le l\le m,l\notin\{i_1,i_2,\ldots,i_r\}}\left\{\max_{k\in C_l}\{T_k\}\right\}>t,\min_{s\in\{i_1,i_2,\ldots,i_r\}}\left\{\max\left\{\max_{k\in C_s,k\ne i,j}\{T_k\},T_i\#S_1,\right.\right.\\&\qquad\left.\left.T_j\#S_2\right\}\right\}>t\Bigg)\\&=\mathrm{P}\Bigg(\min_{1\le l\le m,l\notin\{i_1,i_2,\ldots,i_r\}}\left\{\max_{k\in C_l}\{T_k\}\right\}>t,\max\left\{\min_{s\in\{i_1,i_2,\ldots,i_r\}}\left\{\max_{k\in C_s,k\ne i,j}\{T_k\}\right\},V_1\right\}>t\Bigg)\\&=\Bigg(1-F_{V_1}(t)\,\mathrm{P}\Bigg(\min_{s\in\{i_1,i_2,\ldots,i_r\}}\left\{\max_{k\in C_s,k\ne i,j}\{T_k\}\right\}\le t\,\Big|\\&\qquad\min_{1\le l\le m,l\notin\{i_1,i_2,\ldots,i_r\}}\left\{\max_{k\in C_l}\{T_k\}\right\}>t\Bigg)\Bigg)\mathrm{P}\Bigg(\min_{1\le l\le m,l\notin\{i_1,i_2,\ldots,i_r\}}\left\{\max_{k\in C_l}\{T_k\}\right\}>t\Bigg),\end{aligned}$$

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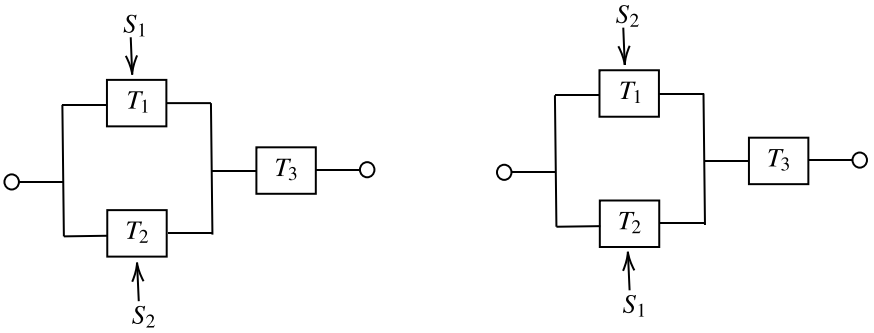


FIGURE 7. Schematic diagrams of systems with lifetimes Z_1 (left) and Z_2 (right) corresponding to Example 4.

and

$$\overline{F}_{Z_2}(t) = \mathrm{P}\Bigg(\min_{1\le l\le m, l\notin\{i_1,i_2,\ldots,i_r\}} \Big\{ \max_{k\in C_l} \{T_k\} \Big\} > t, \max \Big\{ \min_{s\in\{i_1,i_2,\ldots,i_r\}} \Big\{ \max_{k\in C_s, k\neq i,j} \{T_k\} \Big\}, V_1 \Big\} > t \Bigg)$$

$$= \Bigg(1 - F_{V_2}(t)\,\mathrm{P}\Bigg(\min_{s\in\{i_1,i_2,\ldots,i_r\}} \Big\{ \max_{k\in C_s, k\neq i,j} \{T_k\} \Big\} \le t \,\Big|\, \min_{1\le l\le m, l\notin\{i_1,i_2,\ldots,i_r\}} \Big\{ \max_{k\in C_l} \{T_k\} \Big\} > t \Bigg) \Bigg) \mathrm{P}\Bigg(\min_{1\le l\le m, l\notin\{i_1,i_2,\ldots,i_r\}} \Big\{ \max_{k\in C_l} \{T_k\} \Big\} > t \Bigg).$$

Thus, to prove $Z_1 \ge_{st} Z_2$, it suffices to show that $V_1 \ge_{st} V_2$, which we obtain, by using conditions '$1 - \overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 in $(i, u) \in \{1, 2\} \times (0, t]$' and '$T_i \le_{rh} T_j$', from Theorem 6.

In a similar vein we can prove that $Z_1 \le_{st} Z_2$ under the assumptions '$1 - \frac{\overline{F}_{S_i}(t)}{\overline{F}_{S_i}(u)}$ is RR_2 in $(i, u) \in \{1, 2\} \times (0, t]$' and '$T_i \le_{rh} T_j$'. Hence, the results of Theorem 7 of (i) are proved.

The proof of (ii) of Theorem 7 is analogous to the proof of part (i), hence it is omitted. □

To illustrate the application of Theorem 7, we present the following example. Recall that T is said to be a truncated normal random variable with mean zero, variance $\sigma(>0)$, and parameter $\mu = 1/2$, denoted by $T \sim TN(\mu, \sigma)$, if its density function is given by

$$\overline{F}_T(t; \mu, \sigma) = \frac{1}{\mu\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2} \right), \quad t > 0.$$

Example 6. Consider a coherent system made up of three components having lifetimes T_1, T_2, and T_3. Suppose the lifetime of a coherent system is $\tau(\boldsymbol{T}) = \min\{\max\{T_1, T_2\}, T_3\}$. Thus, the minimal cut sets are $\{3\}$ and $\{1, 2\}$ and, consequently, $\mathcal{S}_1 = \mathcal{S}_2 = \{1, 2\}$. Suppose that either relevation S_1 is performed with T_1 and S_2 is performed with T_2 or S_1 is performed with T_2 and S_2 is performed with T_1 which gives that the resultant system has lifetime Z_1 or Z_2, i.e. $Z_1 = \min\{\max\{T_1\#S_1, T_2\#S_2\}, T_3\}$ and $Z_2 = \min\{\max\{T_1\#S_2, T_2\#S_1\}, T_3\}$ (see Figure 7). Further, let $T_i \sim TN(\mu_i, \sigma_i)$ for $i = 1, 2, 3$, where $\sigma_i > 0$, for $i = 1, 2$. In addition, suppose that S_1 and S_2 are independent random variables coming from the Weibull family with $S_1 \sim W(2, 3)$ and $S_2 \sim W(2, 2)$ for $t > 0$. It is easy to verify that when $\sigma_1 \le \sigma_2$, then $T_1 \le_{rh} T_2$. Since $S_1 \le_{rh} S_2$, it implies that condition '$1 - \overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 in $(i, u) \in \{1, 2\} \times (0, t]$' holds true. Thus, the conditions of Theorem 7 (i) are satisfied.

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With regards to case (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$, Wu *et al.* [30] established the usual stochastic order for two relevations allocation (performed one at a time). Their result was further extended by [34, Theorem 2] by establishing the usual stochastic order for two relevations allocation (performed both at the same time). As mentioned previously (Theorem 6), we explore the conjecture of [34, Theorem 2] by employing Theorem 6 and establishing the usual stochastic order for two relevations allocation (performed both at the same time) to a coherent system.

Theorem 8. *Let* $Z_1 \sim \tau\left(T_i\#S_1, T_j\#S_2, \boldsymbol{T}^{(i,j)}\right)$ *and* $Z_2 \sim \tau\left(T_i\#S_2, T_j\#S_1, \boldsymbol{T}^{(i,j)}\right)$. *Suppose* $\mathcal{S}_i \subseteq \mathcal{S}_j$. *If any of the following sets of conditions are satisfied:*

(i) $1-(\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ *is* RR_2 *in* $(i,u) \in \{1,2\} \times (0,t]$, $T_i \leq_{rh} (\geq_{rh}) T_j$ *and* $S_1 \leq_{hr} (\geq_{hr}) S_2$;

(ii) $1-(\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ *is* TP_2 *in* $(i,u) \in \{1,2\} \times (0,t]$, $T_i \geq_{rh} (\leq_{rh}) T_j$ *and* $S_1 \leq_{hr} (\geq_{hr}) S_2$;

then $Z_1 \geq_{st} (\leq_{st}) Z_2$.

Proof. Denote $V_1 = \max\left\{T_i\#S_1, T_j\#S_2\right\}$ and $V_2 = \max\left\{T_i\#S_2, T_j\#S_1\right\}$. First, we prove part (i). In view of [34, Theorem 2], to prove $Z_1 \geq_{st} Z_2$, it is suffices to show that $V_1 \geq_{st} V_2$ and $\mathrm{P}\left(T_j\#S_2 \leq t\right) \leq \mathrm{P}\left(T_j\#S_1 \leq t\right)$. Now, since T_j is non-negative, $S_1 \leq_{hr} S_2$ implies that $\mathrm{P}\left(T_j\#S_2 \leq t\right) \leq \mathrm{P}\left(T_j\#S_1 \leq t\right)$. By using conditions '$1-\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 in $(i,u) \in \{1,2\} \times (0,t]$' and '$T_i \leq_{rh} T_j$', from Theorem 6, we get $V_1 \geq_{st} V_2$. Hence, the result is proved. In the similar vein, one can prove that $Z_1 \leq_{st} Z_2$ under the assumption that '$1-\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 in $(i,u) \in \{1,2\} \times (0,t]$', '$T_i \geq_{rh} T_j$' and '$S_1 \geq_{hr} S_2$'. Hence, the results of part (i) are proved.

