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COMMENTS ON THE SURVEY BY BALAKRISHNAN AND ZHAO

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N. Balakrishnan and Peng Zhao have prepared an outstanding survey [2] of recent results that stochastically compare various order statistics and some ranges based on two collections of independent heterogeneous random variables. Their survey focuses on results for heterogeneous exponential random variables and their extensions to random variables with proportional hazard rates. In addition, some results that stochastically compare order statistics based on heterogeneous gamma, Weibull, geometric, and negative binomial random variables are also given. In particular, the authors of [2] have listed some stochastic comparisons that are based on one heterogeneous collection of random variables, and one homogeneous collection of random variables. Personally, I find these types of comparisons to be quite fascinating. Balakrishnan and Zhao have done a thorough job of listing all the known results of this kind.

In the present discussion note I will point out a general technical idea that is useful for deriving results involving random variables with proportional hazard rates, from results that entail independent heterogeneous exponential random variables. I will also provide some additional references for results that compare order statistics—results that are related to the results surveyed by Balakrishnan and Zhao in [2]. Finally I will draw attention to another set of results—those that stochastically compare convolutions of independent heterogeneous exponential random variables—and I will provide some recent references for the interested reader.

Below, for any absolutely continuous random variable Z we denote by F_Z, $\overline{F}_Z$, and f_Z the distribution, survival, and density functions, respectively. As in [2], the term *increasing* is used for *monotone non-decreasing* and similarly *decreasing* is used for *monotone non-increasing*. All the stochastic orders that are mentioned below are formally defined in [2].

It is worthwhile to point out that a stochastic comparison of orders statistics from a collection of independent heterogeneous exponential random variables, and a stochastic comparison of orders statistics from a collection of independent heterogeneous random variables with proportional hazard rates, are, in some respects, essentially the same comparison. The ideas that underlie the validity, for instance, of Theorems 2.4, 2.6, 2.8, 2.11, 2.13, 2.15, and 2.16 in [2], are essentially all a single idea. More specifically, the reader may look at the

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proofs of Theorem 3.3 in Khaledi and Kochar [5] and of Theorem 2.3 in Kochar and Xu [8] to see applications of essentially the same idea in two special cases. Below I will describe this idea in some generality, and, using it, the results in the theorems in [2], mentioned above, can then be seen to follow at once from the analogous comparisons of order statistics from a collection of independent heterogeneous exponential random variables.

First we recall the following facts. Let $\leq_{\text{order}}$ denote some stochastic order. Many such orders are preserved under some families of transformations. That is, many such orders satisfy

$$X \leq_{\text{order}} Y \Longrightarrow \phi(X) \leq_{\text{order}} \phi(Y), \quad \text{for all } \phi \in \mathcal{F}, \tag{1}$$

where $\mathcal{F}$ is a family of real functions. For example, if $\leq_{\text{order}}$ is $\leq_{\text{st}}$, $\leq_{\text{hr}}$, or $\leq_{\text{lr}}$, then (1) holds for all increasing functions ϕ (see, e.g., Theorems 1.A.3(a), 1.B.2, and 1.C.8 in Shaked and Shanthikumar [15]), and if $\leq_{\text{order}}$ is $\leq_{\text{mrl}}$, $\leq_{\text{disp}}$, or $\leq_{\text{ew}}$ then (1) holds for all increasing convex functions ϕ (see, e.g., Theorems 2.A.19, 3.B.10, and 3.C.4 in Shaked and Shanthikumar [15]).

Now, let $(E_1, E_2, \ldots, E_n)$ be a vector of independent exponential random variables with hazard rate vector $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and let $(E_1^*, E_2^*, \ldots, E_n^*)$ be another vector of independent exponential random variables with hazard rate vector $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$. Suppose that some condition on $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$ has been found under which

$$E_{k:n} \geq_{\text{order}} E_{k:n}^* \tag{2}$$

for some $1 \leq k \leq n$, where $E_{1:n} \leq E_{2:n} \leq \cdots \leq E_{n:n}$ are the order statistics that correspond to $E_1, E_2, \ldots, E_n$ and $E_{1:n}^* \leq E_{2:n}^* \leq \cdots \leq E_{n:n}^*$ are the order statistics that correspond to $E_1^*, E_2^*, \ldots, E_n^*$.

