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Some Inequalities for Whitney–Tutte Polynomials

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We introduce and prove a family of inequalities satisfied by the Whitney rank generating function of a matroid in the positive quadrant of $\mathbb{R}^2$. These can be interpreted as correlation inequalities at those points where the polynomial is known to count the number of independent sets, bases or spanning sets of the matroid. Our proofs also introduce an idea of rank dominating bijections in matroids, which are then used to obtain some simple extensions of the submodular property of matroid ranks.

1. Introduction

The Tutte polynomial of a matroid is a two-variable polynomial well known to have connections to several natural combinatorial counting problems. First studied in its popular form by Tutte, it is based on the Whitney rank generating function for matroids. The *Whitney rank generating function* of a matroid $M(E,\rho)$ is defined by

$$R(M;x,y)=\sum_{X\subseteq E} x^{\rho(E)-\rho(X)}y^{|X|-\rho(X)}, \tag{1.1}$$

while the Tutte polynomial of M is

$$T(M;x,y)=R(M;x-1,y-1). \tag{1.2}$$

In this paper, we introduce and prove a family of inequalities for the Whitney rank generating function in the positive quadrant of $\mathbb{R}^2$. For a matroid $M(E,\rho)$ and $X\subseteq E$, we use $M|X$ to denote the matroid restriction of M to the set X [8, p. 92]. The main result of this paper is as follows.

Theorem 1.1. *Let $M(E,\rho)$ be a matroid, $E_1,E_2\subseteq E$ and $k=\rho(E_1)+\rho(E_2)-\rho(E_1\cup E_2)-\rho(E_1\cap E_2)$. If $|E_1\cap E_2|\leqslant 3$, then for all $x,y\geqslant 0$, $xy<1$,*

$$x^k\cdot R(M|E_1\cup E_2;x,y)\cdot R(M|E_1\cap E_2;x,y)\leqslant R(M|E_1;x,y)\cdot R(M|E_2;x,y). \tag{1.3}$$

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In our proofs, we introduce the idea of a *rank dominating bijection* and an extension of rank submodularity in matroids. We also show that the direction of the inequality is reversed in the region $x, y > 0, xy > 1$ (Proposition 4.2), while (1.3) becomes an equality along the curve $xy = 1$ (Proposition 4.1). These last two results are significantly easier to prove, and are shown to be true for all subsets $E_1, E_2 \subseteq E$.

The motivations for studying such inequalities are multifold. First, several combinatorial inequalities and conjectured inequalities for the Whitney–Tutte polynomials of matroids were discussed by Seymour and Welsh [10]. In particular, matroids with bases that satisfy a particular negative edge association property were called *balanced matroids* by Feder and Mihail [2], and they showed that this class includes all regular matroids. More recently, Grimmet and Winkler [4], Semple and Welsh [9], and Cocks [1] discussed classes of matroids for which the independent and spanning sets exhibit similar negative association properties. It is straightforward to show that if a matroid is balanced, independence-correlated or spanning-correlated, then it satisfies (1.3) for all subsets $E_1, E_2 \subseteq E$ at $(0, 0)$, $(0, 1)$ or $(1, 0)$, respectively. It is also known that the binary matroid known as S_8, which was introduced in [10], does not satisfy inequality (1.3) for some subsets $E_1, E_2 \subseteq E$. (Note that this does not contradict our result as $|E_1 \cap E_2| > 3$ in all such counter-examples.) This provides a theoretical motivation to study the effect of the particular choices of the sets E_1 and E_2 on the validity of (1.3) for matroids.

Second, computing the Tutte polynomial is known to be #P-hard, and Jaeger, Vertigan and Welsh [5] showed that even evaluating the polynomial at a specific point is #P-hard in general at almost all points in $\mathbb{Q}^2$. It is also known that when $x < -1$ or $y < -1$, most of these points do not admit good approximation schemes (see Goldberg and Jerrum [3]). However, the approximability of the polynomial in the positive quadrant of $\mathbb{Q}^2$ is still an open problem. The inequality (1.3) and related results in this paper suggest that for a matroid, $M(E)$, one way to compute upper or lower bounds of Tutte polynomial evaluations at specific points in the region $x, y \geqslant 1$ is by recursively computing these bounds for matroid restrictions of M to some proper subsets of the ground set E. This provides a computational motivation to study such inequalities.

Finally, using the notation $\langle E \rangle$ to denote a graph on the edge set E, recall the following identity for the chromatic polynomial of a graph:

$$P(\langle E_1 \cup E_2 \rangle; \lambda) = \frac{P(\langle E_1 \rangle; \lambda) \cdot P(\langle E_2 \rangle; \lambda)}{P(\langle E_1 \cap E_2 \rangle; \lambda)},$$

for all $E_1, E_2 \subseteq E$ such that $\langle E_1 \cap E_2 \rangle$ is a complete graph. We note that the results in this paper lead to similar inequalities for counting forests, spanning trees and connected subgraphs of a graph. In particular, Theorem 1.1 implies that if $F(\langle E \rangle)$ is the number of forests of graph $\langle E \rangle$, then

$$F(\langle E_1 \cup E_2 \rangle) \leqslant \frac{F(\langle E_1 \rangle) \cdot F(\langle E_2 \rangle)}{F(\langle E_1 \cap E_2 \rangle)},$$

for all $E_1, E_2 \subseteq E$ such that $|E_1 \cap E_2| \leqslant 3$. Similarly, if $S(\langle E \rangle)$ counts either the number of spanning trees or connected subgraphs of $\langle E \rangle$ and $\rho : 2^E \to \mathbb{Z}_{\geqslant 0}$ is its rank function,

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then

$$S(\langle E_1 \cup E_2 \rangle) \leqslant \frac{S(\langle E_1 \rangle) \cdot S(\langle E_2 \rangle)}{S(\langle E_1 \cap E_2 \rangle)},$$

for all modular pairs of subsets $E_1, E_2 \subseteq E$ (that is, subset pairs E_1, E_2 such that $\rho(E_1 \cup E_2) + \rho(E_1 \cap E_2) = \rho(E_1) + \rho(E_2)$) whenever $|E_1 \cap E_2| \leqslant 3$. The family of inequalities discussed in this paper can also be seen as an extension of such relationships to the Whitney–Tutte polynomials of matroids.

The rest of the paper is organized as follows. The next section introduces some necessary definitions and notation, while Section 3 discusses rank dominations in matroids. Section 4 states and proves the main inequality results, and we conclude with a discussion on open questions in the last section. Throughout the paper we assume familiarity with basic matroid concepts and notation [8].

2. Definitions and notation

For three mutually disjoint sets, P_1, P_2, R, we first define a set, $S(P_1, P_2, R)$, of disjoint pairs as follows:

$$S(P_1, P_2, R) = \{(P_1 \cup C, P_2 \cup (R \setminus C)) : C \subseteq R\}. \tag{2.1}$$

Equivalently, the set $S(P_1, P_2, R)$ is the collection of partitions (X, Y) of the set $P_1 \cup P_2 \cup R$ subject to the constraints $P_1 \subseteq X$ and $P_2 \subseteq Y$. There is one such pair (X, Y) for every subset of R, and hence $|S(P_1, P_2, R)| = 2^{|R|}$.

