Ergod. Th. & Dynam. Sys. (2018), **38**, 371–383 doi:10.1017/etds.2016.24
© *Cambridge University Press, 2016*

A circle diffeomorphism with breaks that is absolutely continuously linearizable

ALEXEY TEPLINSKY

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivska Street, Kiev 01601, Ukraine (e-mail: teplinsky@imath.kiev.ua, teplinsky.a@gmail.com)

(*Received* 22 *June* 2015 *and accepted in revised form* 18 *February* 2016)

Abstract. In this paper we answer positively to a question of whether it is possible for a circle diffeomorphism with breaks to be smoothly conjugate to a rigid rotation in the case where its breaks are lying on pairwise distinct trajectories. An example constructed is a piecewise linear circle homeomorphism that has four break points lying on distinct trajectories and whose invariant measure is absolutely continuous with respect to the Lebesgue measure. The irrational rotation number for our example can be chosen to be a Roth number, but not of bounded type.

1. *Introduction*

Any orientation-preserving homeomorphism T of the unit circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ with an irrational rotation number $\rho = \rho(T)$ is uniquely ergodic [**9**], i.e. it possesses a unique invariant probability measure $\mu = \mu(T)$. If T is piecewise C^1 and the total variation of $\log T'$ is finite, then by the renowned Denjoy's theorem it is topologically conjugate to the rigid (linear) rotation $R_\rho : \xi \mapsto \xi + \rho$, i.e. there exists a homeomorphism ϕ (which is unique up to an additive constant) of $\mathbb{T}^1$ such that

$$\phi \circ T \circ \phi^{-1} = R_\rho. \tag{1}$$

The invariant measure μ and the conjugacy (i.e. the linearizing change of variables) ϕ are related by the equality $\phi(\xi_1) - \phi(\xi_0) = \int_{\xi_0}^{\xi_1} d\mu(\xi)$ for any arc $[\xi_0, \xi_1] \subset \mathbb{T}^1$. The question of how smooth this conjugacy ϕ is, depending on the smoothness of T, has been studied extensively since 1960s.

The case where T is a diffeomorphism is covered by Herman's theory and its later developments [**8**, **10**, **11**].

Quite different is the case where T is a diffeomorphism with breaks, i.e. it is piecewise C^1, but not C^1 (and we still assume that the total variation of $\log T'$ is finite). Let us call the 'size of the break' the ratio of derivatives at the break point from the left and from

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

the right. Since the first paper [**4**], in which it was shown that the invariant measure of a circle diffeomorphism with a single break of size not equal to 1 is singular (with respect to the Lebesgue measure ℓ), there have appeared a number of publications proving the singularity of the invariant measure in different cases. Recently, two groups of scientists have independently proved [**1**, **6**] the following most general result: if the product of sizes of all breaks is not equal to 1, then the invariant measure is singular.

The case of unit product of sizes of breaks appeared to be difficult to investigate. Let us notice at this point that several breaks lying on the same trajectory are in a sense equivalent to one single break of a size that is the product of their sizes; therefore, without loss of generality one may consider only diffeomorphisms with breaks that lie on pairwise distinct trajectories. It is shown in [**8**] (although not formulated as a statement) that a piecewise linear diffeomorphism with two breaks (the product of their sizes is equal to 1 automatically) has singular invariant measure unless the break points lie on the same trajectory. In [**1**, **5**], it was proved that for a diffeomorphism with multiple breaks lying on distinct trajectories, the invariant measure is singular provided that the rotation number is of bounded type (i.e. its continued fraction expansion is a bounded sequence). Generally, there was a colloquial belief among specialists on the subject that the invariant measure is singular with respect to ℓ if for at least one trajectory the product of sizes of all breaks it contains is not equal to 1.

In this paper, we disprove that conjecture, presenting an example of a piecewise linear homeomorphism of the unit circle that has four breaks lying on distinct trajectories, but is nevertheless absolutely continuously linearizable. Its rotation number can be chosen to be a Roth number (a Diophantine number with any positive Diophantine exponent), but not of bounded type. The linearizing conjugacy is *essentially* absolutely continuous in the sense that it is not piecewise C^1, possessing an everywhere-dense set of points where it is not differentiable.

The example we construct is also interesting in a framework of rigidity theory for circle diffeomorphisms with breaks [**12**], which asks a more general question of whether two such given maps are smoothly conjugate. It was proved in [**3**] that, in the case of break equivalence of the two maps (a property assuming that a conjugacy sends each break point of the first map to a break point of the second one with the same size of break), the product of sizes of breaks equal to 1 and bounded combinatorics (this notion generalizes bounded type of rotation number), the conjugacy is C^1. On the other hand, it was shown very recently [**2**, **7**] that if two circle diffeomorphisms with breaks with the same irrational rotation number have different products of sizes of breaks, then every conjugacy between them is a singular function. Our example suggests that in the case of the equal products, the condition of generalized break equivalence (i.e. a break equivalence adjusted by a piecewise smooth change of coordinates) is not necessary for absolute continuity of the conjugacy.

The mechanics beyond our example can be explained in the following way. When the breaks of inverse sizes lie on the same trajectory, they annihilate each other. In our case, the trajectories of such break points are distinct but come very close to each other (a kind of a Liouvillian behaviour, although the rotation number can be Diophantine) and therefore manage to compensate each other enough for making the invariant measure

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

absolutely continuous. It was known for a long time (Arnold's counterexample) that Liouvillian effects can result in an invariant measure being singular even for analytic circle diffeomorphisms. In our example it works in the opposite direction: Liouvillian-type effects make the invariant measure absolutely continuous.

In §2, we define our example for given sizes of breaks and rotation number and formulate the theorem about absolute continuity of its invariant measure with respect to ℓ. In §3, we prove the theorem and make some additional comments.

