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DETAILED COMPUTATIONAL ANALYSIS OF QUEUEING-TIME DISTRIBUTIONS OF THE BMAP/G/1 QUEUE USING ROOTS

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Abstract

In this paper we present closed-form expressions for the distribution of the virtual (actual) queueing time for the BMAP/R/1 and BMAP/D/1 queues, where 'R' represents a class of distributions having rational Laplace–Stieltjes transforms. The closed-form analysis is based on the roots of the underlying characteristic equation. Numerical aspects have been tested for a variety of arrival and service-time distributions and results are matched with those obtained using the matrix-analytic method (MAM). Further, a comparative study of computation time of the proposed method with the MAM has been carried out. Finally, we also present closed-form expressions for the distribution of the virtual (actual) system time. The proposed method is analytically quite simple and easy to implement.

Keywords: Batch Markovian arrival process; BMAP; roots; queueing time; system time; virtual waiting time; rational Laplace–Stieltjes transform; matrix exponential; phase type; deterministic

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Secondary 90B22

1. Introduction

The versatile Markovian point process of Neuts [23] and the N-process of Ramaswami [27] were found to be equivalent to the batch Markovian arrival process (BMAP) introduced by Lucantoni *et al.* [21]. For an excellent tutorial on the BMAP/G/1 queue, we refer the reader to [12] and [20].

The Laplace–Stieltjes transform (LST) of the virtual queueing-time distribution of the BMAP/G/1 queue is available at several places in the queueing literature; see, e.g. Ramaswami [27] and Matendo [22]. Broadly speaking, there are two major steps involved in obtaining the queueing-time distribution from its LST. First, determining the unknown boundary vector (usually denoted as vector $\boldsymbol{g}$ in the literature) and secondly, inverting the LST after substituting the vector $\boldsymbol{g}$.

The vector $\boldsymbol{g}$ is usually obtained using the matrix-analytic method (MAM) pioneered by Neuts [24], [25] which involves the solution of $\boldsymbol{gG} = \boldsymbol{g}$ with $\boldsymbol{ge} = 1$, where $\boldsymbol{e}$ is a column

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vector of 1s of order m and the stochastic matrix $\boldsymbol{G}$ is the minimal nonnegative solution of a matrix nonlinear equation, $\boldsymbol{G} = \sum_{k=0}^{\infty} \boldsymbol{A}_k \boldsymbol{G}^k$.

It has been reported in the literature that the evaluation of the matrix $\boldsymbol{G}$ is the most difficult part of the procedure. Several methods have been proposed in the past for the evaluation of $\boldsymbol{G}$; see, e.g. [9]. It is seen that the computation time of matrix $\boldsymbol{G}$ is proportional to the traffic intensity and correlation between the interarrival times, i.e. the higher the traffic intensity or correlation, the higher the computation time; see, e.g. [20, p. 15, lines 4–5]. Problems in evaluation of the boundary vector using the MAM are highlighted at several places in the literature; see, e.g. [17]. The other method to obtain the unknown boundary vector is based on the spectral theory approach which uses eigenvalue algebra. Lee *et al.* [18] considered the BMAP/G/1 queue and obtained the boundary vector and mean queue length using eigenvalues and eigenvectors of the matrix $-\boldsymbol{D}(z)$. Further, Nishimura [26] also obtained the boundary vector using the spectral approach.

Once the unknown vector $\boldsymbol{g}$ is obtained, the next step is to obtain the virtual queueing-time distribution by inverting its LST. For this, one can use various methods based on Fourier series; see [1] and [2]. However, its use in inverting the LST for BMAP/G/1 queues has yet to be demonstrated. An alternative approach is to use the *method of roots* to invert the LST, as demonstrated in this paper.

In this paper we propose an entirely different approach for evaluation of the unknown vector, $\boldsymbol{q}(= (1-\rho)\boldsymbol{g})$ as well as inversion of the LST. More specifically, we provide a comprehensive analysis of the BMAP/G/1 queue in order to obtain waiting-time distributions using the *method of roots*. The evaluation of $\boldsymbol{q}$ is performed by considering just one component of the vector LST instead of the entire vector LST. In particular, we obtain the unknown boundary vector $\boldsymbol{q}$ using roots with nonnegative real parts of the associated characteristic equation (CE) (defined in Section 3.1) and further evaluate virtual queueing-time distributions by inverting the associated vector LST using the roots with negative real parts.

We also present results for the virtual system-time distributions, actual queueing (system)-time distribution of the first customer of an arriving batch, and actual queueing (system)-time distribution of an arbitrary customer of an arriving batch. All these distributions are presented in closed-form expressions in terms of roots of the CE with negative real parts.

The proposed method has been well demonstrated in recent papers by Chaudhry *et al.* [13] and Singh *et al.* [28], where they have carried out an analysis of MAP/R/1 ('R' represents a class of distributions having a rational LST) and MAP/D/1 queues, respectively. The complexity increases manifolds if one considers batches in the arrival process; namely, the BMAP. Therefore, a separate analysis of the BMAP/G/1 queue is needed.

To demonstrate the method for the numerical work we choose service-time distribution as matrix-exponential (ME) which covers a wide variety of distributions that arise in applications. Results for the BMAP/D/1 queue are also presented. In this case, the LST of the deterministic distribution is first approximated as a rational function using the Padé approximation as Padé(ℓ, n), where the degree of the numerator is ℓ and that of the denominator is n $(\ell \leq n)$. Then using the ME representation of the approximated rational LST proposed by Fackrell [14], we obtain the virtual queueing-time distribution and other distributions for the BMAP/D/1 queue. We also obtain the unknown boundary vector (see [20]) using the MAM. Finally, a comparative study is performed in order to compare the computation time of the proposed method (*the roots method*) with the MAM, [20]. It is observed that the computation time of the proposed method is independent of the value of the traffic intensity, ρ, and correlation; while in the case of the MAM, the computation time is dependent on the value of the traffic

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intensity, ρ, as well as correlation. There are faster methods available to calculate the boundary vector and stochastic matrix $\boldsymbol{G}$ (other than [20]), such as the logarithmic reduction algorithm by Latouche and Ramaswami [16], Newton iteration by Latouche [15], cyclic reduction by Bini and Meini [7], [8], and invariant subspace algorithm by Akar and Sohraby [5]. Our goal for this paper is to introduce a method which uses roots of some polynomials in order to calculate the boundary vector using the roots of the said polynomials (CEs), where the computation time is independent of both the traffic intensity as well as the correlation.

The paper is structured as follows. In Section 2 we start with the notation and a preliminary description of the model. In Section 3 we carry out a detailed analysis of queueing-time distributions which also involves a method for determining the unknown vector $\boldsymbol{q}$. Cases of repeated roots are discussed in Section 4. Performance measures and numerical results are presented in Sections 5 and 6, respectively. Finally, Section 7 ends with conclusions and future scope. Analysis of system-time distributions are also briefly presented in the Appendix.

2. Model description and notation

We consider an infinite-buffer single-server queue where arrivals occur according to the BMAP($\boldsymbol{D}_0$, $\boldsymbol{D}_1$, …), where the $\boldsymbol{D}_k$ are square matrices of order m. The (i, j)th element of $\boldsymbol{D}_k$, i.e. $(\boldsymbol{D}_k)_{i,j}$, is a transition rate from state i to state j with an arrival of batch having size k, $k \geq 1$, while the (i, j)th element of $\boldsymbol{D}_0$, i.e. $(\boldsymbol{D}_0)_{i,j}$, $i \neq j$, is a transition rate from state i to state j without an arrival. Let $\overline{\boldsymbol{\pi}}$ be the stationary row vector (order m) of $\boldsymbol{D}$, where $\boldsymbol{D} = \sum_{k=0}^{\infty} \boldsymbol{D}_k$ and $\boldsymbol{D}(z) = \sum_{k=0}^{\infty} \boldsymbol{D}_k z^k$. Note that $\overline{\boldsymbol{\pi}}$ satisfies $\overline{\boldsymbol{\pi}}\boldsymbol{D} = \boldsymbol{0}$ and $\overline{\boldsymbol{\pi}}\boldsymbol{e} = 1$, where $\boldsymbol{0}$ is a row vector of 0s and $\boldsymbol{e}$ is a column vector of 1s of order m, respectively. The mean customer arrival rate, λ^*, and the mean batch arrival rate, λ_g, of the stationary BMAP are given by $\lambda^* = \overline{\boldsymbol{\pi}}(\sum_{k=1}^{\infty} k\boldsymbol{D}_k)\boldsymbol{e}$ and $\lambda_g = \overline{\boldsymbol{\pi}}(\boldsymbol{D} - \boldsymbol{D}_0)\boldsymbol{e} = \overline{\boldsymbol{\pi}}(-\boldsymbol{D}_0)\boldsymbol{e}$, respectively. Also, without loss of generality, we assume that the maximum batch size of an arriving batch is N, i.e. $\boldsymbol{D}_k = \boldsymbol{0}$, $k \geq N + 1$. Service times are independent and identically distributed random variables with distribution function $H(t)$, density function $h(t)$, and LST $h^*(s)$. Let the mean service time be $1/\mu$ and the traffic intensity is $\rho = \lambda^*/\mu < 1$.

We use the word *root* of an equation $f(z) = 0$ and the *zero* of the function $f(z)$, interchangeably in subsequent sections. We assume that $f^{(k)}(\theta)$ is the kth ($k \geq 1$) derivative of $f(s)$ at $s = \theta$ and $f^{(0)}(\theta) \equiv f(\theta)$. We will use $f^{(1)}(\theta)$ and $f'(\theta)$ interchangeably.

3. Queueing-time distributions

In this section we develop a procedure for calculating three different types of queueing-time distribution; namely,

- virtual queueing time,
- actual queueing time of the first customer of an arriving batch,
- actual queueing time of an arbitrary customer of an arriving batch.