For the sake of brevity, henceforth any reference to the set $S(P_1, P_2, R)$ will be understood to imply that the three sets P_1, P_2 and R are mutually disjoint.

Definition (Rank domination and equivalence in matroids). Let $M(E, \rho)$ be a matroid and $P_1, P_2, Q_1, Q_2, R \subseteq E$ such that (P_1, P_2, R) and (Q_1, Q_2, R) are triples of mutually disjoint sets. We say the set $S(P_1, P_2, R)$ is *rank dominated* by $S(Q_1, Q_2, R)$ in M if there exists a bijection $\pi : S(P_1, P_2, R) \to S(Q_1, Q_2, R)$ such that $\rho(W) + \rho(Z) \leqslant \rho(X) + \rho(Y)$ whenever $\pi(W, Z) = (X, Y)$. We write $S(P_1, P_2, R) \leqslant_M S(Q_1, Q_2, R)$ to denote such a relationship, omitting the subscript M when the matroid is clear from the context. The bijection π is called a *rank dominating bijection* in M.

Additionally, if $\rho(W) + \rho(Z) = \rho(X) + \rho(Y)$ whenever $\pi(W, Z) = (X, Y)$, we also say the set $S(P_1, P_2, R)$ is *rank equivalent* to $S(Q_1, Q_2, R)$ in M, and write $S(P_1, P_2, R) \equiv_M S(Q_1, Q_2, R)$ (omitting the subscript M if the matroid is clear from the context).

The submodular property of matroid rank functions provides a simple and very useful example of rank dominations for all matroids.

Example 2.1. *In any matroid $M(E, \rho)$, for all $P_1, P_2 \subseteq E$ we have $S(P_1 \cup P_2, \emptyset, \emptyset) \leqslant S(P_1, P_2, \emptyset)$. Indeed this is equivalent to the submodular property of matroid rank functions, $\rho(P_1 \cup P_2) \leqslant \rho(P_1) + \rho(P_2)$.*

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Considering the last example, we may ask for what sets $P_1, P_2, R \subseteq E$ is $S(P_1 \cup P_2, \emptyset, R) \leqslant S(P_1, P_2, R)$ in $M(E)$? We provide a partial answer to this question in the next section.

Clearly, if $S(P_1, P_2, R) \equiv S(Q_1, Q_2, R)$ in M, then $S(P_1, P_2, R) \leqslant S(Q_1, Q_2, R)$ and $S(Q_1, Q_2, R) \leqslant S(P_1, P_2, R)$ in M. We can immediately verify the following properties of rank domination and equivalence in matroids.

Proposition 2.2. *In any matroid $M(E)$, for all sets $P_1, P_2, P_1', P_2', Q_1, Q_2, Q_1', Q_2', T_1, T_2, R \subseteq E$, we have the following properties.*

(a) **Reflexivity.** $S(P_1, P_2, R) \equiv S(P_1, P_2, R)$. (*Thus rank domination is also reflexive.*)
(b) $S(P_1, P_2, R) \equiv S(P_2, P_1, R)$.
(c) **Transitivity.** *If $S(P_1, P_2, R) \leqslant S(Q_1, Q_2, R)$ and $S(Q_1, Q_2, R) \leqslant S(T_1, T_2, R)$, then $S(P_1, P_2, R) \leqslant S(T_1, T_2, R)$.* (*Hence rank equivalence is also transitive.*)
(d) **Symmetry of $\equiv$.** *If $S(P_1, P_2, R) \equiv S(Q_1, Q_2, R)$, then $S(Q_1, Q_2, R) \equiv S(P_1, P_2, R)$.*
(e) **Antisymmetry of $\leqslant$.** *$S(P_1, P_2, R) \leqslant S(Q_1, Q_2, R)$ and $S(Q_1, Q_2, R) \leqslant S(P_1, P_2, R)$ if and only if $S(P_1, P_2, R) \equiv S(Q_1, Q_2, R)$.*
(f) *Let $S(P_1, P_2, R) \equiv S(P_1', P_2', R)$ and $S(Q_1, Q_2, R) \equiv S(Q_1', Q_2', R)$. If $S(P_1, P_2, R) \leqslant S(Q_1, Q_2, R)$ with a unique rank dominating bijection in M, then also $S(P_1', P_2', R) \leqslant S(Q_1, Q_2, R)$ and $S(P_1, P_2, R) \leqslant S(Q_1', Q_2', R)$ in M with unique rank dominating bijections.*

Proof. **(a)** The identity map is the required rank dominating bijection.

(b) Since $S(P_2, P_1, R) = \{(Y, X) : (X, Y) \in S(P_1, P_2, R)\}$, we use the bijection $\pi(X, Y) = (Y, X)$.

(c) This can be proved using the composition of two rank dominating bijections.

(d) Follows from the definition of rank equivalence.

(e) We only have to show that $S(P_1, P_2, R) \leqslant S(Q_1, Q_2, R)$ and $S(Q_1, Q_2, R) \leqslant S(P_1, P_2, R)$ implies $S(P_1, P_2, R) \equiv S(Q_1, Q_2, R)$. Suppose that $S(P_1, P_2, R) \not\equiv S(Q_1, Q_2, R)$. Then, since $S(P_1, P_2, R) \leqslant S(Q_1, Q_2, R)$, we must have

$$\sum_{(W,Z) \in S(P_1, P_2, R)} (\rho(W) + \rho(Z)) < \sum_{(X,Y) \in S(Q_1, Q_2, R)} (\rho(X) + \rho(Y)),$$

which contradicts $S(Q_1, Q_2, R) \leqslant S(P_1, P_2, R)$.

(f) $S(P_1', P_2', R) \leqslant S(Q_1, Q_2, R)$ in M follows from transitivity. Suppose for a contradiction that there were at least two distinct rank dominating bijections π_1' and π_2' from the set $S(P_1', P_2', R)$ to $S(Q_1, Q_2, R)$. Also, let $\pi : S(P_1, P_2, R) \to S(P_1', P_2', R)$ be a rank dominating bijection in M. Then it is easy to check that $\pi \circ \pi_1'$ and $\pi \circ \pi_2'$ are two distinct rank dominating bijections in M from $S(P_1, P_2, R)$ to $S(Q_1, Q_2, R)$, which is a contradiction.

