On the minimum modulus of integral functions. By Miss M. L. CARTWRIGHT, M.A., Girton College.

[*Received* 23 July, *read* 29 October 1934.]

1·1. Let $f(z)$ be an integral function of order ρ, and let

$$m(r) = \min_{0 \leqslant \theta < 2\pi} |f(re^{i\theta})|, \quad M(r) = \max_{0 \leqslant \theta < 2\pi} |f(re^{i\theta})|.$$

It is well known that, if $\rho < 1$,

$$\overline{\lim_{r \to \infty}} \frac{\log m(r)}{\log M(r)} \geqslant \cos \pi\rho.$$

This result was first conjectured by Littlewood*, who proved it with $\cos 2\pi\rho$ in place of $\cos \pi\rho$; it was afterwards proved independently by Wiman† and Valiron‡ about the same time. Many other authors§ have written on the subject, considering among other things the set of valves of r for which $m(r)$ is comparatively large.

1·2. We denote by

$$E[\phi(r) > \psi(r)]$$

the set of r for which $\phi(r) > \psi(r)$. If E is any measurable set, the upper and lower limits of the ratio

$$\frac{m[E \times (0, r)]}{r}$$

as $r \to \infty$ are called the upper and lower densities of E. Using a modification of Littlewood's method, Besicovitch‖ proved the following results:

I. *If* $0 < \rho < \rho' < 1$, *the upper density of the set*

$$E[m(r) > (M(r))^{\cos \pi\rho'}]$$

is greater than or equal to $1 - \rho/\rho'$.

II. *If* $0 < \rho < \frac{1}{2}$, *the upper density of the set*

$$E[m(r) > e^{r^{\rho - \epsilon}}]$$

is greater than or equal to $1 - 2\rho$ *for every* $\epsilon > 0$.

* J. E. Littlewood, *Proc. London Math. Soc.* (2), 8 (1908), 189–204.

† A. Wiman, *Math. Ann.* 76 (1915), 197–211.

‡ G. Valiron, *Annales Fac. Sc. Toulouse*, 3 (1913), 117–257. See also *Lectures on the general theory of integral functions* (Toulouse) (1923), 125–132, and *Opuscula Mathematica A. Wiman dedicata* (Lund, 1930).

§ References will be found in the last paper mentioned. See also J. M. Whittaker, *Proc. Edinburgh Math. Soc.* (2), 2 (1930), 111–128 (115).

‖ A. S. Besicovitch, *Math. Ann.* 97 (1927), 677–695.

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

III. *If* $\rho = 0$, *the upper density of the set*

$$E[m(r) > (M(r))^{1-\epsilon}]$$

is 1 *for every* $\epsilon > 0$.

He also proved some results about the lower densities of similar sets, but I shall not be concerned with those here.

2·1. My main object here is to prove results about the upper density which are, in some respects, better than I and II; in III the index and density cannot be improved. I shall use methods similar to those of Valiron, depending on the theorems of Phragmèn and Lindelöf.

THEOREM I. *If* $0 < \rho \leqslant 1$, *either the upper density of the set*

$$E[m(r) > (M(r))^{\cos \pi\rho - \epsilon}] \tag{2·11}$$

is equal to 1 *for every* $\epsilon > 0$, *or*

$$\overline{\lim_{r \to \infty}} \frac{\log m(r)}{\log M(r)} = K > \cos \pi\rho.$$

In the latter case, the upper density of the set (2·11) *for* $\epsilon = 0$ *is greater than* $\delta = \delta(K)$, *which is positive for every* $K > \cos \pi\rho$.

Since $\delta(K) \to 0$ as $K \to \cos \pi\rho$, this only gives 0 as the lower bound to the upper density of (2·11) for functions of order $\rho < 1$, and so the result is, in that respect, no better than I. The extension to functions of order 1 is almost trivial, but though the method is more or less applicable to functions of order $\rho > 1$, it does not seem to give the best possible result.

2·2. We need two well-known theorems, the first of which involves a function $V(z)$, having the following properties:

(1) $V(z)$ is regular for $|z| > l$, $|\arg z| \leqslant \pi$.

(2) $V(x) > 0$ for $x > l$.

(3) $\lim_{r \to \infty} \frac{V(r)}{r^{\rho+\epsilon}} = 0$ for every $\epsilon > 0$.