The proof of part (ii) of Theorem 8 is analogous to the proof of part (i), hence it is omitted. □

As observed, one of the conditions of Theorem 8, i.e. $S_1 \geq_{hr} S_2$, is the same as a condition in [34, Theorem 2], where the authors conjectured that $T_i \leq_{rh} T_j$ might be required to prove $F_{T_i\#S_1}(t)F_{T_j\#S_2}(t) \geq F_{T_i\#S_2}(t)F_{T_j\#S_1}(t)$, for $t > 0$. We have found in Theorem 8 that to prove this conjecture we require $T_i \leq_{rh} T_j$, as well as $1-(\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u))$ is TP_2 in $(i,u) \in \{1,2\} \times (0,t]$. Moreover, our proposed Theorems 7 and 8 contribute to the literature by adding some new results on coherent systems, and also, collectively summarizing all the existing results in a comprehensive manner.

The following example illustrates the application of Theorem 8.

Example 7. Consider a coherent system made up of three components having lifetimes T_1, T_2, and T_3. Suppose, the lifetime of a coherent system is $\tau(\boldsymbol{T}) = \max\{\min\{T_1, T_2\}, T_3\}$. We have $\mathcal{S}_1 = \{\{1,3\}\}$ and $\mathcal{S}_3 = \{\{1,3\}, \{2,3\}\}$ and, consequently, $\mathcal{S}_1 \subseteq \mathcal{S}_3$. Let $T_i \sim TN(\mu_i, \sigma_i)$ for $i = 1, 2, 3$, where $\sigma_i > 0$ and $\mu_i > 0$ for $i = 1, 2$. It is easy to verify that when $\sigma_1 \leq \sigma_3$, then $T_1 \leq_{rh} T_3$. Further, suppose that S_1 and S_2 are independent random variables from the Pareto family, with $S_1 \sim \mathrm{Pareto}(\alpha_1, \lambda_1)$ and $S_2 \sim \mathrm{Pareto}(\alpha_2, \lambda_2)$, where $\alpha_1, \alpha_2, \lambda_1, \lambda_2 > 0$, and $t > \min(\lambda_1, \lambda_2)$. It can be seen that if $\lambda_1, \lambda_2 > 0$ and $\alpha_1 = \alpha_2$, then the conditions '$1-\overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ is RR_2 in $(i,u) \in \{1,2\} \times (0,t]$' and '$S_1 \leq_{hr} S_2$' are satisfied. Hence, the conditions of Theorem 8(i) are satisfied.

We provide the following counterexample, which shows that the conditions of Theorem 8 are not sufficient to establish hazard rate order (or reversed hazard rate order), i.e. $Z_1 \geq_{hr} (\leq_{hr}) Z_2$ (or $Z_1 \geq_{rh} (\leq_{rh}) Z_2$). Furthermore, we also show that the conjecture proposed

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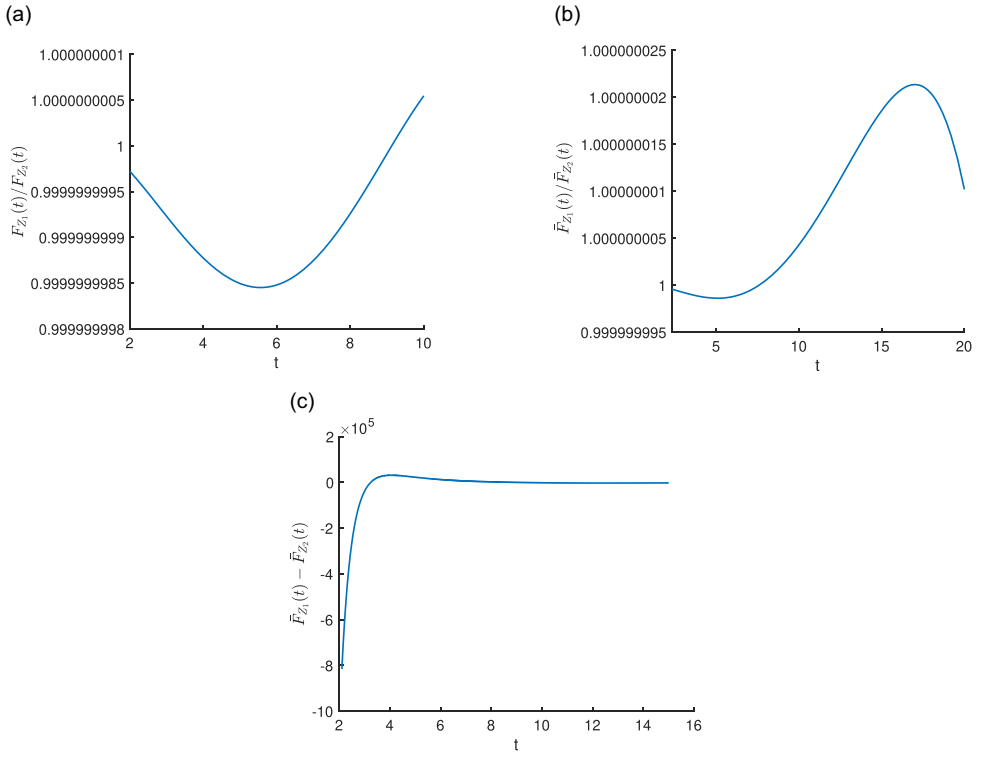


FIGURE 8. (a) Plot of the ratio of distribution functions of Z_1 and Z_2 (i.e. $F_{Z_1}(t)/F_{Z_2}(t)$). (b) Plot of the ratio of survival functions of Z_1 and Z_2 (i.e. $\overline{F}_{Z_1}(t)/\overline{F}_{Z_2}(t)$). (c) Plot of the difference of survival functions of Z_1 and Z_2 (i.e. $\overline{F}_{Z_1}(t) - \overline{F}_{Z_2}(t)$).

in [34, Theorem 2] (i.e. stating that $T_i \leq_{rh} T_j$ might required to prove $F_{T_i\#S_1}(t)F_{T_j\#S_2}(t) \geq F_{T_i\#S_2}(t)F_{T_j\#S_1}(t)$, for $t > 0$), may not hold true for a coherent system.

Counterexample 4. *Under the setup of Example 7, assume that* $\sigma_1 = 2$, $\sigma_2 = 20$, $\sigma_3 = 10$, $\mu_1 = \mu_2 = \mu_3 = 0.5$, $\alpha_1 = \alpha_2 = 2$, $\lambda_1 = 1$, *and* $\lambda_2 = 2$. *Then, the conditions of Theorem 8 are satisfied. However, from Figure 8(a) we can see that the ratio of distribution functions of* Z_1 *and* Z_2 *(i.e.* F_{Z_1}/F_{Z_2}*) is neither increasing nor decreasing. Hence, we can conclude that* $Z_1 \not\geq_{rh} (\not\leq_{rh}) Z_2$. *Similarly, we can see from Figure 8(b) that* $Z_1 \not\geq_{hr} (\not\leq_{hr}) Z_2$, *when* $\sigma_1 = 2$, $\sigma_2 = 20$, $\sigma_3 = 20$, $\mu_1 = \mu_2 = \mu_3 = 0.5$, $\alpha_1 = \alpha_2 = 2$, $\lambda_1 = 1$, *and* $\lambda_2 = 2$ *(i.e. conditions of Theorem 8 are satisfied). Thus, the conditions of Theorem 8 are not sufficient to establish reversed hazard rate and hazard rate orders between* Z_1 *and* Z_2.

Moreover, it is interesting to see that if condition '$1 - \overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ *is* RR_2 *(or* TP_2*) in* $(i, u) \in \{1, 2\} \times (0, t]$*' is relaxed in Theorem 8, then will the result hold or not ? It is easy to see that if* $\sigma_1 = 2$, $\sigma_2 = 10$, $\sigma_3 = 20$, $\mu_1 = \mu_2 = \mu_3 = 0.5$, $\alpha_1 = 5$, $\alpha_2 = 10$, *and* $\lambda_1 = \lambda_2 = 1$, *then* $T_1 \leq_{rh} T_3$ *and* $S_1 \geq_{hr} S_2$ *hold but the condition '*$1 - \overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ *is* RR_2 *(or* TP_2*) in* $(i, u) \in \{1, 2\} \times (0, t]$*' does not hold. Thus, we can see from Figure 8(c) that if* $T_1 \leq_{rh} T_3$ *and* $S_1 \geq_{hr} S_2$ *hold but '*$1 - \overline{F}_{S_i}(t)/\overline{F}_{S_i}(u)$ *is* RR_2 *(or* TP_2*) in* $(i, u) \in \{1, 2\} \times (0, t]$*' does not hold, then neither* $Z_1 \not\geq_{st} Z_2$ *nor* $Z_1 \not\leq_{st} Z_2$. *Hence, the above conjecture given by [34] may not be true for a general coherent system.*

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From the discussion, it is clear that exploring the problem of simultaneously allocating two relevations to a coherent system in terms of reversed hazard rate order is intriguing. This remains an open problem that warrants further investigation.

