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Some two-dimensional punch and crack problems in classical elasticity

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Abstract. General methods for solving two-dimensional punch and crack problems in classical elasticity have been developed by Muskhelishvili and others. An alternative method for solving such problems is presented here. Use is made of suitable integral representations of the complex potentials and each problem is reduced to the solution of an Abel or Fredholm type of integral equation.

1. *Introduction.* In this paper we consider some two-dimensional punch and crack problems in classical elasticity. Many problems of this nature have been solved using the extensive and general methods developed by Muskhelishvili (1) and others, based on the use of complex variable techniques. The method presented in this paper involves a rather different approach and reduces some of these problems to the solution of simple integral equations and real variable integration as compared with the Plemelj formulae and Cauchy integrals which occur in the more general theories mentioned above. The method is similar to that used in Green and Zerna (2) for axially-symmetric problems in three-dimensional elasticity. This three-dimensional work has been extended by Collins (3) in a series of papers to a number of problems and some of their two-dimensional analogues can be solved by the methods of the present paper.

In § 2 the basic formulae are quoted and, after a rearrangment of the complex potentials, the case of a single line crack in an infinite region is considered in § 3. In § 4 a solution is found in closed form to the problem of an infinite row of collinear line cracks, and in § 5 to the case of an infinite row of identical punches acting on a half-space. Koiter (4) has considered an infinite row of collinear cracks in an infinite elastic sheet with constant stresses at infinity and some of his results may be obtained as a particular case of § 3. Koiter (4) also refers to work by Shtaerman ((6), chapter II) on a periodic array of frictionless stamps on a half plane. In § 6 the problem of an infinite row of parallel cracks is examined and reduced to Fredholm integral equations and a solution is found explicitly for widely spaced cracks.

2. *Basic formulae.* We quote the basic equations for two-dimensional isotropic elasticity in the form used by Green and Zerna (2). If (x, y) are rectangular cartesian coordinates and $z = x+iy$, $\bar{z} = x-iy$, then

$$\mu D = \mu(u_x + iu_y) = \kappa\Omega(z) - z\overline{\Omega}'(\bar{z}) - \overline{\omega}'(\bar{z}), \tag{2·1}$$

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$$\left.\begin{aligned}\sigma_{xx}+\sigma_{yy} &= 4\{\Omega'(z)+\overline{\Omega}'(\bar{z})\},\\ \sigma_{xx}+i\sigma_{xy} &= 2\{\Omega'(z)+\overline{\Omega}'(\bar{z})-z\overline{\Omega}''(\bar{z})-\overline{\omega}''(\bar{z})\},\\ \sigma_{yy}-i\sigma_{xy} &= 2\{\Omega'(z)+\overline{\Omega}'(\bar{z})+z\overline{\Omega}''(\bar{z})+\overline{\omega}''(\bar{z})\},\end{aligned}\right\} \tag{2·2}$$

where (u_x, u_y) are components of displacement, $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are components of stress and μ is the rigidity modulus. Also $\kappa = 3-4\eta$ for plane strain and $\kappa = (3-\eta)/(1+\eta)$ for generalized plane stress, η being Poisson's ratio. It is convenient to introduce two other functions $\phi(z)$, $\psi(z)$ by the equations

$$\left.\begin{aligned}\omega'(z) &= z\Omega'(z)-\Omega(z)-\phi(z),\\ \omega'(z) &= -z\Omega'(z)+\Omega(z)+\psi(z),\end{aligned}\right\} \tag{2·3}$$

so that

$$\left.\begin{aligned}\mu D &= \kappa\Omega(z)+\overline{\Omega}(\bar{z})-(z+\bar{z})\,\overline{\Omega}'(\bar{z})+\overline{\phi}(\bar{z}),\\ \sigma_{xx}+i\sigma_{xy} &= 2\{\Omega'(z)+\overline{\Omega}'(\bar{z})-(z+\bar{z})\,\overline{\Omega}''(\bar{z})+\overline{\phi}'(\bar{z})\},\end{aligned}\right\} \tag{2·4}$$

or

$$\left.\begin{aligned}\mu D &= \kappa\Omega(z)-\overline{\Omega}(\bar{z})+(\bar{z}-z)\,\overline{\Omega}'(\bar{z})-\overline{\psi}(\bar{z}),\\ \sigma_{yy}-i\sigma_{xy} &= 2\{\Omega'(z)+\overline{\Omega}'(\bar{z})+(z-\bar{z})\,\overline{\Omega}''(\bar{z})+\overline{\psi}'(\bar{z})\}.\end{aligned}\right\} \tag{2·5}$$

