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INTEGRAL STOCHASTIC ORDERS WITH PARAMETRIC CLASSES OF FUNCTIONS

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Abstract

This paper introduces the general ideas for parametric integral stochastic orders, with which a continuum of parametric functions are defined as a bridge between different classes of non-parametric functions. This approach allows one to identify a parametric function class over which two given random variables may violate the non-parametric stochastic order with specific patterns. The parameter used to name the parametric function class also measures the ratio of dominance violation for the corresponding non-parametric stochastic orders. Our framework, expanding the domain of stochastic orders, covers the existing studies of almost stochastic dominance. This leads to intuitive explanations and simpler proofs of existing results and their extensions.

Keywords: Stochastic orderings; parametric functions; asymmetric relaxation

2010 Mathematics Subject Classification: Primary 60E10

1. Introduction

Stochastic orders are partial orders of distribution functions. Depending on the applications, various comparisons between distributions arise in the analysis of the related stochastic systems. Shaked and Shanthikumar [21] and Müller and Stoyan [17] provide comprehensive discussions of stochastic orders. Among those, many orders can be characterized as *integral stochastic orders* [2]. Specifically, let X_1 and X_2 be two random variables with distribution functions F_{X_1} and F_{X_2}, respectively, and let $\mathcal{G}$ be a class of measurable functions. We say $X_1 \leq_{\mathcal{G}} X_2$ (or $F_{X_1} \leq_{\mathcal{G}} F_{X_2}$) if and only if $\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$ for all $g \in \mathcal{G}$. Commonly studied function classes include the increasing (*inc*), convex (*cx*), and increasing convex (*icx*) functions, which give rise to the characterizations of the usual stochastic order, convex order, and increasing convex orders, respectively. One may also consider multivariate functions to define the comparison between random vectors (e.g. the supermodularity order). In this paper, we focus on the univariate case.

The aforementioned function classes and the corresponding orders are non-parametric. Depending on the applications, one may want to extend a stochastic order through parameterization. Leshno and Levy [7] first parameterize the usual stochastic order and the increasing concave order, which are named, respectively, 'almost first-degree stochastic dominance'

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(ϵ-AFSD) and 'almost second-degree stochastic dominance' (ϵ-ASSD). The term 'almost' indicates that the distributions may violate the non-parametric order but the violation is limited by the parameter ϵ. Subsequently, many scholars, notably Guo et al. [4], Müller et al. [19], Tsetlin et al. [23], and Tzeng et al. [24], examine parametric stochastic orders that are useful for comparing utility functions in decision analysis. Levy et al. [10] report experimental results to demonstrate the violation of the conventional stochastic orders and provide economic interpretations for the almost orders. Studies also showcase the applications to investment valuation [8, 9], insurance [26], and supply chains [3]. Tsetlin et al. [23] introduce the 'almost second degree risk' (ϵ-ASR) to complement some limitations of the ASSD. Müller et al. [18] define $(1+\gamma)$ stochastic dominance (i.e. $(1+\gamma)$-SD), which is in between the first-degree (i.e. the usual stochastic order) and the second-degree stochastic dominance (i.e. the increasing concave order). Huang et al. [5] and Mao et al. [16] named this order 'fractional stochastic dominance' for utility functions with bounded coefficient of risk aversion.

To systematically examine the parametric integral stochastic orders, we parameterize the function classes used to characterize the integral orders. One way to obtain a parametric function class is to impose restrictions on the property of some non-parametric class $\mathcal{G}$. Take the class of increasing functions as an example, i.e. $\mathcal{G} = inc$. The increasingness of a function can be evaluated by $\delta_g(x_1, x_2) = (g(x_2) - g(x_1))/(x_2 - x_1)$ for any $x_1 < x_2$. We may restrict the increasingness with some parameter $\Delta \geq 0$ to obtain subclasses of $\mathcal{G} = inc$:

$$\mathcal{G}(\Delta) = inc(\Delta) = \Big\{ g \in \mathcal{G} \colon 0 \leq \delta_g(x_1, x_2) \leq (1+\Delta) \inf_{x_a < x_b} \delta_g(x_a, x_b),\ \forall x_1 < x_2 \Big\},$$

$$\mathcal{G}^+(\Delta) = inc^+(\Delta) = \big\{ g \in \mathcal{G} \colon 0 \leq \delta_g(x_1, x_2) \leq (1+\Delta)\delta_g(x_3, x_4),\ \forall x_1 < x_2 < x_3 < x_4 \big\},$$

$$\mathcal{G}^-(\Delta) = inc^-(\Delta) = \big\{ g \in \mathcal{G} \colon 0 \leq \delta_g(x_3, x_4) \leq (1+\Delta)\delta_g(x_1, x_2),\ \forall x_1 < x_2 < x_3 < x_4 \big\}.$$

The subclass $\mathcal{G}(\Delta)$ is obtained by restricting the property of $\mathcal{G}$ over the entire domain, while $\mathcal{G}^-(\Delta)$ and $\mathcal{G}^+(\Delta)$ are obtained by restricting the property from, respectively, the left and the right sides of the evaluation point. When varying Δ from zero to infinity, we obtain a continuum of function classes. In particular, $\Delta = 0$ leads to the smallest subclass and $\Delta = \infty$ gives the non-parametric class $\mathcal{G}$. We may also define a subclass with a two-sided restriction, i.e. $\mathcal{G}^{\pm}(\Delta_+, \Delta_-) = \mathcal{G}^+(\Delta_+) \cap \mathcal{G}^-(\Delta_-)$, which with the symmetric restriction reduces to $\mathcal{G}^{\pm}(\Delta, \Delta) = \mathcal{G}(\Delta)$.

The parametric integral stochastic order, associated with the subclasses of functions, is a *relaxation* of the non-parametric order $\mathcal{G}$, e.g. $X_1 \leq_{\mathcal{G}(\Delta)} X_2$ if $X_1 \leq_{\mathcal{G}} X_2$. The amount of relaxation to the non-parametric order can be evaluated through a *violation measure*. We show that the stochastic orders associated with function classes $\mathcal{G}(\Delta)$, $\mathcal{G}^-(\Delta)$, and $\mathcal{G}^+(\Delta)$ can be characterized by the overall, left-side, and right-side violations, respectively, to the distributional property of the order associated with $\mathcal{G}$. In particular, when the function class is restricted by a parameter Δ, the minimum ratio of dominance violation for the corresponding non-parametric order is $(1+\Delta)$.

This approach, leading to simple proofs, allows one to easily see the relationship among various parametric orders proposed in previous studies (see Table 1). We find the equivalence of $\Delta = 1/\epsilon - 2$ to the almost stochastic dominance or $\Delta = 1/\gamma - 1$ to the fractional stochastic dominance. In particular, the AFSD (corresponding to $inc(\Delta)$ functions) and ASSD (corresponding to $inc\text{-}cv(\Delta)$ functions) are characterized by the parametric functions with symmetric two-sided restrictions, which give rise to symmetric two-sided relaxation of the non-parametric integral stochastic orders. The $(1+\gamma)$-SD (corresponding to $inc^-(\Delta)$ functions) is

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TABLE 1. Parametric function classes and almost stochastic orders defined in previous studies.

Function class	Stochastic order	Parameter relation
$inc(\Delta)$	ϵ-AFSD	$\Delta = 1/\epsilon - 2$
$inc^{-}(\Delta)$	$(1+\gamma)$-FSD	$\Delta = 1/\gamma - 1$
$inc^{\pm}(\Delta_+, \Delta_-)$	$(1+\gamma_{cx}, 1+\gamma_{cv})$-SD	$\Delta_+ = 1/\gamma_{cx} - 1$, $\Delta_- = 1/\gamma_{cv} - 1$
$cv(\Delta)$	ϵ-ASR	$\Delta = 1/\epsilon - 2$
inc-cv(Δ)	ϵ-ASSD	$\Delta = 1/\epsilon - 2$
m-cv(Δ)	ϵ-AnR	$\Delta = 1/\epsilon - 2$
m-icv(Δ)	ANSR or ϵ-GAnSD	$\Delta = 1/\epsilon - 2$

characterized by the left-side restriction of increasing functions, leading to the left-side relaxation of the usual stochastic order. The ASR (corresponding to $cv(\Delta)$ functions) is obtained through a symmetric two-sided relaxation of the concave order. For higher-order functions, we show that the almost nth-degree risk (AnR) and the generalized almost nth-degree stochastic dominance (GAnSD) defined by Tsetlin et al. [23] can be obtained by imposing a symmetric two-sided restriction of the m-concave functions and of the m-increasing-concave functions, respectively. Our way of characterization depicts a full picture of parametric integral stochastic orders for functions with mth-order properties (see Table 3 in Section 5).

The remainder of the paper is organized as follows. We introduce the basic concepts and notation in Section 2. The discussion of parameterized usual stochastic order is summarized in Section 3 and that of the second-order stochastic dominance is provided in Section 4. The analysis is extended to higher orders in Section 5. We conclude the study in Section 6.

2. Preliminaries

To systematically characterize the parametric integral stochastic orders with unified notation, we introduce some basic concepts in this section.

As Denuit and Müller [2] suggest, it is technically convenient to consider sufficiently smooth functions to analyze integral stochastic orders. We focus on functions with bounded variation:

$$\overline{\mathcal{G}} = \left\{ g \colon \mathcal{X} \to \mathbb{R} \mid g(x) = h_1(x) - h_2(x),\, x \in \mathcal{X};\, h_1 \text{ and } h_2 \text{ are monotone and bounded over } \mathcal{X} \right\},$$

where $\mathcal{X} = [\underline{x}, \bar{x}]$ is the bounded support. We assume that the functions under consideration are integrable with respect to the measures under consideration (e.g. Lebesgue integrable) over $\mathcal{X}$. This ensures that we can invoke the fundamental theorem of calculus and apply integration by parts. The non-parametric function classes discussed in this paper are summarized in Table 2.

Throughout this paper, we compare distributions with a common bounded support $\mathcal{X}$. Our analysis can be easily extended to distributions with different supports as long as they are bounded. We note that the assumption of bounded support, though commonly adopted, is not innocuous in general; see [20] for example.

Definition 1. (*Integral stochastic order.*) A random variable X_1 is said to be smaller than another random variable X_2 in the $\hat{\mathcal{G}}$-order, i.e. $X_1 \leq_{\hat{\mathcal{G}}} X_2$, if and only if, for any function $g \in \hat{\mathcal{G}} \subset \overline{\mathcal{G}}$, $\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$.

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TABLE 2. Classes of non-parametric functions.

Function class[a]	Symbol	Characterization
Linear	*lin*	$g(x) = a + bx,\ a, b \in \mathbb{R},\ x \in \mathcal{X}$
Increasing (decreasing)	*inc* (*dec*)	$g(x_1) \leq (\geq)\ g(x_2),\ x_1 \leq x_2, x_1, x_2 \in \mathcal{X}$
Quadratic	*quad*	$g(x) = a + bx + cx^2,\ a, b, c \in \mathbb{R},\ x \in \mathcal{X}$
Convex (convex)	*cx* (*cv*)	$\frac{1}{2}(g(x_1) + g(x_2)) \geq (\leq)\ g(\frac{x_1+x_2}{2})\, x_1, x_2 \in \mathcal{X}$
m-convex (concave)	*m-cx* (*cv*)	$g^{(m)}(x) \geq 0,\ x \in \mathcal{X}$
m-increasing convex (concave)	*m-icx*	$g^{(m-1)}[(-1)^{m-1} g^{(m-1)}] \in inc,\ x \in \mathcal{X}$

[a]We may append *i* or *d* to describe monotone subclasses. For example, *ilin* stands for increasingly linear functions, and *dcx* for decreasing convex functions.

By Definition 1, when restrictions are imposed on the function class, the corresponding integral stochastic order is relaxed, suggesting the possibility of violating the distributional condition for the corresponding non-parametric order. Specifically, for $\hat{\mathcal{G}}(\Delta) \subset \mathcal{G}$, two random variables X_1 and X_2 satisfy the parametric order $X_1 \leq_{\hat{\mathcal{G}}(\Delta)} X_2$ may violate the non-parametric order $\mathcal{G}$. We introduce a *violation measure* $v(x; X_1 \leq_{\mathcal{G}} X_2)$ to represent the amount of violation to the distributional characterization of the non-parametric order at value x. This measure should be zero for all $x \in \mathcal{X}$ if $X_1 \leq_{\mathcal{G}} X_2$. When the random variables X_1 and X_2 fail to be ordered by $\mathcal{G}$, we use

$$v_O(X_1 \leq_{\mathcal{G}} X_2) = \int_{\xi \in \mathcal{X}} v(\xi; X_1 \leq_{\mathcal{G}} X_2)\, \mathrm{d}\xi, \tag{1}$$

$$v_{\leq[\geq]}(x; X_1 \leq_{\mathcal{G}} X_2) = \int_{\xi \leq[\geq] x} v(\xi; X_1 \leq_{\mathcal{G}} X_2)\, \mathrm{d}\xi \tag{2}$$

to evaluate the overall and left-side (right-side) partial violations, respectively. We can also measure the non-violation as $nv(x; X_1 \leq_{\mathcal{G}} X_2) = v(x; X_1 \geq_{\mathcal{G}} X_2)$, and define the corresponding aggregate amounts. We will see that the parametric orders can be characterized by using these violation measures. The violation measures are evaluated based on the distributions. We use F_Z and $\bar{F}_Z$ to denote the cumulative distribution and survival functions, respectively, for a random variable Z.

3. Parametric monotone functions

In this section, we summarize the parametric orders by measuring the amount of violation to the usual stochastic order. We consider subclasses of increasing functions by restricting the increasingness of the functions, which leads to relaxations of the usual stochastic order over the entire domain, from the left side, right side, or both sides.