2. *Construction*

Let us fix an arbitrary number $a > 1$ and take an arbitrary increasing sequence of positive integers $k_1, k_2, \ldots, k_n, \ldots$, where we assume $k_1 \geq 2a$, $k_1 \geq 5$, and $\sum_{n=1}^{+\infty} k_n^{-1} < +\infty$. The irrational number given by the continued fraction

$$\rho = [k_1, k_2, \ldots, k_n, \ldots] = 1/(k_1 + 1/(k_2 + 1/(\cdots/(k_n + \cdots)))),$$

which is defined as a limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \ldots, k_n] = 1/(k_1 + 1/(k_2 + 1/(\cdots/k_n)))$, will be the rotation number of our circle homeomorphism. It is known that the mutually prime positive integers p_n and q_n satisfy the recurrence relation

$$p_n = k_n p_{n-1} + p_{n-2}, \quad q_n = k_n q_{n-1} + q_{n-2} \quad n \geq 1,$$

where it is convenient to define $p_0 = 0$, $q_0 = 1$ and $p_{-1} = 1$, $q_{-1} = 0$. Accordingly, the decreasing sequence of quantities $\Delta_n = |q_n\rho - p_n| = (-1)^n(q_n\rho - p_n)$ satisfies the relation

$$\Delta_n = k_{n+2}\Delta_{n+1} + \Delta_{n+2} \quad n \geq -1.$$

LEMMA 1. *There exist a sequence of positive numbers d_n, $n \geq -1$, with $d_{-1} = 1$, which satisfy the relations*

$$d_{2n} = k_{2n+2}d_{2n+1} + d_{2n+2}, \quad d_{2n-1} = k_{2n+1}d_{2n} + d_{2n+1} + (a-1)\delta_{2n}, \tag{2}$$

where

$$\delta_{2n} = d_{2n+2} + a\delta_{2n+4} = \sum_{s=0}^{+\infty} a^s d_{2n+4s+2} < +\infty; \tag{3}$$

and a number $\alpha > 1$ such that the following inequality holds:

$$0 \leq \log \frac{\alpha d_n}{\Delta_n} < \frac{2(a-1)}{k_{n+2}}. \tag{4}$$

Proof. Let us put $d_n^{(0)} = \Delta_n$, $n \geq -1$, and define recursively $d_n^{(r+1)} = k_{n+2}d_{n+1}^{(r)} + d_{n+2}^{(r)} + (a-1)\delta_n^{(r)}$ for $n \geq -1$, $r \geq 0$, with $\delta_n^{(r)} = \sum_{s=0}^{+\infty} a^s d_{n+4s+2}^{(r)}$ for n even, $\delta_n^{(r)} = 0$ for n odd. It is obvious that $d_n^{(r)}$ and $\delta_n^{(r)}$ are non-decreasing with respect to r for any fixed n.

At the moment we cannot state that all defined quantities are finite, but it will be proven shortly. Since we have $\Delta_n = k_{n+2}\Delta_{n+1} + \Delta_{n+2} = (k_{n+2}k_{n+3} + 1)\Delta_{n+2} + k_{n+2}\Delta_{n+3} > k_{n+2}k_{n+3}\Delta_{n+2}$, $n \geq -1$, it follows that $\Delta_{n+4s+2}/\Delta_n < 1/(k_{n+2}\cdots k_{n+4s+3})$, $s \geq 0$, and

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

$$\frac{\delta_n^{(0)}}{\Delta_n} < \sum_{s=0}^{+\infty} \frac{a^s}{k_{n+2}\cdots k_{n+3+4s}} < \frac{2}{k_{n+2}k_{n+3}} \tag{5}$$

(here we used the assumed lower bound $2a$ for all k_n).

It is easy to see from the definition that (as soon as $d_n^{(r)} < +\infty$ for all $n \geq -1$)

$$\frac{d_n^{(r+1)}}{d_n^{(r)}} \leq \sup_{s\geq n+1} \frac{d_s^{(r)}}{d_s^{(r-1)}} \leq \cdots \leq \sup_{s\geq n+r} \frac{d_s^{(1)}}{d_s^{(0)}}.$$

Since $d_s^{(1)} = \Delta_s + (a-1)\delta_s^{(0)}$, $d_s^{(0)} = \Delta_s$, and $(k_n)_n$ is strictly increasing, the two latter displayed inequalities give us an estimate:

$$\frac{d_n^{(r+1)}}{d_n^{(r)}} < 1 + \frac{2(a-1)}{k_{n+r+2}k_{n+r+3}}.$$

It follows that

$$\begin{aligned}\log \frac{d_n^{(r+1)}}{\Delta_n} &= \sum_{s=0}^{r} \log \frac{d_n^{(s+1)}}{d_n^{(s)}} < \sum_{s=0}^{r} \frac{2(a-1)}{k_{n+s+2}k_{n+s+3}} \\ &< 2\sum_{s=0}^{r}\left(\frac{a-1}{k_{n+s+2}} - \frac{a-1}{k_{n+s+3}}\right) < \frac{2(a-1)}{k_{n+2}}.\end{aligned}$$

In particular, all $d_n^{(r)}$ and $\delta_n^{(r)}$ are finite. Moreover, the whole set $\{d_n^{(r)}\}_{n,r}$ is bounded. Therefore, there exist finite limits $d_n^* = \lim_{r\to+\infty} d_n^{(r)}$, $\delta_n^* = \lim_{r\to+\infty} \delta_n^{(r)}$, $n \geq -1$, that satisfy

$$d_n^* = k_{n+2}d_{n+1}^* + d_{n+2}^* + (a-1)\delta_n^*, \quad \delta_{2n}^* = \sum_{s=0}^{+\infty} a^s d_{2n+4s+2}^*.$$

By scaling them all down by putting $d_n = d_n^*/d_{-1}^*$, $n \geq -1$, we obtain the sequence that satisfies the conditions of the lemma with $\alpha = d_{-1}^*$. □

We assume $\delta_n = 0$ for n odd.

Notice that (2) and (3) imply the inequalities

$$\frac{d_{n+2}}{d_n} < \frac{1}{k_{n+2}k_{n+3}}, \quad \frac{\delta_{2n}}{d_{2n}} < \frac{2}{k_{2n+2}k_{2n+3}}. \tag{6}$$

Now let us produce our main example of a piecewise linear circle homeomorphism T through its lift function $f_0 : \mathbb{R} \to \mathbb{R}$, $f_0(x+1) \equiv f_0(x) + 1$, given by

$$f_0(x) = \begin{cases} ax + d_0 & 0 \leq x \leq \delta_0, \\ x + d_0 + (a-1)\delta_0 & \delta_0 \leq x \leq d_0 + a\delta_2, \\ a^{-1}x + (2 - a^{-1})\, d_0 + (a-1)\delta_0 + (a-1)\delta_2 & d_0 + a\delta_2 \leq x \leq d_0 + a\delta_0 + a\delta_2, \\ x + d_0 & d_0 + a\delta_0 + a\delta_2 \leq x \leq 1. \end{cases}$$

(It easily follows from (6) that $d_0 + a\delta_0 + a\delta_2 < 1$.) As one can see, $f_0(1) = 1 + d_0 = f_0(0) + 1$ indeed, and, thus defined, T has four breaks: two of size a and two of size a^{-1}. In the next section we will prove the following result.