The argument for the case $S(P_1, P_2, R) \leqslant S(Q_1', Q_2', R)$ is similar. □

Definition (R-family of minors). Let $M(E)$ be a matroid and $R \subseteq E$. We define the *R-family of minors* of M, $\mathcal{MF}(M, R)$, to be

$$\mathcal{MF}(M, R) = \{M/C \setminus \overline{C} : C \subseteq R, \overline{C} = R \setminus C\}. \tag{2.2}$$

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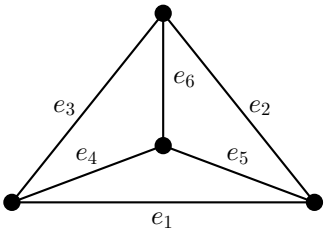


Figure 1. Graph K_4.

That is, the R-family of minors is the set of all minors of M obtained by deleting or contracting every element in R. Note that $|\mathcal{MF}(M,R)| = 2^{|R|}$.

Finally, if N is a minor of matroid $M(E,\rho)$, we write ρ_N to denote the rank function of N.

3. Rank dominations in matroids

The relationship between rank dominations and inequality (1.3) is explained by the following main theorem of the paper.

Theorem 3.1. *Let $M(E,\rho)$ be a matroid, $E_1, E_2 \subseteq E$, and $k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$. If $S(P_1 \cup P_2, \emptyset, R) \leqslant_N S(P_1, P_2, R)$ for all sets $P_1 \subseteq E_1 \setminus E_2, P_2 \subseteq E_2 \setminus E_1$, $R \subseteq E_1 \cap E_2$ and minors $N \in \mathcal{MF}(M, (E_1 \cap E_2) \setminus R)$, then*

$$x^k \cdot R(M|E_1 \cup E_2; x, y) \cdot R(M|E_1 \cap E_2; x, y) \leqslant R(M|E_1; x, y) \cdot R(M|E_2; x, y),$$

whenever $x, y \geqslant 0$ and $xy < 1$.

We note that the above theorem does not rely on the condition $|E_1 \cap E_2| \leqslant 3$, and thus can find wider use. We give a proof of Theorem 3.1 in the next section.

The main result of this section is the following extension to rank submodularity (see Example 2.1), which along with Theorem 3.1 proves Theorem 1.1.

Proposition 3.2. *Let $M(E)$ be a matroid and $P_1, P_2, R \subseteq E$. If $|R| \leqslant 3$ then $S(P_1 \cup P_2, \emptyset, R) \leqslant S(P_1, P_2, R)$ in M.*

We quickly note that, in general, Proposition 3.2 cannot be extended for $|R| \geqslant 4$, and we give the following counter-example when $|R| = 4$.

Example 3.3. *Consider the graphic matroid $M(K_4)$ of the complete graph on four vertices, K_4, with the edges labelled as shown in Figure 1. If we let $P_1 = \{e_1\}$, $P_2 = \{e_6\}$ and $R = \{e_2, e_3, e_4, e_5\}$, then it is easy to check that $S(P_1 \cup P_2, \emptyset, R) \not\leqslant S(P_1, P_2, R)$ in $M(K_4)$.*

Before we can prove Proposition 3.2, we need several other results. We begin with the following simple and useful observation.

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Lemma 3.4. *Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be two sets, and $v : A \cup B \to \mathbb{Z}_{\geqslant 0}$ be a function such that $v(a_1) + v(a_2) \leqslant v(b_1) + v(b_2)$, $|v(a_1) - v(a_2)| \leqslant 1$ and $|v(b_1) - v(b_2)| \leqslant 1$. Then there exists a bijection $\pi : A \to B$ such that $v(a) \leqslant v(b)$ whenever $\pi(a) = b$.*

Furthermore, this bijection is unique if and only if $v(a_1) + v(a_2) = v(b_1) + v(b_2)$ and $|v(a_1) - v(a_2)| = |v(b_1) - v(b_2)| = 1$.

Lemma 3.4 can be verified by case checking, considering the two cases $v(a_1) = v(a_2)$ and $v(a_1) = v(a_2) - 1$ separately, and we skip the details. The first of our rank domination results is an immediate consequence of Lemma 3.4.

Proposition 3.5. *Let $M(E, \rho)$ be a matroid, $r \in E$ and $W, X, Y, Z \subseteq E$. If $S(W, Z, \emptyset) \leqslant_N S(X, Y, \emptyset)$ in all minors $N \in \mathcal{MF}(M, \{r\})$, then $S(W, Z, \{r\}) \leqslant_M S(X, Y, \{r\})$.*

Proof. From (2.1),

$$S(W, Z, \{r\}) = \{(W \cup \{r\}, Z), (W, Z \cup \{r\})\},$$
$$S(X, Y, \{r\}) = \{(X \cup \{r\}, Y), (X, Y \cup \{r\})\}.$$

Also, from (2.2), $\mathcal{MF}(M, \{r\}) = \{M \setminus r, M/r\}$. Let $N_1 = M \setminus r$ and $N_2 = M/r$. Using $\rho_{N_1}(A) = \rho(A)$, $\rho_{N_2}(A) = \rho(A \cup \{r\}) - \rho(\{r\})$ for all $A \subseteq E \setminus \{r\}$, and $S(W, Z, \emptyset) \leqslant_{N_i} S(X, Y, \emptyset)$ for $i = 1, 2$, we get

$$\rho(W) + \rho(Z) \leqslant \rho(X) + \rho(Y), \tag{3.1}$$
$$\rho(W \cup \{r\}) + \rho(Z \cup \{r\}) \leqslant \rho(X \cup \{r\}) + \rho(Y \cup \{r\}). \tag{3.2}$$

Now let

$$a_1 = (W \cup \{r\}, Z), \quad v(a_1) = \rho(W \cup \{r\}) + \rho(Z),$$
$$a_2 = (W, Z \cup \{r\}), \quad v(a_2) = \rho(W) + \rho(Z \cup \{r\}),$$
$$b_1 = (X \cup \{r\}, Y), \quad v(b_1) = \rho(X \cup \{r\}) + \rho(Y),$$
$$b_2 = (X, Y \cup \{r\}), \quad v(b_2) = \rho(X) + \rho(Y \cup \{r\}).$$

Since $\rho(A) \leqslant \rho(A \cup \{r\}) \leqslant \rho(A) + 1$ for all $A \subseteq E \setminus \{r\}$, it is easy to verify $|v(a_1) - v(a_2)| \leqslant 1$ and $|v(b_1) - v(b_2)| \leqslant 1$. Also, adding (3.1) and (3.2) we find $v(a_1) + v(a_2) \leqslant v(b_1) + v(b_2)$. Thus, from Lemma 3.4, there exists a rank dominating bijection $\pi : S(W, Z, \{r\}) \to S(X, Y, \{r\})$ in M. □

We next state the conditions for uniqueness of rank dominating bijections in matroids when $|R| = 1$, which will be useful for establishing many of our rank domination results in this paper.