3.1. Violation to the usual stochastic order

Consider two random variables X_1 and X_2 defined on domain $\mathcal{X}$. If $X_1 \leq_{st} X_2$, then $\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x) \leq 0$ for all $x \in \mathcal{X}$. The *violation to the usual stochastic order* at $x \in \mathcal{X}$ is thus

$$v(x; X_1 \leq_{st} X_2) = (\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x))^+.$$

If the usual stochastic order is violated at x, the amount of violation $v(x; X_1 \leq_{st} X_2)$ is strictly positive. Then the overall, left-side, and right-side violations can be computed through (1)–(2).

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The integral characterization of the usual stochastic order gives rise to the class of all increasing functions, i.e. $X_1 \leq_{st} X_2$, if and only if $\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$ for all $g \in inc$. When the usual stochastic order is violated, the relationship $\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$ fails to hold for some $g \in inc$. To identify the subclass of functions that maintains the usual stochastic order, define

$$g^{(1)}_{\min} = \inf_{x_1,x_2} \{\delta_g(x_1, x_2) \colon \underline{x} \leq x_1 < x_2 \leq \overline{x}\},$$

$$g^{(1)}_{\max} = \sup_{x_1,x_2} \{\delta_g(x_1, x_2) \colon \underline{x} \leq x_1 < x_2 \leq \overline{x}\},$$

$$g^{(1)}_{\min^-}(x) = \inf_{x_1,x_2} \{\delta_g(x_1, x_2) \colon \underline{x} \leq x_1 < x_2 \leq x\},$$

$$g^{(1)}_{\max^-}(x) = \sup_{x_1,x_2} \{\delta_g(x_1, x_2) \colon \underline{x} \leq x_1 < x_2 \leq x\},$$

$$g^{(1)}_{\min^+}(x) = \inf_{x_1,x_2} \{\delta_g(x_1, x_2) \colon x \leq x_1 < x_2 \leq \overline{x}\},$$

$$g^{(1)}_{\max^+}(x) = \sup_{x_1,x_2} \{\delta_g(x_1, x_2) \colon x \leq x_1 < x_2 \leq \overline{x}\},$$

where

$$\delta_g(x_1, x_2) = \frac{g(x_2) - g(x_1)}{x_2 - x_1}.$$

Thus $g^{(1)}_{\min}$ is the 'smallest slope' of function g, and

$$g^{(1)}_{\min^-}(x)\left[g^{(1)}_{\min^+}(x)\right]$$

is the smallest slope of g below (above) x. With these, we can define subclasses of increasing functions parameterized by some $\Delta \in [0, \infty)$:

$$inc(\Delta) = \left\{g \in \overline{\mathcal{G}} \colon 0 \leq g^{(1)}_{\min} \leq \delta_g(x_1, x_2) \leq (1 + \Delta) g^{(1)}_{\min},\ \underline{x} \leq x_1 < x_2 \leq \overline{x}\right\}, \tag{3}$$

$$inc^+(\Delta) = \left\{g \in \overline{\mathcal{G}} \colon 0 \leq g^{(1)}_{\min^+}(x_2) \leq \delta_g(x_1, x_2) \leq (1 + \Delta) g^{(1)}_{\min^+}(x_2),\ \underline{x} \leq x_1 < x_2 \leq \overline{x}\right\}, \tag{4}$$

$$inc^-(\Delta) = \left\{g \in \overline{\mathcal{G}} \colon 0 \leq g^{(1)}_{\min^-}(x_1) \leq \delta_g(x_1, x_2) \leq (1 + \Delta) g^{(1)}_{\min^-}(x_1),\ \underline{x} \leq x_1 < x_2 \leq \overline{x}\right\}. \tag{5}$$

These conditions are equivalent to

$$\delta_g(x_1, x_2) \geq \frac{1}{1+\Delta} g^{(1)}_{\max}, \quad \delta_g(x_1, x_2) \geq \frac{1}{1+\Delta} g^{(1)}_{\max^-}(x_1), \quad \text{and} \quad \delta_g(x_1, x_2) \geq \frac{1}{1+\Delta} g^{(1)}_{\max^+}(x_2),$$

respectively. It is clear that any function within these classes must be continuous. We further observe that the subclasses get larger with a larger Δ. In particular, for $\Delta \in (0, \infty)$,

$$ilin = inc(0) \subset inc(\Delta) \subset inc(\infty) = inc,$$

$$icx = inc^+(0) \subset inc^+(\Delta) \subset inc^+(\infty) = inc,$$

$$icv = inc^-(0) \subset inc^-(\Delta) \subset inc^-(\infty) = inc,$$

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where *ilin* is the class of linearly increasing functions. The parameter Δ specifies the extent to which the increasingness of the function is restricted. The functions in $inc^+(\Delta)$ ($inc^-(\Delta)$) are close to, but can deviate from, increasing convex (concave) functions. Thus the associated integral stochastic orders are weaker than the usual stochastic order, but can be stronger than the increasing convex (concave) order. We can relax the usual stochastic orders by evaluating the overall, left-side, and right-side violations, as described in the next theorem.

Theorem 1. (Parametric usual stochastic order.) *For some $\Delta \in [0, \infty)$, two random variables X_1 and X_2 with support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all*

(i) $g \in inc(\Delta)$ *if and only if*

$$v_O(X_1 \le_{st} X_2) \le \frac{1}{\Delta}(\mathbb{E}[X_2] - \mathbb{E}[X_1]); \tag{6}$$

(ii) $g \in inc^+(\Delta)$ *if and only if*

$$v_{\ge}(x; X_1 \le_{st} X_2) \le \frac{1}{\Delta}\int_x^{\bar{x}} \big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big)\,\mathrm{d}\xi, \quad x \in \mathcal{X}; \tag{7}$$

and

(iii) $g \in inc^-(\Delta)$ *if and only if*

$$v_{\le}(x; X_1 \le_{st} X_2) \le \frac{1}{\Delta}\int_{\underline{x}}^{x} \big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big)\,\mathrm{d}\xi, \quad x \in \mathcal{X}. \tag{8}$$

Proof of Theorem 1. Part (i) is proved by Leshno and Levy [7] and Tsetlin et al. [23]. We first show part (iii). Take $g \in inc^-(\Delta)$. We derive

$$\begin{aligned}
&\mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&= \int_{\xi\in\mathcal{X}} \big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big)^+ \,\mathrm{d}g(\xi) - \int_{\xi\in\mathcal{X}} \big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+ \,\mathrm{d}g(\xi) \\
&\ge \int_{\xi\in\mathcal{X}} g^{(1)}_{\min^-}(\xi)\big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big)^+ \,\mathrm{d}\xi - (1+\Delta)\int_{\xi\in\mathcal{X}} g^{(1)}_{\min^-}(\xi)\big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+ \,\mathrm{d}\xi \\
&= \int_{\xi\in\mathcal{X}} g^{(1)}_{\min^-}(\xi)\big(\big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big) - \Delta\big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+\big)\,\mathrm{d}\xi.
\end{aligned}$$

Because $g^{(1)}_{\min^-}$ is decreasing and bounded by definition, we further deduce

$$\begin{aligned}
&\mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&\ge \int_{\xi\in\mathcal{X}} \left(b_1 - \int_{\xi}^{\bar{x}} \mathrm{d}g^{(1)}_{\min^-}(x)\right)\big(\big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big) - \Delta\big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+\big)\,\mathrm{d}\xi \\
&= b_1 \int_{\xi\in\mathcal{X}} \big(\big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big) - \Delta\big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+\big)\,\mathrm{d}\xi \\
&\quad + \int_{x\in\mathcal{X}} \int_{\underline{x}}^{x} \big(\big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big) - \Delta\big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+\big)\,\mathrm{d}\xi\,\mathrm{d}\big(-g^{(1)}_{\min^-}(x)\big),
\end{aligned}$$

where $b_1 = g^{(1)}_{\min}(\bar{x}) \ge 0$. When (8) holds, the right-hand side is non-negative.

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Now take X_1 and X_2 such that $\mathbb{E}[g(X_1)] \le E[g(X_2)]$ for any $g \in inc^-(\Delta)$. Let

$$g_y(x) = \int_{\underline{x}}^{x} g_y^{(1)}(\xi)\,\mathrm{d}\xi,$$

where

$$g_y^{(1)}(x) = \big(1 + \Delta \mathbb{I}_{\{\bar{F}_{X_1}(x) > \bar{F}_{X_2}(x)\}}\big)\mathbb{I}_{\{x \le y\}} \quad \text{for } x \in \mathcal{X}.$$

It is clear that $g_y^{(1)}(x) \ge 0$ for $x \in \mathcal{X}$, and thus g_y is increasing. Moreover, for $x \le y$,

$$(1+\Delta) g^{(1)}_{y\,\min^-}(x) \ge 1 + \Delta \ge g^{(1)}_{y\,\max^+}(x) \ge g_y^{(1)}(x);$$

for $x > y$,

$$g^{(1)}_{y\,\min^-}(x) = g^{(1)}_{y\,\max^+}(x) = g_y^{(1)}(x) = 0.$$

Thus $g_y \in inc^-(\Delta)$. Moreover, for a random variable Z with distribution F_Z over $\mathcal{X}$,

$$\begin{aligned}
\mathbb{E}[g_y(Z)] &= -\int_{\xi \in \mathcal{X}} \int_{\underline{x}}^{\xi} g_y^{(1)}(x)\,\mathrm{d}x\,\mathrm{d}\bar{F}_Z(\xi) \\
&= -\int_{\xi \in \mathcal{X}} \int_{\underline{x}}^{\xi} \big(1 + \Delta \mathbb{I}_{\{\bar{F}_{X_1}(x) > \bar{F}_{X_2}(x)\}}\big)\mathbb{I}_{\{x \le y\}}\,\mathrm{d}x\,\mathrm{d}\bar{F}_Z(\xi) \\
&= \int_{x \in \mathcal{X}} \mathbb{I}_{\{x \le y\}} \bar{F}_Z(x)\,\mathrm{d}x + \Delta \int_{x \in \mathcal{X}} \mathbb{I}_{\{\bar{F}_{X_1}(x) > \bar{F}_{X_2}(x)\}} \mathbb{I}_{\{x \le y\}} \bar{F}_Z(x)\,\mathrm{d}x \\
&= \int_{\underline{x}}^{y} \bar{F}_Z(x)\,\mathrm{d}x + \Delta \int_{\underline{x}}^{y} \mathbb{I}_{\{\bar{F}_{X_1}(x) > \bar{F}_{X_2}(x)\}} \bar{F}_Z(x)\,\mathrm{d}x.
\end{aligned}$$

Because $g_y \in inc^-(\Delta)$, we deduce

$$\begin{aligned}
&0 \le \mathbb{E}[g_y(X_2)] - \mathbb{E}[g_y(X_1)] \\
&= \int_{\underline{x}}^{y} \big(\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x)\big)\,\mathrm{d}x - \Delta \int_{\underline{x}}^{y} \big(\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x)\big)^+\,\mathrm{d}x,
\end{aligned}$$

yielding (8).

The proof of part (ii) can be carried out similarly. In particular, we can define

$$g_y^{(1)}(x) = \big(1 + \Delta \mathbb{I}_{\{\bar{F}_{X_1}(x) > \bar{F}_{X_2}(x)\}}\big)\mathbb{I}_{\{x \ge y\}}, \quad x \in \mathcal{X}.$$

For $x < y$, $g_y^{(1)}(x) = g^{(1)}_{y\,\min^+}(x) = 0$. For $x \ge y$, $g^{(1)}_{y\,\min^+}(x) \ge 1$ and thus

$$(1+\Delta) g^{(1)}_{y\,\min^+}(x) \ge 1 + \Delta \ge g_y^{(1)}(x),$$

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suggesting $g_y \in inc^+(\Delta)$. For any random variable Z with distribution F_Z, we derive

$$\mathbb{E}[g_y(Z)] = \int_y^{\bar{x}} \bar{F}_Z(x)\,\mathrm{d}x + \Delta \int_y^{\bar{x}} \mathbb{I}_{\{\bar{F}_{X_1}(x) > \bar{F}_{X_2}(x)\}} \bar{F}_Z(x)\,\mathrm{d}x.$$

It follows that

$$\mathbb{E}[g_y(X_2)] - \mathbb{E}[g_y(X_1)] = \int_y^{\bar{x}} (\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x))\,\mathrm{d}x - \Delta \int_y^{\bar{x}} (\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x))^+\,\mathrm{d}x.$$

Because $g_y \in inc^+(\Delta)$, we conclude that (7) holds. □

The proof of Theorem 1 directly utilizes the upper and lower bounds for the slope of function g defined (3)–(4) to evaluate, respectively, the positive part and the negative part of $g(X_1) - g(X_2)$, which gives rise to the distributional characterization of the relaxed usual stochastic orders.

The stochastic order associated with $inc(\Delta)$ is the ϵ-AFSD, first introduced in [7] with $\Delta = 1/\epsilon - 2 \in [0, \infty)$. Leshno and Levy [7] derive Theorem 1(i) for differentiable g and Tsetlin et al. [23] analyze this order for strictly increasing functions. Müller et al. [18] analyze differentiable functions within $inc^-(\Delta)$ and define the $(1+\gamma)$-SD order with $1 + \Delta = 1/\gamma$ interpreted as the index of greediness in the utility theory. This order is sometimes referred to as the 'fractional stochastic dominance' [5, 16]. Müller et al. [18] recognize that this order is in between the usual stochastic order and the increasing concave order, and show that two discrete random variables are $(1+\gamma)$-SD ordered if and only if one can be obtained through a sequence of mean-preserving contraction mass transports from the other one. Klar and Müller [6] connect the $(1+\gamma)$-SD order to the Ω ratio, $\Omega(\xi) = \mathbb{E}[(X-\xi)^+]/\mathbb{E}[(X-\xi)^-]$, used for fund performance. Yang et al. [25], among others, explore useful properties of this parametric order.