3. *Single line crack.* We now suppose that there is a line crack along the real axis between the points $x = -a$ and $x = a$, the region being otherwise unbounded, and we assume that this line crack is opened by equal and opposite normal pressures on each side of the crack. Then the following conditions must be satisfied on $y = 0$:

$$\left.\begin{aligned}\sigma_{yy} &= -f(x)-g(x) && (y=0,\ -a<x<a),\\ \sigma_{xy} &= 0 && (y=0,\ \text{all } x),\\ u_y &= 0 && (y=0,\ |x|>a),\end{aligned}\right\} \tag{3·1}$$

where $f(x)$ is an even function of x and $g(x)$ is an odd function of x. We assume that $f(x)$ and $g(x)$ are sectionally continuous in $0 \leqslant x \leqslant a$. If we use formulae (2·5) the second condition in (3·1) can be satisfied by taking $\psi(z) \equiv 0$ and then, on $y = 0$ (or $z = \bar{z}$), we have

$$\left.\begin{aligned}\sigma_{yy} &= 2\{\Omega'(z)+\overline{\Omega}'(\bar{z})\},\\ 2i\mu u_y &= (\kappa+1)\{\Omega(z)-\overline{\Omega}(\bar{z})\},\end{aligned}\right\} \tag{3·2}$$

provided $y\overline{\Omega}'(\bar{z})$ and $y\overline{\Omega}''(\bar{z})$ tend to zero as $y \to \pm 0$. Since the crack is opened by equal and opposite pressures on each side (which must be such that the crack opens along its entire length), the resultant force over the entire surface of the crack is zero. Hence, if, in addition, the stress and rotation vanish at infinity, we have

$$\Omega'(z) = O(1/z^2) \tag{3·3}$$

for large $|z|$.

In order to solve the problem specified by (3·1), (3·2) and (3·3) we consider the function†

$$\Omega(z) = \int_0^a \frac{F(t)+zG(t)}{\sqrt{(z^2-t^2)}}\,dt, \tag{3·4}$$

† We may add a real constant to Ω which represents a rigid body translation along the x-axis, but we omit this here.

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where $F(t)$, $G(t)$ are real continuous functions of t in the interval $0 \leqslant t \leqslant a$ and

$$\sqrt{(z^2-t^2)} = \xi e^{i\eta} \quad (-\pi \leqslant \eta \leqslant \pi), \tag{3·5}$$

where
$$\xi^2 \cos 2\eta = x^2-y^2-t^2, \quad \xi^2 \sin 2\eta = 2xy. \tag{3·6}$$

An examination of the function (3·5) shows that, on the real axis $y = 0$,

$$\left.\begin{aligned} \sqrt{(z^2-t^2)} &= \sqrt{(x^2-t^2)} && (x > t,\ y \to \pm 0), \\ \sqrt{(z^2-t^2)} &= -\sqrt{(x^2-t^2)} && (x < -t,\ y \to \pm 0), \\ \sqrt{(z^2-t^2)} &= i\sqrt{(t^2-x^2)} && (|x| < t,\ y \to +0), \\ \sqrt{(z^2-t^2)} &= -i\sqrt{(t^2-x^2)} && (|x| < t,\ y \to -0). \end{aligned}\right\} \tag{3·7}$$

The function defined by (3·4) is regular at all points of the z-plane where z^2-t^2 does not vanish, that is everywhere except points on the crack $y = 0$, $|x| \leqslant a$. Since $|z^2-t^2| \geqslant |z|^2-t^2 \geqslant |z|^2-a^2$, for large $|z|$ we have

$$|\Omega(z)| \leqslant \frac{1}{\sqrt{(|z|^2-a^2)}} \int_0^a |F(t)|\, dt + \frac{|z|}{\sqrt{(|z|^2-a^2)}} \int_0^a |G(t)|\, dt \tag{3·8}$$

and (3·3) is satisfied.

Suppose now that t in (3·4) is a complex variable and that $F(t)$, $G(t)$ are regular functions of t in some simply-connected region containing the line segment $0 \leqslant t \leqslant a$ on the real t-axis. The zeros of $\sqrt{(z^2-t^2)}$ are at $\pm(x+iy)$, i.e. in the first and third quadrants of the t-plane when $x > 0$, $y > 0$ or $x < 0$, $y < 0$; and in the second and fourth quadrants of the t-plane when $x > 0$, $y < 0$ or $x < 0$, $y > 0$. In these cases we can displace the path of integration in (3·4) away from the real axis into the fourth and the first quadrants respectively. The function $\Omega(z)$ is then a regular function of z in the neighbourhood of the crack $|x| < a$ on the real z-axis except possibly at $z = \pm a$ and $z = 0$. Thus $\Omega(z)$ and its derivatives tend to finite limits as the point z approaches a point on the crack in any manner through values of y greater or less than zero, except possibly at $z = \pm a$ or $z = 0$. By excluding the points $t = |x|$ $(x \neq \pm a,\ x \neq 0)$ on the real t-axis with small semicircles whose radii tend to zero we can replace the path of integration by one along the real t-axis, as in (3·4). The function $\Omega(z)$ given by (3·4), and its derivatives, are then continuous on to each side $y > 0$, $y < 0$ of the crack, except at $z = \pm a$, $z = 0$. The value of $\Omega(z)$ given by (3·4) is continuous at $z = \pm a$ but not its derivatives.