(4) $\overline{\lim_{r \to \infty}} \frac{V(r)}{r^{\rho-\epsilon}} = \infty$ for every $\epsilon > 0$.

(5) $\lim_{r \to \infty} \frac{V(re^{i\theta})}{V(r)} = e^{\rho\phi i}$ uniformly for $-\pi \leqslant \theta \leqslant \pi$.

THEOREM OF PHRAGMÈN AND LINDELÖF*. *Suppose that* $f(z)$ *is regular and of order* ρ *for* $|z| > l$, $\alpha \leqslant \arg z \leqslant \beta$, *where* $\beta - \alpha < \pi/\rho$. *Suppose also that*

$$h(\theta) = \overline{\lim_{r \to \infty}} \frac{\log |f(re^{i\theta})|}{V(r)} = A \cos \rho\theta + B \sin \rho\theta \tag{2·21}$$

* E. Phragmèn and E. Lindelöf, *Acta Math.* 31 (1908), 381.

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

for $\theta = \alpha$, $\theta = \beta$. *Then*

$$\log |f(re^{i\theta})| < (A \cos \rho\theta + B \sin \rho\theta + \epsilon) V(r)$$

uniformly for $\alpha \leqslant \theta \leqslant \beta$, $r > r_0(\epsilon)$.

THEOREM OF BERNSTEIN*. *Suppose that* $\Phi(z)$ *is regular for* $|z| \leqslant R$, *and that*

$$\log |\Phi(0)| \geqslant -A,$$

$$\log |\Phi(z)| \leqslant A \quad \text{for} \quad |z| \leqslant R.$$

Then, given any two positive numbers $\eta \leqslant \eta_1$, $k < 1$, *we can find a number* $H(\eta, k)$ *such that the set of points for which* $|z| \leqslant kR$ *and the inequality*

$$\log |\Phi(z)| \geqslant -H(\eta, k) A$$

is not satisfied can be included in a system of circles, the sum of whose radii is less than ηR.

2·3. Any integral function of order $\rho < 1$ may be written in the form

$$f(z) = Gz^p \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right);$$

and, without real loss of generality, we may suppose that $G = 1$, $p = 0$. Further we may restrict ourselves to the case in which $z_n > 0$. For if we change the arguments of the zeros, without changing their moduli, so that they all lie on one straight line, we increase $M(r)$ and decrease $m(r)$; and so results found for the particular case, hold *a fortiori* in the general case. In the particular case we obviously have $M(r) = f(r)$ and $m(r) = f(-r)$.

Corresponding to any integral function of finite order ρ, we can construct a proximate order† $\rho(r)$ which is positive, continuous and differentiable on the right and on the left for $r > l$, and such that

$$\lim_{r\to\infty} \rho(r) = \rho, \quad \lim_{r\to\infty} r\rho'(r) \log r = 0,$$

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{\rho(r)}} = 1, \tag{2·31}$$

where $\rho'(r)$ denotes either the right- or left-hand derivative. Further, corresponding to any such proximate order $\rho(r)$, we

* V. Bernstein, *Annali di Mat.* (4), 12 (1933–34), 173–196 (179). This is a generalization of a theorem of Valiron; see *Lectures on the general theory of integral functions*, 89.

† See G. Valiron, *Lectures on the general theory of integral functions*, 64–67. It may be observed that we only require a specially simple case. For since all the zeros lie on the negative real axis, $M(r) = f(r)$ and is obviously a regular function of r for all positive values of r.

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

can construct a function* $V(z)$ satisfying the conditions of § 2·1 and such that

$$(2\cdot32)\quad \lim_{r\to\infty}\frac{V(r)}{r^{\rho(r)}}=1, \quad \lim_{r\to\infty}\frac{V(kr)}{V(r)}=k^{\rho} \text{ for every } k>1.$$

Now suppose that for every $\epsilon>0$ we have

$$(2\cdot33)\qquad \log|f(-r)|<(\cos\pi\rho+\epsilon)V(r)$$

for $r>r_0(\epsilon)$. Applying the theorem of Phragmèn and Lindelöf to $f(z)$ in the angles $-\pi\leqslant\arg z\leqslant 0$, $0\leqslant\arg z\leqslant\pi$, we have

$$(2\cdot34)\qquad \log|f(re^{i\theta})|<(\cos\theta\rho+2\epsilon)V(r)$$

for $|\theta|\leqslant\pi$, $r>r_0(\epsilon)$.