The characterization in (6) for ϵ-AFSD associated with $inc(\Delta)$ functions suggests the possibility of violating the usual stochastic order with the amount of violation bounded by $1/\Delta$ of the mean difference. It is easy to show that (6) implies

$$1 + \Delta \le \frac{nv_O(X_1 \le_{st} X_2)}{v_O(X_1 \le_{st} X_2)},$$

where

$$nv_O(X_1 \le_{st} X_2) = v_O(X_1 \ge_{st} X_2) = \int_{x \in \mathcal{X}} (\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x))^+\,\mathrm{d}x$$

is the overall measure of X_1 dominating X_2 in the usual stochastic order. Thus $(1+\Delta)$ is the *minimum ratio of dominance violation*. Similarly, (7) and (8), respectively, imply

$$1 + \Delta \le \frac{nv_{\ge}(x; X_1 \le_{st} X_2)}{v_{\ge}(x; X_1 \le_{st} X_2)} \quad \text{and} \quad 1 + \Delta \le \frac{nv_{\le}(x; X_1 \le_{st} X_2)}{v_{\le}(x; X_1 \le_{st} X_2)}.$$

3.2. Generalization with two-sided restrictions

We may generalize the discussion in the previous subsection by imposing asymmetric two-sided restrictions to linear functions and define the parametric order that allows for the

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asymmetric two-sided violation to the usual stochastic order. Specifically, define

$$inc^{\pm}(\Delta_+, \Delta_-) = \left\{ g \in \overline{\mathcal{G}} \colon 0 \le \delta_g(x_1, x_2) \le \min\{(1+\Delta_+) g^{(1)}_{\min^+}(x_2), (1+\Delta_-) g^{(1)}_{\min^-}(x_1)\}, \underline{x} \le x_1 < x_2 \le \overline{x} \right\},$$
$$\Delta_+ \in [0, \infty), \Delta_- \in [0, \infty). \tag{9}$$

It is clear that $inc^{\pm}(\Delta_+, \Delta_-) = inc^{+}(\Delta_+) \cap inc^{-}(\Delta_-)$. The stochastic order associated with this class of functions is named as $(1+\gamma_{cx}, 1+\gamma_{cv})$-SD by Müller et al. [18, 19] with $\gamma_{cx} = 1/(1+\Delta_+) \in [0, 1]$ and $\gamma_{cv} = 1/(1+\Delta_-) \in [0, 1]$. The characterization in Theorem 2 below, using the left- and right-side violation measures, facilitates an intuitive proof.

Theorem 2. (Two-sided relaxation of usual stochastic order.) *For some $\Delta_+ \in [0, \infty)$ and $\Delta_- \in [0, \infty)$, two random variables X_1 and X_2 defined on support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all $g \in inc^{\pm}(\Delta_+, \Delta_-)$, if and only if (where $a \wedge b = \min\{a, b\}$), for any $x \in \mathcal{X}$,*

$$\frac{1}{1+\Delta_-}\left(\int_{\underline{x}}^{x} \left(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\right) \mathrm{d}\xi - (\Delta_+ \wedge \Delta_-) v_{\le}(x; X_1 \le_{st} X_2) \right)$$
$$+ \frac{1}{1+\Delta_+}\left(\int_{x}^{\overline{x}} \left(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\right) \mathrm{d}\xi - (\Delta_+ \wedge \Delta_-) v_{\ge}(x; X_1 \le_{st} X_2) \right) \ge 0. \tag{10}$$

Proof of Theorem 2. We first prove the 'if' part of the theorem by showing that (10) implies that $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all $g \in inc^{\pm}(\Delta_+, \Delta_-)$. We derive the result for $\Delta_+ \le \Delta_-$, as that for $\Delta_+ \ge \Delta_-$ follows similarly. When $\Delta_+ \le \Delta_-$, the condition in (10) becomes

$$\Xi(x) \equiv \frac{1}{1+\Delta_-} \int_{\underline{x}}^{x} \left(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\right)^+ \mathrm{d}\xi - \frac{1+\Delta_+}{1+\Delta_-} \int_{\underline{x}}^{x} \left(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\right)^+ \mathrm{d}\xi$$
$$+ \frac{1}{1+\Delta_+} \int_{x}^{\overline{x}} \left(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\right)^+ \mathrm{d}\xi - \int_{x}^{\overline{x}} \left(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\right)^+ \mathrm{d}\xi \ge 0. \tag{11}$$

Take $g \in inc^{\pm}(\Delta_+, \Delta_-) = inc^{+}(\Delta_+) \cap inc^{-}(\Delta_-)$. Then,

$$\delta_g(x_1, x_2) \ge \max\left\{ \frac{1}{1+\Delta_-} g^{(1)}_{\max^+}(x_2), \frac{1}{1+\Delta_+} g^{(1)}_{\max^-}(x_1) \right\}.$$

Because $g^{(1)}_{\max^+}(x)$ is decreasing and $g^{(1)}_{\max^-}(x)$ is increasing, and both attain $g^{(1)}_{\max}$ within $[\underline{x}, \overline{x}]$, we can define

$$x_{\max} = \inf\left\{ x \in \mathcal{X} \colon g^{(1)}_{\max^+}(x) = g^{(1)}_{\max^-}(x) \right\}$$

and

$$x_0 = \sup\left\{ x \in \mathcal{X} \colon \frac{1}{1+\Delta_-} g^{(1)}_{\max^+}(x) > \frac{1}{1+\Delta_+} g^{(1)}_{\max^-}(x) \right\}.$$

We have $x_0 \le x_{\max}$ as $1/(1+\Delta_-) \le 1/(1+\Delta_+)$. We treat the situation where

$$\frac{1}{1+\Delta_-} g^{(1)}_{\max^+}(x_0) = \frac{1}{1+\Delta_+} g^{(1)}_{\max^-}(x_0) \quad \text{for some } x_0 \in [\underline{x}, \overline{x}].$$

The situation for

$$\frac{1}{1+\Delta_-} g^{(1)}_{\max^+}(x) < \frac{1}{1+\Delta_+} g^{(1)}_{\max^-}(x) \quad \text{for all } x \in [\underline{x}, \overline{x}]$$

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can be easily deduced from the argument below.

Note that $g^{(1)}_{\max^-}$ must be strictly increasing at x_0. Because $x_0 \le x_{\max}$, we have $g^{(1)}_{\max^+}(x_0) = g^{(1)}_{\max}$. It follows that

$$g^{(1)}_0 \equiv g^{(1)}_{\max^-}(x_0) = \frac{1+\Delta_+}{1+\Delta_-} g^{(1)}_{\max}.$$

Define

$$\rho(x) = \begin{cases} g^{(1)}_0, & x \le x_0, \\ g^{(1)}_{\max^-}(x), & x_0 < x < x_{\max}, \\ g^{(1)}_{\max}, & x \ge x_{\max}. \end{cases} \tag{12}$$

We have two cases to consider. In the following derivation, we choose a sufficiently small $\epsilon > 0$ to compute the left- or right-side deviation of g at given point x.

(i) For $x \le x_0$, δ_g is lower than

$$g^{(1)}_{\max^-}(x) \le g^{(1)}_{\max^-}(x_0) = g^{(1)}_0$$

as $g^{(1)}_{\max^-}$ is increasing. Moreover, δ_g is higher than

$$\frac{1}{1+\Delta_-} g_{\max^+}(x) = \frac{1}{1+\Delta_-} g^{(1)}_{\max} = \frac{1}{1+\Delta_+} g^{(1)}_0.$$

Therefore,

$$\frac{1}{1+\Delta_+} g^{(1)}_0 \le \delta_g(x-\epsilon, x) \le g^{(1)}_0.$$

(ii) For $x > x_0$, we deduce, because $g \in inc^+(\Delta_+)$ and $g^{(1)}_{\max^-}$ is strictly increasing at x_0,

$$\frac{1}{1+\Delta_+} g^{(1)}_{\max^-}(x) \le \delta_g(x, x+\epsilon) \le g^{(1)}_{\max^-}(x).$$

In particular, $g^{(1)}_{\max^-}(x) = g^{(1)}_{\max}$ for $x \ge x_{\max}$.

Together with (12), we obtain

$$\begin{aligned} \mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] &= \int_{x\in\mathcal{X}} \left(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\right)^+ \mathrm{d}g(\xi) - \int_{x\in\mathcal{X}} \left(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\right)^+ \mathrm{d}g(\xi) \\ &\ge \int_{x\in\mathcal{X}} \Gamma(\xi)\rho(\xi)\,\mathrm{d}\xi, \end{aligned}$$

where

$$\Gamma(\xi) = \frac{1}{1+\Delta_+}(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi))^+ - (\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi))^+.$$

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Now define

$$\mu(x) = \frac{g^{(1)}_{\max^-}(x) - g^{(1)}_0}{g^{(1)}_{\max} - g^{(1)}_0} \quad \text{for } x_0 \le x \le x_{\max}.$$

It is clear that μ is a probability distribution that increases from zero to one. From (11), we have

$$\begin{aligned}
0 &\le g^{(1)}_{\max} \int_{x_0}^{x_{\max}} \Xi(x)\, \mathrm{d}\mu(x) \\
&= \int_{x_0}^{x_{\max}} \Bigg(\frac{1}{1+\Delta_+} g^{(1)}_0 \int_{\underline{x}}^{x} \big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big)^+ \mathrm{d}\xi - g^{(1)}_0 \int_{\underline{x}}^{x} \big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+ \mathrm{d}\xi \\
&\quad + \frac{1}{1+\Delta_+} g^{(1)}_{\max} \int_{x}^{\bar{x}} \big(\bar{F}_{X_2}(\xi) - \bar{F}_{X_1}(\xi)\big)^+ \mathrm{d}\xi - g^{(1)}_{\max} \int_{x}^{\bar{x}} \big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)^+ \mathrm{d}\xi \Bigg) \mathrm{d}\mu(x) \\
&= \int_{x_0}^{x_{\max}} \left(g^{(1)}_0 \int_{\underline{x}}^{x} \Gamma(\xi)\, \mathrm{d}\xi + g^{(1)}_{\max} \int_{x}^{\bar{x}} \Gamma(\xi)\, \mathrm{d}\xi \right) \mathrm{d}\mu(x) \\
&= \int_{\underline{x}}^{\bar{x}} \Gamma(\xi)\rho(\xi)\, \mathrm{d}\xi \le \mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)].
\end{aligned}$$

Thus we obtain $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$.

Now we prove the 'only if' part of the theorem by showing that $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for any $g \in inc^{\pm}(\Delta_+, \Delta_-)$ implies (10). Define

$$g_y(x) = \int_{\underline{x}}^{x} g^{(1)}_y(\xi)\, \mathrm{d}\xi,$$

where

$$g^{(1)}_y(x) = \frac{\big(1 + (\Delta_+ \wedge \Delta_-)\mathbb{I}_{\{\bar{F}_{X_1}(x) \ge \bar{F}_{X_2}(x)\}}\big)\mathbb{I}_{\{x \le y\}}}{1+\Delta_-} + \frac{\big(1 + (\Delta_+ \wedge \Delta_-)\mathbb{I}_{\{\bar{F}_{X_1}(x) \ge \bar{F}_{X_2}(x)\}}\big)\mathbb{I}_{\{x \ge y\}}}{1+\Delta_+}.$$

We can easily verify that $g_y(x) \in inc^{\pm}(\Delta_+, \Delta_-)$. We derive (following the proof of Theorem 1)

$$\begin{aligned}
0 &\le \mathbb{E}[g_y(X_2)] - \mathbb{E}[g_y(X_1)] \\
&= \frac{1}{1+\Delta_-} \left(\int_{\underline{x}}^{y} \big(\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x)\big)\, \mathrm{d}x - (\Delta_+ \wedge \Delta_-) \int_{\underline{x}}^{y} \big(\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x)\big)^+ \mathrm{d}x \right) \\
&\quad + \frac{1}{1+\Delta_+} \left(\int_{y}^{\bar{x}} \big(\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x)\big)\, \mathrm{d}x - (\Delta_+ \wedge \Delta_-) \int_{y}^{\bar{x}} \big(\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x)\big)^+ \mathrm{d}x \right),
\end{aligned}$$

which gives rise to (10). □

It is clear that Theorem 2 generalizes Theorem 1. We recognize that the integral comparison $\mathbb{E}[g(X_1)] - \mathbb{E}[g(X_2)]$ is determined by both the slope δ_g and the distributional difference $\bar{F}_{X_1} -$

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$\bar{F}_{X_2}$ over the entire support $\mathcal{X}$. The functional property described in (9) allows us to bound the positive and the negative part of $g(X_1) - g(X_2)$ with a monotone upper bound of δ_g. When we convert this upper bound into a probability measure for the distributional differences, the characterization in (10) can be easily obtained.

We name several special cases of the two-sided parametric functions.