3.1. Virtual queueing-time distribution

In this section we develop an expression for the virtual queueing-time distribution of the BMAP/G/1 queue which is amenable to computations with ease. First, let $\widetilde{\boldsymbol{W}}_{vq}(x)$ denote the row vector of order m whose jth ($1 \leq j \leq m$) component $\widetilde{W}_{vq,j}(x)$ is the joint probability of a virtual customer who arrives at an arbitrary time and waits at most a time x before entering service and the arrival process is in phase j. Also, let $W_{vq,j}(s)$ (and $\boldsymbol{W}_{vq}(s)$ be the corresponding

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row vector) be the LST of $\widetilde{W}_{vq,j}(x)$ which is given by $W_{vq,j}(s) = \int_0^\infty e^{-sx}\, d\widetilde{W}_{vq,j}(x)$, $\mathrm{Re}(s) \geq 0$. Then, the virtual queueing-time distribution and its LST for the BMAP/G/1 queue are given by $\widetilde{W}_{vq}(x) = \widetilde{\boldsymbol{W}}_{vq}(x)\boldsymbol{e}$ and $W_{vq}(s) = \boldsymbol{W}_{vq}(s)\boldsymbol{e}$, and $\boldsymbol{W}_{vq}(s)$ is given as

$$\boldsymbol{W}_{vq}(s) = s\boldsymbol{q}[s\boldsymbol{I} + \boldsymbol{D}(h^*(s))]^{-1} \quad \text{with } \boldsymbol{W}_{vq}(0) = \overline{\boldsymbol{\pi}}, \tag{1}$$

where $\boldsymbol{q}$ is an unknown vector whose jth element q_j represents the joint probability that an arriving customer is in phase j and the server is idle; see [27], and $\boldsymbol{I}$ is an identity matrix.

Now, in order to obtain the virtual queueing-time distribution from (1), we divide the procedure into two parts; namely, determination of the unknown boundary vector and inverting the LST given in (1).

3.1.1. *Determination of the unknown boundary vector* $\boldsymbol{q}$. The literature on queueing theory shows that distributions having the LST as a rational function cover a wide range of distributions that arise in applications; see [11]. In view of this, we consider those distributions that have a rational LST of the form $h^*(s) = P(s)/Q(s)$, where the degree of the polynomial $Q(s)$ is n and that of the polynomial $P(s)$ is at most n. Our starting point for further analysis is (1). For the $s \neq 0$ case, write (1) as

$$\boldsymbol{W}_{vq}(s)[s\boldsymbol{I} + \boldsymbol{D}(h^*(s))] = s\boldsymbol{q}. \tag{2}$$

Using $\boldsymbol{D}(z) = \sum_{k=0}^{N} \boldsymbol{D}_k z^k$, we can write the (i,k)th element of $\boldsymbol{D}(h^*(s))$ as

$$(\boldsymbol{D}(h^*(s)))_{i,k} = \frac{\sum_{\ell=0}^{N} (\boldsymbol{D}_\ell)_{i,k} P(s)^\ell Q(s)^{N-\ell}}{Q(s)^N}, \qquad 1 \leq i,\ k \leq m.$$

As each element of $\boldsymbol{D}(h^*(s))$ is a rational function whose denominator is fixed as $Q(s)^N$, one can say that each element of $s\boldsymbol{I} + \boldsymbol{D}(h^*(s))$ is also a rational function with the same denominator. We further choose the notation in such a manner that it eases the visibility of the analytical expressions. For that, we first let the (i,k)th element of $s\boldsymbol{I} + \boldsymbol{D}(h^*(s))$ be $g_{i,k}(s)/Q(s)^N$, $g_{i,k}(s)$, $1 \leq i, k \leq m$, is given as

$$g_{i,k}(s) = \begin{cases} sQ(s)^N + \displaystyle\sum_{\ell=0}^{N} (\boldsymbol{D}_\ell)_{i,i} P(s)^\ell Q(s)^{N-\ell}, & i = k, \\ \displaystyle\sum_{\ell=0}^{N} (\boldsymbol{D}_\ell)_{i,k} P(s)^\ell Q(s)^{N-\ell}, & i \neq k. \end{cases} \tag{3}$$

For further analysis, we first determine an analytic expression for each element of $\boldsymbol{W}_{vq}(s)$. Using (3) in (2) and expanding both sides of (2) element-wise, we obtain m simultaneous equations in m unknowns $W_{vq,j}(s)$, $1 \leq j \leq m$, in terms of unknown q_j as

$$g_{1,j}(s)W_{vq,1}(s) + g_{2,j}(s)W_{vq,2}(s) + \cdots + g_{m,j}(s)W_{vq,m}(s) = sQ(s)^N q_j.$$

Solving these simultaneous equations, we obtain the actual expression of $W_{vq,j}(s)$ in a simple form as

$$W_{vq,j}(s) = \frac{sQ(s)^N |\boldsymbol{G}_j(s)|}{|\boldsymbol{G}(s)|} \equiv \frac{sQ(s)^N F_j(s)}{F(s)}, \tag{4}$$

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where $\boldsymbol{G}(s)$ and $\boldsymbol{G}_j(s)$ are square matrices whose (i,k)th elements are given by

$$(\boldsymbol{G}(s))_{i,k} = g_{k,i}(s) \quad \text{and} \quad (\boldsymbol{G}_j(s))_{i,k} = \begin{cases} g_{k,i}(s), & k \neq j, \\ q_i, & k = j,\ 1 \leq i,\ j,k \leq m. \end{cases} \tag{5}$$

That is, $\boldsymbol{G}_j(s)$ is a square matrix whose elements are the same as those of $\boldsymbol{G}(s)$ except for the jth column which is $[q_1 \quad q_2 \quad \cdots \quad q_m]^\top$, where '$\top$' denotes transpose.

Remark 1. As one can see from (3) and (5) that each $(G(s))_{i,k}$ is a polynomial of degree $\ell (1 \leq \ell \leq nN+1)$. This is due to the fact that the degree of the polynomial $Q(s)$ is n and that of the polynomial $P(s)$ is at most n. Hence, for the unique expression of $W_{\mathrm{v}q,j}(s)$, we assume that $|\boldsymbol{G}(s)|$ is a nonzero polynomial in s, i.e. it must have at least one coefficient of powers of s as nonzero.

Remark 2. For ease of notation, we use $F_j(s) = |\boldsymbol{G}_j(s)|$ and $F(s) = |\boldsymbol{G}(s)|$. Letting $\overline{\boldsymbol{F}}(s) = [F_1(s) \quad F_2(s) \quad \cdots \quad F_m(s)]$, the new expression for $\boldsymbol{W}_{\mathrm{v}q}(s)$ is given as

$$\boldsymbol{W}_{\mathrm{v}q}(s) = \frac{sQ(s)^N}{F(s)}\overline{\boldsymbol{F}}(s).$$

Also, we call $F(s)$ the CE associated with the LST of the virtual queueing-time distribution.

Theorem 1. *The degree of the CE, $F(s)$, is at most $mnN+m$ and $F(s)$ has m roots having nonnegative real parts including the root at* 0*, while at most mnN roots having negative real parts.*

Proof. The (i,j)th element of $\boldsymbol{D}(h^*(s))$ is a rational function whose numerator is a polynomial of degree at most nN and whose denominator $Q(s)^N$ is a polynomial of degree nN ($Q(s)$ is of degree n and degree of the $P(s)$ is at most n). From (3), one can immediately say that $g_{i,j}(s)$ is a polynomial of degree at most $nN+1$ in the $i=j$ case, and nN in the $i \neq j$ case. Hence, one can immediately infer that $F(s) \equiv |\boldsymbol{G}(s)|$ is a polynomial of degree at most $m(nN+1)$, i.e. $mnN+m$.

Since $h^*(s) = P(s)/Q(s)$ is an LST of the service-time distribution and is analytic in the region $\mathrm{Re}(s) > 0$, it follows that $Q(s)$ has all its 0s having negative real parts. When $\boldsymbol{D}$ is irreducible and $\rho < 1$, then $|s\boldsymbol{I} + \boldsymbol{D}(h^*(s))| = 0$ has exactly m roots with nonnegative real parts including the root at 0; see [20]. As $|s\boldsymbol{I} + \boldsymbol{D}(h^*(s))| = F(s)/Q(s)^{mN}$, $F(s) = 0$ has exactly m roots with nonnegative real parts including the root at 0. This leads to the fact that $F(s) = 0$ has at most mnN roots with negative real parts. □

3.1.2. *Solving m simultaneous equations.* As $F(s) = 0$ has exactly m roots with nonnegative real parts including the root at 0, we call these roots whose real parts are positive as $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ including multiplicity, if any, and the root at 0 as γ_m. Let us assume that the CE, $F(s) = 0$, has multiple roots $\gamma_1, \gamma_2, \ldots, \gamma_k$ with multiplicity $r_1, r_2, \ldots, r_k$, respectively, such that $r = r_1 + r_2 + \cdots + r_k$ (equal to or less than $m-1$). All other roots $\gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_m$ are distinct roots in $\mathrm{Re}(s) \geq 0$ with $\gamma_m = 0$. One must note that $\gamma_m = 0$ has multiplicity 1 only; see Remark 3 below. We can determine the unknown vector $\boldsymbol{q}$ by considering any one component of $\boldsymbol{W}_{\mathrm{v}q}(s)$, say $W_{\mathrm{v}q,p}(s)$, $1 \leq p \leq m$. As $W_{\mathrm{v}q,p}(s)$ is analytic in $\mathrm{Re}(s) \geq 0$, it implies that

$$F_p^{(j-1)}(\gamma_i) = 0, \qquad j = 1, 2, \ldots, r_i,\ i = 1, 2, \ldots, k, \tag{6}$$

$$F_p(\gamma_i) = 0, \qquad i = r+1, \ldots, m-1, \tag{7}$$

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Further, as $\gamma_m = 0$ is a zero of both the numerator and the denominator of $W_{\mathrm{v}q,p}(s)$, we have, on using $W_{\mathrm{v}q,p}(0) = \overline{\pi}_p$ and the fact that $F(0) = 0$,

$$Q(0)^N F_p(0) = \overline{\pi}_p F'(0). \tag{8}$$

Equations (6)–(8) give m simultaneous equations in m unknowns, q_j. Solving these m equations, we obtain the vector $\boldsymbol{q}$. As the vector $\boldsymbol{q}$ is completely known, $W_{\mathrm{v}q,j}(s)$ given by (4) is completely known.

Remark 3. Now, (8) can be written as $F'(0) = (Q(0))^N F_j(0)/\overline{\pi_j}$. If we assume that $\gamma_m = 0$ has multiplicity 2 then $F'(0) = 0$. This is possible only if either $Q(0) = 0$ or $F_j(0) = 0$. Now, $h^*(0) = 1$ implies that $P(0) = Q(0) \neq 0$, so the only possibility is $F_j(0) = 0$. Now, $\gamma_m = 0$ has multiplicity 2, implies that both the numerator and denominator of (4) must be 0. Hence, the factor $(z - \gamma_m)$ (if it exists) must be canceled before proceeding further.