4. Applications

In this section, we aim to demonstrate the practical application of our theoretical results in reliability engineering. Through a simulation study and real-world examples, we highlight the effect and effectiveness of the theorems we developed. We begin with a simulation study that provides a thorough comparison of the two systems described in Subsection 3.1. By doing so, we study the simulation study at component level (for Theorem 2) and system level (for Theorem 1) in Subsection 4.1. Later, in Subsection 4.2, we conduct a real data analysis that aims to enhance the tensile strength of a high-voltage electricity transmission network. Finally, in Subsection 4.3, we study the allocation of two relevations in a coherent system under a minimal repair policy.

4.1. Simulation study

4.1.1. *An empirical data analysis at component level:* In this subsection, we conduct a simulation study to investigate the implications and performance of Theorem 2. To achieve this, we analyze and assess the behavior of the following systems:

$$W_1 = \max\{\min\{T_1 \# S_1, T_2, T_3\}, T_4\} \quad \text{and} \quad W_2 = \max\{\min\{T_1, T_2 \# S_2, T_3\}, T_4\}.$$

We begin with the assumption that the random variables T_1 and T_2 follow Weibull distributions with the distribution functions:

$$F_{T_1}(t; \alpha, \lambda_1) = \begin{cases} 1 - \mathrm{e}^{-(\lambda_1 t)^{\alpha}}, & \text{if } \mathrm{t} \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$F_{T_2}(t; \alpha, \lambda_2) = \begin{cases} 1 - \mathrm{e}^{-(\lambda_2 t)^{\alpha}}, & \text{if } \mathrm{t} \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha > 0$, $\lambda_i > 0$, $i = 1, 2$. In addition, assume that $T_1 \overset{d}{=} T_4$ and $T_2 \overset{d}{=} T_3$. Further, let two random variables S_1 and S_2 that follow Weibull distributions with distribution functions,

$$F_{S_1}(t; \alpha, \beta_1) = \begin{cases} 1 - \mathrm{e}^{-(\beta_1 t)^{\alpha}}, & \text{if } \mathrm{t} \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$F_{S_2}(t; \alpha, \beta_2) = \begin{cases} 1 - \mathrm{e}^{-(\beta_2 t)^{\alpha}}, & \text{if } \mathrm{t} \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha > 0$, $\beta_i > 0$, $i = 1, 2$. First, we fix the value of α as 2. We now follow a step-by-step procedure.

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TABLE 1. Goodness-of-fit test.

Parameter	Estimate	Statistic	p-value
λ_1	0.3992	0.1152	0.5155
λ_2	0.1993	0.1149	0.5168
β_1	0.2991	0.1363	0.4340
β_2	0.5960	0.3880	0.0778

Step 1. Simulate random numbers from the distribution functions F_{T_i} for $i = 1, 2, 3, 4,$ and G_{S_i} for $i = 1, 2$. We generate 10,000 random numbers from $F_{T_1}(t; 2, 0.4)$, $F_{T_2}(t; 2, 0.2)$, $F_{T_3}(t; 2, 0.2)$, $F_{T_4}(t; 2, 0.4)$, $G_{S_1}(t; 2, 0.3)$, and $G_{S_2}(t; 2, 0.6)$.

Step 2. Use the simulated data to obtain an estimate (see, [36]) of parameters ($\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\beta}_1$ and $\hat{\beta}_2$). The Cramér–von Mises statistic is used in the context of goodness-of-fit tests [14, 15, 32, 33]. It measures the discrepancy between an empirical distribution function (EDF) $K_n(v)$ and a theoretical distribution function $K_{\theta_n}(v)$. The statistic is given by

$$S_n(K) = n \int_0^1 \left(K_n(v) - K_{\theta_n}(v)\right)^2 \, \mathrm{d}K_{\theta_n}(v)$$

where $K_n(v)$ is the EDF based on the sample data and $K_{\theta_n}(v)$ is the theoretical cumulative distribution function (CDF) under the null hypothesis, parameterized by θ_n. The p-values of the goodness-of-fit test are obtained as follows.

- From the estimated parameters ($\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\beta}_1$ and $\hat{\beta}_2$), we obtain estimated F_{T_i} for $i = 1, 2, 3, 4,$ and G_{S_i} for $i = 1, 2$.
- For each F_{T_i}, $i = 1, 2, 3, 4,$ and G_{S_i}, $i = 1, 2$, generate random observations and obtain Cramer–von Mises statistic $S_n(K)$. Repeat this process 10,000 times to obtain 10,000 test statistics.
- Estimate the p-value by calculating the percentage of the above statistics that exceed the value obtained from the simulated data.

The p-values in Table 1 are relatively high, indicating that the simulated data aligns well with the model. From Table 1, it can be seen that the conditions of our Theorem 2 (i.e. $\hat{\lambda}_2 \le \hat{\lambda}_1$, $\hat{\beta}_1 \le \hat{\beta}_2$, and $\hat{\beta}_1 \le \hat{\lambda}_1$, imply '$T_1 \le_{hr} T_2$', '$S_2 \le_{hr} S_1$', and '$T_1 \le_{hr} S_1$', respectively) are satisfied. In addition, from Example 2, we can see that since $\hat{\lambda}_2 \le \hat{\lambda}_1$, i.e. $T_4 \le_{hr} T_3$ implies that condition

$$\frac{\mathrm{P}\left(\max\limits_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l} \{T_m\} > t\right)}{\mathrm{P}\left(\max\limits_{s\in\{i_1,\ldots,i_r\}} \min\limits_{m\in P_s, m\neq i,j} \{T_m\} > t, \max\limits_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l} \{T_m\} \le t\right)}$$

is decreasing in $t > 0$, is satisfied. Thus, all conditions of Theorem 2 are satisfied.

Therefore, we can conclude that $W_1 \ge_{hr} W_2$. To illustrate this, we plot the survival functions of W_1 and W_2, and also plot the ratio of survival functions W_1 and W_2 (i.e. $\overline{F}_{W_1}(t)/\overline{F}_{W_2}(t)$) using

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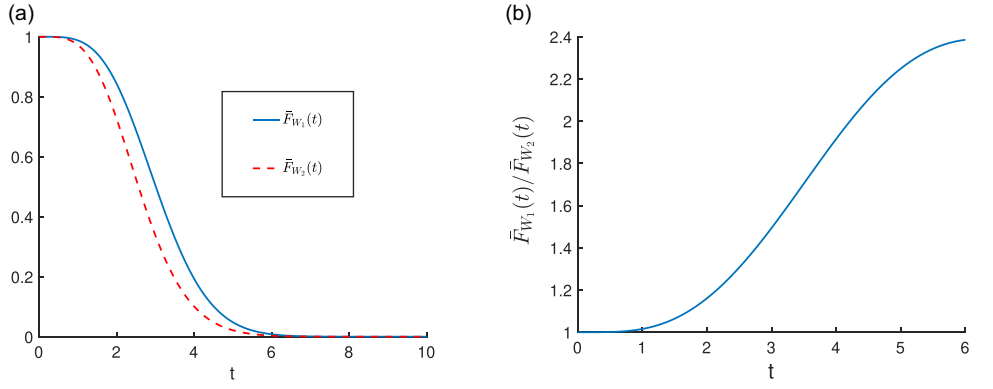


FIGURE 9. (a) Plot of the survival functions of W_1 and W_2 $\left(\text{i.e. } \overline{F}_{W_1}(t) \text{ and } \overline{F}_{W_2}(t)\right)$. (b) Plot of the ratio of survival functions of W_1 and W_2 (i.e. $\overline{F}_{W_1}(t)/\overline{F}_{W_2}(t)$).

the estimated parameters, as shown in Figure 9(a), and (b), respectively. The plot demonstrates that $\overline{F}_{W_1}(t)/\overline{F}_{W_2}(t)$ is increasing in $t > 0$, implies that $W_1 \geq_{hr} W_2$.

4.1.2. *An empirical data analysis at system level:* This subsection presents a simulation study to illustrate the result of Theorem 1. Consider two systems defined by

$$U_1 = \min\{T_1 \# S_1, T_2\}, \quad \text{and} \quad U_2 = \min\{T_2 \# S_2, T_1\},$$

where T_1 and T_2 are component lifetimes, and S_1 and S_2 represent the lifetimes of two available relevations.