- When $\Delta_+ = \Delta_- = \Delta$, we obtain ϵ-AFSD, which corresponds to the symmetric two-sided restriction of increasing functions:
$$inc^{\pm}(\Delta, \Delta) = inc(\Delta).$$
- When $\Delta_- = \infty$ or $\Delta_+ = \infty$, the restriction of increasing function is only one-sided:
$$inc^{\pm}(\Delta, \infty) = inc^{+}(\Delta) \quad \text{and} \quad inc^{\pm}(\infty, \Delta) = inc^{-}(\Delta).$$
- When $\Delta_+ = 0$ or $\Delta_- = 0$, the functions are increasing convex or increasing concave with the two-sided restriction on the increasingness (see our later discussion in Section 4):
$$inc^{\pm}(0, \Delta) = inc(\Delta)\text{-}cx \quad \text{and} \quad inc^{\pm}(\Delta, 0) = inc(\Delta)\text{-}cv.$$

4. Parametric convex and concave functions

In addition to the usual stochastic order, the convex and the concave orders are most studied and applied. These orders are related to the second-order properties of the functions (i.e. convexity and concavity). To name the associated parametric functions, let

$$g^{(2)}_{\min} = \inf\left\{\delta_g^2(x_1, x_2) \colon \underline{x} \le x_1 < x_2 \le \bar{x}\right\} \quad \text{and} \quad g^{(2)}_{\max} = \sup\left\{\delta_g^2(x_1, x_2) \colon \underline{x} \le x_1 < x_2 \le \bar{x}\right\},$$

where

$$\delta_g^2(x_1, x_2) = \frac{\delta_g(x_1, (x_1 + x_2)/2) - \delta_g((x_1 + x_2)/2, x_2)}{\frac{1}{2}(x_2 - x_1)}.$$

The parametric subclasses of functions are defined as

$$cx(\Delta) = \left\{g \in \overline{\mathcal{G}} \colon 0 \le \delta_g^2(x_1, x_2) \le (1 + \Delta) g^{(2)}_{\min},\ \underline{x} \le x_1 < x_2 \le \bar{x}\right\}, \tag{13}$$

$$cv(\Delta) = \left\{g \in \overline{\mathcal{G}} \colon 0 \ge \delta_g^2(x_1, x_2) \ge (1 + \Delta) g^{(2)}_{\max},\ \underline{x} \le x_1 < x_2 \le \bar{x}\right\}. \tag{14}$$

We note that any $g \in cv(\Delta)$ if and only if $-g \in cx(\Delta)$. Moreover, for $\Delta \in (0, \infty)$,

$$quadcx = cx(0) \subset cx(\Delta) \subset cx(\infty) = cx.$$

where *quadcx* is the set of quadratically convex functions.

For two random variables X_1 and X_2 with support $\mathcal{X}$ and $\mathbb{E}[X_1] = \mathbb{E}[X_2]$, define

$$v(x; X_1 \le_{cx} X_2) = \left(\int_x^{\bar{x}} \bar{F}_{X_1}(\xi)\, \mathrm{d}\xi - \int_x^{\bar{x}} \bar{F}_{X_2}(\xi)\, \mathrm{d}\xi\right)^+, \quad x \in \mathcal{X},$$

$$v(x; X_1 \le_{cv} X_2) = \left(\int_{\underline{x}}^{x} F_{X_2}(\xi)\, \mathrm{d}\xi - \int_{\underline{x}}^{x} F_{X_1}(\xi)\, \mathrm{d}\xi\right)^+, \quad x \in \mathcal{X}.$$

as the violations to the convex order and the concave order, respectively, evaluated at x. We first characterize the comparisons between X_1 and X_2 with bounded overall violations.

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Theorem 3. *Two random variables X_1 and X_2 with support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all*

(i) $g \in cx(\Delta)$ *if and only if* $\mathbb{E}[X_1] = \mathbb{E}[X_2]$ *and*

$$v_O(X_1 \le_{cx} X_2) = \int_{x\in\mathcal{X}} \left(\int_x^{\overline{x}} \bar{F}_{X_1}(\xi)\,\mathrm{d}\xi - \int_x^{\overline{x}} \bar{F}_{X_2}(\xi)\,\mathrm{d}\xi \right)^+ \mathrm{d}x \le \frac{1}{2\Delta}(\mathbb{E}[X_2^2] - \mathbb{E}[X_1^2]); \tag{15}$$

and

(ii) $g \in cv(\Delta)$ *if and only if* $\mathbb{E}[X_1] = \mathbb{E}[X_2]$ *and*

$$v_O(X_1 \le_{cv} X_2) = \int_{x\in\mathcal{X}} \left(\int_{\underline{x}}^{x} F_{X_2}(\xi)\,\mathrm{d}\xi - \int_{\underline{x}}^{x} F_{X_1}(\xi)\,\mathrm{d}\xi \right)^+ \mathrm{d}x \le \frac{1}{2\Delta}\big(\mathbb{E}[X_1^2] - \mathbb{E}[X_2^2]\big).$$

Proof of Theorem 3. To show part (i), we take $g \in cx(\Delta)$. Because g is convex and thus almost surely continuous, there exists $g^{(1)} \in inc$ such that

$$g(x) = a_0 + \int_{\underline{x}}^{x} g^{(1)}(\xi)\,\mathrm{d}\xi, \quad x \in \mathcal{X},$$

where $a_0 = g(\underline{x})$. Also define $a_1 = g^{(1)}(\underline{x})$. By (13), we must have

$$g^{(2)}_{\min} \le \delta_{g^{(1)}}(x_1, x_2) \le (1+\Delta) g^{(2)}_{\min} \quad \text{for any } x_1, x_2 \in \mathcal{X}.$$

For any random variable X with support $\mathcal{X}$,

$$\begin{aligned}
\mathbb{E}[g(X)] &= a_0 + \int_{\xi\in\mathcal{X}} g^{(1)}(\xi)\bar{F}_X(\xi)\,\mathrm{d}\xi \\
&= a_0 + \int_{\xi\in\mathcal{X}} \left(a_1 + \int_{\underline{x}}^{\xi} \mathrm{d}g^{(1)}(\xi_0) \right) \bar{F}_X(\xi)\,\mathrm{d}\xi \\
&= a_0 + a_1(\mathbb{E}[X] - \underline{x}) + \int_{\xi_0\in\mathcal{X}} \left(\int_{\xi_0}^{\overline{x}} \bar{F}_X(\xi)\,\mathrm{d}\xi \right) \mathrm{d}g^{(1)}(\xi_0).
\end{aligned}$$

We further note the relation

$$\begin{aligned}
\int_{\xi\in\mathcal{X}} \int_{\xi}^{\overline{x}} \bar{F}_X(x)\,\mathrm{d}x\,\mathrm{d}\xi &= \int_{x\in\mathcal{X}} (x - \underline{x})\bar{F}_X(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{x\in\mathcal{X}} \bar{F}_X(x)\,\mathrm{d}(x-\underline{x})^2 \\
&= \frac{1}{2}\int_{x\in\mathcal{X}} (x-\underline{x})^2 f_X(x)\,\mathrm{d} \\
&= \frac{1}{2}\mathbb{E}[(X-\underline{x})^2].
\end{aligned}$$

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Then, for X_1 and X_2 defined on support $\mathcal{X}$, we have

$$\begin{aligned}
\mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] &= a_1\big(\mathbb{E}[X_2] - \mathbb{E}[X_1]\big) + \int_{\xi\in\mathcal{X}} \left(\int_{\xi}^{\bar{x}} \big(\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x)\big)\, \mathrm{d}x \right) \mathrm{d}g^{(1)}(\xi) \\
&= a_1\big(\mathbb{E}[X_2] - \mathbb{E}[X_1]\big) + \int_{\xi\in\mathcal{X}} \left(\int_{\xi}^{\bar{x}} \big(\bar{F}_{X_2}(x) - \bar{F}_{X_1}(y)\big)\, \mathrm{d}x \right)^{+} \mathrm{d}g^{(1)}(\xi) \\
&\quad - \int_{\xi\in\mathcal{X}} \left(\int_{\xi}^{\bar{x}} \big(\bar{F}_{X_1}(x) - \bar{F}_{X_2}(y)\big)\, \mathrm{d}x \right)^{+} \mathrm{d}g^{(1)}(\xi) \\
&\geq a_1\big(\mathbb{E}[X_2] - \mathbb{E}[X_1]\big) + \int_{\xi\in\mathcal{X}} g^{(2)}_{\min} \left(\int_{\xi}^{\bar{x}} \big(\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x)\big)\, \mathrm{d}x \right)^{+} \mathrm{d}\xi \\
&\quad - \int_{\xi\in\mathcal{X}} (1+\Delta) g^{(2)}_{\min} \left(\int_{\xi}^{\bar{x}} \big(\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x)\big)\, \mathrm{d}x \right)^{+} \mathrm{d}\xi \\
&= a_1\big(\mathbb{E}[X_2] - \mathbb{E}[X_1]\big) + g^{(2)}_{\min} \left(\int_{x\in\mathcal{X}} (x - \underline{x}) \big(\bar{F}_{X_2}(x) - \bar{F}_{X_1}(x)\big)\, \mathrm{d}x \right. \\
&\quad \left. - \Delta \int_{\xi\in\mathcal{X}} \left(\int_{\xi}^{\bar{x}} \big(\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x)\big)\, \mathrm{d}x \right)^{+} \mathrm{d}\xi \right) \\
&= a_1\big(\mathbb{E}[X_2] - \mathbb{E}[X_1]\big) + g^{(2)}_{\min} \left(\frac{1}{2}\big(\mathbb{E}[(X_2 - \underline{x})^2] - \mathbb{E}[(X_1 - \underline{x})^2]\big) \right. \\
&\quad \left. - \Delta \int_{\xi\in\mathcal{X}} \left(\int_{\xi}^{\bar{x}} \big(\bar{F}_{X_1}(x) - \bar{F}_{X_2}(x)\big)\, \mathrm{d}x \right)^{+} \mathrm{d}\xi \right).
\end{aligned}$$

When $\mathbb{E}[X_1] = \mathbb{E}[X_2]$ and (15) holds, the right-hand side is non-negative.

To show the reverse relation, define

$$g^{(2)}(x) = 1 + \Delta \mathbb{I}_{\left\{ \int_x^{\bar{x}} \bar{F}_{X_1}(\xi)\, \mathrm{d}\xi > \int_x^{\bar{x}} \bar{F}_{X_2}(\xi)\, \mathrm{d}\xi \right\}}.$$

It is easy to verify that

$$g(x) = \int_{\underline{x}}^{x} \int_{\underline{x}}^{\xi} g^{(2)}(\xi_0)\, \mathrm{d}\xi_0\, \mathrm{d}\xi \in cx(\Delta)$$

as defined in (13). We deduce

$$\begin{aligned}
0 &\leq \mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&= \int_{\xi\in\mathcal{X}} \int_{\xi}^{\bar{x}} \big(\bar{F}_{X_2}(\xi_0) - \bar{F}_{X_1}(\xi_0)\big)\, \mathrm{d}\xi_0 g^{(2)}(\xi)\, \mathrm{d}\xi \\
&= \int_{\xi\in\mathcal{X}} \left(1 + \Delta \mathbb{I}_{\left\{ \int_{\xi}^{\bar{x}} \bar{F}_{X_1}(x)\, \mathrm{d}x > \int_{\xi}^{\bar{x}} \bar{F}_{X_2}(x)\, \mathrm{d}x \right\}}\right) \int_{\xi}^{\bar{x}} \big(\bar{F}_{X_2}(\xi_0) - \bar{F}_{X_1}(\xi_0)\big)\, \mathrm{d}\xi_0\, \mathrm{d}\xi \\
&= \frac{1}{2}\big(\mathbb{E}[(X_2 - \underline{x})^2] - \mathbb{E}[(X_1 - \underline{x})^2]\big) - \Delta \int_{\xi\in\mathcal{X}} \left(\int_{\underline{x}}^{\xi} \big(\bar{F}_{X_1}(\xi_0) - \bar{F}_{X_2}(\xi_0)\big)\, \mathrm{d}\xi_0 \right)^{+} \mathrm{d}\xi.
\end{aligned}$$

Because $\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$ for all $g \in cx(\Delta)$, we have $\mathbb{E}[X_1] = \mathbb{E}[X_2]$, suggesting (15).

The proof of part (ii) follows a similar argument. □

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By Theorem 3, the parametric convex (concave) functions lead to a bounded violation to the convex (concave) order. The bound is determined by the difference in the second moment or the variance (given the equal mean). Tsetlin et al. [23] consider $cv(\Delta)$ functions that are strictly concave and name the associated order as the 'almost second degree risk' (ϵ-ASR) with $\Delta = 1/\epsilon - 2$.

We can also define the left-side and right-side violations to the convex and concave orders, and show a result similar to Theorem 2. We will not state this result, but postpone the discussion to the general mth-order functions in Section 5.

4.1. Parametric increasing convex or concave functions

In Section 3 we discussed two subclasses of increasing functions, $inc^+(\Delta)$ and $inc^-(\Delta)$, which are *superclasses* of increasing convex (*icx*) and increasing concave (*icv*) functions, respectively. In this subsection we analyze *subclasses* of *icx* and *icv*, and characterize the corresponding orders. For that, we can parameterize either the increasingness or the convexity (concavity) of the functions. Specifically, define

$$\begin{aligned}
&inc\text{-}cx(\Delta) = inc \cap cx(\Delta)\\
&\qquad = \left\{g \in \overline{\mathcal{G}}\colon \delta_g(x_1, x_2) \geq 0,\, 0 \leq \delta_g^2(x_1, x_2) \leq (1+\Delta) g^{(2)}_{\min},\ \underline{x} \leq x_1 < x_2 \leq \overline{x}\right\},\\
&inc\text{-}cv(\Delta) = inc \cap cv(\Delta)\\
&\qquad = \left\{g \in \overline{\mathcal{G}}\colon \delta_g(x_1, x_2) \geq 0,\ 0 \geq \delta_g^2(x_1, x_2) \geq (1+\Delta) g^{(2)}_{\max},\ \underline{x} \leq x_1 < x_2 \leq \overline{x}\right\},\\
&inc(\Delta)\text{-}cx = inc(\Delta) \cap cx\\
&\qquad = \left\{g \in \overline{\mathcal{G}}\colon 0 \leq \delta_g(x_1, x_2) \leq (1+\Delta) g^{(1)}_{\min},\ \delta_g^2(x_1, x_2) \geq 0,\ \underline{x} \leq x_1 < x_2 \leq \overline{x}\right\},\\
&inc(\Delta)\text{-}cv = inc(\Delta) \cap cv\\
&\qquad = \left\{g \in \overline{\mathcal{G}}\colon 0 \leq \delta_g(x_1, x_2) \leq (1+\Delta) g^{(1)}_{\min},\ \delta_g^2(x_1, x_2) \leq 0,\ \underline{x} \leq x_1 < x_2 \leq \overline{x}\right\}.
\end{aligned}$$

Certainly $inc\text{-}cx(\Delta)$ ($inc\text{-}cv(\Delta)$) and $inc(\Delta)\text{-}cx$ ($inc(\Delta)\text{-}cv$) are different, though their intersection is non-empty. In particular, for $\Delta \in (0, \infty)$,

$$\left.\begin{aligned} iquadcx &= inc\text{-}cx(0) \subset inc\text{-}cx(\Delta) \subset inc\text{-}cx(\infty)\\ ilin &= inc(0)\text{-}cx \subset inc(\Delta)\text{-}cx \subset inc(\infty)\text{-}cx \end{aligned}\right\} = icx,$$

$$\left.\begin{aligned} iquadcv &= inc\text{-}cv(0) \subset inc\text{-}cv(\Delta) \subset inc\text{-}cv(\infty)\\ ilin &= inc(0)\text{-}cv \subset inc(\Delta)\text{-}cv \subset inc(\infty)\text{-}cv \end{aligned}\right\} = icv,$$

where *iquadcx* and *iquadcv* are the sets of increasing functions that are quadratically convex and concave, respectively. Theorem 4 below provides the distributional characterizations for the orders associated with the above function classes.