Proposition 3.6. *Let $W, X, Y, Z \subseteq E$, $r \in E$ and $S(W, Z, \{r\}) \leqslant S(X, Y, \{r\})$ in $M(E, \rho)$. Then the rank dominating bijection from $S(W, Z, \{r\})$ to the set $S(X, Y, \{r\})$ in M is unique if and only if*

(a) $\rho(W \cup \{r\}) + \rho(Z) + \rho(W) + \rho(Z \cup \{r\}) = \rho(X \cup \{r\}) + \rho(Y) + \rho(X) + \rho(Y \cup \{r\})$,
(b) $|\rho(W \cup \{r\} + \rho(Z) - \rho(W) - \rho(Z \cup \{r\})| = 1$, *and*
(c) $|\rho(X \cup \{r\} + \rho(Y) - \rho(X) - \rho(Y \cup \{r\})| = 1$.

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Proof. Let

$$\begin{aligned} a_1 &= (W \cup \{r\}, Z), \quad v(a_1) = \rho(W \cup \{r\}) + \rho(Z), \\ a_2 &= (W, Z \cup \{r\}), \quad v(a_2) = \rho(W) + \rho(Z \cup \{r\}), \\ b_1 &= (X \cup \{r\}, Y), \quad v(b_1) = \rho(X \cup \{r\}) + \rho(Y), \\ b_2 &= (X, Y \cup \{r\}), \quad v(b_2) = \rho(X) + \rho(Y \cup \{r\}), \end{aligned}$$

As in our last proof, it can be checked that $|v(a_1) - v(a_2)| \leqslant 1$ and $|v(b_1) - v(b_2)| \leqslant 1$. Also, since $S(W, Z, \{r\}) \leqslant S(X, Y, \{r\})$, we have $v(a_1) + v(a_2) \leqslant v(b_1) + v(b_2)$. So by the uniqueness condition of Lemma 3.4, there is a unique rank dominating bijection between sets $S(W, Z, \{r\})$ and $S(X, Y, \{r\})$ in M if and only if $v(a_1) + v(a_2) = v(b_1) + v(b_2)$, $|v(a_1) - v(a_2)| \leqslant 1$ and $|v(b_1) - v(b_2)| \leqslant 1$. □

Our next series of results extends Proposition 3.5 from $|R| = 1$ to $|R| \leqslant 3$ (Lemma 3.10). We begin by identifying some conditions when different minors of a matroid have identical rank dominating bijections.

Recall that the *closure* of $X \subseteq E$ in the matroid $M(E, \rho)$ is the set $\mathrm{cl}_M(X) = \{e : \rho(X \cup \{e\}) = \rho(X)\}$.

Lemma 3.7. *Let $N_1 = M/C_1 \setminus C_1'$ and $N_2 = M/C_2 \setminus C_2'$ be two minors of matroid $M(E, \rho)$ for sets $C_1, C_1', C_2, C_2' \subseteq E$ with $C_1 \subseteq C_2$. If there is an $r \in E$ and subsets $W, X, Y, Z \subseteq E$ such that $S(W, Z, \{r\}) \leqslant_{N_i} S(X, Y, \{r\})$ for $i = 1, 2$, then there exists a bijection $\pi_r : S(W, Z, \{r\}) \to S(X, Y, \{r\})$ that is rank dominating in both N_1 and N_2.*

Proof. Suppose for a contradiction that there is no such common rank dominating bijection $\pi_r : S(W, Z, \{r\}) \to S(X, Y, \{r\})$ in minors N_1 and N_2.

Let $\pi_1, \pi_2 : S(W, Z, \{r\}) \to S(X, Y, \{r\})$ be rank dominating bijections in the matroids N_1 and N_2, respectively, with $\pi_1 \neq \pi_2$. As $|S(W, Z, \{r\})| = 2$, there are exactly two bijections between the sets $S(W, Z, \{r\})$ and $S(X, Y, \{r\})$. Clearly then π_1 and π_2 are the unique dominating bijections from $S(W, Z, \{r\})$ to $S(X, Y, \{r\})$ in N_1 and N_2, respectively.

Then, by Propositions 3.6(b) and 3.6(c), we have for $i = 1, 2$:

$$\begin{aligned} |\rho_{N_i}(W \cup \{r\}) + \rho_{N_i}(Z) - \rho_{N_i}(W) - \rho_{N_i}(Z \cup \{r\})| &= 1, \\ |\rho_{N_i}(X \cup \{r\}) + \rho_{N_i}(Y) - \rho_{N_i}(X) - \rho_{N_i}(Y \cup \{r\})| &= 1. \end{aligned}$$

Using Propositions 2.2(b) and 2.2(f), we can choose W, X, Y and Z such that, without loss of generality, the rank function of the minor N_1 satisfies

$$\rho_{N_1}(W \cup \{r\}) + \rho_{N_1}(Z) - \rho_{N_1}(W) - \rho_{N_1}(Z \cup \{r\}) = 1, \tag{3.3}$$

$$\rho_{N_1}(X \cup \{r\}) + \rho_{N_1}(Y) - \rho_{N_1}(X) - \rho_{N_1}(Y \cup \{r\}) = 1. \tag{3.4}$$

Now there are two possible cases for the minor N_2, given that π_2 is unique and $\pi_2 \neq \pi_1$.

Case 1. The sets W, X, Y, Z in N_2 satisfy

$$\begin{aligned} &\rho_{N_2}(W \cup \{r\}) + \rho_{N_2}(Z) - \rho_{N_2}(W) - \rho_{N_2}(Z \cup \{r\}) = 1, \\ &\rho_{N_2}(X) + \rho_{N_2}(Y \cup \{r\}) - \rho_{N_2}(X \cup \{r\}) - \rho_{N_2}(Y) = 1. \end{aligned} \tag{3.5}$$

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From (3.4) we can deduce $\rho_{N_1}(Y \cup \{r\}) = \rho_{N_1}(Y)$ and so $r \in \mathrm{cl}_{N_1}(Y)$. Equivalently, $r \in \mathrm{cl}_M(Y \cup C_1)$. But (3.5) gives $\rho_{N_2}(Y \cup \{r\}) = \rho_{N_2}(Y) + 1$ and thus $r \notin \mathrm{cl}_M(Y \cup C_2)$, which is a contradiction as $C_1 \subseteq C_2$.

Case 2. The sets W, X, Y, Z in N_2 satisfy

$$\begin{aligned} \rho_{N_2}(W) + \rho_{N_2}(Z \cup \{r\}) - \rho_{N_2}(W \cup \{r\}) - \rho_{N_2}(Z) = 1, \\ \rho_{N_2}(X \cup \{r\}) + \rho_{N_2}(Y) - \rho_{N_2}(X) - \rho_{N_2}(Y \cup \{r\}) = 1. \end{aligned} \tag{3.6}$$

An analogous argument using (3.3) and (3.6) on the set Z yields a contradiction in this case.

Hence it must be that at least one of π_1 and π_2 is not unique and there is a common rank dominating bijection $\pi_r : S(W, Z, \{r\}) \to S(X, Y, \{r\})$ in minors N_1 and N_2. □

For our next result we need the following two well-known properties of the closure set of a matroid.