3.1.3. *Inverting the LST* $\boldsymbol{W}_{\mathrm{v}q}(s)$. Once the unknown vector $\boldsymbol{q}$ is completely known, $\boldsymbol{W}_{\mathrm{v}q}(s)$ can be inverted by using numerical inversion techniques. Now, $W_{\mathrm{v}q,j}(s)$, given in (4) is an analytic rational function with completely known polynomials both in the numerator (after substituting the values of the q_j) and the denominator. We need to invert this LST using partial fractions using the poles of $W_{\mathrm{v}q,j}(s)$ which are the zeros of $F(s)$ with negative real parts. As stated in Theorem 1, $F(s)$ has at most mnN roots with negative real parts. In total, we have t (less than or equal to mnN) roots of $F(s)$ each with negative real parts; namely, $\psi_i (1 \leq i \leq t)$. This leads to

$$W_{\mathrm{v}q,j}(s) = \bar{a}_{0,j} + \sum_{i=1}^{t} \frac{\bar{a}_{i,j}}{s - \psi_i}, \qquad j = 1, 2, \ldots, m, \tag{9}$$

where the $\bar{a}_{i,j}$ are constants to be determined. In the above equation, we assume that all ψ_i are distinct. However, if some roots are repeated then using the theory of partial fraction, (9) can be suitably modified; see Section 4. Now, taking the inverse Laplace transform of (9), we obtain

$$w_{\mathrm{v}q,j}(x) = \bar{a}_{0,j}\delta(x) + \sum_{i=1}^{t} \bar{a}_{i,j} \mathrm{e}^{\psi_i x}, \qquad j = 1, 2, \ldots, m, \tag{10}$$

where $\delta(x)$ is the Dirac delta function. From (10), we can express $\widetilde{W}_{\mathrm{v}q,j}(x)$ as

$$\widetilde{W}_{\mathrm{v}q,j}(x) = \int_0^x w_{\mathrm{v}q,j}(\tau)\,\mathrm{d}\tau = \bar{a}_{0,j} + \sum_{i=1}^{t} \left(\frac{\bar{a}_{i,j}}{\psi_i} \mathrm{e}^{\psi_i x} - \frac{\bar{a}_{i,j}}{\psi_i} \right). \tag{11}$$

Using (4), (9), the fact that $W_{\mathrm{v}q,j}(0) = \overline{\pi}_j$, and the usual procedure to find the constants, we obtain the values of the $\bar{a}_{i,j}$, $1 \leq j \leq m$. These are given by

$$\bar{a}_{i,j} = \begin{cases} \overline{\pi}_j + \displaystyle\sum_{r=1}^{t} \frac{\bar{a}_{r,j}}{\psi_r}, & i = 0, \\ \dfrac{\psi_i Q(\psi_i)^N F_j(\psi_i)}{F'(\psi_i)}, & i = 1, 2, \ldots, t. \end{cases} \tag{12}$$

Here, for each $j = 1, 2, \ldots, m$, one has to calculate all $\bar{a}_{i,j}$ for $i = 1, 2, \ldots, t$ and then $\bar{a}_{0,j}$ in terms of the $\bar{a}_{i,j}$. Using the value of $\bar{a}_{0,j}$ from (12) in (11), we obtain

$$\widetilde{W}_{\mathrm{v}q,j}(x) = \overline{\pi}_j + \sum_{i=1}^{t} \frac{\bar{a}_{i,j}}{\psi_i} \mathrm{e}^{\psi_i x}. \tag{13}$$

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Hence, the distribution of the virtual queueing time is obtained as

$$\widetilde{W}_{vq}(x) = \widetilde{\boldsymbol{W}}_{vq}(x)\boldsymbol{e}, \tag{14}$$

where $\widetilde{\boldsymbol{W}}_{vq}(x) = \overline{\boldsymbol{\pi}} + \sum_{i=1}^{t} \bar{\boldsymbol{a}}_i (e^{\psi_i x}/\psi_i)$ with $\bar{\boldsymbol{a}}_i = [\bar{a}_{i,1} \quad \bar{a}_{i,2} \quad \cdots \quad \bar{a}_{i,m}]$ being a row vector of order m which can be calculated directly as

$$\bar{\boldsymbol{a}}_i = \begin{cases} \overline{\boldsymbol{\pi}} + \displaystyle\sum_{r=1}^{t} \frac{\bar{\boldsymbol{a}}_r}{\psi_r}, & i = 0, \\ \dfrac{\psi_i Q(\psi_i)^N}{F'(\psi_i)} \overline{\boldsymbol{F}}(\psi_i), & i = 1, 2, \ldots, t, \end{cases} \tag{15}$$

where $\overline{\boldsymbol{F}}(\psi_i) = [F_1(\psi_i) \quad F_2(\psi_i) \quad \cdots \quad F_m(\psi_i)]$, i.e. $\overline{\boldsymbol{F}}(s)$ at $s = \psi_i$.

Remark 4. If we let $x = 0$ in (13), we obtain $\widetilde{W}_{vq,j}(0) = \overline{\pi}_j + \sum_{i=1}^{t} \bar{a}_{i,j}/\psi_i$ which is nothing but $\bar{a}_{0,j}$. That is, $\widetilde{\boldsymbol{W}}_{vq}(0) = \bar{\boldsymbol{a}}_0$. Also, we know that $\widetilde{\boldsymbol{W}}_{vq}(0) = \boldsymbol{q}$ (see [27]). Hence, we can check the accuracy of the constants $\bar{\boldsymbol{a}}_i$ by using $\boldsymbol{q} \cong \bar{\boldsymbol{a}}_0 = \overline{\boldsymbol{\pi}} + \sum_{i=1}^{t} \bar{\boldsymbol{a}}_i/\psi_i$.

3.2. Actual queueing-time distribution of the first customer of an arriving batch

Let $\widetilde{W}_{fq,j}(x)$ denote the joint probability that the first customer of an arriving batch has to wait at most a time x before entering service, and that immediately after that arrival epoch, the arrival process is in phase j. Also, let $W_{fq,j}(s)$ be the LST of $\widetilde{W}_{fq,j}(x)$. Further, let $\widetilde{\boldsymbol{W}}_{fq}(x)$ and $\boldsymbol{W}_{fq}(s)$ denote the row vectors of order m whose jth components are $\widetilde{W}_{fq,j}(x)$ and $W_{fq,j}(s)$, respectively. Then, the LST of the actual queueing-time distribution of the first customer of an arriving batch is given by $W_{fq}(s) = \boldsymbol{W}_{fq}(s)\boldsymbol{e}$, where $\boldsymbol{W}_{fq}(s) = (1/\lambda_g)\boldsymbol{W}_{vq}(s)[\boldsymbol{D} - \boldsymbol{D}_0]$; see [22, Equation 6]. Using $\boldsymbol{W}_{vq}(s)$ obtained in last section (see Remark 2), and $\boldsymbol{D}_1' = \boldsymbol{D} - \boldsymbol{D}_0$, we obtain

$$\boldsymbol{W}_{fq}(s) = \frac{1}{\lambda_g} \frac{s Q(s)^N}{F(s)} \overline{\boldsymbol{F}}(s) \boldsymbol{D}_1'. \tag{16}$$

Here, one can see that $\boldsymbol{W}_{fq}(s)$ is a vector whose each element is again a rational function, with known polynomials. The denominator of each component is the same, i.e. $F(s)$, and we have the same roots $\psi_i (1 \le i \le t)$ with negative real parts. Further, we use these roots to make partial fractions and obtain

$$\boldsymbol{W}_{fq}(s) = \bar{\boldsymbol{b}}_0 + \sum_{i=1}^{t} \frac{\bar{\boldsymbol{b}}_i}{s - \psi_i}, \tag{17}$$

where the $\bar{\boldsymbol{b}}_i$ are constant row vectors of order m having jth element as $\bar{b}_{i,j}$. Taking the inverse Laplace transform of (17) (element-wise), we obtain

$$\boldsymbol{w}_{fq}(x) = \bar{\boldsymbol{b}}_0 \delta(x) + \sum_{i=1}^{t} \bar{\boldsymbol{b}}_i e^{\psi_i x}. \tag{18}$$

From (18), we can obtain the distribution $\widetilde{W}_{fq}(x)$ of the queueing time of the first customer of an arriving batch as $\widetilde{W}_{fq}(x) = \widetilde{\boldsymbol{W}}_{fq}(x)\boldsymbol{e}$, where $\widetilde{\boldsymbol{W}}_{fq}(x)$ is obtained as

$$\widetilde{\boldsymbol{W}}_{fq}(x) = \int_0^x \boldsymbol{w}_{fq}(\tau)\, d\tau = \bar{\boldsymbol{b}}_0 + \sum_{i=1}^{t} \left(\frac{\bar{\boldsymbol{b}}_i}{\psi_i} e^{\psi_i x} - \frac{\bar{\boldsymbol{b}}_i}{\psi_i} \right). \tag{19}$$

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Using (16), (17), the fact that $\boldsymbol{W}_{\mathrm{f}q}(0) = (1/\lambda_g)\boldsymbol{W}_{\mathrm{v}q}(0)\boldsymbol{D}_1' = (1/\lambda_g)\overline{\boldsymbol{\pi}}\boldsymbol{D}_1'$, and the usual procedure to find the vector constants, we obtain the values of the $\bar{\boldsymbol{b}}_i$ $(0 \le i \le t)$. These are given by

$$\bar{\boldsymbol{b}}_i = \begin{cases} \dfrac{1}{\lambda_g}\overline{\boldsymbol{\pi}}\boldsymbol{D}_1' + \displaystyle\sum_{r=1}^{t}\frac{\bar{\boldsymbol{b}}_r}{\psi_r}, & i = 0, \\ \dfrac{1}{\lambda_g}\dfrac{\psi_i Q(\psi_i)^N}{F'(\psi_i)}\overline{\boldsymbol{F}}(\psi_i)\boldsymbol{D}_1', & i = 1, 2, \ldots, t. \end{cases} \tag{20}$$

Using (15), i.e. $\bar{\boldsymbol{a}}_i = (\psi_i Q(\psi_i)^N / F'(\psi_i))\overline{\boldsymbol{F}}(\psi_i)$, we write the vectors $\bar{\boldsymbol{b}}_i$ in terms of the vectors $\bar{\boldsymbol{a}}_i$ as

$$\bar{\boldsymbol{b}}_i = \frac{1}{\lambda_g}\bar{\boldsymbol{a}}_i\boldsymbol{D}_1', \qquad i = 0, 1, 2, \ldots, t.$$

Using the value of $\bar{\boldsymbol{b}}_0$ from (20) in (19), we obtain the distribution of the actual queueing time of the first customer of an arriving batch as

$$\widetilde{W}_{\mathrm{f}q}(x) = \widetilde{\boldsymbol{W}}_{\mathrm{f}q}(x)\boldsymbol{e},$$

where $\widetilde{\boldsymbol{W}}_{\mathrm{f}q}(x) = (1/\lambda_g)\overline{\boldsymbol{\pi}}\boldsymbol{D}_1' + \sum_{i=1}^{t}(\bar{\boldsymbol{b}}_i/\psi_i)\mathrm{e}^{\psi_i x}$.