Consider now the vector $(X_1, X_2, \ldots, X_n)$ of independent random variables with proportional hazard rate vector $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and the vector $(X_1^*, X_2^*, \ldots, X_n^*)$ of independent random variables with proportional hazard rate vector $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$. Formally, let $\overline{F}$ be an absolutely continuous survival function, and suppose that X_i and X_i^* have the following survival functions

$$\overline{F}_{X_i}(x) = \left[\overline{F}(x)\right]^{\lambda_i}, \quad x \in \mathbb{R},$$

and

$$\overline{F}_{X_i^*}(x) = \left[\overline{F}(x)\right]^{\lambda_i^*}, \quad x \in \mathbb{R},$$

$i = 1, 2, \ldots, n$.

Next consider the transformation (which depends on $\overline{F}$)

$$\psi_F(x) = \overline{F}^{-1}(\exp\{-x\}), \quad x \in \mathbb{R}.$$

Note that $\psi_F(x)$ is an increasing function of x. For $i = 1, 2, \ldots, n$ define,

$$\widehat{X}_i = \psi_F(E_i) = \overline{F}^{-1}(\exp\{-E_i\}).$$

We see that

$$\begin{aligned} P\{\widehat{X}_i > x\} &= P\left\{\overline{F}^{-1}(\exp\{-E_i\}) > x\right\} \\ &= P\{E_i > -\log \overline{F}(x)\} \\ &= \left[\overline{F}(x)\right]^{\lambda_i}, \quad x \in \mathbb{R}. \end{aligned}$$

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That is, $(\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_n) =_{\rm st} (X_1, X_2, \ldots, X_n)$, or, equivalently,

$$(X_1, X_2, \ldots, X_n) =_{\rm st} (\psi_F(E_1), \psi_F(E_2), \ldots, \psi_F(E_n)). \tag{3}$$

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the order statistics that correspond to $(X_1, X_2, \ldots, X_n)$. Since ψ_F is increasing we get from (3) that

$$(X_{1:n}, X_{2:n}, \ldots, X_{n:n}) =_{\rm st} (\psi_F(E_{1:n}), \psi_F(E_{2:n}), \ldots, \psi_F(E_{n:n})). \tag{4}$$

Denoting by $X^*_{1:n} \le X^*_{2:n} \le \cdots \le X^*_{n:n}$ the order statistics that correspond to $(X^*_1, X^*_2, \ldots, X^*_n)$, we also have

$$(X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n}) =_{\rm st} (\psi_F(E^*_{1:n}), \psi_F(E^*_{2:n}), \ldots, \psi_F(E^*_{n:n})). \tag{5}$$

Note that the same transformation ψ_F is used in both (4) and (5).

Since ψ_F is an increasing function we see from (4), (5), and (1) that

$$\text{if} \quad E_{k:n} \ge_{\rm st} E^*_{k:n} \quad \text{then} \quad X_{k:n} \ge_{\rm st} X^*_{k:n};$$
$$\text{if} \quad E_{k:n} \ge_{\rm hr} E^*_{k:n} \quad \text{then} \quad X_{k:n} \ge_{\rm hr} X^*_{k:n};$$

and

$$\text{if} \quad E_{k:n} \ge_{\rm lr} E^*_{k:n} \quad \text{then} \quad X_{k:n} \ge_{\rm lr} X^*_{k:n}.$$

Thus, Theorems 2.4, 2.8, and 2.13 in [2] follow from Theorem 2.3, (2.11), and Theorem 2.12 in [2], respectively.

The derivative of ψ_F is given by

$$\psi'_F(x) = \frac{\exp\{-x\}}{f\left(\overline{F}^{-1}(\exp\{-x\})\right)}, \quad x \in \mathbb{R},$$

where f is the density function that is associated with $\overline{F}$. Note that $\psi'_F(x)$ is increasing in x (that is, ψ_F is convex), if and only if,

$$\frac{y}{f\left(\overline{F}^{-1}(y)\right)} \quad \text{is decreasing in } y;$$

that is, if and only if,

$$\frac{\overline{F}(y)}{f(y)} \quad \text{is increasing in } y;$$

that is, if, and only if, F is DFR. In other words, ψ_F is increasing convex, if and only if, F is DFR. We thus see from (4), (5), and (1), assuming that F is DFR, that

$$\text{if} \quad E_{k:n} \ge_{\rm mrl} E^*_{k:n} \quad \text{then} \quad X_{k:n} \ge_{\rm mrl} X^*_{k:n};$$
$$\text{if} \quad E_{k:n} \ge_{\rm disp} E^*_{k:n} \quad \text{then} \quad X_{k:n} \ge_{\rm disp} X^*_{k:n};$$

and

$$\text{if} \quad E_{k:n} \ge_{\rm ew} E^*_{k:n} \quad \text{then} \quad X_{k:n} \ge_{\rm ew} X^*_{k:n}.$$