Lemma 3.8. *Let $M(E)$ be a matroid.*

(a) *If $X \subseteq E$, $x \in E$ and $y \in \mathrm{cl}_M(X \cup \{x\}) \setminus cl_M(X)$, then $x \in \mathrm{cl}_M(X \cup \{y\})$ (Closure Exchange property* [8, p. 29]).

(b) *If $X \subseteq E$ and $x \in \mathrm{cl}_M(X)$, then $\mathrm{cl}_M(X \cup \{x\}) = \mathrm{cl}_M(X)$* ([8, p. 31]).

Lemma 3.9. *Let $M(E, \rho)$ be a matroid and $R, W, X, Y, Z \subseteq E$. If $1 \leqslant |R| \leqslant 3$ and $S(W, Z, \emptyset) \leqslant_N S(X, Y, \emptyset)$ in all minors $N \in \mathcal{MF}(M, R)$, then there exists an $r \in R$ and a bijection $\pi_r : S(W, Z, \{r\}) \to S(X, Y, \{r\})$ such that π_r is rank dominating in every minor $N_r \in \mathcal{MF}(M, R \setminus \{r\})$.*

Proof. Let $s \in R$. Note that

$$\mathcal{MF}(M, R) = \bigcup_{N_s \in \mathcal{MF}(M, R \setminus \{s\})} \mathcal{MF}(N_s, \{s\}).$$

Thus, by repeatedly applying Proposition 3.5 for each minor $N_s \in \mathcal{MF}(M, R \setminus \{s\})$, we find that $S(W, Z, \{s\}) \leqslant_{N_s} S(X, Y, \{s\})$. Also, since $|S(W, Z, \{s\})| = 2$, there are at most two choices for the corresponding rank dominating bijection in each of the minors, N_s.

We look at the three different cases of $|R|$ individually.

Case $|R| = 1$. The only minor in $\mathcal{MF}(M, R \setminus \{s\})$ is M itself and the result follows trivially by letting $r = s$.

Case $|R| = 2$. Let $R = \{s, t\}$. The two minors in $\mathcal{MF}(M, R \setminus \{s\})$ are $N_1 = M \setminus t$ and $N_2 = M/t$. Taking $C_1 = \emptyset$, $C_1' = \{t\}$, $C_2 = \{t\}$, $C_2' = \emptyset$ and $r = s$, the result follows from Lemma 3.7 in this case.

Case $|R| = 3$. Let $R = \{s, t, u\}$. In this case,

$$\mathcal{MF}(M, R \setminus \{s\}) = \{M \setminus t \setminus u, M/t \setminus u, M/u \setminus t, M/t/u\}. \tag{3.7}$$

Suppose minors $N_3 = M/t \setminus u$ and $N_4 = M/u \setminus t$ have a common rank dominating bijection from the set $S(W, Z, \{s\})$ to $S(X, Y, \{s\})$. Then, using Lemma 3.7 on the remaining

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five pairs of minors in (3.7), we can check that all minors in (3.7) will have a common rank dominating bijection between these two sets. That is, the choice $r = s$ will satisfy the lemma.

However, if these two minors, N_3 and N_4, did not have such a common rank dominating bijection, then it must be that they both have unique rank dominating bijections between sets $S(W, Z, \{s\})$ and $S(X, Y, \{s\})$. In such a case, we show that the choice $r = t$ will always satisfy the lemma.

Using Propositions 3.6(b) and 3.6(c), we obtain, for $i = 3, 4$:

$$|\rho_{N_i}(W \cup \{s\}) + \rho_{N_i}(Z) - \rho_{N_i}(W) - \rho_{N_i}(Z \cup \{s\})| = 1, \tag{3.8}$$

$$|\rho_{N_i}(X \cup \{s\}) + \rho_{N_i}(Y) - \rho_{N_i}(X) - \rho_{N_i}(Y \cup \{s\})| = 1. \tag{3.9}$$

First, using Propositions 2.2(b) and 2.2(f), without loss of generality, we can choose W, X, Y and Z such that the minor N_3 satisfies

$$\rho_{N_3}(W \cup \{s\}) + \rho_{N_3}(Z) - \rho_{N_3}(W) - \rho_{N_3}(Z \cup \{s\}) = 1, \tag{3.10}$$

$$\rho_{N_3}(X \cup \{s\}) + \rho_{N_3}(Y) - \rho_{N_3}(X) - \rho_{N_3}(Y \cup \{s\}) = 1. \tag{3.11}$$

However, given that the minors N_3 and N_4 do not have a common rank dominating bijection between $S(W, Z, \{s\})$ and $S(X, Y, \{s\})$, there are two different possibilities for the minor N_4.

Subcase 1. The rank function of N_4 is such that

$$\begin{aligned} &\rho_{N_4}(W \cup \{s\}) + \rho_{N_4}(Z) - \rho_{N_4}(W) - \rho_{N_4}(Z \cup \{s\}) = 1, \\ &\rho_{N_4}(X) + \rho_{N_4}(Y \cup \{s\}) - \rho_{N_4}(X \cup \{s\}) - \rho_{N_4}(Y) = 1. \end{aligned} \tag{3.12}$$

We now consider the minor family,

$$\mathcal{MF}(M, R \setminus \{t\}) = \{M \setminus s \setminus u, M/s \setminus u, M/u \setminus s, M/s/u\}.$$

Applying the same arguments used above for $\mathcal{MF}(M, R \setminus \{s\})$, we note that the choice $r = t$ will satisfy our lemma if the two minors $N_5 = M/s \setminus u$ and $N_6 = M/u \setminus s$ have a common rank dominating bijection from the set $S(W, Z, \{t\})$ to $S(X, Y, \{t\})$. Suppose for a contradiction that the minors N_5 and N_6 also fail to have such a common rank dominating bijection. This implies that these minors have unique rank dominating bijections between $S(W, Z, \{t\})$ and $S(X, Y, \{t\})$. Now, applying Proposition 3.6(c) on minors N_5 and N_6, we have the following four different scenarios.

Scenario 1. We have

$$\begin{aligned} &\rho_{N_5}(X \cup \{t\}) + \rho_{N_5}(Y) - \rho_{N_5}(X) - \rho_{N_5}(Y \cup \{t\}) = 1, \\ &\rho_{N_6}(X \cup \{t\}) + \rho_{N_6}(Y) - \rho_{N_6}(X) - \rho_{N_6}(Y \cup \{t\}) = 1. \end{aligned} \tag{3.13}$$

There are two possibilities in this scenario.

First, it could be that $t \in \mathrm{cl}_M(Y)$. Then, from (3.11) we know that $s \in \mathrm{cl}_M(Y \cup \{t\}) = \mathrm{cl}_M(Y)$ (using Lemma 3.8(b)), which contradicts (3.12).