3.3. Actual queueing-time distribution of an arbitrary customer of an arriving batch

Let $\widetilde{W}_{\mathrm{a}q,j}(x)$ denote the joint probability that a random (arbitrary) customer in a batch has to wait at most a time x before entering service, and that the arrival process is in phase j. Also, let $W_{\mathrm{a}q,j}(s)$ be the LST of $\widetilde{W}_{\mathrm{a}q,j}(x)$. Further, let $\widetilde{\boldsymbol{W}}_{\mathrm{a}q}(x)$ and $\boldsymbol{W}_{\mathrm{a}q}(s)$ denote the row vectors of order m whose jth components are $\widetilde{W}_{\mathrm{a}q,j}(x)$ and $W_{\mathrm{a}q,j}(s)$, respectively. Then, the LST of the actual queueing-time distribution of a random customer of an arriving batch is given by $W_{\mathrm{a}q}(s) = \boldsymbol{W}_{\mathrm{a}q}(s)\boldsymbol{e}$, where $\boldsymbol{W}_{\mathrm{a}q}(s) = (1/\lambda^*(1 - h^*(s)))\boldsymbol{W}_{\mathrm{v}q}(s)[\boldsymbol{D} - \boldsymbol{D}(h^*(s))]$; see [20, p 11, Equation (16)]. Just observing this equation does not give a clear picture about the denominator of each component of $\boldsymbol{W}_{\mathrm{a}q}(s)$. Hence, we first need to simplify this expression. Now, $\boldsymbol{D} - \boldsymbol{D}(h^*(s))$ can be simplified and, therefore, $\boldsymbol{W}_{\mathrm{a}q}(s)$ can be expressed as

$$\boldsymbol{W}_{\mathrm{a}q}(s) = \frac{1}{\lambda^*}\boldsymbol{W}_{\mathrm{v}q}(s)\left(\sum_{k=1}^{N}\boldsymbol{D}_k\sum_{i=0}^{k-1}(h^*(s))^i\right).$$

Using $h^*(s) = P(s)/Q(s)$ and after little more manipulation, we have

$$\boldsymbol{W}_{\mathrm{a}q}(s) = \frac{1}{\lambda^*}\frac{1}{Q(s)^{N-1}}\boldsymbol{W}_{\mathrm{v}q}(s)\sum_{k=1}^{N}\boldsymbol{D}_k\sum_{i=0}^{k-1}P(s)^i Q(s)^{N-1-i}.$$

Again using the expression for $\boldsymbol{W}_{\mathrm{v}q}(s)$ (see Remark 2), we obtain

$$\boldsymbol{W}_{\mathrm{a}q}(s) = \frac{1}{\lambda^*}\frac{sQ(s)}{F(s)}\overline{\boldsymbol{F}}(s)\sum_{k=1}^{N}\boldsymbol{D}_k\sum_{i=0}^{k-1}P(s)^i Q(s)^{N-1-i}. \tag{21}$$

Here, one can see that $\boldsymbol{W}_{\mathrm{a}q}(s)$ is a vector whose each element is again a rational function, with known polynomials. The denominator of each component is the same, i.e. $F(s)$, and we have

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the same roots $\psi_i (1 \le i \le t)$ with negative real parts. Further, we use these roots to make partial fractions and obtain

$$\boldsymbol{W}_{aq}(s) = \bar{\boldsymbol{c}}_0 + \sum_{i=1}^{t} \frac{\bar{\boldsymbol{c}}_i}{s - \psi_i}, \tag{22}$$

where the $\bar{\boldsymbol{c}}_i$ are constant row vectors of order m having jth element as $\bar{c}_{i,j}$. Taking the inverse Laplace transform of (22) (element-wise), we obtain

$$\boldsymbol{w}_{aq}(x) = \bar{\boldsymbol{c}}_0 \delta(x) + \sum_{i=1}^{t} \bar{\boldsymbol{c}}_i \mathrm{e}^{\psi_i x}. \tag{23}$$

From (23), we can obtain the distribution $\widetilde{W}_{aq}(x)$ of the actual queueing time of a random customer of an arriving batch as $\widetilde{W}_{aq}(x) = \widetilde{\boldsymbol{W}}_{aq}(x)\boldsymbol{e}$, where $\widetilde{\boldsymbol{W}}_{aq}(x)$ is obtained as

$$\widetilde{\boldsymbol{W}}_{aq}(x) = \int_0^x \boldsymbol{w}_{aq}(\tau)\,\mathrm{d}\tau = \bar{\boldsymbol{c}}_0 + \sum_{i=1}^{t} \left(\frac{\bar{\boldsymbol{c}}_i}{\psi_i} \mathrm{e}^{\psi_i x} - \frac{\bar{\boldsymbol{c}}_i}{\psi_i} \right). \tag{24}$$

Using (21), (22), $\boldsymbol{W}_{aq}(0) = (1/\lambda^*)\boldsymbol{W}_{vq}(0)\sum_{k=1}^{N} \boldsymbol{D}_k \sum_{i=0}^{k-1} h^*(0)^i = (1/\lambda^*)\overline{\boldsymbol{\pi}}\sum_{k=1}^{N} k\boldsymbol{D}_k$, and the usual procedure to find the vector constants, we obtain the values of the $\bar{\boldsymbol{c}}_i$ $(0 \le i \le t)$. These are given by

$$\bar{\boldsymbol{c}}_i = \begin{cases} \dfrac{1}{\lambda^*}\overline{\boldsymbol{\pi}} \displaystyle\sum_{k=1}^{N} k\boldsymbol{D}_k + \sum_{r=1}^{t} \frac{\bar{\boldsymbol{c}}_r}{\psi_r}, & i = 0, \\ \dfrac{1}{\lambda^*} \dfrac{\psi_i Q(\psi_i)}{F'(\psi_i)} \overline{\boldsymbol{F}}(\psi_i) \displaystyle\sum_{k=1}^{N} \boldsymbol{D}_k \sum_{i=0}^{k-1} P(\psi_i)^i Q(\psi_i)^{N-1-i}, & i = 1, 2, \ldots, t. \end{cases} \tag{25}$$

Since $\bar{\boldsymbol{a}}_i = (\psi_i Q(\psi_i)^N / F'(\psi_i))\overline{\boldsymbol{F}}(\psi_i)$ (see (15)), we can write the vectors $\bar{\boldsymbol{c}}_i$ in terms of the vectors $\bar{\boldsymbol{a}}_i$ as

$$\bar{\boldsymbol{c}}_i = \frac{1}{\lambda^*}\bar{\boldsymbol{a}}_i \sum_{k=1}^{N} \boldsymbol{D}_k \sum_{\ell=0}^{k-1} \left(\frac{P(\psi_i)}{Q(\psi_i)} \right)^{\ell} = \frac{1}{\lambda^*(1 - h^*(\psi_i))} \bar{\boldsymbol{a}}_i [\boldsymbol{D} - \boldsymbol{D}(h^*(\psi_i))].$$

Hence, the $\bar{\boldsymbol{c}}_i$ can be written as

$$\bar{\boldsymbol{c}}_i = \begin{cases} \dfrac{1}{\lambda^*}\overline{\boldsymbol{\pi}} \displaystyle\sum_{k=1}^{N} k\boldsymbol{D}_k + \sum_{r=1}^{t} \frac{\bar{\boldsymbol{c}}_r}{\psi_r}, & i = 0, \\ \dfrac{\bar{\boldsymbol{a}}_i [\boldsymbol{D} - \boldsymbol{D}(h^*(\psi_i))]}{\lambda^*(1 - h^*(\psi_i))} & i = 1, 2, \ldots, t. \end{cases}$$

Using the value of $\bar{\boldsymbol{c}}_0$ from (25) in (24), we obtain the actual queueing-time distribution of a random customer of an arriving batch as

$$\widetilde{W}_{aq}(x) = \widetilde{\boldsymbol{W}}_{aq}(x)\boldsymbol{e},$$

where $\widetilde{\boldsymbol{W}}_{aq}(x) = (1/\lambda^*)\overline{\boldsymbol{\pi}}\sum_{k=1}^{N} k\boldsymbol{D}_k + \sum_{i=1}^{t} (\bar{\boldsymbol{c}}_i/\psi_i)\mathrm{e}^{\psi_i x}$.