Assume that the lifetimes T_1, T_2, S_1, and S_2 are mutually independent and exponentially distributed. Let $T_1 \sim \text{Exp}(\lambda_1)$, $T_2 \sim \text{Exp}(\lambda_2)$, $S_1 \sim \text{Exp}(\beta_1)$, and $S_2 \sim \text{Exp}(\beta_2)$, with survival functions

$$\bar{F}_{T_1}(t) = \mathrm{e}^{-\lambda_1 t}, \quad \bar{F}_{T_2}(t) = \mathrm{e}^{-\lambda_2 t}, \quad \bar{F}_{S_1}(t) = \mathrm{e}^{-\beta_1 t},$$

$$\bar{F}_{S_2}(t) = \mathrm{e}^{-\beta_2 t}, \quad \text{where } \lambda_1, \lambda_2, \beta_1, \text{ and } \beta_2 > 0.$$

If $\lambda_1 \geq \lambda_2$, $\beta_1 \leq \beta_2$, and $\beta_1 \leq \lambda_1$ then the conditions $T_1 \leq_{hr} T_2$, $S_1 \geq_{hr} S_2$, and $S_1 \geq_{hr} T_1$ hold true, respectively. Consequently, the conditions of Theorem 1 are satisfied. Thus, the values of the parameters are fixed (such that the conditions of Theorem 1 are satisfied) as

$$\lambda_1 = 0.07, \quad \lambda_2 = 0.05, \quad \beta_1 = 0.03, \quad \beta_2 = 0.04.$$

To examine the behavior of the systems U_1 and U_2, we simulate 1,000 observations from each distribution. The samples are generated using the inverse transform method based on the derived forms of the CDFs for each system. The Kaplan–Meier estimator (Kaplan and Meier [18]) is used to estimate the survival functions of U_1 and U_2. Figure 10(a) illustrates that the system U_1 tends to exhibit higher survival probabilities than U_2 across time. Moreover, the ratio of the survival functions plotted in Figure 10(b) shows a non-decreasing trend. This empirical observation provides numerical support for the hazard rate ordering $U_1 \geq_{hr} U_2$, thereby validating the theoretical result established in Theorem 1.

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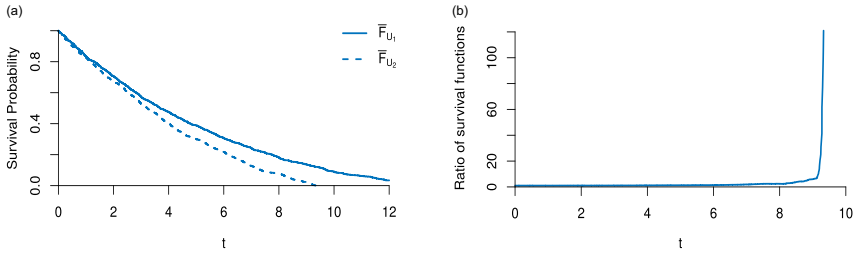


FIGURE 10. (a) Plot of the survival functions of U_1 and U_2 $\left(\text{i.e. } \overline{F}_{U_1}(t) \text{ and } \overline{F}_{U_2}(t)\right)$. (b) Plot of the ratio of survival functions of U_1 and U_2 (i.e. $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$).

4.2. A real data analysis

A cable is usually made up of several wires and, over time, the tensile strength of a cable generally diminishes due to the wear and aging of its internal parts. As a result, improving the tensile strength of cables is an essential and fascinating challenge, with the goal of maximizing their reliability and maintaining their operational status for as long as possible. To address this issue, a common strategy is to introduce redundancies, such as relevation, at the component level.

Based on the data recorded by Hand *et al.* [17], the tensile strengths of wires (refer to Table 2) were examined in a study that included a collection of cables taken from various high-voltage electricity transmission networks. A cable typically comprises multiple wires, and the strength of the structured cable can be viewed as a k-out-of-n system, with each wire regarded as an individual component. There are nine cables, and each cable contains twelve wires. This dataset has been widely used in various research papers on reliability optimization, including [16, 26, 32, 33]. Now, let us consider a cable composed of two wires connected in series, namely Wire 7 (lifetime is denoted by T_1) and Wire 5 (lifetime is denoted by T_2), and two available relevations to perform, namely Wire 4 (lifetime is denoted by S_1) and Wire 6 (lifetime is denoted by S_2). In addition, assume that either the relevation S_1 is performed with T_1 or S_2 is performed with T_2, which gives a resultant system has lifetime U_1 or U_2, i.e. $U_1 = \min\{T_1\#S_1, T_2\}$ and $U_2 = \min\{T_1, T_2\#S_2\}$. Thus, we want to make a comparison between $U_1 = \min\{T_1\#S_1, T_2\}$ and $U_2 = \min\{T_1, T_2\#S_2\}$.

To start, we use Gamma, Log-Normal, Exponential, Weibull, and Pareto distributions to model the tensile strengths of these four wires (Wires 4, 5, 6, and 7). We estimated the parameters of these four wires based on the method of maximum-likelihood estimation [36], wherein the values of the criterion such as the Akaike information criterion (AIC; cf. [1]) and Bayesian information criterion (BIC; cf. [28]) are also estimated. It can be seen from Table 3 that the best-fitted distribution is Log-Normal for Wire 5, Wire 6, and Wire 7. Overall, it should be also noted that the best-fit distribution is Log-Normal (since AIC and BIC of Log-Normal are lower compared with other distributions) of all wires. Thus, we can conclude that the best-fitted distribution is Log-Normal for all four wires. From Table 3, for Log-Normal distribution, it can be seen that the estimated standard deviations are approximately the same (≈ 0.02, say $\hat{\sigma}$) for T_1, T_2, S_1, and S_2. Now, suppose, the density functions of the Log-Normal distribution of T_1, T_2, S_1, and S_2 are denoted as $f_{T_1}(t;\mu_1,\sigma), f_{T_2}(t;\mu_2,\sigma), f_{S_1}(t;\mu_3,\sigma)$, and $f_{S_2}(t;\mu_4,\sigma)$, respectively, where $\sigma > 0$, $\mu_{i's} > 0$, for $i = 1, 2, 3, 4$, and $t > 0$. Suppose the maximum-likelihood estimates of μ_i and σ are $\hat{\mu}_i$ and $\hat{\sigma}$, respectively, for Log-normal distribution. Since, it can be seen from Table 3 that $\hat{\mu}_1 \le \hat{\mu}_2$, $\hat{\mu}_4 \le \hat{\mu}_3$, and $\hat{\mu}_1 \le \hat{\mu}_3$, which imply '$T_1 \le_{hr} T_2$', '$S_2 \le_{hr} S_1$',

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TABLE 2. Dataset for wire measurements.

Dataset	Tensile strength of wires								
Wire 4	338	348	328	350	340	336	341	345	348
Wire 5	330	337	338	335	350	339	336	350	355
Wire 6	334	328	332	332	346	340	342	348	351
Wire 7	335	328	335	328	345	342	347	341	333

and '$T_1 \leq_{hr} S_1$', respectively. Hence, we can see that the conditions of our Theorem 1 are satisfied for the real data set. It can be seen from Figure 11 that the ratio of survival functions of U_1 and U_2 (i.e. $\overline{F}_{U_1}/\overline{F}_{U_2}$) is increasing. Thus, $U_1 \geq_{hr} U_2$.

4.3. Minimal repair policy

In reliability engineering, the minimal repair policy is a highly cost-effective approach, frequently employed to extend the lifetime of a system. In minimal repair, a failed component is replaced with a redundant one that has the same age and distribution as the original component at the time of failure. Suppose that the survival functions of random variables T and S are the same (say $\overline{F}$), then relevation is reduced to minimal repair policy, as follows:

$$\overline{F}_{T\#S}(t) = \overline{F}(t) + \overline{F}(t) \int_0^t \frac{1}{\overline{F}(x)} \, \mathrm{d}F(x) = \overline{F}(t)[1 - \ln(\overline{F}(t))].$$

Barlow and Hunter [4] first introduced the notion of minimal repair. Note that the minimal repair is a special case of relevation [31] and is also an alternate to standby redundancy allocation. Minimal repair is employed in various real-world situations. For instance, in the case of heart failure caused by coronary occlusion, cardiopulmonary resuscitation (CPR) can revive the patient. If the revival occurs without significant damage to vital organs, CPR can be considered a minimal repair [6]. For more details on minimal repair, the reader may refer to [2, 3, 10, 24, 31, 34, 35] and others.

Belzunce *et al.* [10] studied the optimal allocation policy for two minimal repairs (performed one at a time) to a two-component series system. They proved that $U_1 \geq_{st} U_2$, if $T_1 \leq_{st} T_2$ and $T_i \overset{d}{=} S_i$ for $i = 1, 2$ (see [10, Lemma 2]). Later, Yan *et al.* [31] provided a necessary and sufficient condition to prove $U_1 \geq_{hr} U_2$ for a two-component series system. They established that $U_1 \geq_{hr} U_2$ if and only if $r_{T_1}(t)/(1 - \ln(\overline{F}_{T_1}(t))) \geq r_{T_2}(t)/(1 - \ln(\overline{F}_{T_2}(t)))$, where r_{T_i} is the hazard rate of random variable T_i for $i = 1, 2$. Moreover, in Theorem 1, we studied the optimal allocation of two relevations (performed one at a time) in the two-component series system, in the sense of hazard rate order. As we know, minimal repair is a special case of relevation, in light of this, it is interesting to determine the sufficient conditions in the case of minimal repair. The following theorem provides sufficient conditions to establish $U_1 \geq_{hr} (\leq_{hr}) U_2$ in case of the minimal repair policy (i.e. $T_i \overset{d}{=} S_i$ for $i = 1, 2$).