Alternatively, we can have $t \notin \mathrm{cl}_M(Y)$. Since from (3.13) we know $t \in \mathrm{cl}_M(Y \cup \{u\})$, it must be that $u \in \mathrm{cl}_M(Y \cup \{t\})$ (using Lemma 3.8(a)), and we can apply Lemma 3.8(b)

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to get $\mathrm{cl}_M(Y \cup \{t\}) = \mathrm{cl}_M(Y \cup \{t,u\}) = \mathrm{cl}_M(Y \cup \{u\})$. But (3.11) gives $s \in \mathrm{cl}_M(Y \cup \{t\}) = \mathrm{cl}_M(Y \cup \{u\})$, which contradicts (3.12).

Scenario 2. We have

$$\rho_{N_5}(X \cup \{t\}) + \rho_{N_5}(Y) - \rho_{N_5}(X) - \rho_{N_5}(Y \cup \{t\}) = 1, \tag{3.14}$$
$$\rho_{N_6}(X) + \rho_{N_6}(Y \cup \{t\}) - \rho_{N_6}(X \cup \{t\}) - \rho_{N_6}(Y) = 1. \tag{3.15}$$

Then, from (3.14) we know $t \notin \mathrm{cl}_M(X \cup \{s\})$, while (3.15) implies $t \in \mathrm{cl}_M(X \cup \{u\})$. Hence, applying Lemma 3.8(a), $u \in \mathrm{cl}_M(X \cup \{t\})$, and further using Lemma 3.8(b), we can deduce $\mathrm{cl}_M(X \cup \{t\}) = \mathrm{cl}_M(X \cup \{t,u\}) = \mathrm{cl}_M(X \cup \{u\})$. But from (3.12) we find $s \in \mathrm{cl}_M(X \cup \{u\}) = \mathrm{cl}_M(X \cup \{t\})$, which contradicts (3.11).

Scenarios 3 and 4. We have

$$\rho_{N_5}(X) + \rho_{N_5}(Y \cup \{t\}) - \rho_{N_5}(X \cup \{t\}) - \rho_{N_5}(Y) = 1, \tag{3.16}$$
$$|\rho_{N_6}(X) + \rho_{N_6}(Y \cup \{t\}) - \rho_{N_6}(X \cup \{t\}) - \rho_{N_6}(Y)| = 1.$$

From (3.11) we know that $s \in \mathrm{cl}_M(Y \cup \{t\})$, while (3.16) implies $t \notin \mathrm{cl}_M(Y \cup \{s\})$. Hence, using Lemma 3.8(a), $s \notin \mathrm{cl}_M(Y \cup \{t\}) \setminus \mathrm{cl}_M(Y)$ and it must be that $s \in \mathrm{cl}_M(Y)$, which contradicts (3.12).

Subcase 2. The minor N_4 is such that

$$\rho_{N_4}(W) + \rho_{N_4}(Z \cup \{s\}) - \rho_{N_4}(W \cup \{s\}) - \rho_{N_4}(Z) = 1, \tag{3.17}$$
$$\rho_{N_4}(X \cup \{s\}) + \rho_{N_4}(Y) - \rho_{N_4}(X) - \rho_{N_4}(Y \cup \{s\}) = 1.$$

An analogous argument using (3.10) and (3.17), and applying Proposition 3.6(b) on minors N_5 and N_6, obtains the contradictions for this subcase.

Consequently we can deduce that if the minors N_3 and N_4 do not have the required common rank dominating bijection between sets $S(W,Z,\{s\})$ and $S(X,Y,\{s\})$ to satisfy the lemma, the minors N_5 and N_6 must have a common rank dominating bijection between sets $S(W,Z,\{t\})$ and $S(X,Y,\{t\})$. The result then follows by choosing $r = t$. □

We are now ready to state the main rank domination lemma.

Lemma 3.10 (Rank Domination Lemma). *Let $M(E,\rho)$ be a matroid and $R,T,W,X,Y,Z \subseteq E$. If $|R| \leqslant 3$ and $\pi : S(W,Z,T) \to S(X,Y,T)$ is a rank dominating bijection in every minor $N \in \mathcal{MF}(M,R)$, then $S(W,Z,R \cup T) \leqslant_M S(X,Y,R \cup T)$.*

Proof. The lemma is trivially true when R is empty.

Let $1 \leqslant k \leqslant 3$ be a positive integer, and assume the lemma is true whenever $|R| < k$. Now, suppose $|R| = k$.

Let $(A,B) \in S(W,Z,T)$ and $(C,D) = \pi(A,B)$. Clearly, from what is given, $S(A,B,\emptyset) \leqslant_N S(C,D,\emptyset)$ in every minor $N \in \mathcal{MF}(M,R)$. As $1 \leqslant k \leqslant 3$, by Lemma 3.9 we know there exists an $r \in R$ and a bijection, $\pi^r_{A,B} : S(A,B,\{r\}) \to S(C,D,\{r\})$, that is rank dominating in every minor $N_r \in \mathcal{MF}(M, R \setminus \{r\})$.

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We can now apply our inductive hypothesis to obtain $S(A,B,R) \leqslant_M S(C,D,R)$. Let $\pi_{A,B} : S(A,B,R) \to S(C,D,R)$ be a rank dominating bijection in M. It follows that the bijection $\pi_R : S(W,Z,R \cup T) \to S(X,Y,R \cup T)$ defined by

$$\pi_R = \bigcup_{(A,B) \in S(W,Z,T)} \pi_{A,B}$$

is rank dominating in M. □

We are finally ready to prove Proposition 3.2.

Proof of Proposition 3.2. The submodularity of matroid ranks implies that the bijection $\pi : S(P_1 \cup P_2, \emptyset, \emptyset) \to S(P_1, P_2, \emptyset)$ defined by $\pi = \{(P_1 \cup P_2, \emptyset) \mapsto (P_1, P_2)\}$ is rank dominating in all minors $N \in \mathcal{MF}(M,R)$. The result then follows from Lemma 3.10. □

4. The inequalities

Our first observation in this section follows directly from (1.1), the definition of the Whitney rank generating function.

Proposition 4.1. *Let $M(E,\rho)$ be a matroid, $E_1, E_2 \subseteq E$, and $k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$. Then, at all points on the curve $xy = 1$ in $\mathbb{R}^2$,*

$$x^k \cdot R(M|E_1 \cup E_2; x, y) \cdot R(M|E_1 \cap E_2; x, y) = R(M|E_1; x, y) \cdot R(M|E_2; x, y).$$

Proof. When $xy = 1$, (1.1) becomes $R(M; x, y) = x^{\rho(E)}(1+y)^{|E|}$, and the result follows from $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$. □

This identity changes to an inequality in the region $xy > 1$ in the positive quadrant of $\mathbb{R}^2$. We omit the straightforward proof of this result, which together with Proposition 4.1 is equivalent to Theorem 5.14 in [10]. Nevertheless, we believe that this alternative form of the inequality may be more amenable to recursively computing lower bounds for Tutte polynomial evaluations of matroids.