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4. Virtual queueing-time distribution when $F(s) = 0$ has multiple roots in Re(s) < 0

Now, we assume that $F(s) = 0$ has multiple roots $\psi_1, \psi_2, \ldots, \psi_f$ each with multiplicity $k_1, k_2, \ldots, k_f$ in $\mathrm{Re}(s) < 0$. As in (9), we have t roots of $F(s) = 0$ with negative real parts, i.e. $\sum_{i=1}^{f} k_i = t$. This leads us to

$$W_{vq,j}(s) = \overline{a}_{0,0,j} + \sum_{i=1}^{f}\sum_{r=1}^{k_i} \frac{\overline{a}_{i,r,j}}{(s-\psi_i)^r}, \qquad j = 1, 2, \ldots, m. \tag{26}$$

Now, taking the inverse Laplace transform of (26), we obtain

$$w_{vq,j}(x) = \overline{a}_{0,0,j}\delta(x) + \sum_{i=1}^{f}\sum_{r=1}^{k_i} \frac{\overline{a}_{i,r,j}x^{r-1}\mathrm{e}^{\psi_i x}}{(r-1)!}, \qquad j = 1, 2, \ldots, m. \tag{27}$$

From (27), we can express $\widetilde{W}_{vq,j}(x)$ as

$$\widetilde{W}_{vq,j}(x) = \int_0^x w_{vq,j}(\tau)\,\mathrm{d}\tau = \overline{a}_{0,0,j} + \sum_{i=1}^{f}\sum_{r=1}^{k_i}\left(\sum_{\ell=1}^{r-1} \frac{(-1)^{\ell+1}\overline{a}_{i,r,j}x^{\ell}\psi_i^{\ell}\mathrm{e}^{\psi_i x}}{\ell!\,(-\psi_i)^r} + \frac{\overline{a}_{i,r,j}}{(-\psi_i)^r}\right). \tag{28}$$

Using (4), (26), the fact that $W_{vq,j}(0) = \overline{\pi}_j$, and the usual procedure to find the constants, we obtain the values of the $\overline{a}_{i,r,j}$ $(1 \le j \le m)$. These are given by

$$\overline{a}_{i,r,j} = \begin{cases} \overline{\pi}_j - \displaystyle\sum_{w=1}^{f}\sum_{u=1}^{k_w} \frac{\overline{a}_{w,u,j}}{(-\psi_w)^u}, & i = 0,\ r = 0, \\ \dfrac{1}{(k_i - r)!}\displaystyle\lim_{s\to\psi_i} \frac{\mathrm{d}^{k_i - r}}{\mathrm{d}s^{k_i - r}}\left(\frac{(s-\psi_i)^{k_i} s\,Q(s)^N F_j(s)}{F(s)}\right), & i = 1, \ldots, f,\ r = 1, \ldots, k_i. \end{cases} \tag{29}$$

Using the value of $\overline{a}_{0,0,j}$ from (29) in (28), we obtain

$$\widetilde{W}_{vq,j}(x) = \overline{\pi}_j + \sum_{i=1}^{f}\sum_{r=1}^{k_i}\sum_{\ell=1}^{r-1} \frac{(-1)^{\ell+1}\overline{a}_{i,r,j}x^{\ell}\psi_i^{\ell}\mathrm{e}^{\psi_i x}}{\ell!\,(-\psi_i)^r}.$$

Finally, the virtual queueing-time distribution is given as

$$\widetilde{W}_{vq}(x) = \widetilde{\boldsymbol{W}}_{vq}(x)\boldsymbol{e}, \tag{30}$$

where $\widetilde{\boldsymbol{W}}_{vq}(x) = \overline{\boldsymbol{\pi}} + \sum_{i=1}^{f}\sum_{r=1}^{k_i}\overline{\boldsymbol{a}}_{i,r}(\sum_{\ell=1}^{r-1} - x^{\ell}/\ell!\,(-\psi_i)^{r-\ell})\mathrm{e}^{\psi_i x}$ with $\bar{\boldsymbol{a}}_{i,r}(r = 1, \ldots, k_i$ and $i = 1, \ldots, f)$ being a row vector of order m whose jth element is $\overline{a}_{i,r,j}$. If we let $f = t, k_i = 1(i = 1, \ldots, f)$, and $\bar{\boldsymbol{a}}_{i,1} \equiv \bar{\boldsymbol{a}}_i$, then (30) reduces to (14). It may be remarked that the virtual system-time and other distributions, namely, the actual queueing (system)-time distribution of the first or an arbitrary customer of an arriving batch, can be evaluated in a similar way.

5. Performance measures

The mean virtual queueing delay can be obtained as $W_{vq} = \boldsymbol{W}_{vq}\boldsymbol{e}$, where

$$\boldsymbol{W}_{vq} = (-1)\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{W}_{vq}(s)\bigg|_{s=0} = \sum_{i=1}^{t}\frac{\overline{\boldsymbol{a}}_i}{(-\psi_i)^2}.$$

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Similarly, the mean actual queueing delay of the first and an arbitrary customer of an arriving batch can be obtained as $W_{fq} = \boldsymbol{W}_{fq}\boldsymbol{e}$ and $W_{aq} = \boldsymbol{W}_{aq}\boldsymbol{e}$, respectively, where $\boldsymbol{W}_{fq} = \sum_{i=1}^{t} \overline{\boldsymbol{b}}_i/(-\psi_i)^2$ and $\boldsymbol{W}_{aq} = \sum_{i=1}^{t} \overline{\boldsymbol{c}}_i/(-\psi_i)^2$.

6. Numerical results

We have carried out extensive numerical work based on the procedure discussed in this paper by considering different arrival matrices BMAP($\boldsymbol{D}_0, \boldsymbol{D}_1, \ldots, \boldsymbol{D}_N$) and service-time distributions ME($\boldsymbol{\beta}, \boldsymbol{T}, \boldsymbol{\eta}$). Further, we have tested our procedure for different values of ($\boldsymbol{D}_0, \boldsymbol{D}_1, \ldots, \boldsymbol{D}_N$) with reasonable positive lag-1 correlation (see, e.g. [12]). Several outputs were generated and the results match exactly against some existing results in special cases. We have also matched our results using the MAM given in [19]. All the calculations were performed on a PC having configurations as Intel® Core™i5-2400 CPU processor running at 3.10GHz with 4.00GB of RAM, using MAPLE® 13. Further, though all the numerical results were carried out in 256 precision, they are reported here in four decimal places due to space constraints. Some sample outputs are shown in the following tables, where the results are given for the BMAP/R/1 and BMAP/D/1 queues.

Example 1. (*The BMAP/R/1 queue.*) For the first example, we choose the arrival matrices

$$\boldsymbol{D}_0 = \begin{bmatrix} -0.170\,300\,0 & 0.144\,100\,0 & 0.013\,100\,0 & 0.000\,000\,0 \\ 0.013\,100\,0 & -0.196\,500\,0 & 0.000\,000\,0 & 0.013\,100\,0 \\ 0.144\,100\,0 & 0.000\,000\,0 & -0.183\,400\,0 & 0.013\,100\,0 \\ 0.013\,100\,0 & 0.013\,100\,0 & 0.013\,100\,0 & -1.244\,500\,0 \end{bmatrix},$$

$$\boldsymbol{D}_1 = \begin{bmatrix} 0.000\,000\,0 & 0.000\,000\,0 & 0.002\,777\,2 & 0.000\,000\,0 \\ 0.030\,549\,2 & 0.002\,777\,2 & 0.002\,777\,2 & 0.000\,000\,0 \\ 0.002\,777\,2 & 0.000\,000\,0 & 0.002\,777\,2 & 0.000\,000\,0 \\ 0.000\,000\,0 & 0.000\,000\,0 & 0.002\,777\,2 & 0.252\,725\,2 \end{bmatrix},$$

$$\boldsymbol{D}_2 = \begin{bmatrix} 0.000\,000\,0 & 0.000\,000\,0 & 0.002\,855\,8 & 0.000\,000\,0 \\ 0.031\,413\,8 & 0.002\,855\,8 & 0.002\,855\,8 & 0.000\,000\,0 \\ 0.002\,855\,8 & 0.000\,000\,0 & 0.002\,855\,8 & 0.000\,000\,0 \\ 0.000\,000\,0 & 0.000\,000\,0 & 0.002\,855\,8 & 0.259\,877\,8 \end{bmatrix},$$

$$\boldsymbol{D}_3 = \begin{bmatrix} 0.000\,000\,0 & 0.000\,000\,0 & 0.002\,240\,1 & 0.000\,000\,0 \\ 0.024\,641\,1 & 0.002\,240\,1 & 0.002\,240\,1 & 0.000\,000\,0 \\ 0.002\,240\,1 & 0.000\,000\,0 & 0.002\,240\,1 & 0.000\,000\,0 \\ 0.000\,000\,0 & 0.000\,000\,0 & 0.002\,240\,1 & 0.203\,849\,1 \end{bmatrix},$$

$$\boldsymbol{D}_4 = \begin{bmatrix} 0.000\,000\,0 & 0.000\,000\,0 & 0.003\,550\,1 & 0.000\,000\,0 \\ 0.039\,051\,1 & 0.003\,550\,1 & 0.003\,550\,1 & 0.000\,000\,0 \\ 0.003\,550\,1 & 0.000\,000\,0 & 0.003\,550\,1 & 0.000\,000\,0 \\ 0.000\,000\,0 & 0.000\,000\,0 & 0.003\,550\,1 & 0.323\,059\,1 \end{bmatrix},$$

$$\boldsymbol{D}_5 = \begin{bmatrix} 0.000\,000\,0 & 0.000\,000\,0 & 0.001\,676\,8 & 0.000\,000\,0 \\ 0.018\,444\,8 & 0.001\,676\,8 & 0.001\,676\,8 & 0.000\,000\,0 \\ 0.001\,676\,8 & 0.000\,000\,0 & 0.001\,676\,8 & 0.000\,000\,0 \\ 0.000\,000\,0 & 0.000\,000\,0 & 0.001\,676\,8 & 0.152\,588\,8 \end{bmatrix},$$

with $\boldsymbol{D}_k = \boldsymbol{0}, k \geq 6$, having lag-1 correlation of 0.3993. These lead to $\overline{\boldsymbol{\pi}} = [0.4298\ 0.3459\ 0.1103\ 0.1140]$, $\lambda_g = 0.2049$, and $\lambda^* = 0.5910$. We choose an arbitrary service-time distribution having density $h(t) = (1 + 1/4\pi^2)(1 - \cos(2\pi t))\mathrm{e}^{-t}$ and its LST, $h^*(s)$,

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given by $h^*(s) = (4\pi^2+1)/(s^3+3s^2+(4\pi^2+3)s+4\pi^2+1)$, $1/\mu = 1.0494$, $\rho = 0.6202$. From the above, we have $m = 4$, $N = 5$, and $n = 3$. Using the procedure given in this paper, we calculate the various queueing-time distributions for different values of x which are given in Tables 1–3. One may note that $\widetilde{\boldsymbol{W}}_{\mathrm{v}q}(0) = \boldsymbol{q}$ and $\widetilde{\boldsymbol{W}}_{\mathrm{v}q}(\infty) = \overline{\boldsymbol{\pi}}$, respectively, which should be the case. This also acts as an internal check for our results.