Thus, Theorems 2.6, 2.15, 2.16, and 2.20 in [2] follow from Theorems 2.5, 2.14, (2.14), and Theorem 2.19 in [2], respectively. Also, the general results in Theorems 2.7 and 2.11

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in [2] are essentially equivalent to the more specific results that entail only heterogeneous exponential random variables.

Some related results involving comparisons of order statistics based on samples of heterogeneous exponential random variables, that are worth mentioning in the present discussion note, are the following. The multivariate likelihood ratio order comparison of the vectors of the normalized spacings associated with heterogeneous exponential random variables is derived in Kochar and Rojo [7]. Various stochastic comparisons (with respect to $\leq_{\mathrm{st}}$, $\leq_{\mathrm{hr}}$, $\leq_{\mathrm{rh}}$, $\leq_{\mathrm{lr}}$, $\leq_{\mathrm{mrl}}$, $\leq_{\mathrm{disp}}$, and $\leq_{\mathrm{ew}}$) of spacings that are based on collections of independent heterogeneous exponential random variables are described in Kochar and Korwar [11] and in Kochar and Xu [10]. Păltănea [14] has obtained stochastic comparisons, in the sense of $\leq_{\mathrm{st}}$ and $\leq_{\mathrm{hr}}$, of mixtures of order statistics that are associated with heterogeneous exponential random variables. Positive dependence stochastic comparison of the copulas of the extreme order statistics (that is, the smallest and the largest), that are associated with various independent heterogeneous exponential random variables, is given in Genest, Kochar, and Xu [4]. Theorem 6.B.13 of Shaked and Shanthikumar [15] shows that the vector of order statistics, that correspond to a collection of independent heterogeneous exponential random variables, conditionally increases in the value of the first order statistic. In the area of reliability theory, where the smallest and the largest order statistics correspond to series and parallel systems, Zhao, Chan, and Ng [20] derived optimal allocation of redundancies of items with heterogeneous exponential lifetimes, with respect to the orders $\leq_{\mathrm{st}}$ and $\leq_{\mathrm{lr}}$.

In the review paper [2], Balakrishnan and Zhao study stochastic comparisons of order statistics based on independent heterogeneous exponential, gamma, geometric, negative binomial, and other random variables. A class of results that may deserve a similar study is the one that stochastically compares convolutions that are made up of independent heterogeneous random variables. Recently, a large number of papers that derive such results have appeared. I cannot describe here all the results that have been obtained, but below I will direct the reader to some references that involve stochastic comparisons of sums of independent heterogeneous exponential random variables. Comparisons of two sums of independent heterogeneous gamma random variables (and, in particular, exponential random variables), with respect to the orders $\leq_{\mathrm{st}}$, $\leq_{\mathrm{hr}}$, $\leq_{\mathrm{lr}}$, $\leq_{\mathrm{mrl}}$, $\leq_{\mathrm{disp}}$, and $\leq_{*}$, are derived in Bon and Păltănea [3], in Yu [16], in Zhao and Balakrishnan [18,19], in Zhao, Li, and Balakrishnan [21], and in Zhao [17]. Furthermore, comparisons of sums of heterogeneous exponential random variables with respect to $\leq_{\mathrm{cx}}$, $\leq_{\mathrm{disp}}$, $\leq_{\mathrm{ew}}$, and $\leq_{\mathrm{icx}}$ are given in Kochar and Ma [6], in Păltănea [13], in Mao, Hu, and Zhao [12], and in Kochar and Xu [9]. An interesting comparison, with respect to the order $\leq_{\mathrm{icx}}$, of average of exponential random variables with the largest among them, can be found in Argon and Andradóttir [1].

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