Proposition 4.2. *Let $M(E,\rho)$ be a matroid, $E_1, E_2 \subseteq E$, and $k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$. Then, whenever $x, y > 0$ and $xy > 1$,*

$$x^k \cdot R(M|E_1 \cup E_2; x, y) \cdot R(M|E_1 \cap E_2; x, y) \geqslant R(M|E_1; x, y) \cdot R(M|E_2; x, y).$$

However, as discussed in Section 1, the validity of inequality (1.3) in the region $xy < 1$ of the positive quadrant of $\mathbb{R}^2$ is affected by the choice of the sets E_1 and E_2. Our main result, Theorem 3.1, is that, where applicable, matroid rank dominations can be an effective tool in establishing inequality (1.3) for this region. We now give our proof of Theorem 3.1.

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Proof of Theorem 3.1. Let $f(x,y) = x^k \cdot R(M|E_1 \cup E_2; x, y) \cdot R(M|E_1 \cap E_2; x, y) - R(M|E_1; x, y) \cdot R(M|E_2; x, y)$. We show that $f(x,y) \leqslant 0$ under the stated conditions.

We look at each of the following four regions separately. The proof for Region 1 is algebraic while the proofs for Regions 2, 3 and 4 are injective.

Region 1 ($x, y > 0, xy < 1$). Since $x, y \neq 0$, we can write

$$f(x,y) = x^{\rho(E_1)+\rho(E_2)} \cdot g(x,y), \tag{4.1}$$

where

$$g(x,y) = \sum_{(W,Z)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}} (xy)^{-\rho(W)-\rho(Z)} y^{|W|+|Z|} - \sum_{(X,Y)\in 2^{E_1}\times 2^{E_2}} (xy)^{-\rho(X)-\rho(Y)} y^{|X|+|Y|}.$$

To apply rank dominations, we split each of the two sums into smaller sums on pairs (W,Z) (and (X,Y), respectively), according to $W \cup Z$ and $W \cap Z$ (and $X \cup Y$ and $X \cap Y$, respectively). Using $|W| + |Z| = |W \cup Z| + |W \cap Z|$, we can write

$$g(x,y) = \sum_{(A,B)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}} y^{|A|+|B|} \cdot h_{A,B}(x,y), \tag{4.2}$$

where

$$h_{A,B}(x,y) = \sum_{(W,Z)\in S^1_{A,B}} (xy)^{-\rho(W)-\rho(Z)} - \sum_{(X,Y)\in S^2_{A,B}} (xy)^{-\rho(X)-\rho(Y)},$$

where $S^1_{A,B} = \{(W,Z) : (W,Z) \in 2^{E_1\cup E_2} \times 2^{E_1\cap E_2}, W \cup Z = A, W \cap Z = B\}$ and $S^2_{A,B} = \{(X,Y) : (X,Y) \in 2^{E_1} \times 2^{E_2}, X \cup Y = A, X \cap Y = B\}$.

Now let

$$P^1_{A,B} = A \cap (E_1 \setminus E_2), \quad P^2_{A,B} = A \cap (E_2 \setminus E_1), \quad R_{A,B} = (A \setminus B) \cap (E_1 \cap E_2),$$
$$\overline{R_{A,B}} = (E_1 \cap E_2) \setminus R_{A,B} \quad \text{and} \quad \overline{B} = \overline{R_{A,B}} \setminus B.$$

It can be readily seen that the minor $N_B = M/B \setminus \overline{B} \in \mathcal{MF}(M, \overline{R_{A,B}})$, and so

$$S(P^1_{A,B} \cup P^2_{A,B}, \emptyset, R_{A,B}) \leqslant_{N_B} S(P^1_{A,B}, P^2_{A,B}, R_{A,B}).$$

Since $\rho_{N_B}(T) = \rho(T \cup B) - \rho(B)$ for all $T \subseteq E \setminus \overline{R_{A,B}}$, it follows that there is a bijection $\pi_{A,B} : S^1_{A,B} \to S^2_{A,B}$ such that $\rho(W) + \rho(Z) \leqslant \rho(X) + \rho(Y)$ whenever $\pi_{A,B}(W,Z) = (X,Y)$. Thus we get

$$h_{A,B}(x,y) = \sum_{\substack{(W,Z)\in S^1_{A,B} \\ (X,Y)=\pi_{A,B}(W,Z)}} \left((xy)^{-\rho(W)-\rho(Z)} - (xy)^{-\rho(X)-\rho(Y)}\right). \tag{4.3}$$

It is now easy to see that each term of the sum in (4.3) is at most zero when $x, y > 0, xy < 1$. This, along with (4.2) and (4.1), proves $f(x,y) \leqslant 0$ in this region.

Region 2 ($x > 0, y = 0$). In this region,

$$f(x,0) = x^{\rho(E_1)+\rho(E_2)} \cdot g(x), \tag{4.4}$$

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where

$$g(x) = \sum_{\substack{(W,Z)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}\\ \rho(W)=|W|,\rho(Z)=|Z|}} x^{-|W|-|Z|} - \sum_{\substack{(X,Y)\in 2^{E_1}\times 2^{E_2}\\ \rho(X)=|X|,\rho(Y)=|Y|}} x^{-|X|-|Y|}.$$

Again we split the two sums, obtaining

$$g(x) = \sum_{(A,B)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}} x^{-|A|-|B|}\cdot(|S^3_{A,B}| - |S^4_{A,B}|), \tag{4.5}$$

where $S^3_{A,B} = \{(W,Z) : (W,Z) \in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2} W\cup Z = A, W\cap Z = B, \rho(W) = |W|, \rho(Z) = |Z|\}$ and $S^4_{A,B} = \{(X,Y) : (X,Y)\in 2^{E_1}\times 2^{E_2}, X\cup Y = A, X\cap Y = B, \rho(X) = |X|, \rho(Y) = |Y|\}$.

Clearly $S^3_{A,B}\subseteq S^1_{A,B}$. Let $(W,Z)\in S^3_{A,B}$ and $(X,Y) = \pi_{A,B}(W,Z)$. Then $\rho(X)+\rho(Y)\geqslant \rho(W)+\rho(Z) = |W|+|Z| = |X|+|Y|$. However, since $0\leqslant\rho(X)\leqslant|X|$ and $0\leqslant\rho(Y)\leqslant|Y|$, this implies $\rho(X) = |X|$ and $\rho(Y) = |Y|$. In other words $(X,Y)\in S^4_{A,B}$, and $\pi_{A,B}$ induces an injective map from $S^3_{A,B}$ to $S^4_{A,B}$. Thus $|S^3_{A,B}|\leqslant|S^4_{A,B}|$, and $f(x,0)\leqslant 0$ in this case from (4.5) and (4.4).