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

THEOREM 1. *The circle diffeomorphism with breaks T defined above has rotation number ρ; its four break points lie on pairwise distinct trajectories and its invariant measure μ is absolutely continuous with respect to the Lebesgue measure ℓ.*

In what follows, we will represent a continuous piecewise linear function defined on a segment by indicating the values of its derivative over those subsegments where it differs from 1, rather than expressions for the function itself; this makes the formulas much more intuitive. (Such information together with a value at a single point lets one restore the whole function.) For example:

$$f_0'(x) = \begin{cases} a & \text{for } x \in (0,\, \delta_0), \\ a^{-1} & \text{for } x \in (d_0 + a\delta_2,\, d_0 + a\delta_0 + a\delta_2), \\ 1 & \text{elsewhere on } [0,1], \end{cases}$$

and this information together with the value $f_0(0) = d_0$ determines f_0 on [0, 1].

3. *Proof of the claim*

3.1. *Renormalization basics.* For analyzing metrical properties of the dynamical system induced on $\mathbb{T}^1$ by the homeomorphism T, we will apply the renormalization approach (described in detail in [**12**], for example) as follows.

Given a circle homeomorphism T with irrational rotation number ρ, one may 'mark' a point $\xi_0 \in \mathbb{T}^1$, consider its trajectory $\xi_i = \xi_i(\xi_0) = T^i\xi_0 \in \mathbb{T}^1$, $i \in \mathbb{Z}$, and pick out of it the sequence of the 'dynamical convergents' $\{\xi_{q_n}\}_n$ indexed by the denominators of the consecutive rational convergents to ρ. The dynamical convergents approach the marked point ξ_0, alternating their order in the following way:

$$\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \cdots < \xi_{q_{2m+1}} < \cdots < \xi_0 < \cdots < \xi_{q_{2m}} < \cdots < \xi_{q_2} < \xi_{q_0}$$

(the point $\xi_{q_{-1}}$ on the circle coincides with ξ_0; here it may be seen as $\xi_{q_{-1}} = \xi_0 - 1$). Define the nth 'fundamental segment' $\Delta_0^{(n)} = \Delta_0^{(n)}(\xi_0)$ as the circle arc $[\xi_0, \xi_{q_n}]$ if n is even and as $[\xi_{q_n}, \xi_0]$ if n is odd. The iterates T^{q_n} and $T^{q_{n-1}}$ restricted to $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ respectively are two continuous components of the first-return map for T on the segment $\overline{\Delta}_0^{(n-1)} = \Delta_0^{(n-1)} \cup \Delta_0^{(n)}$. The consecutive images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ until the return to $\overline{\Delta}_0^{(n-1)}$ cover the whole circle without overlapping beyond their endpoints, thus forming the nth 'dynamical partition' $\mathcal{P}_n(\xi_0) = \{\Delta_i^{(n-1)},\, 0 \le i < q_n\} \cup \{\Delta_i^{(n)},\, 0 \le i < q_{n-1}\}$ of $\mathbb{T}^1$, where $\Delta_i^{(n)} = \Delta_i^{(n)}(\xi_0)$ stands for $T^i\Delta_0^{(n)}$.

Given $n \ge 0$, one may consider the pair of functions $(f_{n,\xi_0},\, g_{n,\xi_0})$, called sometimes the nth 'pre-renormalization' of T, which are the mappings T^{q_n} and $T^{q_{n-1}}$ restricted to $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, respectively, in the coordinate system with the origin at ξ_0:

$$f_{n,\xi_0} = \pi_{\xi_0} \circ T^{q_n} \circ \pi_{\xi_0}^{-1}, \quad g_{n,\xi_0} = \pi_{\xi_0} \circ T^{q_{n-1}} \circ \pi_{\xi_0}^{-1},$$

where π_{ξ_0} is a linear mapping in a neighbourhood of ξ_0 that sends ξ_0 to 0 and changes neither length nor orientation. We will omit the index ξ_0 when it is clear which point is marked. Both f_n and g_n are strictly increasing continuous functions defined on the segments $[-\ell(\Delta_0^{(n-1)}),\, 0]$ and $[0,\, \ell(\Delta_0^{(n)})]$ respectively for n even and

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

on $[0, \ell(\Delta_0^{(n-1)})]$ and $[-\ell(\Delta_0^{(n)}), 0]$ respectively for n odd. Since $f_n(0) = (-1)^n \ell(\Delta_0^{(n)})$, $g_n(0) = (-1)^{n-1}\ell(\Delta_0^{(n-1)})$ and $f_n(g_n(0)) = g_n(f_n(0))$, the functions f_n and g_n are said to form a 'commuting pair'. (Notice that our pairs (f_n, g_n) differ from those in [**12**] by having not been rescaled.)

This definition is slightly ambiguous for $n = 0$, in which case we define f_0 on the segment $[-1, 0]$ representing the unit circle $\mathbb{T}^1$ cut by the marked point ξ_0 (one can write $\Delta_0^{(-1)} = [\xi_0 - 1, \xi_0]$) so that f_0 is a lift of T onto $\mathbb{R}$, shifted to the marked point, while $g_0 = \mathbb{I}\mathrm{d} - 1$ defined on $[0, f_0(0)]$.

Notice that g_{n+1} is f_n restricted to a smaller segment, while $f_{n+1} = f_n^{k_{n+1}} \circ g_n$, and the 'renormalization height' k_{n+1} is the largest integer k such that $f_n^k(g_n(0))$ has the same sign as $g_n(0)$ (it is also a partial quotient in the continued fraction $\rho = [k_1, k_2, \ldots, k_n, \ldots]$), $n \geq 0$.

In this paper, we shall also consider the 'backward dynamical partition segments' $\nabla_i^{(n)} = T^{-i}\nabla_0^{(n)}$, where $\nabla_0^{(n)} = \nabla_0^{(n)}(\xi_0)$ is the circle arc $[\xi_0, \xi_{-q_n}]$ if n is odd and $[\xi_{-q_n}, \xi_0]$ if n is even, and define the 'backward pre-renormalization' as the pair of functions

$$\check{f}_n = \pi_{\xi_0} \circ T^{-q_n} \circ \pi_{\xi_0}^{-1}, \quad \check{g}_n = \pi_{\xi_0} \circ T^{-q_{n-1}} \circ \pi_{\xi_0}^{-1}$$

restricted to $\nabla_0^{(n-1)}$ and $\nabla_0^{(n)}$ respectively. These functions have properties similar to f_n and g_n, in particular $\check{f}_{n+1} = \check{f}_n^{k_{n+1}} \circ \check{g}_n$.