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Theorem 4. *Two random variables X_1 and X_2 on support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all*

(i) *(increasing and Δ-convex (convex) function)*

(i-a) $g \in$ *inc-cx*(Δ) *if and only if* $\mathbb{E}[X_1] \le \mathbb{E}[X_2]$ *and*

$$v_O(X_1 \le_{cx} X_2) \le \frac{1}{2\Delta}(\mathbb{E}[(X_2 - \underline{x})^2] - \mathbb{E}[(X_1 - \underline{x})^2]), \tag{16}$$

(i-b) $g \in$ *inc-cv*(Δ) *if and only if* $\mathbb{E}[X_1] \le \mathbb{E}[X_2]$ *and*

$$v_O(X_1 \le_{cv} X_2) \le \frac{1}{2\Delta}(\mathbb{E}[(\bar{x} - X_1)^2] - \mathbb{E}[(\bar{x} - X_2)^2]);$$

and

(ii) *(Δ-increasing and convex (concave) function)*

(ii-a) $g \in$ *inc*(Δ)*-cx if and only if*

$$\int_{\underline{x}}^{\bar{x}} \big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)\,\mathrm{d}\xi \le \frac{1}{\Delta}\big(\mathbb{E}[X_2] - \mathbb{E}[X_1]\big) \quad \textit{for all } x \in \mathcal{X},$$

(ii-b) $g \in$ *inc*(Δ)*-cv if and only if*

$$\int_{\underline{x}}^{x} \big(\bar{F}_{X_1}(\xi) - \bar{F}_{X_2}(\xi)\big)\,\mathrm{d}\xi \le \frac{1}{\Delta}\big(\mathbb{E}[X_2] - \mathbb{E}[X_1]\big) \quad \textit{for all } x \in \mathcal{X}.$$

Proof of Theorem 4. Part (i-a) follows from the proof of Theorem 3(i) by imposing the condition $\mathbb{E}[X_1] \le \mathbb{E}[X_2]$ and using the fact that

$$g(x) = \int_{\underline{x}}^{x} \int_{\underline{x}}^{\xi} g^{(2)}(\xi_0)\,\mathrm{d}\xi_0\,\mathrm{d}\xi$$

is increasing. Part (i-b) can be derived similarly, and it is shown for twice-differentiable g by Tzeng et al. [24]. We remark that the result can be generalized when the supports of X_1 and X_2 are different in view of the relations

$$\int_{\underline{x}_i}^{\bar{x}_i} \int_{x}^{\bar{x}_i} \bar{F}_{X_i}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \mathbb{E}[(X_i - \underline{x}_i)^2]$$

and

$$\int_{\underline{x}_i}^{\bar{x}_i} \int_{\underline{x}_i}^{x} F_{X_i}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \mathbb{E}[(\bar{x}_i - X_i)^2],$$

where $\underline{x}_i$ and $\bar{x}_i$, respectively, are the minimum and maximum realizations of X_i, $i = 1, 2$.

To see part (ii-a), we note from (10) that *inc*(Δ)*-cx* $=$ *inc*$(\Delta) \cap cx =$ *inc*$^{\pm}(0, \Delta)$. The relation in (16) can be derived from Theorem 2, which also implies $\mathbb{E}[X_2] \ge \mathbb{E}[X_1]$. Part (ii-b) follows from an argument similar to that of part (ii-a). □

The *inc-cv*(Δ) functions lead to the 'almost second-degree stochastic dominance' (ϵ-ASSD) introduced in [7] and further discussed in [10] with the relation $\Delta = 1/\epsilon - 2$.

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4.2. Other integral orders

There are other ways of defining integral stochastic orders and parameterizing the orders, depending on the applications. For example, Tsetlin et al. [23] consider restrictions on both the first-order and the second-order properties by considering the following function class:

$$inc(\Delta_1)\text{-}cv(\Delta_2) = inc(\Delta_1) \cap cx(\Delta_2).$$

They name the corresponding order as the 'generalized almost second-degree stochastic dominance' (GASSD). They show that $X_1 \leq_{\text{GASSD}} X_2$ if and only if $\mathbb{E}[(\bar{x} - X_1)^2] \geq \mathbb{E}[(\bar{x} - X_2)^2]$ and

$$\max_{h(x)} \int_{x\in\mathcal{X}} h(x)\big(F^{(2)}_{X_1}(x) - F^{(2)}_{X_2}(x)\big)\,\mathrm{d}x \leq \frac{1}{\Delta_1}(\mathbb{E}[(\bar{x} - X_1)^2] - \mathbb{E}[(\bar{x} - X_2)^2]),$$

where

$$h(x) \geq 0,\ \int_{x\in\mathcal{X}} h(x)\,\mathrm{d}x = 1,\ \sup_{x\in\mathcal{X}}\{h(x)\} \leq (1+\Delta_2)\inf_{x\in\mathcal{X}}\{h(x)\}.$$

This characterization can be derived by modifying the proof of Theorem 3(ii). Specifically, we recognize that the function class is closed under scaling, i.e. $g \in inc(\Delta_1)\text{-}cv(\Delta_2)$ implies $\alpha g \in inc(\Delta_1)\text{-}cv(\Delta_2)$. Moreover, any increasing concave function with bounded first- and second-order deviations can be approximated by twice-differentiable functions. For twice-differentiable $g \in inc(\Delta_1)\text{-}cv(\Delta_2)$ with appropriate scaling such that $g^{(1)}_{\max} - g^{(1)}_{\min} = 1$, we have

$$g^{(1)}_{\min^+}(\underline{x}) = g^{(1)}_{\min} = g^{(1)}_{\max} - 1 \leq (1+\Delta_1)g^{(1)}_{\min},$$

implying

$$g^{(1)}_{\min^+}(\underline{x}) = g^{(1)}_{\min} \geq \frac{1}{\Delta_1}.$$

Moreover,

$$g^{(1)}_{\max} - g^{(1)}_{\min} = \int_{\xi\in\mathcal{X}} g^{(2)}(\xi)\,\mathrm{d}\xi.$$

One may similarly define the $inc(\Delta_1)\text{-}cx(\Delta_2)$ class, and obtain, by modifying the proof of Theorem 3(i), the characterization that $\mathbb{E}[(X_1 - \underline{x})^2] \leq \mathbb{E}[(X_2 - \underline{x})^2]$ and

$$\max_{h(x)} \int_{x\in\mathcal{X}} h(x)\big(\bar{F}^{(2)}_{X_2}(x) - \bar{F}^{(2)}_{X_1}(x)\big)\,\mathrm{d}x \leq \frac{1}{\Delta_1}(\mathbb{E}[(X_2 - \underline{x})^2] - \mathbb{E}[(X_1 - \underline{x})^2]),$$

where

$$h(x) \geq 0,\ \int_{x\in\mathcal{X}} h(x)\,\mathrm{d}x = 1,\ \sup_{x\in\mathcal{X}}\{h(x)\} \leq (1+\Delta_2)\inf_{x\in\mathcal{X}}\{h(x)\}.$$

As we can see, the distributional characterizations for $inc(\Delta_1)\text{-}cv(\Delta_2)$ and $inc(\Delta_1)\text{-}cx(\Delta_2)$, involving maximization over a functional space, may not be easy to verify in general.

A similar analysis can be used to characterize the 'almost risk-averse stochastic dominance' (ARSD) defined by Luo and Tan [14]. They consider the following subclass of increasing concave (twice-differentiable) functions parameterized by $\kappa \geq 0$:

$$\left\{0 \leq g^{(1)}(x) \leq (1+\Delta)g^{(1)}_{\min},\ 0 \leq \frac{-g^{(2)}(x)}{g^{(1)}(x)} \leq \frac{\kappa}{1+\Delta},\ \Delta \in [0,\infty)\right\},$$

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and obtain the characterization of $\mathbb{E}[X_1] \le \mathbb{E}[X_2]$ and

$$v_O(X_1 \le_{icv} X_2) = \int_{x\in\mathcal{X}} \left(\int_{\underline{x}}^{x} F_{X_2}(\xi)\,\mathrm{d}\xi - \int_{\underline{x}}^{x} F_{X_1}(\xi)\,\mathrm{d}\xi \right)^+ \mathrm{d}x \le \frac{1}{\kappa}(\mathbb{E}[X_2] - \mathbb{E}[X_1]).$$

In a related discussion, Huang et al. [5] define the 'fractional degree stochastic dominance', denoted by $(1+c)$-SD, for the following class of functions:

$$\left\{ g \in \overline{\mathcal{G}} \colon\ g^{(1)}(x) \ge 0,\ \ \frac{-g^{(2)}(x)}{g^{(1)}(x)} \ge -\left(\frac{1}{c} - 1\right),\ x \in \mathcal{X} \right\}.$$

They reach the characterization

$$\int_{\underline{x}}^{x} (F_{X_2}(\xi) - F_{X_1}(\xi))\,\mathrm{d}h_c(\xi) \ge 0,$$

where

$$h_c(\xi) = \begin{cases} \mathrm{e}^{(1/c-1)\xi}, & 0 < c < 1, \\ x, & c = 1. \end{cases}$$

They show that ϵ-AFSD implies $(1+c)$-SD order for appropriately chosen c (as a function of ϵ). Thus this order is more general than ϵ-AFSD.

5. Parametric higher-orders functions

In this section we generalize the discussion to higher orders. We refer the reader to Table 3 for the relationships between the orders analyzed in the previous sections and the higher orders introduced below. To analyze the higher orders, we consider the class of functions $\overline{\mathcal{G}}^m$ with well-defined mth derivative over the support $\mathcal{X}$. Specifically, we have

$$g^{(n)}(x) = a_n + \int_{\underline{x}}^{x} g^{(n+1)}(\xi)\,\mathrm{d}\xi = b_n - \int_{x}^{\overline{x}} g^{(n+1)}(\xi)\,\mathrm{d}\xi, \quad n = 0, 1, \ldots, m-1,$$

with $g^{(0)} = g$. For any function $h\colon \mathcal{X} \to \mathbb{R}$, define

$$\begin{aligned} h_{\min} &= \inf\{h(\xi)\colon \xi \in \mathcal{X}\}, & h_{\max} &= \sup\{h(\xi)\colon \xi \in \mathcal{X}\}, \\ h_{\min^+}(x) &= \inf\{h(\xi)\colon \xi \ge x\}, & h_{\max^+}(x) &= \sup\{h(\xi)\colon \xi \ge x\}, \\ h_{\min^-}(x) &= \inf\{h(\xi)\colon \xi \le x\}, & h_{\max^-}(x) &= \sup\{h(\xi)\colon \xi \le x\}. \end{aligned}$$

In our discussion below, h may be the mth-order local deviation of some function g of interest.

5.1. Parametric m-convex and m-concave functions

The m-convex and m-concave functions, respectively, are defined by (e.g. [21])

$$\begin{aligned} m\text{-}cx &= \left\{ g \in \overline{\mathcal{G}}^m \colon\ g^{(m)}(x) \ge 0,\ x \in \mathcal{X} \right\}, \\ m\text{-}cv &= \left\{ g \in \overline{\mathcal{G}}^m \colon\ (-1)^{m-1} g^{(m)}(x) \ge 0,\ x \in \mathcal{X} \right\}. \end{aligned}$$

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TABLE 3. Relationship to higher-order functions.