Region 3 ($x = 0, y > 0$). First note that if $k > 0$, the theorem is trivially true in this region. Suppose $k = 0$. Then

$$f(0,y) = y^{-\rho(E_1)-\rho(E_2)}\cdot g(y), \tag{4.6}$$

where

$$g(y) = \sum_{\substack{(W,Z)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}\\ \rho(W)=\rho(E_1\cup E_2)\\ \rho(Z)=\rho(E_1\cap E_2)}} y^{|W|+|Z|} - \sum_{\substack{(X,Y)\in 2^{E_1}\times 2^{E_2}\\ \rho(X)=\rho(E_1),\rho(Y)=\rho(E_2)}} y^{|X|+|Y|}.$$

Once again rearranging the sums, we find

$$g(y) = \sum_{(A,B)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}} y^{|A|+|B|}\cdot(|S^5_{A,B}| - |S^6_{A,B}|), \tag{4.7}$$

where $S^5_{A,B} = \{(W,Z) : (W,Z)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}, W\cup Z = A, W\cap Z = B, \rho(W) = \rho(E_1\cup E_2), \rho(Z) = \rho(E_1\cap E_2)\}$ and $S^6_{A,B} = \{(X,Y) : (X,Y)\in 2^{E_1}\times 2^{E_2}, X\cup Y = A, X\cap Y = B, \rho(X) = \rho(E_1), \rho(Y) = \rho(E_2)\}$.

Let $(W,Z)\in S^5_{A,B}$ and $(X,Y) = \pi_{A,B}(W,Z)$. Then, as $k = 0$, we know that $\rho(X)+\rho(Y)\geqslant\rho(W)+\rho(Z) = \rho(E_1\cup E_2)+\rho(E_1\cap E_2) = \rho(E_1)+\rho(E_2)$. But $0\leqslant\rho(X)\leqslant\rho(E_1)$ and $0\leqslant\rho(Y)\leqslant\rho(E_2)$, and so it must be that $\rho(X) = \rho(E_1)$ and $\rho(Y) = \rho(E_2)$. Again we see that $\pi_{A,B}$ yields an injection from $S^5_{A,B}$ to $S^6_{A,B}$. The result then follows from (4.7) and (4.6).

Region 4 ($x = 0, y = 0$). At the point $(0,0)$, once again we only need to deal with the case $k = 0$. We then find

$$f(0,0) = \sum_{(A,B)\in 2^{E_1\cup E_2}\times 2^{E_1\cap E_2}} (|S^7_{A,B}| - |S^8_{A,B}|), \tag{4.8}$$

where $S^7_{A,B} = S^3_{A,B}\cap S^5_{A,B}$ and $S^8_{A,B} = S^4_{A,B}\cap S^6_{A,B}$.

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Now let $(W,Z) \in S^7_{A,B}$ and $(X,Y) = \pi_{A,B}(W,Z)$. By the arguments in the previous two cases, we know that $(X,Y) \in S^4_{A,B}, S^6_{A,B}$ and hence $(X,Y) \in S^8_{A,B}$. As before $\pi_{A,B}$ gives an injective map from $S^7_{A,B}$ to $S^8_{A,B}$, and hence the result. □

It is worth noting that the above proof argument for Region 1 also shows $f(x,y) = 0$ when $xy = 1$, and $f(x,y) \geqslant 0$ when $x, y > 0$ and $xy > 1$, for the sets E_1 and E_2. Furthermore, the arguments used in Regions 2, 3 and 4 show that the matroid rank dominating bijections provide an injective proof of inequalities on the number of independent sets, spanning sets and bases of matroids, obtained from (1.3) at the points $(1,0)$, $(0,1)$ and $(0,0)$, respectively.

5. Conclusion and open questions

We introduced rank dominations in matroids as a simple extension of the well-known property of rank submodularity, and used it to establish an inequality for the Whitney rank generating function in the positive quadrant of $\mathbb{R}^2$. Empirical evidence from graphs with up to seven vertices suggests that Proposition 3.2 may be true whenever R is independent, which would imply inequality (1.3) whenever $E_1 \cap E_2$ is independent. We make the following conjecture.

Conjecture 5.1. *In any matroid $M(E)$, given $P_1, P_2, R \subseteq E$, if R is independent in M, then $S(P_1 \cup P_2, \emptyset, R) \leqslant_M S(P_1, P_2, R)$.*

We observe that it is enough to verify Conjecture 5.1 for those cases where $P_1 \cup P_2$ is independent and contained in the closure, $\mathrm{cl}_M(R)$, of R. Conjecture 5.1 has been shown to be true when $R \subseteq \mathrm{cl}_M(P_1 \cup P_2)$ by Noble [7]. Thus, for the class of graphic matroids, Conjecture 5.1 can be equivalently stated as follows.

Conjecture 5.2. *Let $M(G)$ be the cycle matroid of a graph G with edge set E and $P_1, P_2, R \subseteq E$. Then $S(P_1 \cup P_2, \emptyset, R) \leqslant_M S(P_1, P_2, R)$ whenever R is a spanning tree and $P_1 \cup P_2$ is a forest of G.*

We know that for the class of uniform matroids, $S(P_1 \cup P_2, \emptyset, R) \leqslant S(P_1, P_2, R)$ for all $P_1, P_2, R \subseteq E$. We can also show that the series-parallel networks are the only minor closed class of binary matroids where $S(P_1 \cup P_2, \emptyset, R) \leqslant S(P_1, P_2, R)$ for all $P_1, P_2, R \subseteq E$ [6]. Finding a characterization of minor closed classes of non-binary matroids with this property is an interesting open problem. Indeed it is an immediate consequence of Theorem 3.1 that inequality (1.3) is valid for all sets $E_1, E_2 \subseteq E$ in such matroids.

We stress that the rank domination conditions for M and its minors are sufficient but not strictly necessary for the inequality of Theorem 3.1. There are known examples where the inequality remains true even when there is no rank dominating bijection in M for some sets $P_1 \subseteq E_1 \setminus E_2, P_2 \subseteq E_2 \setminus E_1$ and $R \subseteq E_1 \cap E_2$. For example, despite the observation in Example 3.3, we can computationally verify the validity of (1.3) when

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$E_1 = \{e_1, e_2, e_3, e_4, e_5\}$ and $E_2 = \{e_2, e_3, e_4, e_5, e_6\}$ in $M(K_4)$. Thus, identifying the necessary conditions for (1.3) is our next, though probably difficult, unsolved problem.

We say a matroid $M(E, \rho)$ is *Whitney-correlated* if it satisfies (1.3) for all subsets $E_1, E_2 \subseteq E$. Clearly all such matroids are also balanced. We know that uniform matroids, series-parallel networks and $M(K_4)$ are Whitney-correlated. It seems likely that there could be matroids that are balanced yet not Whitney-correlated. However, we have no examples of such matroids and finding one is an open problem. The question whether all graphic matroids are Whitney-correlated may also be of particular interest.

Finally, one of our original motivations to study these inequalities was to obtain efficiently computable non-trivial upper bounds for the number of independent sets, bases and spanning sets of matroids. Nonetheless this task still remains open, and we conclude by posing the question of finding an efficient algorithm to compute such upper bounds for these combinatorial counting problems.

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