Example 2. (*The BMAP/D/1 queue.*) For the second example, we choose the arrival matrices the same as in Example 1 and assume that the service-time distribution is deterministic with LST $h^*(s) = \mathrm{e}^{-s}$, $1/\mu = 1$, $\rho = 0.5910$. As $h^*(s)$ is a transcendental function, we find a rational function approximation to $h^*(s)$. Generally, in the literature it is usual to approximate the LST of the deterministic distribution with the LST of an Erlang(k) distribution, i.e. $(k/(s+k))^k$ with high k. We approximate e^{-s} with a Padé approximation as Padé(4, 5); see [4] (i.e. rational approximation with degree of numerator and denominator as 4 and 5, respectively),

$$h^*(s) \approx \frac{15120 - 6720s + 1260s^2 - 120s^3 + 5s^4}{15120 + 8400s + 2100s^2 + 300s^3 + 25s^4 + s^5}.$$

TABLE 1: Values of virtual queueing-time distribution for different values of x. Note that $\boldsymbol{W}_{\mathrm{v}q} = [18.5100\ 14.4309\ 5.4093\ 10.6912]$, $W_{\mathrm{v}q} = 49.0415$.

x	$\widetilde{W}_{\mathrm{v}q,1}(x)$	$\widetilde{W}_{\mathrm{v}q,2}(x)$	$\widetilde{W}_{\mathrm{v}q,3}(x)$	$\widetilde{W}_{\mathrm{v}q,4}(x)$	$\widetilde{W}_{\mathrm{v}q}(x)$
0	$0.1679 = q_1$	$0.1714 = q_2$	$0.0377 = q_3$	$0.0028 = q_4$	$0.3798 = \boldsymbol{qe}$
10	0.2564	0.2102	0.0595	0.0133	0.5394
50	0.3162	0.2568	0.0770	0.0478	0.6978
100	0.3625	0.2931	0.0906	0.0747	0.8209
500	0.4288	0.3451	0.1100	0.1134	0.9973
1000	$0.4298 = \overline{\pi}_1$	$0.3459 = \overline{\pi}_2$	$0.1103 = \overline{\pi}_3$	$0.1140 = \overline{\pi}_4$	$1.0000 = \overline{\boldsymbol{\pi}}\boldsymbol{e}$

TABLE 2: Values of actual queueing-time distribution of the first customer of an arriving batch for different values of x. Note that $\boldsymbol{W}_{\mathrm{f}q} = [10.4969\ 0.9228\ 3.1361\ 62.2149]$, $W_{\mathrm{f}q} = 76.7707$.

x	$\widetilde{W}_{\mathrm{f}q,1}(x)$	$\widetilde{W}_{\mathrm{f}q,2}(x)$	$\widetilde{W}_{\mathrm{f}q,3}(x)$	$\widetilde{W}_{\mathrm{f}q,4}(x)$	$\widetilde{W}_{\mathrm{f}q}(x)$
0	0.1230	0.0110	0.0243	0.0162	0.1744
10	0.1517	0.0134	0.0345	0.0773	0.2770
50	0.1855	0.0164	0.0446	0.2778	0.5244
100	0.2119	0.0187	0.0525	0.4349	0.7181
500	0.2498	0.0221	0.0638	0.6601	0.9957
1000	0.2503	0.0221	0.0639	0.6636	1.0000

TABLE 3: Values of actual queueing-time distribution of an arbitrary customer of an arriving batch for different values of x. Note that $\boldsymbol{W}_{\mathrm{a}q} = [10.8281\ 0.9521\ 3.2207\ 63.0928]$, $W_{\mathrm{a}q} = 78.0937$.

x	$\widetilde{W}_{\mathrm{a}q,1}(x)$	$\widetilde{W}_{\mathrm{a}q,2}(x)$	$\widetilde{W}_{\mathrm{a}q,3}(x)$	$\widetilde{W}_{\mathrm{a}q,4}(x)$	$\widetilde{W}_{\mathrm{a}q}(x)$
0	0.0426	0.0038	0.0084	0.0056	0.0605
10	0.1497	0.0133	0.0339	0.0691	0.2660
50	0.1846	0.0163	0.0443	0.2724	0.5177
100	0.2114	0.0187	0.0523	0.4317	0.7141
500	0.2497	0.0221	0.0638	0.6601	0.9956
1000	0.2503	0.0221	0.0639	0.6636	1.0000

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From this, we obtain $m = 4$, $N = 5$, and $n = 5$. Following the procedure discussed in this paper, we calculate the vector $\boldsymbol{q}$ and virtual queueing-time distributions for different values of x. These results are given in Table 4.

Example 3. (*The BMAP/PH/1 queue.*) For the third example, we choose the arrival matrices of the BMAP such that the mean arrival rate and traffic intensity are too small. This example demonstrates that even if the roots of the characteristics equation are close enough, we can still find them and evaluate the boundary vector as well as queueing-time distributions with the same ease. We choose the arrival matrices of the BMAP (with correlation = 0.13) having representation $\{\boldsymbol{D}_N,\ N = 0, 1, \ldots, 10\}$ as

$$\boldsymbol{D}_0 = \mathcal{C}\begin{bmatrix} -1.2500 & 0.0500 & 0.0700 \\ 0.0800 & -0.2700 & 0.0400 \\ 0.1000 & 0.0400 & -0.3000 \end{bmatrix}, \quad \boldsymbol{D}_1 = \mathcal{C}\begin{bmatrix} 0.1111 & 0.0055 & 0.0077 \\ 0.0011 & 0.0066 & 0.0088 \\ 0.0022 & 0.0110 & 0.0044 \end{bmatrix},$$

$$\boldsymbol{D}_2 = \mathcal{C}\begin{bmatrix} 0.1212 & 0.0060 & 0.0084 \\ 0.0012 & 0.0072 & 0.0096 \\ 0.0024 & 0.0120 & 0.0048 \end{bmatrix}, \quad \boldsymbol{D}_4 = \mathcal{C}\begin{bmatrix} 0.1313 & 0.0065 & 0.0091 \\ 0.0013 & 0.0078 & 0.0104 \\ 0.0026 & 0.0130 & 0.0052 \end{bmatrix},$$

$$\boldsymbol{D}_7 = \mathcal{C}\begin{bmatrix} 0.0202 & 0.0010 & 0.0014 \\ 0.0002 & 0.0012 & 0.0016 \\ 0.0004 & 0.0020 & 0.0008 \end{bmatrix}, \quad \boldsymbol{D}_9 = \mathcal{C}\begin{bmatrix} 0.1414 & 0.0070 & 0.0098 \\ 0.0014 & 0.0084 & 0.0112 \\ 0.0028 & 0.0140 & 0.0056 \end{bmatrix},$$

$$\boldsymbol{D}_{10} = \mathcal{C}\begin{bmatrix} 0.0101 & 0.0005 & 0.0007 \\ 0.0001 & 0.0006 & 0.0008 \\ 0.0002 & 0.0010 & 0.0004 \end{bmatrix},$$

$$\boldsymbol{D}_6 = \boldsymbol{D}_3 = \boldsymbol{D}_1, \qquad \boldsymbol{D}_5 = \boldsymbol{D}_2, \quad \text{and} \quad \boldsymbol{D}_8 = \boldsymbol{D}_4,$$

and the phase-type (PH) service-time distributions having representation PH($\boldsymbol{\beta}$, $\boldsymbol{T}$) as

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} -4.12 & 3.5 \\ 2.5 & -5.25 \end{bmatrix}.$$

Now changing the value of $\mathcal{C}$ will give different values of ρ. As the value of $\mathcal{C}$ decreases, the closeness within the roots of the γ_i (with nonnegative real parts) and the ψ_i (with negative real parts) increases. In order to demonstrate this fact let $\mathcal{C} = 10^{-80}$. This gives us $\overline{\boldsymbol{\pi}} = [0.3029\ 0.3654\ 0.3317]$, $\lambda_g = 4.5014 \times 10^{-81}$, $\lambda^* = 2.2507 \times 10^{-80}$, $1/\mu = 0.6793$, and $\rho = 1.5290 \times 10^{-80}$. This queueing system depicts the extreme condition where the traffic intensity is too low to represent any realistic situation. We choose this example to demonstrate

TABLE 4: Values of virtual queueing-time distribution for different values of x. Note that $\boldsymbol{W}_{\mathrm{v}q} = [14.8056\ 11.4983\ 4.4112\ 9.3388]$, $W_{\mathrm{v}q} = 41.0539$.

x	$\widetilde{W}_{\mathrm{v}q,1}(x)$	$\widetilde{W}_{\mathrm{v}q,2}(x)$	$\widetilde{W}_{\mathrm{v}q,3}(x)$	$\widetilde{W}_{\mathrm{v}q,4}(x)$	$\widetilde{W}_{\mathrm{v}q}(x)$
0	$0.1799 = q_1$	$0.1858 = q_2$	$0.0405 = q_3$	$0.0028 = q_4$	$0.4090 = \boldsymbol{qe}$
10	0.2737	0.2238	0.0637	0.0145	0.5756
50	0.3333	0.2704	0.0815	0.0525	0.7376
100	0.3769	0.3045	0.0945	0.0803	0.8561
500	0.4294	0.3455	0.1101	0.1138	0.9988
1000	$0.4298 = \overline{\pi}_1$	$0.3459 = \overline{\pi}_2$	$0.1103 = \overline{\pi}_3$	$0.1140 = \overline{\pi}_4$	$1.0000 = \overline{\boldsymbol{\pi}}\boldsymbol{e}$

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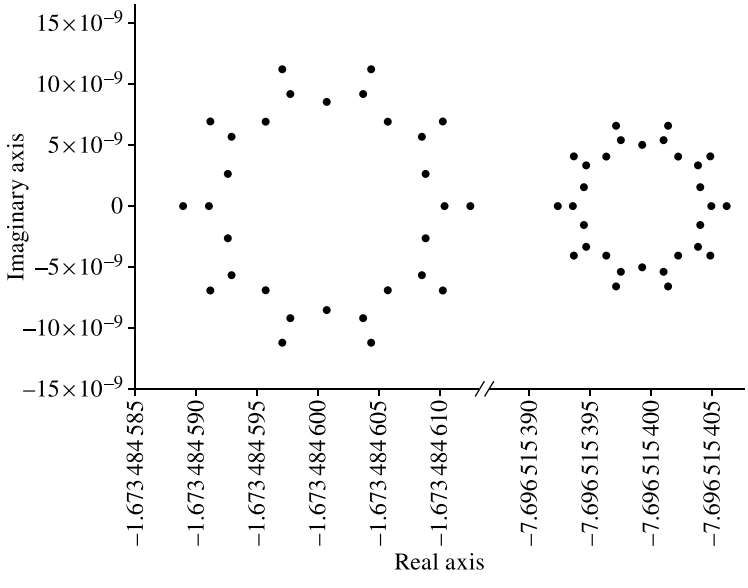


FIGURE 1: Location of the roots of $F(s)$ having $\mathrm{Re}(s) < 0$, i.e. $\psi_1, \psi_2, \ldots, \psi_{60}$.

that even in this extreme case we can find the value of the unknown vector $\boldsymbol{q}$ and the queueing-time distributions.