It follows from (2·31) and (2·32) that, given any $\epsilon>0$, we can find a sequence of values of $r=R_1, R_2, \ldots, R_m\to\infty$, such that

$$\log|f(R_m)|>(1-\epsilon)V(R_m).$$

Let
$$\log F(z^2)=\log f(z)-V(z).$$

Then $F(z)$ is regular for $|z|>l$, $|\arg z|\leqslant\frac{1}{2}\pi$, and

$$\log|F(R_m^2)|>-\epsilon V(R_m),$$

$$\log|F(z)|<-2\epsilon V(2R_m)$$

for $|z-R_m^2|\leqslant R_m^2-l$. Hence, applying Bernstein's theorem, we have

$$\log|F(z)|\geqslant-\epsilon H(\eta,k)V(2R_m)$$

for $|z-R_m^2|\leqslant k(R_m^2-l)$, except in circles the sum of whose radii is less than ηR_m^2. Given any $\delta>0$, $\zeta>0$, we can choose k so that the circle $|z-R_m^2|\leqslant k|R_m^2-l|$ covers the sector

$$|\arg z|\leqslant\tfrac{1}{2}\pi-\delta, \quad \zeta R_m^2\sin\delta\leqslant|z|\leqslant R_m^2\sin\delta.$$

Hence, choosing η, k and ϵ in a suitable manner, we have the following result†: given any positive numbers ϵ_1, δ_1, η_1, $\zeta_1<1$, we can find a sequence $R_1', R_2', \ldots, R_m'\to\infty$, such that

$$(2\cdot35)\qquad \log|f(re^{i\theta})|>(\cos\theta\rho-\epsilon_1)V(r)$$

for $\zeta_1 R_m'\leqslant r\leqslant R_m'$, $|\theta|\leqslant\pi-\delta_1$ except in a set of circles the sum of whose radii is at most $\eta_1 R_m'$.

Let $r'e^{i(\pi-\delta_1)}$ be any point for which (2·35) is satisfied in the interval $\zeta_1 R_m'\leqslant r'\leqslant R_m'$, and apply Bernstein's theorem to

$$\log|f(z)|-\cos\pi\rho V(r')$$

* *Loc. cit.* 128–130. See also V. Bernstein, *Reale Accad. d'Italia*, 1 (1933), 339–401.

† Cf. M. L. Cartwright, *Proc. London Math. Soc.* (2) (in the press).

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

in the circle $|z+r'e^{i\delta_1}| \leqslant 2r' \sin \delta_1$. In virtue of (2·34), (2·35) and (2·32), we have

$$\log |f(r'e^{i(\pi-\delta_1)})| - \cos \pi\rho\, V(r') > -(\epsilon_1 + \epsilon(\delta_1))\, V(r'),$$

$$\log |f(z)| - \cos \pi\rho\, V(r') < -(\epsilon_1 + \epsilon(\delta_1))\, V(r'),$$

where $\epsilon(\delta_1) \to 0$ as $\delta_1 \to 0$, for $|z+r'e^{i\delta_1}| \leqslant 2r' \sin \delta_1$ provided that $m > m_0(\epsilon_1)$. Hence, taking $k = \frac{3}{4}$, we have

$$\log |f(z)| - \cos \pi\rho\, V(r') > -(\epsilon_1 + \epsilon(\delta_1))\, H(\eta_2)\, V(r')$$

for $|z+r'e^{i\delta_1}| \leqslant \frac{3}{2} r' \sin \delta_1$, except in circles the sum of whose radii is $\frac{3}{2}\eta_2 r' \sin \delta_1$.