3.2. *Measurements.* Now, let us turn to the specific homeomorphism T that was defined at the end of §2 (notice that the meaning of f_0 there is consistent with §3.1 for $\xi_0 = 0$, though the expressions were written for the segment $[0, 1]$ rather than $[-1, 0]$).

PROPOSITION 1. *For every $n \geq 0$ we have $\ell(\Delta_0^{(n)}) = d_n$, and for every $n \geq 1$,*

$$g'_{2n-1}(x) = 1 \quad \textit{everywhere on } [-d_{2n-1}, 0],$$

$$f'_{2n-1}(x) = \begin{cases} a & \textit{for } x \in (0, \delta_{2n}), \\ a^{-1} & \textit{for } x \in (\delta_{2n-2}, \delta_{2n-2} + a\delta_{2n}), \\ 1 & \textit{elsewhere on } [0, d_{2n-2}], \end{cases}$$

$$f'_{2n}(x) = \begin{cases} a^{-1} & \textit{for } x \in (-d_{2n-1} + a\delta_{2n+2}, -d_{2n-1} + a\delta_{2n+2} + a\delta_{2n}), \\ 1 & \textit{elsewhere on } [-d_{2n-1}, 0], \end{cases}$$

$$g'_{2n}(x) = \begin{cases} a & \textit{for } x \in (0, \delta_{2n}), \\ 1 & \textit{elsewhere on } [0, d_{2n}]. \end{cases}$$

The rotation number of T is $\rho = [k_1, k_2, \ldots, k_n, \ldots]$.

Remark 1. When we state that a formula like those above holds, it is a part of our statement that the segments in it are positioned correctly (for example, the formula for f_{2n-1} includes the claim that the segments $(0, \delta_{2n})$ and $(\delta_{2n-2}, \delta_{2n-2} + a\delta_{2n})$ do not overlap and are both contained within the segment $[0, d_{2n-2}]$). Such correctness is usually easy to check by (6) and the assumed lower bounds on k_n.

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

Proof. On each induction step $m \geq 1$, one must follow the sequence of points

$$g_{m-1}(0),\ f_{m-1}(g_{m-1}(0)),\ \dots,\ f_{m-1}^{k}(g_{m-1}(0)),\ \dots$$

until the moment right before it jumps over zero, and at each moment k one must calculate the function $f_{m-1}^{k} \circ g_{m-1}$ by composing a previously calculated $f_{m-1}^{k-1} \circ g_{m-1}$ with f_{m-1} on an appropriate segment. The last moment will be the renormalization height k_m (a partial quotient in the continued fraction expansion for the rotation number of T), while the function $f_{m-1}^{k_m} \circ g_{m-1}$ will be the new f_m.

For $m = 1$, the segment $[g_0(0),\ f_0(g_0(0))]$ has length d_0. The length of the segment $[f_0(g_0(0)),\ f_0^2(g_0(0))]$ is $d_0 + (a-1)\delta_0$ because of stretching by the 'upper tooth' of derivative $f_0'(x) = a$ on $(-1,\ -1+\delta_0) \subset [g_0(0),\ f_0(g_0(0))] = [-1,\ -1+d_0]$. The length of $[f_0^{k-1}(g_0(0)),\ f_0^{k}(g_0(0))]$ for $k > 2$ is again d_0 because of shrinking back by the 'lower tooth' of derivative $f_0'(x) = a^{-1}$ on $(-1+d_0+a\delta_2,\ -1+d_0+a\delta_0+a\delta_2) \subset [f_0(g_0(0)),\ f_0^2(g_0(0))] = [-1+d_0,\ -1+2d_0+\delta_0]$. The renormalization height is k_1, and $f_1(0) = f_0^{k_1}(g_0(0)) = -d_1$, both by (2). The two mentioned teeth of derivative not equal to 1 partially annihilate each other, and their leftovers appear as the appropriate teeth in the new $f_1 = f_0^{k_1} \circ g_0$ on $[0,\ d_0]$.

For $m = 2n$, $n \geq 1$, the lengths of all segments $[f_{2n-1}^{k}(g_{2n-1}(0)),\ f_{2n-1}^{k-1}(g_{2n-1}(0))]$ is equal to d_{2n-1}. The renormalization height is k_{2n}, and $f_{2n}(0) = f_{2n-1}^{k_{2n}}(g_{2n-1}(0)) = d_{2n}$, both by (2). The only 'tooth' of derivative not equal to 1, which is $f_{2n-1}'(x) = a^{-1}$ on $(\delta_{2n-2},\ \delta_{2n-2}+a\delta_{2n}) \subset [d_{2n},\ d_{2n}+d_{2n-1}] = [f_{2n-1}^{k_{2n}}(g_{2n-1}(0)),\ f_{2n-1}^{k_{2n}-1}(g_{2n-1}(0))]$, appears at the last moment, $k = k_{2n}$, and is being transferred to the segment $[-d_{2n-1},\ 0]$ where the new f_{2n} is defined. It will be exactly the tooth $f_{2n}'(x) = a^{-1}$ on $(-d_{2n-1}+a\delta_{2n+2},\ -d_{2n-1}+a\delta_{2n+2}+a\delta_{2n})$ because of the relation $\delta_{2n-2} - d_{2n} = a\delta_{2n+2}$ implied by (3).