	$inc(\Delta)$	$inc^{+}(\Delta)$	$inc^{-}(\Delta)$	$cx(\Delta)$	$cv(\Delta)$	$inc\text{-}cx(\Delta)$	$inc\text{-}cv(\Delta)$	$inc(\Delta)\text{-}cx$	$inc(\Delta)\text{-}cv$
$m\text{-}cx(\Delta)$	$m=1$			$m=2$					
$m\text{-}cx^{+}(\Delta)$		$m=1$							
$m\text{-}cx^{-}(\Delta)$			$m=1$						
$m\text{-}cx^{\pm}(\Delta,0)$								$m=1$	
$m\text{-}cv(\Delta)$	$m=1$				$m=2$				
$m\text{-}cv^{+}(\Delta)$		$m=1$							
$m\text{-}cv^{-}(\Delta)$			$m=1$						
$m\text{-}cv^{\pm}(0,\Delta)$									$m=1$
$m\text{-}icx(\Delta)$	$m=1$					$m=2$			
$m\text{-}icx^{+}(\Delta)$		$m=1$							
$m\text{-}icx^{-}(\Delta)$			$m=1$						
$m\text{-}icx^{\pm}(\Delta,0)$								$m=1$	
$m\text{-}icx^{\pm}(0,\Delta)$									$m=1$
$m\text{-}icv(\Delta)$	$m=1$						$m=2$		
$m\text{-}icv^{+}(\Delta)$		$m=1$							
$m\text{-}icv^{-}(\Delta)$			$m=1$						
$m\text{-}icv^{\pm}(\Delta,0)$									$m=1$
$m\text{-}icv^{\pm}(0,\Delta)$								$m=1$	

A random variable X_1 is smaller than another random variable X_2 in the m-convex (m-concave) order, i.e. $X_1 \le_{m\text{-}cx} (\le_{m\text{-}cv}) X_2$, if and only if $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all $g \in m\text{-}cx$ ($m\text{-}cv$). An equivalent condition is that $\mathbb{E}[X_1^n] = \mathbb{E}[X_2^n]$, $n = 1, 2, \ldots, m-1$, and

$$\bar{F}_{X_1}^{(m-1)}(x) \le \bar{F}_{X_2}^{(m-1)}(x) \left[F_{X_1}^{(m-1)}(x) \ge F_{X_2}^{(m-1)}(x)\right], \quad x \in \mathcal{X},$$

where

$$\bar{F}_X^{(n)}(x) = \int_x^{\bar{x}} \bar{F}_X^{(n-1)}(\xi)\, \mathrm{d}\xi,$$

with

$$\bar{F}_X^{(0)} = \bar{F}_X \left[F_X^{(n)}(x) = \int_{\underline{x}}^{x} F_X^{(n-1)}(\xi)\, \mathrm{d}\xi \text{ with } F_X^{(0)} = F_X\right].$$

It is straightforward to verify that

$$\bar{F}_X^{(n)}(x) = \frac{1}{n!}\mathbb{E}[((X-x)^+)^n] \quad \text{and} \quad F_X^{(n)}(x) = \frac{1}{n!}\mathbb{E}[((x-X)^+)^n], \quad x \in \mathcal{X}, \quad n = 1, 2, \ldots.$$

We can define the restricted function classes as

$$m\text{-}cx(\Delta) = \left\{g \in \overline{\mathcal{G}}^m : 0 \le g^{(m)}(x) \le (1+\Delta) g_{\min}^{(m)}, x \in \mathcal{X}\right\},$$

$$m\text{-}cv(\Delta) = \left\{g \in \overline{\mathcal{G}}^m : g = (-1)^{m-1}\hat{g},\ 0 \le \hat{g}^{(m)}(x) \le (1+\Delta)\hat{g}_{\min}^{(m)},\ x \in \mathcal{X}\right\}.$$

For an odd m, $m\text{-}cx(\Delta) = m\text{-}cv(\Delta)$, e.g. $1\text{-}cx(\Delta) = 1\text{-}cv(\Delta) = inc(\Delta)$. For an even m, $g \in m\text{-}cx(\Delta)$ is equivalent to $-g \in m\text{-}cv(\Delta)$, e.g. $g \in 2\text{-}cx(\Delta) = cx(\Delta)$, if and only if $-g \in 2\text{-}cv(\Delta) = cv(\Delta)$.

The violations to the m-cx and m-cv orders, respectively, are defined as

$$v(x; X_1 \le_{m\text{-}cx} X_2) = \left(\bar{F}_{X_1}^{(m-1)}(x) - \bar{F}_{X_2}^{(m-1)}(x)\right)^+ = \left((-1)^m\left(F_{X_1}^{(m-1)}(x) - F_{X_2}^{(m-1)}(x)\right)\right)^+,$$

$$v(x; X_1 \le_{m\text{-}cv} X_2) = \left(F_{X_2}^{(m-1)}(x) - F_{X_1}^{(m-1)}(x)\right)^+.$$

With these measures, we characterize the stochastic orders associated with $m\text{-}cx(\Delta)$ and $m\text{-}cv(\Delta)$ functions in the next theorem.

Theorem 5. (Higher-order functions.) *Two random variables X_1 and X_2 with support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all*

(i) *$g \in m\text{-}cx(\Delta)$ if and only if*

$$\mathbb{E}[X_1^k] = \mathbb{E}[X_2^k], \quad k = 1, 2, \ldots, m-1,$$

and

$$v_O(X_1 \le_{m\text{-}cx} X_2) = \int_{x\in\mathcal{X}} \left(\bar{F}_{X_1}^{(m-1)}(x) - \bar{F}_{X_2}^{(m-1)}(x)\right)^+ \mathrm{d}x \le \frac{1}{\Delta m!}(\mathbb{E}[X_2^m] - \mathbb{E}[X_1^m]); \quad (17)$$

and

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(ii) $g \in m\text{-}cv(\Delta)$ *if and only if*

$$\mathbb{E}[X_1^k] = \mathbb{E}[X_2^k], \quad k = 1, 2, \ldots, m-1,$$

and

$$v_O(X_1 \leq_{m\text{-}cv} X_2) = \int_{x\in\mathcal{X}} \left(F_{X_2}^{(m-1)}(x) - F_{X_1}^{(m-1)}(x)\right)^+ \mathrm{d}x \leq \frac{(-1)^m}{\Delta m!}(\mathbb{E}[X_1^m] - \mathbb{E}[X_2^m]). \tag{18}$$

Proof of Theorem 5. We prove part (i), and the proof of part (ii) follows similarly. Take an integer n and an nth differentiable function g. We have

$$\begin{aligned}
\mathbb{E}[g(X)] &= a_0 - \int_{x\in\mathcal{X}} \int_{\underline{x}}^{x} g^{(1)}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\bar{F}_X(x) \\
&= a_0 + \int_{\xi\in\mathcal{X}} g^{(1)}(\xi)\bar{F}_X(\xi)\,\mathrm{d}\xi \\
&= a_0 + a_1\big(\mathbb{E}[X] - \underline{x}\big) + \int_{\xi\in\mathcal{X}} \bar{F}_X^{(1)}(\xi) g^{(2)}(\xi)\,\mathrm{d}\xi \\
&= a_0 + a_1\big(\mathbb{E}[X] - \underline{x}\big) + \int_{x\in\mathcal{X}} \bar{F}_X^{(1)}(x)\left(a_2 + \int_{\underline{x}}^{x} g^{(3)}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}x \\
&= a_0 + a_1\big(\mathbb{E}[X] - \underline{x}\big) + \frac{a_2}{2}\mathbb{E}[(X-\underline{x})^2] + \int_{x\in\mathcal{X}} \bar{F}_X^{(2)}(x) g^{(3)}(x)\,\mathrm{d}x \\
&= \cdots \\
&= \sum_{k=0}^{n} \frac{a_k}{k!}\mathbb{E}[(X-\underline{x})^k] + \int_{x\in\mathcal{X}} \bar{F}_X^{(n)}(x) g^{(n+1)}(x)\,\mathrm{d}x,
\end{aligned}$$

where $a_k = g^{(k)}(\underline{x})$ for $k = 1, 2, \ldots, n$. We deduce

$$\begin{aligned}
&\mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&= \sum_{k=0}^{n} \frac{a_k}{k!}\mathbb{E}[(X_2-\underline{x})^k - (X_1-\underline{x})^k] + \int_{x\in\mathcal{X}} \big(\bar{F}_{X_2}^{(n)}(x) - \bar{F}_{X_1}^{(n)}(x)\big) g^{(n+1)}(x)\,\mathrm{d}x.
\end{aligned} \tag{19}$$

For any $g \in m\text{-}cx(\Delta)$, and two random variables X_1 and X_2 with

$$\mathbb{E}[X_1^k] = \mathbb{E}[X_2^k], \quad k = 1, 2, \ldots, m-1,$$

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we can rewrite (19) as

$$\begin{aligned}
&\mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&= \int_{x\in\mathcal{X}} \big(\bar{F}_{X_2}^{(m-1)}(x) - \bar{F}_{X_1}^{(m-1)}(x)\big)^+ g^{(m)}(x)\,\mathrm{d}x - \int_{x\in\mathcal{X}} \big(\bar{F}_{X_1}^{(m-1)}(x) - \bar{F}_{X_2}^{(m-1)}(x)\big)^+ g^{(m)}(x)\,\mathrm{d}x \\
&\geq g_{\min}^{(m)} \int_{x\in\mathcal{X}} \big(\bar{F}_{X_2}^{(m-1)}(x) - \bar{F}_{X_1}^{(m-1)}(x)\big)^+ \mathrm{d}x - (1+\Delta) g_{\min}^{(m)} \int_{x\in\mathcal{X}} \big(\bar{F}_{X_1}^{(m-1)}(x) - \bar{F}_{X_2}^{(m-1)}(x)\big)^+ \\
&= \frac{g_{\min}^{(m)}}{m!}(\mathbb{E}[(X_2-\underline{x})^m] - \mathbb{E}[(X_1-\underline{x})^m]) - g_{\min}^{(m)} \Delta v_O(x; X_1 \leq_{m\text{-}cx} X_2).
\end{aligned}$$

Thus, when X_1 and X_2 satisfy (17), we must have $\mathbb{E}[g(X_2)] \geq \mathbb{E}[g(X_1)]$.

Now take X_1 and X_2 such that $\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$ for all $g \in m\text{-}cx(\Delta)$. We first note that $\mathbb{E}[X_1^k] = \mathbb{E}[X_2^k]$ for $k = 1, 2, \ldots, m-1$ because $g(x) = x^k$ and $g(x) = -x^k$ both have $g^{(m)}(x) = 0$ and thus both belong to $m\text{-}cx(\Delta)$. Now take

$$g^{(m)}(x) = 1 + \Delta \mathbb{I}_{\{\bar{F}_{X_1}^{(m-1)}(x) > \bar{F}_{X_2}^{(m-1)}(x)\}}$$

and

$$g^{(k)}(x) = \int_{\underline{x}}^{x} g^{(k+1)}(\xi)\,\mathrm{d}\xi \quad \text{for } k = 0, 1, \ldots, m-1.$$

It is easy to verify $g \in m\text{-}cx(\Delta)$. We deduce from (19) that

$$\begin{aligned}
0 &\leq \mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&= \int_{x\in\mathcal{X}} \big(\bar{F}_{X_2}^{(m-1)}(x) - \bar{F}_{X_1}^{(m-1)}(x)\big) g^{(m)}(x)\,\mathrm{d}x \\
&= \int_{x\in\mathcal{X}} \big(\bar{F}_{X_2}^{(m-1)}(x) - \bar{F}_{X_1}^{(m-1)}(x)\big)\Big(1 + \Delta \mathbb{I}_{\{\bar{F}_{X_1}^{(m-1)}(x) > \bar{F}_{X_2}^{(m-1)}(x)\}}\Big)\,\mathrm{d}x \\
&= \frac{1}{m!}(\mathbb{E}[X_2^m] - \mathbb{E}[X_1^m]) - \Delta \int_{x\in\mathcal{X}} \big(\bar{F}_{X_1}^{(m-1)}(x) - \bar{F}_{X_2}^{(m-1)}(x)\big)^+ \mathrm{d}x,
\end{aligned}$$

which gives (17). □

Tsetlin et al. [23] name the 'almost nth-degree risk' (ϵ-AnR) order corresponding to $m\text{-}cv(\Delta)$ functions. They derive (18) and $F_{X_1}^{(k)}(\bar{x}) = F_{X_2}^{(k)}(\bar{x})$ for $k = 1, 2, \ldots, m-1$. The latter condition leads to equal kth moments as the random variables have the same upper support. Mao et al. [15] identify several invariant properties the order associated with $m\text{-}cv$ functions and view the $m\text{-}cv(\Delta)$ functions as an interpolation between the non-parametric function classes.

To extend the analysis to functions with asymmetric two-sided restrictions, we define

$$\begin{aligned}
&m\text{-}cx^{\pm}(\Delta_+, \Delta_-) \\
&= \big\{g \in \overline{\mathcal{G}}^m : 0 \leq g^{(m)}(x) \leq \min\big\{(1+\Delta_+) g_{\min^+}^{(m)}(x), (1+\Delta_-) g_{\min^-}^{(m)}(x)\big\},\ x \in \mathcal{X}\big\},
\end{aligned}$$

$$\begin{aligned}
&m\text{-}cv^{\pm}(\Delta_+, \Delta_-) \\
&= \big\{g \in \overline{\mathcal{G}}^m : \hat{g} = (-1)^m g,\ 0 \leq \hat{g}^{(m)}(x) \leq \min\big\{(1+\Delta_+) \hat{g}_{\min^+}^{(m)}(x), (1+\Delta_-) \hat{g}_{\min^-}^{(m)}(x)\big\},\ x \in \mathcal{X}\big\}.
\end{aligned}$$

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We can characterize the distributional properties of orders associated with the $(\Delta+, \Delta-)$ m-convex (concave) functions using an argument similar to that of Theorem 2.