Theorem 9. *Let* $U_1 = \min\{T_1 \# T_1, T_2\}$ *and* $U_2 = \min\{T_1, T_2 \# T_2\}$. *Suppose* H_{T_1} *and* H_{T_2} *are cumulative hazard rates of random variables* T_1 *and* T_2, *respectively. If* H_{T_1}/H_{T_2} *is increasing (decreasing), and* $T_1 \leq_{hr} (\geq_{hr}) T_2$ *then* $U_1 \geq_{hr} (\leq_{hr}) U_2$.

Proof. First, we prove that $U_1 \geq_{hr} U_2$ under the conditions H_{T_1}/H_{T_2} is increasing, and $T_1 \leq_{hr} T_2$. To prove $U_1 \geq_{hr} U_2$, it suffices to show that $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$ is increasing in $t > 0$.

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TABLE 3. Estimate of parameters of four wires.

Wire	Distribution	MLE	AIC	BIC
Wire 4	Gamma	shape = 2321.367, rate = 6.7967	63.73970	64.13415
Wire 4	LNorm	**meanlog = 5.8333, sdlog = 0.0195**	**63.70862**	**64.10307**
Wire 4	Exp	rate = 0.0029	125.00319	125.20041
Wire 4	Weibull	**shape = 64.8032, scale = 344.6259**	**62.57137**	**62.96582**
Wire 4	Pareto	shape = 5725063.5725, scale = 1952741996.3093	127.00320	127.39765
Wire 5	Gamma	**shape = 1869.757, rate = 5.482**	**66.79049**	**67.18494**
Wire 5	LNorm	**meanlog = 5.8319, sdlog = 0.0232**	**66.75677**	**67.15122**
Wire 5	Exp	rate = 0.0029	124.97975	125.17697
Wire 5	Weibull	shape = 43.4577, scale = 345.1539	68.51233	68.90678
Wire 5	Pareto	shape = 4217915.3987, scale = 1439034532.0315	126.97975	127.37420
Wire 6	Gamma	**shape = 1712.0152, rate = 5.0488**	**66.25260**	**66.64705**
Wire 6	LNorm	**meanlog = 5.8264, sdlog = 0.0226**	**66.16431**	**66.55876**
Wire 6	Exp	0.0029	124.87980	125.07702
Wire 6	Weibull	shape = 49.1605, scale = 342.9849	66.74408	67.13853
Wire 6	Pareto	shape = 4778084.8466, scale = 1620415101.2033	126.87980	127.27425
Wire 7	Gamma	**shape = 2232.0617, rate = 6.6202**	**63.58830**	**63.98275**
Wire 7	LNorm	**meanlog = 5.8202, sdlog = 0.0196**	**63.48042**	**63.87486**
Wire 7	Exp	rate = 0.003	124.76743	124.96465
Wire 7	Weibull	shape = 57.3725, scale = 340.3374	63.90417	64.29862
Wire 7	Pareto	shape = 2982905.8774, scale = 1005566065.5832	126.76743	127.16188

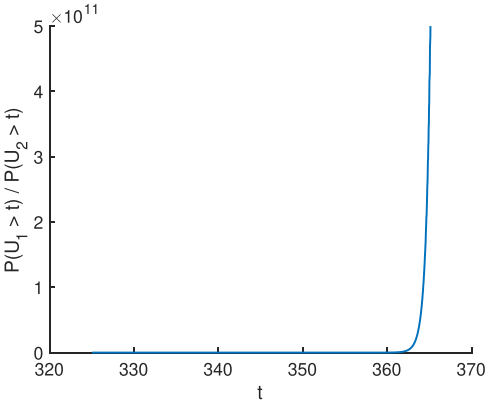


FIGURE 11. Plot of the ratio of survival functions of U_1 and U_2 (i.e. $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$).

We have

$$\overline{F}_{U_1}(t) = \left(\overline{F}_{T_1}(t) + \overline{F}_{T_1}(t)\int_0^t \frac{f_{T_1}(u)}{\overline{F}_{T_1}(u)}\,\mathrm{d}u\right)\overline{F}_{T_2}(t) = \overline{F}_{T_1}(t)\big[1 - \ln(\overline{F}_{T_1}(t))\big]\overline{F}_{T_2}(t), \quad t > 0, \tag{12}$$

and

$$\overline{F}_{U_2}(t) = \left(\overline{F}_{T_2}(t) + \overline{F}_{T_2}(t)\int_0^t \frac{f_{T_2}(u)}{\overline{F}_{T_2}(u)}\,\mathrm{d}u\right)\overline{F}_{T_1}(t) = \overline{F}_{T_2}(t)\big[1 - \ln(\overline{F}_{T_2}(t))\big]\overline{F}_{T_1}(t), \quad t > 0. \tag{13}$$

From (12) and (13), it can be seen that

$$\frac{\overline{F}_{U_1}(t)}{\overline{F}_{U_2}(t)} = \frac{1 - \ln(\overline{F}_{T_1}(t))}{1 - \ln(\overline{F}_{T_2}(t))} = \frac{g(t) + g_1(t)}{g(t) + g_2(t)}, \quad t > 0, \tag{14}$$

where $g(t) = 1$, $g_1(t) = -\ln(\overline{F}_{T_1}(t))$ and $g_2(t) = -\ln(\overline{F}_{T_2}(t))$. To prove $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$ is increasing in $t > 0$, it suffices to show that

$$\frac{g(t) + g_1(t)}{g(t) + g_2(t)} \quad \text{is increasing in } t > 0. \tag{15}$$

Now, $g_1(t)/g_2(t) = -\ln(\overline{F}_{T_1}(t))/-\ln(\overline{F}_{T_2}(t)) = H_{T_1}(t)/H_{T_2}(t)$ is increasing in $t > 0$, where H_{T_1} and H_{T_2} are cumulative hazard rates of random variables T_1 and T_2, respectively. Further, to prove $g(t)/(g_2(t) - g_1(t)) = 1/\ln\left(\overline{F}_{T_1}(t)/\overline{F}_{T_2}(t)\right)$ is increasing, it suffices to show that $\ln(\overline{F}_{T_1}(t)/\overline{F}_{T_2}(t))$ is decreasing in $t > 0$:

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$$\left[\ln\left(\frac{\overline{F}_{T_1}(t)}{\overline{F}_{T_2}(t)}\right)\right]' \overset{sgn}{=} -\frac{f_{T_1}(t)}{\overline{F}_{T_1}(t)} + \frac{f_{T_2}(t)}{\overline{F}_{T_2}(t)}$$
$$= \frac{f_{T_2}(t)}{\overline{F}_{T_2}(t)} - \frac{f_{T_1}(t)}{\overline{F}_{T_1}(t)}$$
$$\leq 0.$$

The last inequality is due to $T_1 \leq_{hr} T_2$ (i.e. $f_{T_2}(t)/\overline{F}_{T_2}(t) \leq f_{T_1}(t)/\overline{F}_{T_1}(t)$ for all $t > 0$). Thus, $g(t)/(g_2(t) - g_1(t))$ is increasing in $t > 0$. From Lemma 1, it is easy to see that $(g(t) + g_1(t))/(g(t) + g_2(t))$ is increasing in $t > 0$, since $g_1(t)/g_2(t)$ and $g(t)/(g_2(t) - g_1(t))$ are increasing in $t > 0$. Hence, $U_1 \geq_{hr} U_2$ holds true under the assumptions H_{T_1}/H_{T_2} is increasing in $t > 0$, and $T_1 \leq_{hr} T_2$.

Similarly, one can prove that $U_1 \leq_{hr} U_2$ holds true under the assumptions H_{T_1}/H_{T_2} is decreasing in $t > 0$, and $T_1 \geq_{hr} T_2$. □

Note that the conditions in Theorem 9 are easy to prove and satisfied by some general families of distribution. For instance, see the following examples which satisfy by some common families of distributions, such as Weibull and Pareto.

Example 8.

(a) Weibull. Let $T_1 \sim W(2, 3)$ and $T_2 \sim W(2, 2)$. Here, $\overline{F}_{T_2}(t)/\overline{F}_{T_1}(t) = e^{t^2(3^2-2^2)}$ is increasing in $t > 0$. It implies that $T_1 \leq_{hr} T_2$. Now, $H_{T_1}(t)/H_{T_2}(t) = \ln(\overline{F}_{T_1}(t))/\ln(\overline{F}_{T_2}(t)) = 4/9$ is nondecreasing in $t > 0$.