For $m = 2n+1$, $n \geq 1$, it is similar to the case of $m = 1$: all the segments $[f_{2n}^{k-1}(g_{2n}(0)),\ f_{2n}^{k}(g_{2n}(0))]$, $k \geq 1$, but one have lengths d_{2n} with the only difference being that the exceptional segment of length $d_{2n} + (a-1)\delta_{2n}$ appears for $k = 1$ rather than for $k = 2$. The upper tooth $g_{2n}'(x) = a$ on $(0,\ \delta_{2n})$ interacts with the lower tooth $f_{2n}' = a^{-1}$ on $(-d_{2n-1}+a\delta_{2n+2},\ -d_{2n-1}+a\delta_{2n+2}+a\delta_{2n}) \subset [g_{2n}(0),\ f_{2n}(g_{2n}(0))] = [-d_{2n-1},\ -d_{2n-1}+d_{2n}+(a-1)\delta_{2n}]$ giving birth to the appropriate teeth in new f_{2n+1}. The renormalization height is k_{2n+1}, and $f_{2n+1}(0) = f_{2n}^{k_{2n+1}}(g_{2n}(0)) = -d_{2n+1}$, both by (2). □

PROPOSITION 2. *For every $n \geq 0$, we have $\ell(\nabla_0^{(n)}) = d_n$,*

$$\check{g}_{2n}'(x) = 1 \quad \textit{everywhere on } [-d_{2n},\ 0],$$

$$\check{f}_{2n}'(x) = \begin{cases} a^{-1} & \textit{for } x \in (d_{2n},\ d_{2n}+a\delta_{2n}), \\ a & \textit{for } x \in (2d_{2n}+(a-1)\delta_{2n}+a\delta_{2n+2},\ 2d_{2n}+a\delta_{2n}+a\delta_{2n+2}), \\ 1 & \textit{elsewhere on } [0,\ d_{2n-1}], \end{cases}$$

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

$$\check{f}'_{2n+1}(x) = \begin{cases} a^{-1} & \text{for } x \in (-d_{2n+1},\, -d_{2n+1} + a\delta_{2n+2}), \\ a & \text{for } x \in (-d_{2n+1} + a\delta_{2n+2} + \delta_{2n} - \delta_{2n+2},\, -d_{2n+1} + a\delta_{2n+2} + \delta_{2n}), \\ 1 & \text{elsewhere on } [-d_{2n},\, 0], \end{cases}$$

$$\check{g}'_{2n+1}(x) = 1 \quad \text{everywhere on } [0,\, d_{2n+1}].$$

Proof. First we calculate $\check{f}_0$ as f_0^{-1} restricted to $[0, 1]$; then we analyze consecutive iterates similarly to the proof of Proposition 1. □

Define the sequence of sets $\Theta_n = \bigcup_{i \in N_n} \nabla_i^{(n)} \subset \mathbb{T}^1$, $n \geq 0$, formed of backward dynamical partition segments, inductively as follows: let $N_0 = \{0 \leq i \leq k_1 - 3\}$ and $N_n = \{i + q_{n-1} + jq_n : i \in N_{n-1}, 0 \leq j \leq k_{n+1} - 3\}$, $n \geq 1$. It is easy to see that $N_n \subset \{0 \leq i < q_{n+1}\}$ and $\Theta_{n+1} \subset \Theta_n$, $n \geq 0$. Saying it in words, we consecutively cut away little pieces of the circle nearby every preimage of the break point ξ_0.

LEMMA 2. *For every $i \in N_n$, $n \geq 0$, we have $\ell(\nabla_i^{(n)}) = d_n$.*

Proof. The definition of N_n implies that for every $i \in N_n$ there exist integers $0 \leq j_{m+1} \leq k_{m+1} - 3$, $0 \leq m \leq n$, such that $i = \sum_{m=0}^{n} (q_{m-1} + j_{m+1}q_m)$. Accordingly, $\nabla_i^{(n)} = (\check{f}_0^{j_1} \circ \check{g}_0) \circ \cdots \circ (\check{f}_n^{j_{n+1}} \circ \check{g}_n)\nabla_0^{(n)}$. In view of Proposition 2, it is easy to check that consecutive images of $\nabla_0^{(n)}$ in the latter sequence never fall onto the teeth of the derivatives of $\check{f}_m$ and $\check{g}_m$ not equal to 1. □

PROPOSITION 3. *The limit set $\Theta = \bigcap_{n \geq 0} \Theta_n$ has positive Lebesgue measure.*

Proof. It follows from Lemma 2 that $\ell(\Theta_n) = d_n \cdot \#(N_n)$, and by definition of N_n we have $\#(N_n) = (k_{n+1} - 2) \cdot \#(N_{n-1})$. Hence, $\ell(\Theta_n) = (((k_{n+1} - 2)d_n)/d_{n-1})\ell(\Theta_{n-1})$. The relations (2) and (6) and the increasing of k_n imply that $d_{n-1} < (k_{n+1} + 1)d_n$ and therefore $\ell(\Theta_n) > (1 - 3/(k_{n+1} + 1))\ell(\Theta_{n-1}) > \exp(-6/k_{n+1})\ell(\Theta_{n-1})$ for large enough n. The statement follows from assumed convergence of the series $\sum_n k_n^{-1}$. □

PROPOSITION 4. *Let $\eta_0 \in \Theta$. Then for every $n \geq 0$, we have $\ell(\Delta_0^{(n)}(\eta_0)) = d_n$,*

$$f'_{2n,\eta_0}(x) = \begin{cases} a & \text{for } x \in (b_{2n},\, b_{2n} + \delta_{2n}), \\ a^{-1} & \text{for } x \in (b_{2n} + d_{2n} + a\delta_{2n+2},\, b_{2n} + d_{2n} + a\delta_{2n} + a\delta_{2n+2}), \\ 1 & \text{elsewhere on } [-d_{2n-1},\, 0], \end{cases}$$

$$g'_{2n,\eta_0}(x) = 1 \quad \text{everywhere on } [0,\, d_{2n}],$$

$$g'_{2n+1,\eta_0}(x) = 1 \quad \text{everywhere on } [-d_{2n+1},\, 0],$$

$$f'_{2n+1,\eta_0}(x) = \begin{cases} a & \text{for } x \in (b_{2n+1},\, b_{2n+1} + \delta_{2n+2}), \\ a^{-1} & \text{for } x \in (b_{2n+1} + \delta_{2n},\, b_{2n+1} + \delta_{2n} + a\delta_{2n+2}), \\ 1 & \text{elsewhere on } [0,\, d_{2n}], \end{cases}$$

with certain values $b_{2n} \in (-d_{2n-1},\, f_{2n}^{k_{2n+1}-2}(-d_{2n-1}))$, $b_{2n+1} \in [f_{2n+1}^{k_{2n+2}-2}(d_{2n}),\, d_{2n})$.