Theorem 6. (Two-sided relaxation of m-convex and m-concave order.) *Given $\Delta_+ > 0$ and $\Delta_- > 0$, two random variables X_1 and X_2 defined on support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all*

(i) *$g \in m\text{-}cx^{\pm}(\Delta_+, \Delta_-)$ if and only if*

$$\mathbb{E}[X_1^k] = \mathbb{E}[X_2^k] \quad \text{for } k = 1, 2, \ldots, m-1,$$

and for any $x \in \mathcal{X}$,

$$\frac{1}{1+\Delta_-}\left(\int_{\underline{x}}^{x} \left(\bar{F}_{X_2}^{(m-1)}(\xi) - \bar{F}_{X_1}^{(m-1)}(\xi)\right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\le}(x; X_1 \le_{m\text{-}cx} X_2)\right)$$
$$+ \frac{1}{1+\Delta_+}\left(\int_{x}^{\bar{x}} \left(\bar{F}_{X_2}^{(m-1)}(\xi) - \bar{F}_{X_1}^{(m-1)}(\xi)\right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\ge}(x; X_1 \le_{m\text{-}cx} X_2)\right) \ge 0; \tag{20}$$

and

(ii) *$g \in m\text{-}cv^{\pm}(\Delta_+, \Delta_-)$ if and only if*

$$\mathbb{E}[X_1^k] = \mathbb{E}[X_2^k] \quad \text{for } k = 1, 2, \ldots, m-1,$$

and for any $x \in \mathcal{X}$,

$$\frac{1}{1+\Delta_-}\left(\int_{\underline{x}}^{x} \left(F_{X_1}^{(m-1)}(\xi) - F_{X_2}^{(m-1)}(\xi)\right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\le}(x; X_1 \le_{m\text{-}cv} X_2)\right)$$
$$+ \frac{1}{1+\Delta_+}\left(\int_{x}^{\bar{x}} \left(F_{X_1}^{(m-1)}(\xi) - F_{X_2}^{(m-1)}(\xi)\right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\ge}(x; X_1 \le_{m\text{-}cv} X_2)\right) \ge 0. \tag{21}$$

With Theorem 6, we can easily deduce the violation conditions for functions with one-sided restrictions that lead to the $m\text{-}cx^{+}(\Delta)$, $m\text{-}cx^{-}(\Delta)$, $m\text{-}cv^{+}(\Delta)$, and $m\text{-}cv^{-}(\Delta)$ functions. We note two other special cases (see Table 3).

- When $\Delta_+ = 0$ for m-convex functions, we obtain the function class $m\text{-}cx^{\pm}(0, \Delta) = m\text{-}cx(\Delta) \cap (m+1)\text{-}cx$. The characterization in (20) becomes

$$\int_{x}^{\bar{x}} \left(\bar{F}_{X_1}^{(m-1)}(\xi) - \bar{F}_{X_2}^{(m-1)}(\xi)\right) \mathrm{d}\xi \le \frac{1}{m!\Delta}\left(\mathbb{E}[X_2^m] - \mathbb{E}[X_1^m]\right), \quad x \in \mathcal{X}.$$

  For $m = 1$, we obtain the $inc(\Delta)\text{-}cx$ order discussed in Theorem 4(ii-a).

- When $\Delta_- = 0$ for m-concave functions, we obtain the function class $m\text{-}cv^{\pm}(\Delta, 0) = m\text{-}cv(\Delta) \cap (m+1)\text{-}cv$. The characterization in (21) becomes

$$\int_{\underline{x}}^{x} \left(F_{X_2}^{(m-1)}(\xi) - F_{X_1}^{(m-1)}(\xi)\right) \mathrm{d}\xi \le \frac{(-1)^{m-1}}{m!\Delta}\left(\mathbb{E}[X_2^m] - \mathbb{E}[X_1^m]\right), \quad x \in \mathcal{X}.$$

  For $m = 1$, we obtain the $inc(\Delta)\text{-}cv$ order discussed in Theorem 4(ii-b).

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5.2. Parametric monotone higher orders

The monotone higher-order function classes are defined as

$$m\text{-}icx = \{g \in \mathcal{G}^m \colon g^{(k)}(\underline{x}) = a_k \geq 0,\ k = 1, 2, \ldots, m-1;\ g^{(m)}(x) \geq 0,\ x \in \mathcal{X}\}$$

and

$$m\text{-}icv = \{g \colon g = \hat{g},\ (-1)^{m-1}\hat{g} \in m\text{-}icx\}.$$

A random variable X_1 is smaller than another random variable X_2 in the m-increasing convex (concave) order, i.e. $X_1 \leq_{m\text{-}icx} [\leq_{m\text{-}icv}] X_2$, if and only if

$$\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)] \quad \text{for all } g \in m\text{-}icx\ (m\text{-}icv).$$

We define the subclasses of functions

$$m\text{-}icx(\Delta) = \left\{ g \in \mathcal{G}^m \colon \begin{array}{l} g^{(k)}(\underline{x}) = a_k \geq 0,\ k = 1, 2, \ldots, m-1; \\ 0 \leq g^{(m)}(x) \leq (1+\Delta) g^{(m)}_{\min},\ x \in \mathcal{X} \end{array} \right\},$$

$$m\text{-}icv(\Delta) = \left\{ g \in \mathcal{G}^m \colon \begin{array}{l} (-1)^{k-1} g^{(k)}(\overline{x}) = (-1)^{k-1} b_k \geq 0,\ k = 1, 2, \ldots, m-1; \\ \hat{g}(x) = (-1)^{m-1} g(x), \\ 0 \leq \hat{g}^{(m)}(x_2) \leq (1+\Delta)\hat{g}^{(m)}_{\min},\ x \in \mathcal{X} \end{array} \right\}.$$

Theorem 7. (Symmetric relaxation of *m-icx* and *m-icv* order.) *Two random variables X_1 and X_2 on support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$ for all*

(i) *$g \in m\text{-}icx(\Delta)$ if and only if*

$$\mathbb{E}[(X_1 - \underline{x})^k] \leq \mathbb{E}[(X_2 - \underline{x})^k], \quad k = 1, 2, \ldots, m-1,$$

and

$$v_O(X_1 \leq_{m\text{-}cx} X_2) = \int_{x \in \mathcal{X}} \left(\bar{F}^{(m-1)}_{X_1}(x) - \bar{F}^{(m-1)}_{X_2}(x)\right)^+ \mathrm{d}x \leq \frac{1}{\Delta m!}(\mathbb{E}[(X_2 - \underline{x})^m] - \mathbb{E}[(X_1 - \underline{x})^m]);$$

and

(ii) *$g \in m\text{-}icv(\Delta)$ if and only if*

$$\mathbb{E}[(\overline{x} - X_1)^k] \geq \mathbb{E}[(\overline{x} - X_2)^k], \quad k = 1, 2, \ldots, m-1,$$

and

$$v_O(X_1 \leq_{m\text{-}cv} X_2) = \int_{x \in \mathcal{X}} \left(F^{(m-1)}_{X_2}(x) - F^{(m-1)}_{X_1}(x)\right)^+ \mathrm{d}x \leq \frac{1}{\Delta m!}(\mathbb{E}[(\overline{x} - X_1)^m] - \mathbb{E}[(\overline{x} - X_2)^m]).$$

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Proof of Theorem 7. The result follows with a slight modification of the proof of Theorem 5 by including the condition $\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k]$ in part (i) and $\mathbb{E}[(\overline{x} - X_1)^k] \ge \mathbb{E}[(\overline{x} - X_2)^k]$ in part (ii) for $k = 1, 2, \ldots, m-1$. □

The *m-icv*(Δ) function leads to the 'almost *N*th-degree stochastic dominance' (ANSD) named by Tzeng et al. [24] and the 'generalized almost *n*th-degree stochastic dominance' (ϵ-GAnSD) named by Tsetlin et al. [23]. Lu et al. [13] show the correspondence of this order to the higher-order Ω ratio.

It is clear that the monotone orders have the same violation measure as the non-monotone orders. The main difference lies in the additional conditions on the ordered moments of the random variables. The moment conditions may or may not be needed for functions with one-sided restrictions, depending on whether the restriction is on the left or the right side. Specifically, for *m*-convex functions, we define the subclasses with one-sided restrictions as

$$m\text{-}icx^{+}(\Delta) = \left\{ g \in \overline{\mathcal{G}}^m : \begin{array}{l} g^{(k)}(\underline{x}) = a_k \ge 0,\ k = 1, 2, \ldots, m-1; \\ 0 \le g^{(m)}(x) \le (1+\Delta) g^{(m)}_{\min^+}(x),\ x \in \mathcal{X} \end{array} \right\},$$

$$m\text{-}icx^{-}(\Delta) = \left\{ g \in \overline{\mathcal{G}}^m : \begin{array}{l} g^{(k)}(\underline{x}) = a_k \ge 0,\ k = 1, 2, \ldots, m-1; \\ 0 \le g^{(m)}(x) \le (1+\Delta) g^{(m)}_{\min^-}(x),\ x \in \mathcal{X} \end{array} \right\}.$$

We also have $m\text{-}icx^{\pm}(\Delta_+, \Delta_-) = m\text{-}icx^{+}(\Delta_+) \cap m\text{-}icx^{-}(\Delta_-)$.

Theorem 8. (Asymmetric relaxations of *m-icx* order.) *Two random variables X_1 and X_2 defined on support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all*

(i) $g \in m\text{-}icx^{+}(\Delta)$ *for some* $\Delta \in [0, \infty)$ *if and only if*

$$v_{\ge}(x; X_1 \le_{m\text{-}cx} X_2) \le \frac{1}{\Delta} \int_x^{\overline{x}} \left(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(\xi) \right) \mathrm{d}\xi, \quad x \in \mathcal{X}; \tag{22}$$

(ii) $g \in m\text{-}icx^{-}(\Delta)$ *for some* $\Delta \in [0, \infty)$ *if and only if*

$$\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k] \quad \text{for } k = 1, 2, \ldots, m-1,$$

and

$$v_{\le}(x; X_1 \le_{m\text{-}cx} X_2) \le \frac{1}{\Delta} \int_{\underline{x}}^{x} \left(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(\xi) \right) \mathrm{d}\xi, \quad x \in \mathcal{X}; \tag{23}$$

and

(iii) $g \in m\text{-}icx^{\pm}(\Delta_+, \Delta_-)$ *for some* $\Delta_+, \Delta_- \in [0, \infty)$ *if and only if*

$$\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k] \quad \text{for } k = 1, 2, \ldots, m-1,$$

and for any $x \in \mathcal{X}$,

$$\frac{1}{1+\Delta_-} \left(\int_{\underline{x}}^{x} \left(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(\xi) \right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\le}(x; X_1 \le_{m\text{-}cx} X_2) \right)$$

$$+ \frac{1}{1+\Delta_+} \left(\int_{x}^{\overline{x}} \left(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(\xi) \right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\ge}(x; X_1 \le_{m\text{-}cx} X_2) \right) \ge 0. \tag{24}$$

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Proof of Theorem 8. To show part (i), we first note that the left-hand side of (22) is non-negative, implying $\bar{F}^{(m)}_{X_1}(x) \le \bar{F}^{(m)}_{X_2}(x)$ for all $x \in \mathcal{X}$. We deduce $X_1 \le_{(m+1)\text{-}icx} X_2$. It follows that $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for any $g \in (m+1)\text{-}icx$, and $g(x) = (x - \underline{x})^k \in (m+1)\text{-}icx$ for $k = 1, 2, \ldots, m$, we must have $\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k]$.

Since $g^{(m)}$ has bounded variation, we derive, using (19) in the proof of Theorem 5,

$$
\begin{aligned}
&\mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&= \sum_{k=0}^{m-1} \frac{a_k}{k!} \mathbb{E}[(X_2 - \underline{x})^k - (X_1 - \underline{x})^k] + \int_{x \in \mathcal{X}} \big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big) g^{(m)}(x)\, \mathrm{d}x \\
&\ge \int_{x \in \mathcal{X}} g^{(m)}_{\min^+}(x) \Big(\big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big) - \Delta \big(\bar{F}^{(m-1)}_{X_1}(x) - \bar{F}^{(m-1)}_{X_2}(x)\big)^+ \Big)\, \mathrm{d}x \\
&= \int_{x \in \mathcal{X}} \left(g^{(m)}_{\min^+}(\underline{x}) + \int_{\underline{x}}^{x} \mathrm{d} g^{(m)}_{\min^+}(\xi) \right) \\
&\quad \times \Big(\big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big) - \Delta \big(\bar{F}^{(m-1)}_{X_1}(x) - \bar{F}^{(m-1)}_{X_2}(x)\big)^+ \Big)\, \mathrm{d}x \\
&= g^{(m)}_{\min^+}(\underline{x}) \int_{\xi \in \mathcal{X}} \Big(\big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big) - \Delta \big(\bar{F}^{(m-1)}_{X_1}(x) - \bar{F}^{(m-1)}_{X_2}(x)\big)^+ \Big)\, \mathrm{d}x \\
&\quad + \int_{\xi \in \mathcal{X}} \int_{\xi}^{\overline{x}} \Big(\big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big) - \Delta \big(\bar{F}^{(m-1)}_{X_1}(x) - \bar{F}^{(m-1)}_{X_2}(x)\big)^+ \Big)\, \mathrm{d}x\, \mathrm{d} g^{(m)}_{\min^+}(\xi).
\end{aligned}
$$

Because $g^{(m)}_{\min^+}$ is increasing and $g^{(m)}_{\min^+}(\underline{x}) \ge 0$, the right-hand side is non-negative when (22) holds.

Next we show that $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all $g \in m\text{-}icx^+(\Delta)$ implies (22). Take g_y such that

$$
g^{(m)}_y(x) = \Big(1 + \Delta \mathbb{I}_{\left\{\bar{F}^{(m-1)}_{X_1}(x) > \bar{F}^{(m-1)}_{X_2}(x)\right\}}\Big) \mathbb{I}_{\{x \ge y\}},
$$

and

$$
g^{(k)}_y(x) = \int_{\underline{x}}^{x} g^{(k+1)}_y(\xi)\, \mathrm{d}\xi \quad \text{for } k = 0, 1, \ldots, m-1.
$$

It is clear that for $x \ge y$, $g^{(m)}_{y\,\min^+}(x) \ge 1$ and thus

$$
g^{(m)}_y(x) \le (1 + \Delta) \le (1 + \Delta) g^{(m)}_{y\,\min^+}(x).
$$

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For $x < y$, $g^{(m)}_{y\,\min^+}(x) = g_y(x) = 0$. Thus $g_y \in m\text{-}icx^+(\Delta)$. Using (19) in the proof of Theorem 5 with $a_k = g_y^{(k)}(\underline{x}) = 0$ for $k = 0, 1, \ldots, m-1$, we derive

$$\begin{aligned}
&\mathbb{E}[g_y(X_2)] - \mathbb{E}[g_y(X_1)] \\
&= \int_{x\in\mathcal{X}} \big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big) g^{(m)}_y(x)\,\mathrm{d}x \\
&= \int_y^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big)\,\mathrm{d}x + \Delta \int_y^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big)\mathbb{I}_{\left\{\bar{F}^{(m-1)}_{X_1}(x) > \bar{F}^{(m-1)}_{X_2}(x)\right\}}\,\mathrm{d}x \\
&= \int_y^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_2}(x) - \bar{F}^{(m-1)}_{X_1}(x)\big)\,\mathrm{d}x - \Delta \int_y^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_1}(x) - \bar{F}^{(m-1)}_{X_2}(x)\big)^+\,\mathrm{d}x.
\end{aligned}$$

We deduce that $\mathbb{E}[g_y(X_1)] \le \mathbb{E}[g_y(X_2)]$ implies (22). We obtain part (i).