(b) Pareto. Let $T_1 \sim Pareto(3, 2)$, and $T_2 \sim Pareto(1, 2)$. It is easy to see that condition '$T_1 \leq_{hr} T_2$' holds true, since $\overline{F}_{T_2}(t)/\overline{F}_{T_1}(t) = t^2$ is increasing in $t > 0$. In addition, one can see that $H_{T_1}(t)/H_{T_2}(t)$ is increasing in $t > 0$.

Note that [10, Lemma 2] showed that $U_1 \geq_{st} U_2$ if $T_1 \leq_{st} T_2$ in a two-component series system under the minimal repair policy, i.e. $T_i \overset{d}{=} S_i$ for $i = 1, 2$. In Theorem 9, we showed that $U_1 \geq_{hr} (\leq_{hr}) U_2$ if $T_1 \leq_{hr} (\geq_{hr}) T_2$ and H_{T_1}/H_{T_2} is increasing (decreasing) in $t > 0$. Thus, it is evident that when we move from a weaker stochastic ordering, i.e. $U_1 \geq_{st} U_2$ to a stronger stochastic ordering, i.e. $U_1 \geq_{hr} U_2$, $T_1 \leq_{st} T_2$ gets strengthen to $T_1 \leq_{hr} T_2$ and an additional condition $H_{T_1}(t)/H_{T_2}(t)$ is increasing in $t > 0$, is also needed for the result to hold true. This highlights the trade-off between the set of sufficient conditions when we move from a weaker to a stronger stochastic ordering.

The following example illustrates the application of Theorem 9.

Example 9. Suppose that the random variables $T_1 \sim \text{Pareto}(\alpha_1, \lambda_1)$ and $T_2 \sim \text{Pareto}(\alpha_2, \lambda_2)$, where $\alpha_1, \alpha_2, \lambda_1, \lambda_2 > 0$, and $t > \min(\lambda_1, \lambda_2)$. It is easy to see that if $\lambda_1 \geq \lambda_2 \geq 1$ then $H_{T_1}(t)/H_{T_2}(t)$ is increasing in $t > 0$, since $[H_{T_1}(t)/H_{T_2}(t)]' \overset{sgn}{=} (1/t)(\ln\lambda_1 - \ln\lambda_2) \geq 0$, for all $t > 0$ and $\lambda_1 \geq \lambda_2 \geq 1$. Moreover, if $\alpha_1 \geq \alpha_2$ then $T_1 \leq_{hr} T_2$ holds true. Thus, the conditions of Theorem 9 are satisfied. In particular, if $\lambda_1 = 4$, $\lambda_2 = 2$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.1$ (implying that $T_1 \leq_{hr} T_2$ and $H_{T_1}(t)/H_{T_2}(t)$ is increasing in $t > 0$) then from Figure 12(a) we can see that the ratio of survival functions of U_1 and U_2 (i.e. $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$) is increasing in $t > 0$.

The following counterexample shows that the condition $H_{T_1}(t)/H_{T_2}(t)$ is increasing in $t > 0$ in Theorem 9, is not sufficient to prove $U_1 \geq_{hr} U_2$.

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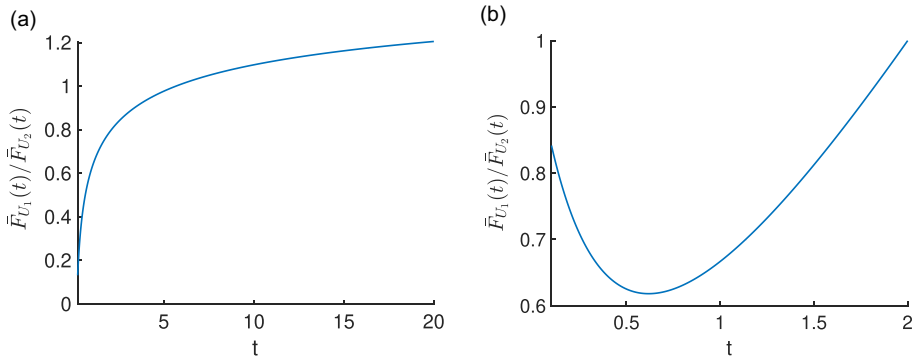


FIGURE 12. Plot of the ratio of survival functions of U_1 and U_2 (i.e. $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$).

Counterexample 5. *Suppose that the random variables* $T_1 \sim W(1, 2)$ *and* $T_2 \sim W(2, 1)$. *It is easy to see that* $H_{T_1}(t)/H_{T_2}(t) = 0.25 \times t$ *is increasing in* $t > 0$. *However,* $T_1 \leq_{hr} T_2$ *does not hold true, since* $\overline{F}_{T_1}(t)/\overline{F}_{T_2}(t) = \mathrm{e}^{t(t-2)}$ *is neither increasing nor decreasing in* $t > 0$. *From Figure 12(b), it can be seen that the ratio of survival functions of* U_1 *and* U_2 *(i.e.* $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$*) is neither increasing nor decreasing in* $t > 0$ *(i.e.* $U_1 \not\geq_{hr} U_2$*). Thus, we can conclude that to prove* $U_1 \geq_{hr} U_2$, *condition* $H_{T_1}(t)/H_{T_2}(t)$ *is increasing in* $t > 0$, *is not sufficient.*

In Theorem 9, we established the result for minimal repairs in a series system, in the sense of hazard rate order. In light of this, it is interesting to determine sufficient conditions for minimal repairs in a coherent system in the sense of hazard rate order. In this regard, Yan *et al.* [31] provide an 'if and only if' condition to establish $W_1 \geq_{hr} W_2$ for minimal repair policy for a coherent system. In the following theorem, we give sufficient conditions to establish hazard rate order for a coherent system in the case of minimal repairs.

Theorem 10. *Let* $W_1 \sim \tau\left(T_i \# T_i, T_j, \boldsymbol{T}^{(i,j)}\right)$ *and* $W_2 \sim \tau\left(T_i, T_j \# T_j, \boldsymbol{T}^{(i,j)}\right)$. *Consider two locations* $i, j \in \{1, \ldots, n\}$ *such that* $i < j$ *and* $\{i, j\} \subseteq P_r$, *for* $r = i_1, \ldots, i_r$, *where* $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$, *i.e.* $\mathcal{A}_i = \mathcal{A}_j$. *Suppose*

$$\frac{\mathrm{P}\left(\max\limits_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l}\{T_m\} > t\right)}{\mathrm{P}\left(\max\limits_{s\in\{i_1,\ldots,i_r\}} \min\limits_{m\in P_s, m\neq i,j}\{T_m\} > t, \max\limits_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min\limits_{m\in P_l}\{T_m\} \leq t\right)}$$

is decreasing in $t > 0$. *If* $\overline{F}_{U_2}(t) - \overline{F}_{U_1}(t)$ *is decreasing (increasing) and negative (positive) in* $t > 0$ *then* $W_1 \geq_{hr} (\leq_{hr}) W_2$.

Proof. The proof is analogous to Theorem 2, hence it is omitted. □

It is worth mentioning here that the condition $\overline{F}_{U_2}(t) - \overline{F}_{U_1}(t)$ is decreasing (increasing) and negative (positive) in $t > 0$ implies that $\overline{F}_{U_1}(t)/\overline{F}_{U_2}(t)$ is increasing (decreasing) in $t > 0$, i.e. $U_1 \geq_{hr} (\leq_{hr}) U_2$ (the result established in Theorem 9).

Note that establishing the stochastic comparison between V_1' and V_2' (for the series system with two revelations), V_1 and V_2 (for the parallel system with two revelations), and Z_1 and Z_2 (for a coherent system with two revelations) under a minimal repair policy is trivial, since both systems have same lifetimes.

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Up to now, we focused on the optimal allocation policy for two minimal repairs (performed one at a time). In real-life situations, reliability engineers might face the problem of where to allocate multiple minimal repairs to a series or coherent system. This has caught the attention of the researchers over the past decade. In this regard, Arriaza *et al.* [3], Navarro *et al.* [24], and Navarro and Fernández-Martnez [25] studied the allocation of r number of minimal repairs to the same components in coherent systems by using redundancy distortion function approach. Moreover, the majority of the results focused either on identical components or on the same number of minimal repairs performed to series and coherent systems. In light of this, we focus on the optimal allocation policy for multiple minimal repairs (not necessarily the same number of repairs) on a single node (or component) within coherent systems with nonidentical components. To begin with, we develop the result of a series system and then later generalize it to a coherent system. Suppose $T_i \#^r T_i$ denote the lifetime of the component i (having lifetime T_i) after allocating $r\,(>0)$ minimal repairs. Then, the survival function of $T_i \#^r T_i$ is given by

$$\mathrm{P}(T_i \#^r T_i > t) = \overline{F}_{T_i}(t) \sum_{k=0}^{r} \frac{1}{k!} [-\ln \overline{F}_{T_i}(t)]^k,$$

where $\overline{F}_{T_i}$ is the survival function of T_i (see, [25]). Suppose, we have a two-component series system with component lifetimes T_1 and T_2. Furthermore, assume that either k_1 minimal repairs are performed on T_1 or k_2 are on T_2. That is, the resultant series systems have lifetimes $U_1^* = \min\left\{T_1 \#^{k_1} T_1, T_2\right\}$ and $U_2^* = \min\left\{T_1, T_2 \#^{k_2} T_2\right\}$. In addition, suppose the system consists of nonidentical components. With these assumptions, we are now ready to state the following theorem for the series system.