Now, we have $m(= 3)$ roots with nonnegative real parts; namely, $\{\gamma_1, \gamma_2, \gamma_3\}$ and $mnN(= 60)$ roots with negative real parts; namely, $\{\psi_1, \psi_2, \ldots, \psi_{60}\}$. One can see from Table 5 that all roots are close enough so that one can regard them as repeated roots, but using MAPLE we are able to identify them as distinct roots and, hence, calculate them accurately. The three roots

TABLE 5: Roots of $F(s)$ having $\mathrm{Re}(s) \geq 0$, i.e. $\gamma_1, \gamma_2, \gamma_3$ and $\mathrm{Re}(s) < 0$, i.e. $\psi_1, \psi_2, \ldots, \psi_{60}$.

i	γ_i	$\lvert F(\gamma_i)\rvert$
1	$3.2\underbrace{000\cdots000}_{\text{78 zeros}}494 \times 10^{-81}$	1×10^{-463}
2	$3.9\overbrace{000\cdots000}^{\text{78 zeros}}271 \times 10^{-81}$	1×10^{-463}
3	0	0
i	ψ_i	$\lvert F(\psi_i)\rvert$
1	$-1.673\,484\,588\,9$	2×10^{-211}
2	$-1.673\,484\,591\,1$	1×10^{-211}
3, 4	$-1.673\,484\,591\,2 \pm 6.934\,368\,562\,8 \times 10^{-9}i$	2×10^{-211}
⋮	⋮	⋮
30	$-1.673\,484\,612\,5$	2×10^{-211}
31	$-7.696\,515\,392\,3$	8×10^{-190}
32	$-7.696\,515\,393\,6$	3×10^{-189}
33, 34	$-7.696\,515\,393\,7 \pm 4.075\,874\,724\,8 \times 10^{-9}i$	7×10^{-190}
⋮	⋮	⋮
60	$-7.696\,515\,406\,2$	1×10^{-189}

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(with $\mathrm{Re}(s) \geq 0$) are distinct roots, however the first 81 decimals of each root are the same. The traffic intensity, ρ, is of the order 10^{-80} and this situation is highly unrealistic. For all the realistic situations (when $\rho = 0.2$ or 0.8, etc.) the roots $\{\gamma_1, \gamma_2, \gamma_3\}$ are always distinct and isolated, being a reasonable distance apart from each other. Using these roots we calculated the boundary vector $\boldsymbol{q}$, which is given as $\boldsymbol{q} = [0.3029\ 0.3654\ 0.3317]$. One must note that $\boldsymbol{q}$ and $\overline{\boldsymbol{\pi}}$ give the same value to four decimal places but $\overline{\boldsymbol{\pi}}\boldsymbol{e} = 1$, while $\boldsymbol{q}\boldsymbol{e} = 1 - 1.5290 \times 10^{-80} = 1 - \rho$.

One can also observe that all roots (with $\mathrm{Re}(s) < 0$) are centered about two points in the complex plane; see Figure 1. The roots are accurate enough as the value of $|F(\psi_i)|$ is of the order 10^{-200} for each $i = 1, 2, \ldots, 60$; see Table 5. Based on our numerical experience of working with the calculations of roots, we can say that the roots never get so close for all the realistic situations. Location of the roots of the CE (for this and other examples as well) is well structured and predictable, so that numerical problems are not likely to occur.

6.1. Evaluation of the boundary vector using the MAM

We have evaluated and matched our results against those obtained using the MAM, [19]. In the first example, $h^*(s)$ is a rational function having all its poles in $\mathrm{Re}(s) < 0$. So, we can write this distribution as a matrix exponential, $\mathrm{ME}(\boldsymbol{\beta}, \boldsymbol{T}, \boldsymbol{\eta})$. Since there is no unique representation of the ME, we present three possible representations as

$$\boldsymbol{\beta} = \begin{bmatrix} 1 + 4\pi^2 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(1 + 4\pi^2) & -(3 + 4\pi^2) & -3 \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

see [6],

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} 0 & -(1 + 4\pi^2) & (1 + 4\pi^2) \\ 3 & 2 & -6 \\ 2 & 2 & -5 \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};$$

see [10], and

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} 2\pi i - 1 & 0 & 0 \\ 0 & -2\pi i - 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} \dfrac{-(1 + 4\pi^2)}{8\pi^2} \\ \dfrac{-(1 + 4\pi^2)}{8\pi^2} \\ \dfrac{(1 + 4\pi^2)}{4\pi^2} \end{bmatrix};$$

see [3]. One must also note that the representation of the ME distribution is clearly not the PH distribution. To calculate the boundary vector we first calculate the $\boldsymbol{A}_k$ and then follow the iterative process to calculate the stochastic matrix $\boldsymbol{G}$ which is given by

$$\boldsymbol{G} = \begin{bmatrix} 0.8640 & 0.1166 & 0.0184 & 0.0009 \\ 0.1016 & 0.8774 & 0.0145 & 0.0065 \\ 0.1259 & 0.0211 & 0.8468 & 0.0061 \\ 0.2372 & 0.2315 & 0.0726 & 0.4586 \end{bmatrix}.$$

Solving $\boldsymbol{g}\boldsymbol{G} = \boldsymbol{g}$ and $\boldsymbol{g}\boldsymbol{e} = 1$, we obtain $\boldsymbol{g} = [0.4420\ 0.4514\ 0.0993\ 0.0073]$. With $\rho = 0.6202$, our boundary vector $\boldsymbol{q}$ matches exactly with $(1 - \rho)\boldsymbol{g}$ which is given in the first row of Table 1.

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For the second example, we have approximated $h^*(s)$ as a rational function having all its poles in $\mathrm{Re}(s) < 0$, we can also write this distribution as $\mathrm{ME}(\boldsymbol{\beta}, \boldsymbol{T}, \boldsymbol{\eta})$ as

$$\boldsymbol{\beta} = \begin{bmatrix} 15120 & -6720 & 1260 & -120 & 5 \end{bmatrix},$$

$$\boldsymbol{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -15120 & -8400 & -2100 & -300 & -25 \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

see [14]. We have used the proposed ME representation of deterministic distribution and calculated the $\boldsymbol{A}_k$ and the matrix $\boldsymbol{G}$, which is given as

$$\boldsymbol{G} = \begin{bmatrix} 0.8555 & 0.1246 & 0.0191 & 0.0008 \\ 0.1061 & 0.8716 & 0.0146 & 0.0077 \\ 0.1351 & 0.0156 & 0.8420 & 0.0073 \\ 0.2925 & 0.2873 & 0.0906 & 0.3296 \end{bmatrix}.$$

Solving $\boldsymbol{g}\boldsymbol{G} = \boldsymbol{g}$ and $\boldsymbol{g}\boldsymbol{e} = 1$, we have $\boldsymbol{g} = [0.4400\ 0.4542\ 0.0990\ 0.0068]$. Now our boundary vector $\boldsymbol{q}$ matches exactly with $0.4090\boldsymbol{g}$, i.e. $(1 - \rho)\boldsymbol{g}$.

6.2. Computation time

In this section we discuss the strengths and advantages of our method in comparison to the MAM. We do this by comparing the computation time in calculating the boundary vector. Whereas there are three steps involved in the computation of the boundary vector in our approach, there are two steps in the MAM. All the calculations presented in this subsection are performed on a PC having the same configuration as stated at the start of Section 6.

Steps involved in the method of roots.

Step 1. Calculation of $F(s)$ and $F_1(s)$ (choose any one of $F_j(s)$, $j = 1, 2, \ldots, m$). This involves determination of the determinant of matrices $\boldsymbol{G}(s)$ and $\boldsymbol{G}_1(s)$ (see (4)), both of order m having polynomial elements.

Step 2. Calculation of m zeros of $F(s)$ with nonnegative real parts.

Step 3. Calculation of vector $\boldsymbol{q}$. This involves solving m simultaneous equations to obtain q_i, $1 \leq i \leq m$.

Steps involved in the MAM.

Step 1. Evaluation of matrixes $\boldsymbol{A}_k$ and stochastic matrix $\boldsymbol{G}$, using an iterative process.

Step 2. Calculation of vector $\boldsymbol{g}$. This involves solving m simultaneous equations to obtain g_i, $1 \leq i \leq m$.

To perform the comparison we consider one example with the BMAP having representation $\{\boldsymbol{D}_N,\ N = 0, 1, \ldots, 10\}$, exactly the same as considered in Example 3, and PH service-time distributions having representation $\mathrm{PH}(\boldsymbol{\beta}, \boldsymbol{T})$ as

$$\boldsymbol{\beta} = \begin{bmatrix} 0.25 & 0.75 \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} -10.0 & 6.0 \\ 2.0 & -12.0 \end{bmatrix}.$$

Thus, $\lambda^* = 2.2507 \times \mathcal{C}$, $1/\mu = 0.125$, and $\rho = 0.2813 \times \mathcal{C}$. The above representation has a correlation of 0.13. Now changing the value of $\mathcal{C}$, we obtain $\rho \in (0, 1)$. We compute the computation times of the proposed method and the MAM, which are shown in Figure 2. We can see from Figure 2 that the computation time for the proposed method is almost constant

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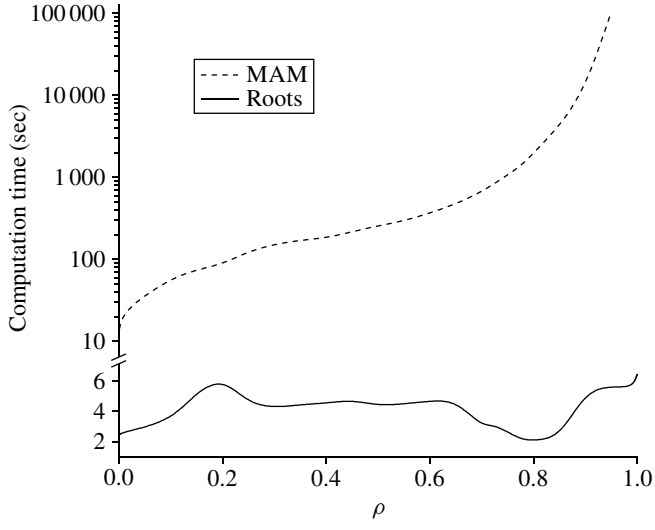


FIGURE 2: Computation time when ρ varies.