Now each circle of the form $|z+r'e^{i\delta_1}| \leqslant \frac{3}{2} r' \sin \delta_1$ covers a piece of the negative real axis of length greater than

$$\tfrac{1}{2}\sqrt{5}r' \sin \delta_1 > r' \sin \delta_1,$$

including the point $z = -r'$. Hence, given any $\zeta_1 > 0$, $\delta_1 > 0$, we can choose η_1 so small that the negative real axis from $(\zeta_1 + \eta_1) R_m'$ to $(1-\eta_1) R_m'$ can be covered by circles of this form, and the excluded portions will then be less than $\frac{3}{2}\eta_2$ times the whole length, i.e. less than $\frac{3}{2}\eta_2(1-\zeta_1-2\eta_1) R_m'$. It follows that, given any $\epsilon_2 > 0$, $\eta_3 > 0$, $\zeta_2 > 0$, we can choose ϵ_1, δ_1, ζ_1, η_1 and η_2, so that

$$\log m(r) = \log |f(re^{i\eta})| > (\cos \pi\rho - \epsilon_2)\, V(r)$$

for $\zeta_2 R_m'' \leqslant r \leqslant R_m''$, except in intervals of length less than $\eta_3 R_m''$, where $R_1'' < R_2'' < \ldots < R_m'' \to \infty$. That is to say, the result holds in a set of r of upper density 1.

If (2·33) is not satisfied, we may suppose that

$$\overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\eta})|}{V(r)} = \cos \pi\rho + B \sin \pi\rho, \tag{2·36}$$

where $B > 0$; and, by the theorem of Phragmèn and Lindelöf, we have

$$\log |f(re^{i\theta})| < (\cos \theta\rho + B |\sin \theta\rho| + \epsilon)\, V(r) \tag{2·37}$$

for every $\epsilon > 0$, provided that $|\theta| \leqslant \pi$, $r > r_0(\epsilon)$. Let

$$F(z, R_m) = f(z) \exp\{-(\cos \pi\rho + B \sin \pi\rho)\, V(R_m)\}.$$

Then it follows from (2·36) that, given any $\epsilon > 0$, we can find a sequence $R_1, R_2, \ldots, R_m \to \infty$, such that

$$\log |F(z, R_m)| > -\epsilon V(R_m);$$

and it follows from (2·37) that

$$\log |F(z, R_m)| < \{(1+B)\,\epsilon(\delta) + 2\epsilon\}\, V(R_m),$$

where $\epsilon(\delta) \to 0$ as $\delta \to 0$, for $|z+R_m| \leqslant \delta R_m$, $R_m > 2r_0(\epsilon)$. Hence, given any two positive numbers η and $k < 1$, we have

$$\log |F(z, R_m)| > -\{(1+B)\,\epsilon(\delta) + 3\epsilon\}\, H(\eta, k)\, V(R_m),$$

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

for $|z+R_m| \leqslant k\delta R_m$, except in circles the sum of whose radii is less than $\eta\delta R_m$. From this we obtain

$$(2{\cdot}38)\quad \log|f(re^{i\pi})| > \{\cos\pi\rho + B\sin\pi\rho - ((1+B)\,\epsilon(\delta)+3\epsilon)\,H(\eta,k)\} \times V(R_m) > \cos\pi\rho\, V(r),$$

provided that $\delta = \delta(B, H)$ and $\epsilon = \epsilon(B, H)$ are chosen sufficiently small, for $R_m(1-k\delta) \leqslant r \leqslant R_m(1+k\delta)$, except in a set of measure less than $\eta\delta R_m$. The set for which (2·38) holds is of measure greater than $(2k-\eta)\,\delta R_m$. Hence, choosing $k=\eta=\frac{1}{2}$, we find that (2·38) holds in a set of upper density $\frac{1}{2}\delta$, where $\delta>0$ for every $B>0$.

2·4. What we have proved so far may be stated as follows:

If $f(z)$ is an integral function of order ρ, where $0<\rho<1$, and of proximate order $\rho(r)$, and if the zeros all lie on the negative real axis, then either the upper density of the set

$$E[\log m(r) > (\cos\pi\rho - \epsilon)\, r^{\rho(r)}]$$

is 1 for every $\epsilon>0$, or else

$$(2{\cdot}41)\qquad \overline{\lim_{r\to\infty}} \frac{\log m(r)}{r^{\rho(r)}} = \cos\pi\rho + B\sin\pi\rho > \cos\pi\rho.$$

In the latter case the upper density of the set

$$E[\log m(r) > \cos\pi\rho\, r^{\rho(r)}]$$

is $\delta=\delta(B)>0$ for every $B>0$.

If $0<\rho<\frac{1}{2}$, $\cos\pi\rho>0$, and so this is more than we require.