Theorem 11. *Consider a two-component series system with nonidentical components having lifetime T_1 and T_2. Let $U_1^* = \min\left\{T_1 \#^{k_1} T_1, T_2\right\}$ and $U_2^* = \min\left\{T_1, T_2 \#^{k_2} T_2\right\}$. If $k_1 \geq (\leq) k_2$ and $T_1 \leq_{st} (\geq_{st}) T_2$ then $U_1^* \geq_{st} (\leq_{st}) U_2^*$.*

Proof. First, we prove that $U_1^* \geq_{st} U_2^*$ under the conditions $k_1 \geq k_2$ and $T_1 \leq_{st} T_2$. To prove it, we first prove for case (i) $k_1 = k_2 = r$ (say) and then for case (i) $k_1 > k_2$.

Case (i): $k_1 = k_2 = r$. We have

$$\mathrm{P}\big(U_1^* > t\big) = \overline{F}_{T_1}(t)\overline{F}_{T_2}(t) \sum_{k=0}^{r} \frac{1}{k!}\big(-\ln \overline{F}_{T_1}(t)\big)^k, \tag{16}$$

and

$$\mathrm{P}\big(U_2^* > t\big) = \overline{F}_{T_1}(t)\overline{F}_{T_2}(t) \sum_{k=0}^{r} \frac{1}{k!}\big(-ln\overline{F}_{T_2}(t)\big)^k. \tag{17}$$

Thus, from (16) and (17), we have

$$\mathrm{P}\big(U_1^* > t\big) - \mathrm{P}\big(U_2^* > t\big) = \overline{F}_{T_1}(t)\overline{F}_{T_2}(t)\left[\sum_{k=0}^{r} \frac{1}{k!}\big(-\ln \overline{F}_{T_1}(t)\big)^k - \big(-\ln \overline{F}_{T_2}(t)\big)^k\right],$$

$$\geq 0.$$

The last inequality holds due to the fact that $T_1 \leq_{st} T_2$, implies that $\big[-ln\overline{F}_{T_1}(t)\big]^k \geq \big[-ln\overline{F}_{T_2}(t)\big]^k$.

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Case (ii): $k_1 > k_2$. We have

$$\mathrm{P}\big(U_1^* > t\big) = \overline{F}_{T_1}(t)\overline{F}_{T_2}(t) \sum_{k=0}^{k_1} \frac{1}{k!}\big(-\ln \overline{F}_{T_1}(t)\big)^k, \tag{18}$$

and

$$\mathrm{P}\big(U_2^* > t\big) = \overline{F}_{T_1}(t)\overline{F}_{T_2}(t) \sum_{k=0}^{k_2} \frac{1}{k!}\big(-\ln \overline{F}_{T_2}(t)\big)^k. \tag{19}$$

Thus, from (18) and (19), we have

$$\begin{aligned}\mathrm{P}\big(U_1^* > t\big) - \mathrm{P}\big(U_2^* > t\big) = \overline{F}_{T_1}(t)\overline{F}_{T_2}(t)\Bigg[\sum_{k=0}^{k_2} \frac{1}{k!}\Big(\big(-\ln \overline{F}_{T_1}(t)\big)^k - \big(-\ln \overline{F}_{T_2}(t)\big)^k\Big) \\ + \sum_{k=k_2+1}^{k_1} \frac{1}{k!}\big(-\ln \overline{F}_{T_1}(t)\big)^k\Bigg], \\ \geq 0.\end{aligned}$$

The last inequality holds due to the fact that $T_1 \leq_{st} T_2$.

Similarly, we can prove $U_1^* \leq_{st} U_2^*$ under the conditions $k_1 \leq k_2$ and $T_1 \geq_{st} T_2$. □

Note that in Theorem 11 if we assume that $k_1 = k_2$, then one can obtain the conditions of [25, Proposition 1(i)] (see Remark 4 there).

In the context of a coherent system, Arriaza *et al.* [3] studied the optimal repair policy for any coherent system with identically distributed components, where r minimal repairs are performed to the system. Later, Navarro and Fernández-Martnez [25] established the usual stochastic order for r minimal repairs to a coherent system having nonidentical components. In particular, suppose a system has n components with lifetimes $T_1, T_2, \ldots, T_n$, and r minimal repairs are available to be performed (either perform at T_i or T_j, i.e. the resultant systems having lifetimes $\tau\left(T_i \#^r T_i, T_j, \boldsymbol{T}^{(i,j)}\right)$ and $\tau\left(T_i, T_j \#^r T_j, \boldsymbol{T}^{(i,j)}\right)$). Then, they studied the optimal repair policy for coherent systems, focusing either on identical components or on the same number of minimal repairs performed to the same node (or component). One natural question arises here that if we perform different numbers of minimal repairs to T_i and T_j, and the system has nonidentical components, then what should be the sufficient conditions to prove $W_1^* \geq_{st} (\leq_{st}) W_2^*$, denoted as $W_1^* \sim \tau\left(T_i \#^{k_1} T_i, T_j, \boldsymbol{T}^{(i,j)}\right)$ and $W_2^* \sim \tau\left(T_i, T_j \#^{k_2} T_j, \boldsymbol{T}^{(i,j)}\right)$? To answer this question, we provide the following results which establish the usual stochastic order between W_1^* and W_2^* for cases (i) $\mathcal{A}_i = \mathcal{A}_j$ and (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$.

Theorem 12. *Let* $W_1^* \sim \tau\left(T_i \#^{k_1} T_i, T_j, \boldsymbol{T}^{(i,j)}\right)$ *and* $W_2^* \sim \tau\left(T_i, T_j \#^{k_2} T_j, \boldsymbol{T}^{(i,j)}\right)$. *Consider two locations* $i, j \in \{1, \ldots, n\}$ *such that* $i < j$ *and* $\{i, j\} \subseteq P_r$, *for* $r = i_1, \ldots, i_r$, *where* $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$, *i.e.* $\mathcal{A}_i = \mathcal{A}_j$. *If* $k_1 \geq (\leq)\, k_2$ *and* $T_i \leq_{st} (\geq_{st})\, T_j$ *then* $W_1^* \geq_{st} (\leq_{st})\, W_2^*$.

Proof. To prove the result, let us consider two series systems having lifetime $U_1^* = \min\{T_i \#^{k_1} T_i, T_j\}$ and $U_2^* = \min\left\{T_i, T_j \#^{k_2} T_j\right\}$. First, we prove that $W_1^* \geq_{hr} W_2^*$ under the assumptions $k_1 \geq k_2$ and $T_i \leq_{hr} T_j$. Then, for some minimal path set P_r such that $\{i, j\} \subseteq P_r$, the lifetimes of systems W_1^* and W_2^* can be expressed as

$$W_1^* = \max\left\{\min\left\{\max_{s\in\{i_1,\ldots,i_r\}} \min_{m\in P_s, m\neq i,j} \{T_m\}, T_i \#^{k_1} T_i, T_j\right\}, \max_{1\leq l\leq k, l\notin\{i_1,\ldots,i_r\}} \min_{m\in P_l} \{T_m\}\right\},$$

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and

$$W_2^* = \max\left\{\min\left\{\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\}, T_i, T_j \#^{k_2} T_j\right\}, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\}\right\}.$$

Thus, we get

$$\begin{aligned}\overline{F}_{W_1^*}(t) &= 1 - \mathrm{P}\Bigg(\min\left\{\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\}, T_i \#^{k_1} T_i, T_j\right\} \le t,\\ &\qquad \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\Bigg)\\ &= \mathrm{P}\left(U_1^* > t\right)\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right)\\ &\quad + \mathrm{P}\left(\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} > t\right), \quad t > 0,\end{aligned}$$

and

$$\begin{aligned}\overline{F}_{W_2^*}(t) &= \mathrm{P}\left(U_2^* > t\right)\mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right)\\ &\quad + \mathrm{P}\left(\max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} > t\right), \quad t > 0.\end{aligned}$$

Now, we have

$$\overline{F}_{W_1^*}(t) - \overline{F}_{W_2^*}(t) = \left(\mathrm{P}\left(U_1^* > t\right) - \mathrm{P}\left(U_2^* > t\right)\right) \times \delta(t), \tag{20}$$

where

$$\delta(t) = \mathrm{P}\left(\max_{s\in\{i_1,\ldots,i_r\}}\min_{m\in P_s, m\neq i,j}\{T_m\} > t, \max_{1\le l\le k, l\notin\{i_1,\ldots,i_r\}}\min_{m\in P_l}\{T_m\} \le t\right).$$

Thus, from (20), it can be seen that to prove $W_1^* \ge_{st} (\le_{st}) W_2^*$, it is equivalent to prove $U_1^* \ge_{st} (\le_{st}) U_2^*$. Hence, the result follows from Theorem 11. □

In Theorem 12, we established the usual stochastic order between W_1^* and W_2^* for case (i) $\mathcal{A}_i = \mathcal{A}_j$. Now, we investigate case (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$, to establish the usual stochastic order between W_1^* and W_2^* in the following theorem.