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

Proof. It is easy to follow the sequence of iterates like we did in the proof of Proposition 1, this time using η_0 as the marked point. What is important here are the bounds on values $(b_n)_{n\geq 0}$ that follow directly from our construction of the set Θ. Indeed, b_n is nothing else but a coordinate of an appropriate preimage of the break point ξ_0 in vicinity of η_0. Since passing from Θ_{n-1} to Θ_n, we have cut away from every $\nabla_i^{(n-1)}$, $i \in N_{n-1}$, three adjoint segments $\nabla^{(n)}_{i+q_{n-1}+(k_{n+1}-2)q_n}$, $\nabla^{(n)}_{i+q_{n-1}+(k_{n+1}-1)q_n}$, and $\nabla_i^{(n+1)}$; therefore $b_{2n} \notin (f_{2n}^{k_{2n+1}-2}(-d_{2n-1}), 0)$ and $b_{2n+1} \notin (0, f_{2n+1}^{k_{2n+2}-2}(d_{2n}))$. □

3.3. *Differentiability.* In what follows, all the renormalization structures are counted from η_0 as the marked point, and we omit the letter η_0 for f_n, g_n, $\mathcal{P}_n$ and $\Delta_i^{(n)}$.

PROPOSITION 5. *The conjugacy ϕ from (1) is differentiable at every point $\eta_0 \in \Theta$.*

First we need to prove the following lemma about the lengths of segments from dynamical partitions of several deeper levels lying within the fundamental segments. Denote by $\Gamma_{n,s}$ the set of all segments $\Delta_i^{(n+s-1)} \in \mathcal{P}_{n+s}$, which lie within the segment $\Delta_0^{(n-1)}$.

LEMMA 3. $(\ell(\Delta_i^{(n+s-1)}))/d_{n+s-1} \to 1$ *as* $n \to +\infty$ *uniformly in* i *for all* $\Delta_i^{(n+s-1)} \in \Gamma_{n,s}$, $0 \leq s \leq 2$, *and for all those* $\Delta_i^{(n+2)} \in \Gamma_{n,3}$, *which lie within* $\Delta^{(n)}_{q_{n+1}-q_{n-1}}$.

Proof. In this proof, for simplicity we identify a set M on the circle in the vicinity of η_0 with its projection $\pi_{\eta_0} M$ to the real line, where f_n and g_n operate.

For $s = 0$ the statement is trivial, since $\Gamma_{n,0} = \{\Delta_0^{(n-1)}\}$, and $\ell(\Delta_0^{(n-1)}) = d_{n-1}$ by Proposition 4.

It is easy to see that $\Gamma_{n,s} = \Gamma_{n+2,s-2} \cup \bigcup_{j=0}^{k_{n+1}-1} f_n^j \circ g_n(\Gamma_{n+1,s-1})$. We will use this recurrence relation in order to estimate the lengths of $\Delta_i^{(n+s-1)} \in \Gamma_{n,s}$ for $s \geq 1$. As seen from Proposition 4, g_n does not distort the length of any segment, while f_n may once stretch and once shrink it, with the maximal possible change in length equal to $(a-1)\delta_n$ for n even and $(a-1)\delta_{n+1}$ for n odd.

For $s = 1$, the segments in $\Gamma_{n,1}$ are the images of $\Delta_0^{(n)}$ under application of g_n (once) and then f_n (from 0 to $k_{n+1}-1$ times). Using Proposition 4, one can easily check that all of them have length d_n for n odd, while for n even, all but one have length d_n, with the exceptional one having length $d_n + (a-1)\delta_n$.

For $s = 2$, the segments in $\Gamma_{n,2} \setminus \Gamma_{n+2,0}$ are the images of the segments in $\Gamma_{n+1,1}$ under $f_n^j \circ g_n$, $0 \leq j < k_{n+1}$. Proposition 4 allows one to calculate that their lengths are either d_{n+1} or $d_{n+1} + (a-1)\delta_n$ for n even, and either d_{n+1} or $d_{n+1} + (a-1)\delta_{n+1}$ for n odd.

For $s = 3$, the segments in $\Gamma_{n,3} \setminus \Gamma_{n+2,1}$ are the images of the segments in $\Gamma_{n+1,2}$ under $f_n^j \circ g_n$, $0 \leq j < k_{n+1}$. If n is odd, then their lengths differ from d_{n+2} by not more than $2(a-1)\delta_{n+1}$. If n is even, then this difference formally can reach $(a-1)\delta_{n+2} + (a-1)\delta_n$, but let us look closer at this case. Notice that for n even, the teeth of the derivative of f_n not equal to 1 are positioned in such a way that the maximal possible change in length of a segment under their *cumulative* action (i.e. the action of f_n^2 on $\Delta_i^{(n)} \in \mathcal{P}_{n+1}$ such that $b_n \in \Delta_i^{(n)}$) is $(a-1)\delta_{n+2}$ rather than $(a-1)\delta_n$. In fact, there are exactly two segments

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

in $\Gamma_{n,3}$ that are possibly stretched by more than $2(a-1)\delta_{n+2}$: those whose preimages under f_n intersect the upper tooth $(b_{2n}, b_{2n}+\delta_{2n})$ (while those segments themselves intersect the lower tooth and are being shrunk back on the next application of f_n). It also follows from Proposition 4 that these two exceptional segments do not intersect $[f_n^{k_{n+1}-1}(-d_{n-1}), 0] = \Delta^{(n)}_{q_{n+1}-q_{n-1}}$.

Since $\delta_n = o(d_{n+1})$ (in fact $\delta_{2n}/d_{2n+2} \to 1$ as $n \to +\infty$), the estimates we proved imply the statement of the lemma. □

Proof of Proposition 5. We will show directly that $\ell([\phi(\eta), \phi(\eta_0)])/\ell([\eta, \eta_0]) \to \alpha$ as $\eta \to \eta_0$, and therefore $\phi'(\eta_0) = \alpha$ for every $\eta_0 \in \Theta$ (during this proof, by $[\xi, \zeta]$ we denote the smallest of the two arcs $[\xi, \zeta]$ and $[\zeta, \xi]$ on $\mathbb{T}^1$).

As η tends to η_0 from the left (from the right), it passes through the sequence of segments $\dot\Delta_0^{(n-1)} = \Delta_0^{(n-1)} \setminus \Delta_0^{(n+1)} \cup \{\eta_{q_{n+1}}\}$ with n even (odd), which in turn consist of segments from the dynamical partitions of the next levels. According to the decomposition $\dot\Delta_0^{(n-1)} = \bigcup_{j_0=0}^{k_{n+1}-1}(\Delta^{(n+2)}_{q_{n-1}+(k_{n+1}-1-j_0)q_n} \cup \bigcup_{j_1=0}^{k_{n+2}-1} \Delta^{(n+1)}_{q_{n-1}+(k_{n+1}-j_0)q_n+j_1q_{n+1}})$, consider three cases.