To see part (ii), we note that the left-hand side of (23) is non-negative, which implies (by setting $x = \bar{x}$) $\mathbb{E}[(X_1 - \underline{x})^m] \le \mathbb{E}[(X_2 - \underline{x})^m]$. Using an argument similar to that for part (i), we can show that (23), together with the condition $\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k]$ for $k = 1, 2, \ldots, m-1$, implies $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all $g \in m\text{-}icx^-(\Delta)$. To show the reverse relation, we note that $g(x) = (x - \underline{x})^k \in m\text{-}icx^-(\Delta)$ for any $k = 1, 2, \ldots, m-1$, leading to $\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k]$ for $k = 1, 2, \ldots, m-1$. Then we take

$$g^{(m)}_y(x) = \left(1 + \Delta \mathbb{I}_{\left\{\bar{F}^{(m-1)}_{X_1}(x) > \bar{F}^{(m-1)}_{X_2}(x)\right\}}\right)\mathbb{I}_{\{x \le y\}}$$

and apply an argument similar to that for part (i).

Next we prove part (iii). For the 'if' part, we show that (24) and the condition $\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k]$ for $k = 1, 2, \ldots, m-1$, imply $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all $g \in m\text{-}icx^{\pm}(\Delta_+, \Delta_-)$. We focus on the case where $\Delta_+ \le \Delta_-$. Then (24) becomes

$$\begin{aligned}
\Xi(x) \equiv{}& \frac{1}{1+\Delta_-}\int_{\underline{x}}^{x} \big(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(\xi)\big)^+\,\mathrm{d}\xi - \frac{1+\Delta_+}{1+\Delta_-}\int_{\underline{x}}^{x} \big(\bar{F}^{(m-1)}_{X_1}(\xi) - \bar{F}^{(m-1)}_{X_2}(\xi)\big)^+\,\mathrm{d}\xi \\
&+ \frac{1}{1+\Delta_+}\int_{x}^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(\xi)\big)^+\,\mathrm{d}\xi - \int_{x}^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_1}(\xi) - \bar{F}^{(m-1)}_{X_2}(\xi)\big)^+\,\mathrm{d}\xi \ge 0.
\end{aligned}$$

We derive

$$\begin{aligned}
&\mathbb{E}[g(X_2)] - \mathbb{E}[g(X_1)] \\
&= \sum_{k=0}^{m} \frac{a_k}{k!}\big(\mathbb{E}[(X_2 - \underline{x})^k] - \mathbb{E}[(X_1 - \underline{x})^k]\big) + \int_{\underline{x}}^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(\xi)\big) g^{(m)}(\xi)\,\mathrm{d}\xi \\
&\ge \int_{\underline{x}}^{\bar{x}} \big(\bar{F}^{(m-1)}_{X_2}(\xi) - \bar{F}^{(m-1)}_{X_1}(x)\big) g^{(m)}(\xi)\,\mathrm{d}\xi \\
&\ge \int_{x\in\mathcal{X}} \Gamma(\xi)\rho(\xi)\,\mathrm{d}\xi \\
&= g^{(m)}_{\max} \int_{x_0}^{x_{\max}} \Xi(x)\,\mathrm{d}\mu(x) \\
&\ge 0,
\end{aligned}$$

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where

$$\Gamma(\xi) = \frac{1}{1+\Delta_+}\left(\bar{F}_{X_2}^{(m-1)}(\xi) - \bar{F}_{X_1}^{(m-1)}(\xi)\right)^+ - \left(\bar{F}_{X_1}^{(m-1)}(\xi) - \bar{F}_{X_2}^{(m-1)}(\xi)\right)^+$$

and

$$\mu(x) = \frac{g_{\max^-}^{(m)}(x) - g_0^{(m)}}{g_{\max}^{(m)} - g_0^{(m)}} \quad \text{for } x_0 \le x \le x_{\max},$$

with $x_{\max}$ and x_0 defined similarly to those in the proof of Theorem 6.

To prove the 'only if' part, we first note that $g(x) = (x - \underline{x})^k \in m\text{-}icx^{\pm}(\Delta_+, \Delta_-)$ and thus $\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k]$ for $k = 1, 2, \ldots, m-1$. Define

$$g_y^{(m)}(x) = \frac{\left(1 + (\Delta_+ \wedge \Delta_-)\mathbb{I}_{\left\{\bar{F}_{X_1}^{(m-1)}(x) \ge \bar{F}_{X_2}^{(m-1)}(x)\right\}}\right)^{\mathbb{I}_{\{x \le y\}}}}{1+\Delta_-} + \frac{\left(1 + (\Delta_+ \wedge \Delta_-)\mathbb{I}_{\left\{\bar{F}_{X_1}^{(m-1)}(x) \ge \bar{F}_{X_2}^{(m-1)}(x)\right\}}\right)^{\mathbb{I}_{\{x \ge y\}}}}{1+\Delta_+},$$

and

$$g_y^{(k)}(x) = \int_{\underline{x}}^{x} g_y^{(k+1)}(\xi)\, \mathrm{d}\xi \quad \text{for } k = 0, 1, \ldots, m-1.$$

We can easily verify that $g_y \in m\text{-}icx^{\pm}(\Delta_+, \Delta_-)$. We can derive (25) with the relation $\mathbb{E}[g_y(X_2)] \ge \mathbb{E}[g_y(X_1)]$. □

By Theorem 8, the condition of ordered moments, that is,

$$\mathbb{E}[(X_1 - \underline{x})^k] \le \mathbb{E}[(X_2 - \underline{x})^k] \quad \text{for } k = 1, 2, \ldots, m-1,$$

is not needed for the parametric order associated with the right-side violation. This is because the violation measure in (22) is non-negative, and thus $X_1 \le_{(m+1)\text{-}icx} X_2$, which implies the order of the moments.

In the special case of $\Delta_- = 0$, we obtain the function class $m\text{-}icx^{\pm}(\Delta, 0) = m\text{-}icx(\Delta) \cap (m+1)\text{-}icx$. By (24), the distributional characterization of this order is

$$\int_{\underline{x}}^{x} \left(\bar{F}_{X_1}^{(m-1)}(\xi) - \bar{F}_{X_2}^{(m-1)}(\xi)\right) \mathrm{d}\xi \le \frac{1}{m!\Delta}\left(\mathbb{E}[(X_2 - \underline{x})^m] - \mathbb{E}[(X_1 - \underline{x})^m]\right), \quad x \in \mathcal{X}.$$

For $m = 1$, $1\text{-}icx^{\pm}(\Delta, 0) = 1\text{-}cx^{\pm}(\Delta, 0)$, and we recover the $inc(\Delta)\text{-}cx$ function discussed in Theorem 4(i-a). For higher orders, however, $m\text{-}icx^{\pm}(\Delta, 0) \ne m\text{-}cx^{\pm}(\Delta, 0)$ in general due to the different conditions on the moments.

For m-concave functions, we define the subclasses with one-sided restrictions as

$$m\text{-}icv^{-}(\Delta) = \left\{ g \in \overline{\mathcal{G}}^m : \begin{array}{c} (-1)^{k-1} g^{(k)}(\overline{x}) = (-1)^{k-1} b_k \ge 0,\ k = 1, 2, \ldots, m-1, \\ \hat{g}(x) = (-1)^{m-1} g(x), \\ 0 \le \hat{g}_{\min^-}^{(m)}(x_1) \le \hat{g}^{(m)}(x_2) \le (1+\Delta)\hat{g}_{\min^-}^{(m)}(x_1),\ \underline{x} \le x_1 < x_2 \le \overline{x} \end{array} \right\},$$

$$m\text{-}icv^{+}(\Delta) = \left\{ g \in \overline{\mathcal{G}}^m : \begin{array}{c} (-1)^{k-1} g^{(k)}(\overline{x}) = (-1)^{k-1} b_k \ge 0,\ k = 1, 2, \ldots, m-1, \\ \hat{g}(x) = (-1)^{m-1} g(x), \\ 0 \le \hat{g}_{\min^+}^{(m)}(x_2) \le \hat{g}^{(m)}(x_1) \le (1+\Delta)\hat{g}_{\min^+}^{(m)}(x_2),\ \underline{x} \le x_1 < x_2 \le \overline{x} \end{array} \right\}.$$

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We also define $m\text{-}icv^{\pm}(\Delta_+, \Delta_-) = m\text{-}icv^{+}(\Delta_+) \cap m\text{-}icv^{-}(\Delta_-)$. We can obtain the following characterization using an argument similar to Theorem 8.

Theorem 9. (Asymmetric relaxation of *m-icv* order.) *Two random variables X_1 and X_2 defined on support $\mathcal{X}$ satisfy $\mathbb{E}[g(X_1)] \le \mathbb{E}[g(X_2)]$ for all*

(i) $g \in m\text{-}icv^{+}(\Delta)$ *for some* $\Delta \in [0, \infty)$ *if and only if*

$$\mathbb{E}[(\bar{x} - X_1)^k] \ge \mathbb{E}[(\bar{x} - X_2)^k] \quad \text{for } k = 1, 2, \ldots, m-1,$$

and

$$v_{\ge}(x; X_1 \le_{m\text{-}cv} X_2) \le \frac{1}{\Delta} \int_x^{\bar{x}} \left(F_{X_1}^{(m-1)}(\xi) - F_{X_2}^{(m-1)}(\xi)\right) \mathrm{d}\xi, \quad x \in \mathcal{X};$$

(ii) $g \in m\text{-}icv^{-}(\Delta)$ *for some* $\Delta \in [0, \infty)$ *if and only if*

$$v_{\le}(x; X_1 \le_{m\text{-}cv} X_2) \le \frac{1}{\Delta} \int_{\underline{x}}^{x} \left(F_{X_1}^{(m-1)}(\xi) - F_{X_2}^{(m-1)}(\xi)\right) \mathrm{d}\xi, \quad x \in \mathcal{X};$$

and

(iii) $g \in m\text{-}icv^{\pm}(\Delta_+, \Delta_-)$ *for some* $\Delta_+, \Delta_- \in [0, \infty)$ *if and only if*

$$\mathbb{E}[(\bar{x} - X_1)^k] \ge \mathbb{E}[(\bar{x} - X_1)^k] \quad \text{for } k = 1, 2, \ldots, m-1,$$

and for any $x \in \mathcal{X}$,

$$\frac{1}{1+\Delta_-}\left(\int_{\underline{x}}^{x} \left(F_{X_2}^{(m-1)}(\xi) - F_{X_1}^{(m-1)}(\xi)\right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\le}(x; X_1 \le_{m\text{-}cv} X_2)\right)$$
$$+ \frac{1}{1+\Delta_+}\left(\int_{x}^{\bar{x}} \left(F_{X_2}^{(m-1)}(\xi) - F_{X_1}^{(m-1)}(\xi)\right) \mathrm{d}\xi - (\Delta_- \wedge \Delta_+) v_{\ge}(x; X_1 \le_{m\text{-}cv} X_2)\right) \ge 0. \tag{25}$$

6. Concluding remark

We present a unified approach to parameterizing functions to characterize parametric integral stochastic orders. The analysis of first-order (i.e. increasingness) and second-order (i.e. convexity and concavity) functional properties is generalized to higher-order ones to characterize various stochastic orders. The parameter used to restrict functional properties also quantifies the violation to the distributional properties of the associated non-parametric orders. Among the functions discussed, $inc(\Delta)$ and $inc^{-}(\Delta)$ are most analyzed with their applications to utility functions in decision analysis. Closure properties and applications of many parametric stochastic orders are yet to be explored.

Although we compare random variables X_1 and X_2 with a common support, this assumption is not essential to our analysis. In particular, Theorems 4, 7, and 8 can be easily modified for random variables with different supports using the relations

$$\bar{F}_{X_i}^{(n-1)}(\underline{x}_i) = \mathbb{E}[(X_i - \underline{x}_i)^n] \quad \text{and} \quad F_{X_i}^{(n-1)}(\bar{x}_i) = \mathbb{E}[(\bar{x}_i - X_i)^n],$$

where $\underline{x}_i$ and $\bar{x}_i$, respectively, are the minimum and maximum realizations of X_i, $i = 1, 2$.

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There are certainly other ways of defining integral stochastic orders as discussed in Section 4.2. For example, Azmoodeh and Hür [1] consider a form of $icv^{-}(\Delta)$ with $\Delta(x)$, $x \in \mathcal{X}$. Light and Perlroth [11] name the α, $[\underline{x}, \overline{x}]$-concave order by considering $g \in inc$ with $(g(\overline{x}) - g(x))^{1/\alpha}$ being convex. One may also look into alternative ways of parameterizing other function classes (e.g. interpolation between linear and log-concave functions).

There can be ways to parameterize an order that does not result in an integral form. For example, Stoyanov et al. [22] propose metric (distance) representation to measure the degree of dominance or violation of the usual stochastic orders and the increasing concave order. Lizyayev and Ruszczyński [12] analyze almost ordered random variables with one being the sum of the other and some random variable with a bounded mean. Methods for systematically parameterizing non-integral orders are important directions for future research.

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