If $\frac{1}{2}\leqslant\rho<1$, it follows from the theorem of Phragmèn and Lindelöf that

$$(2{\cdot}42)\qquad \overline{\lim_{r\to\infty}} \frac{\log|f(\pm ir)|}{r^{\rho(r)}} \geqslant \cos\frac{\pi\rho}{2}.$$

We write $$F(z^2) = f(z)f(-z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{z_n^2}\right),$$

and $$M(r,F) = \max_{0\leqslant\theta\leqslant 2\pi}|F(re^{i\theta})|,\ m(r,F) = \min_{0\leqslant\theta\leqslant 2\pi}|F(re^{i\theta})|.$$

Then $F(z)$ is of order $\frac{1}{2}\rho$, and since $z_n>0$, $M(r,F)=|F(-r)|$ $m(r,F)=|F(r)|$. It follows from (2·42) that for every $\epsilon>0$,

$$\log M(r^2,F) > \left(2\cos\frac{\pi\rho}{2}-\epsilon\right) r^{\rho(r)},$$

for a sequence of values of r tending to ∞. Hence, we can construct the proximate order $\rho_1(r)$ of $F(z)$ so that

$$\lim_{r\to\infty} 2\cos\frac{\pi\rho}{2}\, r^{\rho(r)-2\rho_1(r)} \leqslant 1,$$

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

and so, by the theorem which we have proved, either

$$\log|F(r^2)| = \log|f(r)f(-r)| > \left(2\cos^2\frac{\pi\rho}{2} - \epsilon\right) r^{\rho(r)}$$

for every $\epsilon > 0$ in a set of upper density 1, or else

$$\overline{\lim_{r\to\infty}} \frac{\log|F(r^2)|}{2\cos(\pi\rho/2)\, r^{\rho(r)}} = \cos\frac{\pi\rho}{2} + B\sin\frac{\pi\rho}{2} > \cos\frac{\pi\rho}{2}.$$

In the first case

$$\log|f(-r)| > \left(2\cos^2\frac{\pi\rho}{2} - \epsilon\right) r^{\rho(r)} - \log|f(r)|$$

$$> (\cos\pi\rho - \epsilon)\log|f(r)|$$

in a set of upper density 1. In the second case, we have, by similar reasoning,

$$\log|f(-r)| > \cos\pi\rho \log|f(r)|$$

in a set of r of density $\delta = \delta(B)$ which is positive for every $B > 0$. This is the required result.

If $\rho = 1$, we may, without loss of generality, consider

$$f(z) = e^{cz} \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right) e^{-z/z_n},$$

but we can no longer reduce the proof to the case in which $z_n > 0$. Suppose first that $n(r) = o(r^{\rho(r)})$, where $\rho(r)$ is the proximate order of $f(z)$, and $n(r)$ is the number of zeros of $f(z)$ for which $|z| \leqslant r$. Then the behaviour of $f(z)$ is dominated by the series

$$S(r) = c - \sum_{|z_n|\leqslant r} \frac{1}{z_n}.$$

By methods which I have used elsewhere*, it may be shown that (1) given any $\epsilon > 0$, we can choose $r_0(\epsilon)$ so that

$$\log|f(re^{i\theta})| < \Re\,(S(r)\, e^{i\theta})\, r + \epsilon r^{\rho(r)} \tag{2·43}$$

uniformly for $r \geqslant r_0$, (2) given any $\epsilon > 0$, $\eta > 0$, we can choose $R_0(\epsilon, \eta)$ so that

$$\log|f(re^{i\theta})| > \Re\,(S(r)\, e^{i\theta})\, r - \epsilon r^{\rho(r)} \tag{2·44}$$

for $0 < r < R$, $R > R_0(\epsilon, \eta)$, except in a set of r of measure less than ηR. It follows from (2·43) and the definition of $\rho(r)$ that

$$\overline{\lim_{r\to\infty}} \left\{ \max_{0\leqslant\theta\leqslant 2\pi} \Re\,(S(r)e^{i\theta})\, r \right\} \Big/ r^{\rho(r)} = \overline{\lim_{r\to\infty}} \frac{|S(r)|\, r}{r^{\rho(r)}} = 1.$$

Hence (2·43) and (2·44) give

$$\log m(r) > (-1 - \epsilon)\log M(r)$$

in a set of r of upper density 1.