(2–5 secs), while the computation time for the MAM increases as a function of the traffic intensity ρ. Similarly, we consider different sets of the BMAP representation having the same value of λ^* but with different correlation values. With the same service-time distribution as above and having traffic intensity $\rho = 0.35$, we again computed the computation times for the proposed method and for the MAM, which are shown in Figure 3. Almost the same observations can be drawn from Figure 3; that the computation time for the proposed method is almost constant (6–7 secs), while the computation time for the MAM increases as a function of the correlation. This shows that the proposed method is stable and its computation time is independent of both the traffic intensity and correlation.

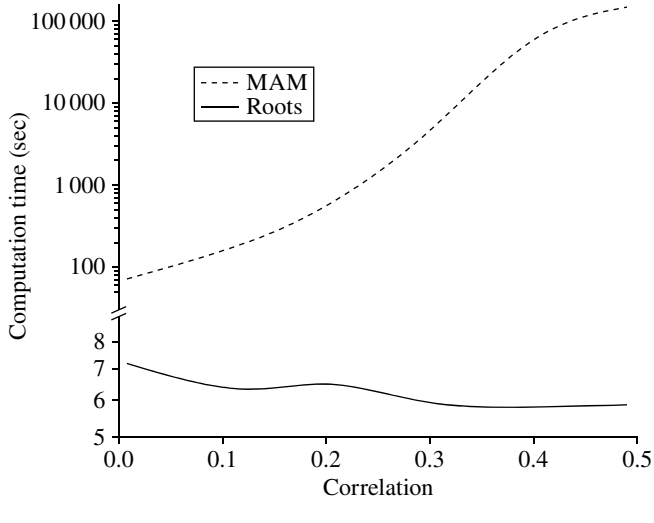


FIGURE 3: Computation time when the correlation varies.

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7. Conclusions and future scope

In this paper we have presented a procedure to evaluate the boundary probabilities as well as the virtual queueing (system)-time distribution, actual queueing (system)-time distribution of the first customer of an arriving batch, and actual queueing (system)-time distribution of an arbitrary customer of an arriving batch for the BMAP/G/1 queue. Motivation behind the procedure is to acquaint researchers and practitioners with an alternative procedure which is easy to understand and implement and computationally less intensive. Our method has skipped the evaluation of the $\boldsymbol{A}_k$. We calculated the unknown vector $\boldsymbol{q}$ using only one phase of the LST which reduces the labor in dealing with the entire LST. The proposed method can be applied to obtain computational results for more complex models such as bulk arrival and/or bulk service queues.

Appendix: System-time distributions

Similar to Section 3, we can calculate three different types of system-time distribution; namely,

- virtual system time, $\widetilde{W}_{\mathrm{v}s}(x)$,
- actual system time of the first customer of an arriving batch, $\widetilde{W}_{\mathrm{f}s}(x)$,
- actual system-time of an arbitrary customer of an arriving batch, $\widetilde{W}_{\mathrm{a}s}(x)$.

TABLE 6: Various formulae for system-time distributions.

	Virtual ($\cdot = \mathrm{v}$)	First customer of batch ($\cdot = \mathrm{f}$)	Arbitrary customer of batch ($\cdot = \mathrm{a}$)
$\boldsymbol{W}_{\cdot}(s) =$	$\dfrac{sP(s)Q(s)^{N-1}}{F(s)}\overline{\boldsymbol{F}}(s)$	$\dfrac{sP(s)Q(s)^{N-1}}{\lambda_g F(s)}\overline{\boldsymbol{F}}(s)\boldsymbol{D}_1'$	$\dfrac{sP(s)}{\lambda^* F(s)}\overline{\boldsymbol{F}}(s)\sum_{k=1}^{N}\boldsymbol{D}_k \times\sum_{i=0}^{k-1}P(s)^i Q(s)^{N-1-i}$
$\boldsymbol{W}_{\cdot}(s) =$	$\bar{\boldsymbol{u}}_0+\sum_{i=1}^{t}\dfrac{\bar{\boldsymbol{u}}_i}{s-\psi_i}$	$\bar{\boldsymbol{v}}_0+\sum_{i=1}^{t}\dfrac{\bar{\boldsymbol{v}}_i}{s-\psi_i}$	$\bar{\boldsymbol{\omega}}_0+\sum_{i=1}^{t}\dfrac{\bar{\boldsymbol{\omega}}_i}{s-\psi_i}$
$\boldsymbol{w}_{\cdot}(x) =$	$\bar{\boldsymbol{u}}_0\delta(x)+\sum_{i=1}^{t}\bar{\boldsymbol{u}}_i\mathrm{e}^{\psi_i x}$	$\bar{\boldsymbol{v}}_0\delta(x)+\sum_{i=1}^{t}\bar{\boldsymbol{v}}_i\mathrm{e}^{\psi_i x}$	$\bar{\boldsymbol{\omega}}_0\delta(x)+\sum_{i=1}^{t}\bar{\boldsymbol{\omega}}_i\mathrm{e}^{\psi_i x}$
$\widetilde{\boldsymbol{W}}_{\cdot}(x) =$	$\overline{\boldsymbol{\pi}}+\sum_{i=1}^{t}\dfrac{\bar{\boldsymbol{u}}_i}{\psi_i}\mathrm{e}^{\psi_i x}$	$\dfrac{1}{\lambda_g}\overline{\boldsymbol{\pi}}\boldsymbol{D}_1'+\sum_{i=1}^{t}\dfrac{\bar{\boldsymbol{v}}_i}{\psi_i}\mathrm{e}^{\psi_i x}$	$\dfrac{1}{\lambda^*}\overline{\boldsymbol{\pi}}\sum_{k=1}^{N}k\boldsymbol{D}_k+\sum_{i=1}^{t}\dfrac{\bar{\boldsymbol{\omega}}_i}{\psi_i}\mathrm{e}^{\psi_i x}$
	$\bar{\boldsymbol{u}}_0=\overline{\boldsymbol{\pi}}+\sum_{r=1}^{t}\dfrac{\bar{\boldsymbol{u}}_r}{\psi_r}$	$\bar{\boldsymbol{v}}_0=\dfrac{1}{\lambda_g}\overline{\boldsymbol{\pi}}\boldsymbol{D}_1'+\sum_{r=1}^{t}\dfrac{\bar{\boldsymbol{v}}_r}{\psi_r}$	$\bar{\boldsymbol{\omega}}_0=\dfrac{1}{\lambda^*}\overline{\boldsymbol{\pi}}\sum_{k=1}^{N}k\boldsymbol{D}_k+\sum_{r=1}^{t}\dfrac{\bar{\boldsymbol{\omega}}_r}{\psi_r}$
	$\bar{\boldsymbol{u}}_i=\bar{\boldsymbol{a}}_i h(\psi_i),\ i=1,2,\ldots,t$	$\bar{\boldsymbol{v}}_i=\bar{\boldsymbol{b}}_i h(\psi_i),\ i=1,2,\ldots,t$	$\bar{\boldsymbol{\omega}}_i=\bar{\boldsymbol{c}}_i h(\psi_i),\ i=1,2,\ldots,t$

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TABLE 7: Virtual system-time distribution. Note that $\boldsymbol{W}_{vs} = [18.9610\ 14.7939\ 5.5251\ 10.8110]$, $W_{vs} = 77.8201$.

x	$\widetilde{W}_{vs,1}(x)$	$\widetilde{W}_{vs,2}(x)$	$\widetilde{W}_{vs,4}(x)$	$\widetilde{W}_{vs,4}(x)$	$\widetilde{W}_{vs}(x)$
0	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.2536	0.2085	0.0586	0.0122	0.5329
50	0.3150	0.2558	0.0767	0.0470	0.6944
100	0.3618	0.2925	0.0904	0.0743	0.8189
500	0.4288	0.3451	0.1100	0.1134	0.9972
1000	$0.4298 = \overline{\pi}_1$	$0.3459 = \overline{\pi}_2$	$0.1103 = \overline{\pi}_3$	$0.1140 = \overline{\pi}_4$	$1.0000 = \overline{\boldsymbol{\pi}}\boldsymbol{e}$

TABLE 8: Actual system-time distribution of the first customer of an arriving batch. Note that $\boldsymbol{W}_{fs} = [10.7596\ 0.9460\ 3.2032\ 62.9113]$, $W_{fs} = 50.0909$.

x	$\widetilde{W}_{fs,1}(x)$	$\widetilde{W}_{fs,2}(x)$	$\widetilde{W}_{fs,4}(x)$	$\widetilde{W}_{fs,4}(x)$	$\widetilde{W}_{fs}(x)$
0	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1504	0.0133	0.0341	0.0708	0.2686
50	0.1848	0.0164	0.0444	0.2735	0.5191
100	0.2115	0.0187	0.0524	0.4324	0.7150
500	0.2497	0.0221	0.0638	0.6601	0.9957
1000	0.2503	0.0221	0.0639	0.6636	1.0000

TABLE 9: Actual system-time distribution of an arbitrary customer of an arriving batch. Note that $\boldsymbol{W}_{as} = [11.0908\ 0.9753\ 3.2878\ 63.7892]$, $W_{as} = 79.1431$.

x	$\widetilde{W}_{as,1}(x)$	$\widetilde{W}_{as,2}(x)$	$\widetilde{W}_{as,4}(x)$	$\widetilde{W}_{as,4}(x)$	$\widetilde{W}_{as}(x)$
0	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1478	0.0131	0.0333	0.0625	0.2567
50	0.1839	0.0163	0.0441	0.2680	0.5123
100	0.2110	0.0187	0.0522	0.4291	0.7110
500	0.2497	0.0221	0.0638	0.6600	0.9956
1000	0.2503	0.0221	0.0639	0.6636	1.0000

Just changing the suffix of the notation used in queueing-time distributions, we can define the same notation for system-time distributions, e.g. the LST of the virtual system-time distribution is given by $W_{vs}(s) = \boldsymbol{W}_{vs}(s)\boldsymbol{e}$, where $\boldsymbol{W}_{vs}(s) = \boldsymbol{W}_{vq}(s)h^*(s)$. Similarly, other quantities can be defined. We present various formulae related to system-time distributions in Table 6. In continuation of Example 1, we also calculate the various system-time distributions for different values of x, given in Tables 7–9.

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