Theorem 13. *Let* $W_1^* \sim \tau\left(T_i \#^{k_1} T_i, T_j, \boldsymbol{T}^{(i,j)}\right)$ *and* $W_2^* \sim \tau\left(T_i, T_j \#^{k_2} T_j, \boldsymbol{T}^{(i,j)}\right)$. *Suppose* $\mathcal{A}_i \subseteq \mathcal{A}_j$. *If* $k_1 \ge (\le) k_2$ *and* $T_i \le_{st} (\ge_{st}) T_j$ *then* $W_1^* \ge_{st} (\le_{st}) W_2^*$.

Proof. The idea of the proof is borrowed from [30, Theorem 2.1]. To establish that $W_1^* \ge_{st} (\le_{st}) W_2^*$, it suffices to prove that $\mathrm{P}(T_i > t)\,\mathrm{P}\left(T_j \#^{k_2} T_j > t\right) \le (\ge) \mathrm{P}(T_j > t)\,\mathrm{P}\left(T_i \#^{k_1} T_i > t\right)$, i.e. $U_1^* \ge_{st} (\le_{st}) U_2^*$. Consequently, the result follows from Theorem 11. □

5. Conclusion

In this article, we have studied the optimal allocation policy of two relevations to a coherent system in the sense of the usual stochastic order and hazard rate order. For this purpose,

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we have focused on cases (i) $\mathcal{A}_i = \mathcal{A}_j$, (ii) $\mathcal{S}_i = \mathcal{S}_j$, (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$, and (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$ (discussed in Section 1) and established stochastic comparison results for a coherent system, by building on the results of series or parallel systems. In particular, for case (i) $\mathcal{A}_i = \mathcal{A}_j$, we studied the optimal allocation policies of two relevations performed one at a time to a series system and later generalized to a general coherent system in the sense of hazard rate order in Theorems 1 and 2, respectively. By doing so, we answer the call for future research on hazard rate order by Belzunce *et al.* [10]. For cases (i) $\mathcal{A}_i = \mathcal{A}_j$ and (iii) $\mathcal{A}_i \subseteq \mathcal{A}_j$, we provided additional set of sufficient conditions to those obtained in [34] (see Theorems 4 and 5). We further contribute to the literature by studying the optimal allocation of two relevations (performed at the same time) to a parallel system and a coherent system for cases (ii) $\mathcal{S}_i = \mathcal{S}_j$ and (iv) $\mathcal{S}_i \subseteq \mathcal{S}_j$ (see Theorems 6, 7, and 8). By establishing new results on the allocation of two relevations, either performed one at a time or simultaneously to a coherent system, we not only extend the findings of [10, 30], but also contribute to the results of [34] (see Theorems 1, 2, 6, 7, and 8).

In addition, we examined the minimal repair policy in the case of a general coherent system. In real-life situations, reliability engineers often face the problem of how to allocate multiple minimal repairs within a series or coherent system. This issue has garnered considerable attention from researchers over the past decade. In this context, Arriaza *et al.* [3], Navarro *et al.* [24], and Navarro and Fernández-Martnez [25] investigated the allocation of r minimal repairs to the same components in coherent systems using the redundancy distortion function approach. Moreover, most existing studies have focused either on systems with identical components or on scenarios involving the same number of minimal repairs applied to series and coherent systems. Motivated by these limitations, we studied the optimal allocation policy for multiple minimal repairs, possibly differing in number, assigned to a single node (or component) in coherent systems with nonidentical components.

Specifically, we studied the problem of minimal repair allocation policy to one node (that is, to one component) in a series (coherent) system and made the stochastic comparisons in the sense of hazard rate order (see Theorems 9 and 10). Moreover, we also established the case for multiple minimal repairs to one node in a series system and then later generalized it to a coherent system in the sense of the usual stochastic order (see Theorems 11, 12, and 13).

Our study offers valuable insights for future research. In Theorems 1 and 2, we established results for the hazard rate order in a series system and a general coherent system. It is interesting to extend this investigation to the problem of simultaneously allocating two relevations to a coherent system in the sense of the hazard rate order. To establish the hazard rate order for a single relevation allocation problem, we employed Lemma 1. A similar type of lemma is required to derive results for the simultaneous allocation of two relevations in the hazard rate order. Specifically, while Lemma 1 requires that the function $(g(t) + g_1(t))/(g(t) + g_2(t))$ be increasing in $t > 0$, we now seek a sufficient condition under which the function $(g_1(t) + g_2(t))/(g_3(t) + g_4(t))$ is increasing in $t > 0$. This remains an open problem that warrants further investigation. Moreover, we considered cases (i)–(iv) out of six possible cases (as discussed in Section 1); the remaining two cases, (v) and (vi), remain open for exploration. It is important to mention that, in the existing literature, the results for cases (v) and (vi) impose rather restrictive conditions on the Birnbaum reliability importance measure of the components. In addition, several distinct subcases need to be considered in order to establish the results for cases (v) and (vi) [30, 34]. Further, it would be interesting to explore relevation allocation policies for coherent systems, particularly in the case of three or more than three relevations. To the best of the authors' knowledge, most existing results have been

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established under the assumption of independent components; therefore, exploring relevation allocation policy in the presence of dependency is also of significant interest.

To underscore the practical relevance of our findings, we included a simulation-based study and a real data analysis that affirm the applicability and effect of our theoretical results. Moreover, we illustrate our findings with a range of examples and counterexamples. Thus, we believe our research provides valuable insights into the optimal allocation strategies in coherent systems, contributing to both theoretical understanding and practical applications in stochastic orderings.

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

Data

The data related to the tensile strength of wires, presented in Section 4, can be found in the book by Hand *et al.* [17].

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Appendix A.

Proof of Lemma 2.3. Since, $g_1(t)/g_2(t)$ and $g(t)/(g_2(t) - g_1(t))$ are increasing in $t > 0$, for all $t_2 \geq t_1 > 0$ such that t_1 and $t_2 \in t$, then we have

$$\frac{g_1(t_1)}{g_2(t_1)} \leq \frac{g_1(t_2)}{g_2(t_2)} \tag{A1}$$

and

$$\frac{g(t_1)}{g_2(t_1) - g_1(t_1)} \leq \frac{g(t_2)}{g_2(t_2) - g_1(t_2)}. \tag{A2}$$

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Let, $\phi(t) = (g(t) + g_1(t))/(g(t) + g_2(t))$. To prove $\phi(t)$ is increasing in $t > 0$, it suffices to prove that $\phi(t_2) \geq \phi(t_1)$ for all $t_2 \geq t_1 > 0$. Now, for all $t_2 \geq t_1 > 0$, we have

$$\begin{aligned}
\phi(t_2) - \phi(t_1) &= \frac{g(t_2) + g_1(t_2)}{g(t_2) + g_2(t_2)} - \frac{g(t_1) + g_1(t_1)}{g(t_1) + g_2(t_1)} \\
&\overset{sgn}{=} g(t_1)g(t_2) + g(t_2)g_2(t_1) + g_1(t_2)g(t_1) + g_1(t_2)g_2(t_1) \\
&\quad - g(t_1)g(t_2) - g(t_1)g_2(t_2) - g_1(t_1)g(t_2) - g_1(t_1)g_2(t_2) \\
&= g(t_2)g_2(t_1) + g_1(t_2)g(t_1) + g_1(t_2)g_2(t_1) - g(t_1)g_2(t_2) - g_1(t_1)g(t_2) - g_1(t_1)g_2(t_2) \\
&= \left[g(t_2)g_2(t_1) + g_1(t_2)g(t_1) - g(t_1)g_2(t_2) - g_1(t_1)g(t_2)\right] \\
&\qquad + \left[g_1(t_2)g_2(t_1) - g_1(t_1)g_2(t_2)\right] \\
&\geq \left[g(t_2)g_2(t_1) + g_1(t_2)g(t_1) - g(t_1)g_2(t_2) - g_1(t_1)g(t_2)\right] \\
&\geq 0.
\end{aligned}$$

The first inequality is due to (A1) and the last inequality is due to (A2). Thus, we have that $\phi(t)$ is increasing in $t > 0$, which further implies that $(g(t) + g_1(t))/(g(t) + g_2(t))$ is increasing in $t > 0$. □

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