Case 1: $\eta \in \Delta^{(n+1)}_{q_{n-1}+(k_{n+1}-j_0)q_n+j_1q_{n+1}}$, $1 \le j_0 \le k_{n+1}-1$ (we have excluded $j_0 = 0$, which will be Case 3 below), $0 \le j_1 \le k_{n+2}-1$. On one hand, the segment $[\eta, \eta_0]$ contains j_0 segments from $\Gamma_{n,1}$ (namely $\Delta^{(n)}_{q_{n-1}+iq_n}$, $k_{n+1}-j_0 \le i \le k_{n+1}-1$) and j_1+1 segments from $\Gamma_{n,2}$ (namely, $\Delta^{(n+1)}_{q_{n-1}+(k_{n+1}-j_0)q_n+iq_{n+1}}$, $0 \le i < j_1$, and $\Delta_0^{(n+1)}$). On another hand, it is contained in the union of j_0 segments from $\Gamma_{n,1}$ (the same as previously) and j_1+2 segments from $\Gamma_{n,2}$ (with $\Delta^{(n+1)}_{q_{n-1}+(k_{n+1}-j_0)q_n+j_1q_{n+1}}$ added). By Lemma 3 and the limit $d_n/\Delta_n \to 1/\alpha$ implied by Lemma 1, the following estimate holds with some $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$:

$$e^{-\varepsilon_n}\alpha^{-1}(j_0\Delta_n + (j_1+1)\Delta_{n+1}) \le \ell([\eta, \eta_0]) \le \alpha^{-1}(j_0\Delta_n + (j_1+2)\Delta_{n+1})e^{\varepsilon_n}.$$

Since $\ell(\phi(\Delta_i^{(n)})) = \Delta_n$ for every i, $n \ge 0$, we similarly obtain that

$$j_0\Delta_n + (j_1+1)\Delta_{n+1} \le \ell([\phi(\eta), \phi(\eta_0)]) \le j_0\Delta_n + (j_1+2)\Delta_{n+1}.$$

Taking the ratio of the two latter estimates, we find that

$$\begin{aligned}\left|\log \frac{\ell([\phi(\eta), \phi(\eta_0)])}{\alpha\ell([\eta, \eta_0])}\right| &\le \varepsilon_n + \log \frac{j_0\Delta_n + (j_1+2)\Delta_{n+1}}{j_0\Delta_n + (j_1+1)\Delta_{n+1}} \\ &\le \varepsilon_n + \frac{\Delta_{n+1}}{j_0\Delta_n + (j_1+1)\Delta_{n+1}} \\ &\le \varepsilon_n + \frac{\Delta_{n+1}}{\Delta_n} \le \varepsilon_n + \frac{1}{k_{n+2}} \to 0 \quad n \to +\infty.\end{aligned}$$

Case 2: $\eta \in \Delta^{(n+2)}_{q_{n-1}+(k_{n+1}-1-j_0)q_n}$, $0 \le j_0 \le k_{n+1}-1$. By similar consideration we get the estimates

$$\begin{aligned}e^{-\varepsilon_n}\alpha^{-1}(j_0\Delta_n + (k_{n+2}+1)\Delta_{n+1}) \le \ell([\eta, \eta_0]) \le \alpha^{-1}((j_0+1)\Delta_n + \Delta_{n+1})e^{\varepsilon_n}, \\ j_0\Delta_n + (k_{n+2}+1)\Delta_{n+1} \le \ell([\phi(\eta), \phi(\eta_0)]) \le (j_0+1)\Delta_n + \Delta_{n+1},\end{aligned}$$

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$, which imply

$$\left|\log \frac{\ell([\phi(\eta), \phi(\eta_0)])}{\alpha \ell([\eta, \eta_0])}\right| \le \varepsilon_n + \frac{\Delta_n - k_{n+2}\Delta_{n+1}}{j_0\Delta_n + (k_{n+2}+1)\Delta_{n+1}} \le \varepsilon_n + \frac{\Delta_{n+2}}{\Delta_n} \le \varepsilon_n + \frac{1}{k_{n+2}k_{n+3}},$$

with the same conclusion.

Case 3: $\eta \in \Delta^{(n+1)}_{q_{n-1}+k_{n+1}q_n+j_1q_{n+1}} = \Delta^{(n+1)}_{(j_1+1)q_{n+1}}$, $0 \le j_1 \le k_{n+2}-1$. The segment $[\eta, \eta_0]$ contains j_1+1 segments from $\Gamma_{n,2}$ (namely, $\Delta^{(n+1)}_{iq_{n+1}}$, $0 \le i \le j_1$), and some $0 \le j_2 \le k_{n+3}-1$ segments from $\Gamma_{n,3}$ (namely, $\Delta^{(n+2)}_{(j_1+2)q_{n+1}+iq_{n+2}}$, $k_{n+3}-j_2 \le i < k_{n+3}$). On the other hand, it is contained within the union of j_1+1 segments from $\Gamma_{n,2}$ (the same as previously), j_2+1 segments from $\Gamma_{n,3}$ (with added $\Delta^{(n+2)}_{(j_1+2)q_{n+1}+(k_{n+3}-j_2-1)q_{n+2}}$) and $\Delta^{(n+3)}_{(j_1+1)q_{n+1}}$ from $\Gamma_{n,4}$. Notice that all the segments from $\Gamma_{n,3}$ mentioned here do lie within $\Delta^{(n)}_{q_{n+1}-q_{n-1}}$, so that the uniform limit in Lemma 3 holds for them. For the tiny segment $\Delta^{(n+3)}_{(j_1+1)q_{n+1}}$ it will suffice to use a rough estimate $\ell(\Delta^{(n+3)}_{(j_1+1)q_{n+1}}) \le a^2 d_{n+3}$ that follows from the fact that it is an image of $\Delta^{(n+3)}_0$ under $f_n^{k_{n+1}-1} \circ g_n \circ f_{n+1}^{j_1} \circ g_{n+1}$, where only one step of applying f_{n+1} and only one step of applying f_n can stretch the length (and with the factor of not more than a) according to Proposition 4. We reach the estimates