* See M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1932), 212. See Theorem V.

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

Suppose now that

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r^{\rho(r)}} = A > 0,$$

and write, as above,

$$F(z^2) = f(z)f(-z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{z_n^2}\right),$$

$$m(r, F) = \min_{0\leqslant\theta\leqslant 2\pi} |F(re^{i\theta})|.$$

Further, let $n(r, F)$ denote the number of zeros of $F(z)$ for which $|z| \leqslant r$; then it is easy to show that we can construct $\rho_1(r)$ so that

$$2A = \overline{\lim_{r\to\infty}} \frac{n(r^2, F)}{r^{\rho(r)}} < \lim_{r\to\infty} r^{2\rho_1(r)-\rho(r)}, \tag{2·45}$$

where $\rho_1(r)$ is the proximate order of

$$F_1(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{|z_n|^2}\right).$$

By the theorem which we have proved either

$$\log m(r^2, F) > -\epsilon r^{2\rho_1(r)} > -2\epsilon \log M(r^2, F) \tag{2·46}$$

in a set of upper density 1 for every $\epsilon > 0$, or

$$\overline{\lim_{r\to\infty}} \frac{\log m(r^2, F)}{r^{2\rho_1(r)}} = B > 0, \tag{2·47}$$

and the upper density of the set $E[m(r^2, F) > 1]$ is $\delta = \delta(B) > 0$ for every $B > 0$. Since

$$\log m(r) > \log m(r^2, F) - \log M(r),$$

and, by (2·45), $\log M(r^2, F) < 2\log M(r) < 2r^{2\rho_1(r)}/A$ for $r > r_0$, the required result follows from (2·46) and (2·47).

2·5. For $\rho > 1$, we may still use methods similar to those for $\rho = 1$, but it seems very unlikely that they will give the best possible results. We can write

$$f(z) = \exp(c_1 z + \ldots + c_p z^p) \prod_{n=1}^{\infty}\left(1+\frac{z}{z_n}\right)\exp\left\{\frac{z}{z_n}+\frac{z^2}{2z_n}+\ldots+\frac{z^p}{pz_n^p}\right\},$$

where p is the integral part of ρ. Let

$$F(z^q) = f(z)f(wz)\ldots f(w^q z),$$

where $w^q = 1$, and q is an integer such that $p+1 \leqslant q < 2\rho$. Then

$$F(z^q) = \prod_{n=1}^{\infty}\left(1-\frac{z^q}{z_n^q}\right),$$

and $F(z)$ is of order ρ_1, such that $\rho_1 = \rho/q$ and $\frac{1}{2} < \rho_1 < 1$. Let

$$M(r, F) = \max|F(re^{i\theta})|, \quad m(r, F) = \min|F(re^{i\theta})|.$$

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press

Then $$\overline{\lim_{r\to\infty}} \frac{\log m(r, F)}{\log M(r, F)} \geqslant \cos \pi\rho_1,$$

and, since $\frac{1}{2} < \rho_1 < 1$, $\cos \pi\rho_1 < 0$, it follows that

$$\log m(r) + (q-1) \log M(r) \geqslant \log m(r^q, F)$$
$$\geqslant (\cos \pi\rho_1 - \epsilon) \log M(r^q, F)$$
$$\geqslant (\cos \pi\rho_1 - \epsilon)\, q \log M(r)$$

for a sequence of values of r tending to infinity. Hence

$$\overline{\lim_{r\to\infty}} \frac{\log m(r)}{\log M(r)} = H(\rho) \geqslant q \cos \frac{\pi\rho}{q} - q + 1,$$

where $p + 1 \leqslant q < 2\rho$. The right-hand side tends to $-\infty$ as $\rho \to \infty$ since $\cos(\pi\rho/q) < 0$; but I cannot construct any function for which $H(\rho) < -1$. When $0 < \rho \leqslant 1$, the functions for which $H(\rho)$ is least are functions whose zeros lie on one straight line, and when $\rho = 1$, e^z has $H(\rho) = -1$. When $\rho > 1$, such functions have $H(\rho) = -1$, but it is possible that an uneven distribution of zeros in varying directions may lead to functions for which $H(\rho) < -1$.

https://doi.org/10.1017/S0305004100012652 Published online by Cambridge University Press