$$\begin{aligned} e^{-\varepsilon_n}\alpha^{-1}((j_1+1)\Delta_{n+1} + j_2\Delta_{n+2}) &\le \ell([\eta, \eta_0]) \\ &\le \alpha^{-1}((j_1+1)\Delta_{n+1} + (j_2+1)\Delta_{n+2} + a^2\Delta_{n+3})e^{\varepsilon_n} \\ &\le \alpha^{-1}((j_1+1)\Delta_{n+1} + (j_2+1)\Delta_{n+2})(1 + a^2/k_{n+2})e^{\varepsilon_n}, \end{aligned}$$

with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to +\infty$, and

$$\begin{aligned} (j_1+1)\Delta_{n+1} + j_2\Delta_{n+2} &\le \ell([\phi(\eta), \phi(\eta_0)]) \\ &\le (j_1+1)\Delta_{n+1} + (j_2+1)\Delta_{n+2} + \Delta_{n+3}, \\ &\le ((j_1+1)\Delta_{n+1} + (j_2+1)\Delta_{n+2})(1 + 1/k_{n+2}). \end{aligned}$$

Finally, we obtain

$$\left|\log \frac{\ell([\phi(\eta), \phi(\eta_0)])}{\alpha \ell([\eta, \eta_0])}\right| \le \varepsilon_n + \frac{a^2}{k_{n+2}} + \frac{\Delta_{n+2}}{(j_1+1)\Delta_{n+1} + j_2\Delta_{n+2}} \le \varepsilon_n + \frac{a^2}{k_{n+2}} + \frac{1}{k_{n+3}},$$

which tends to zero as well. □

3.4. *Conclusions.*

Proof of Theorem 1. The theorem contains three statements, which we will prove now.

(1) The rotation number of T is ρ, by Proposition 1.

(2) Four break points lie on pairwise distinct trajectories, by Proposition 4. Indeed, by the construction, the break points of f_n are nothing else but the projections of first preimages of break points of T within the segment $\Delta^{(n-1)}_0$. If two of the latter points lay on the same trajectory, then for large enough n their first preimages within $\Delta^{(n-1)}_0$ would coincide, which contradicts the fact that f_n has four distinct break points for any n.

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

(3) For a circle diffeomorphism with breaks, its invariant measure μ can be either absolutely continuous, or singular with respect to ℓ (see, for example, [**9**], where this result is proved for pure diffeomorphisms and is easily extended to those with breaks). If μ is singular, then the conjugacy ϕ is a singular function, and $\phi' = 0$ almost everywhere with respect to ℓ, which contradicts Propositions 3 and 5. Therefore, μ is absolutely continuous. □

Remark 2. The linearizing change of variables ϕ in (1) is absolutely continuous and therefore possesses a density $h \in L_1(\mathbb{T}^1)$. This density is positive almost everywhere with respect to ℓ because of the fact that T is ergodic with respect to ℓ (see [**9**]).

Remark 3. The conjugacy ϕ is *essentially* absolutely continuous, i.e. it is not piecewise C^1, moreover, there is an everywhere-dense subset of $\mathbb{T}^1$ over which ϕ is not differentiable. Indeed, it follows from (1) that if, for some $\xi \in \mathbb{T}^1$, any two of three derivatives $\phi'(\xi)$, $\phi'(T(\xi))$ and $T'(\xi)$ exist, then the third one exists as well, and therefore T for any of its break points ξ_{br} is necessarily non-differentiable either along the whole set $T^i(\xi_{br})$, $i \geq 1$, or along the whole set $T^i(\xi_{br})$, $i \leq 0$, or over both of these sets, and both of them are everywhere dense in $\mathbb{T}^1$.

Remark 4. An irrational number ρ is said to belong to the Diophantine class D_δ if there exists a constant $C > 0$ such that $|\rho - p/q| \geq Cq^{-2-\delta}$ for any rational number p/q. The set of Roth numbers is $R = \bigcap_{\delta>0} D_\delta$. The numbers from D_0 are those whose continued fraction expansion is formed by a bounded sequence $(k_n)_n$; therefore they are called 'numbers of bounded type'. The restrictions we have imposed on $(k_n)_n$ allow one to choose $\rho \in R$ (for example, $k_n = (n+5)^2$ with $a = 2$ will work), but not $\rho \in D_0$.

Acknowledgements. The author is grateful to A. Dzhalilov for useful discussions and to a referee for helpful suggestions.

References

[**1**] A. Adouani and H. Marzougui. Singular measures for class P-circle homeomorphisms with several break points. *Ergod. Th. & Dynam. Sys.* **34** (2014), 423–456.
[**2**] A. Adouani. Conjugation between circle maps with severeal break points. *Ergod. Th. & Dynam. Sys.*, to appear, doi:10.1017/etds.2015.32.
[**3**] K. Cunha and D. Smania. Rigidity for piecewise smooth homeomorphisms on the circle. *Adv. Math.* **250** (2014), 193–226.
[**4**] A. A. Dzhalilov and K. M. Khanin. On an invariant measure for homeomorphisms of a circle with a point of break. *Funct. Anal. Appl.* **32** (1998), 153–161.
[**5**] A. Dzhalilov and I. Liousse. Circle homeomorphisms with two break points. *Nonlinearity* **19** (2006), 1951–1968.
[**6**] A. A. Dzhalilov, D. Mayer and U. A. Safarov. Piecewise-smooth circle homeomorphisms with several break points. *Izv. Math.* **76** (2012), 94–112.
[**7**] A. Dzhalilov, D. Mayer and U. Safarov. Conjugacies between P-homeomorphisms with several breaks. *Preprint*, 2014, arXiv:1408.5732 [math.DS], http://iopscience.iop.org/article/10.1088/0951-7715/28/7/2441.
[**8**] M.-R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. Inst. Hautes Études Sci.* **49** (1979), 5–233.

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press

[**9**] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1997.
[**10**] Y. Katznelson and D. Ornstein. The differentiability of the conjugation of certain diffeomorphisms of the circle. *Ergod. Th. & Dynam. Sys.* **9** (1989), 643–680.
[**11**] K. Khanin and A. Teplinsky. Herman's theory revisited. *Invent. Math.* **178** (2009), 333–344.
[**12**] K. Khanin and A. Teplinsky. Renormalization horseshoe and rigidity for circle diffeomorphisms with breaks. *Comm. Math. Phys.* **320** (2013), 347–377.

https://doi.org/10.1017/etds.2016.24 Published online by Cambridge University Press