If
$$F(t) = O(t), \quad G(t) = o(1) \tag{3·9}$$

as $t \to 0$, then, from (3·4) we can show that $\Omega(z)$, $\Omega'(z)$, $z\Omega''(z)$, $\bar{z}\Omega''(z)$ have unique finite values as $|z| \to 0$ in any manner. If we relax the condition on $G(t)$ to

$$G(t) = O(1) \tag{3·10}$$

as $t \to 0$ then $\Omega(z)$, $z\Omega'(z)$, $z\Omega''(z)$ have unique finite values, as $|z| \to 0$ in any manner, but $\Omega'(z)$, $\bar{z}\Omega''(z)$ do not tend to definite limits. However, this only arises when $g(x)$ has a finite discontinuity at $x = 0$ and some difficulty must be expected at the point $z = 0$ of the crack in this case.

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With the help of (3·7) we see that the expression (3·4) for $\Omega(z)$ satisfies the last condition in (3·1) for $y = 0$, $|x| > a$. Using (3·2) and (3·7) the first condition in (3·1) then yields the integral equations

$$4\frac{d}{dx}\int_0^x \frac{F(t)\,dt}{\sqrt{(x^2-t^2)}} = -f(x) \quad (0 \leqslant x \leqslant a), \tag{3·11}$$

$$4\frac{d}{dx}\int_0^x \frac{xG(t)\,dt}{\sqrt{(x^2-t^2)}} = -g(x) \quad (0 \leqslant x \leqslant a). \tag{3·12}$$

The solution of these equations is straightforward (see, for example, (2), p. 175). Thus

$$F(t) = -\frac{1}{2\pi}\frac{d}{dt}\int_0^t \frac{x\,dx}{\sqrt{(t^2-x^2)}}\int_0^x f(u)\,du,$$

$$G(t) = -\frac{1}{2\pi}\frac{d}{dt}\int_0^t \frac{dx}{\sqrt{(t^2-x^2)}}\int_0^x g(u)\,du,$$

and, changing the order of integration, we have

$$\begin{aligned} F(t) &= -\frac{1}{2\pi}\frac{d}{dt}\int_0^t f(u)\sqrt{(t^2-u^2)}\,du \\ &= -\frac{t}{2\pi}\int_0^t \frac{f(u)\,du}{\sqrt{(t^2-u^2)}}, \end{aligned} \tag{3·13}$$

and

$$\begin{aligned} G(t) &= -\frac{1}{2\pi}\frac{d}{dt}\int_0^t g(u)\,[\tfrac{1}{2}\pi - \sin^{-1}(u/t)]\,du \\ &= -\frac{1}{2\pi t}\int_0^t \frac{ug(u)\,du}{\sqrt{(t^2-u^2)}}. \end{aligned} \tag{3·14}$$

If $f(u)$, $g(u)$ are sectionally continuous then $F(t)$ and $G(t)$ are continuous. If $f(u)$, $g(u)$ are continuous functions in the region $0 \leqslant u \leqslant a$ the results (3·13) and (3·14) can be replaced by

$$\left.\begin{aligned} F(t) &= -\frac{t}{2\pi}\int_0^{\frac{1}{2}\pi} f(t\sin\theta)\,d\theta, \\ G(t) &= -\frac{1}{2\pi}\int_0^{\frac{1}{2}\pi} g(t\sin\theta)\sin\theta\,d\theta. \end{aligned}\right\} \tag{3·15}$$

The normal displacement of the surfaces of the crack may be obtained from (3·1), (3·2), (3·4) and (3·7). Thus, when $y \to +0$,

$$\frac{\mu u_y}{\kappa+1} = -\int_{|x|}^a \frac{F(t)+xG(t)}{\sqrt{(t^2-x^2)}}\,dt \quad (|x| \leqslant a). \tag{3·16}$$

If $f(x) = p$, $g(x) = 0$, where p is constant, then

$$F(t) = -pt/4, \quad G(t) = 0, \tag{3·17}$$

and

$$\Omega(z) = -\tfrac{1}{4}p[z - \sqrt{(z^2-a^2)}] \tag{3·18}$$

a well-known result. (See, for example, (2), p. 279.)

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Again, if $f(x) = \frac{1}{2}p$, $g(x) = \frac{1}{2}p$, so that the crack is opened by uniform pressure p over one-half $0 \leqslant x \leqslant a$, and zero pressure over the rest of the crack $-a \leqslant x < 0$, we have

$$F(t) = -pt/8, \quad G(t) = -p/(4\pi). \tag{3·19}$$

The normal displacement of the crack is then given by (3·16) in the form

$$\begin{aligned}\frac{\mu u_y}{\kappa+1} &= \frac{p}{8\pi}\int_{|x|}^{a} \frac{\pi t + 2x}{\sqrt{(t^2-x^2)}}\,dt \quad (|x| \leqslant a)\\ &= \frac{p}{8\pi}[\pi\sqrt{(a^2-x^2)} + 2x\cosh^{-1}(a/|x|)] \quad (|x| \leqslant a).\end{aligned} \tag{3·20}$$

4. *Infinite row of collinear line cracks.* We consider an infinite row of equal line cracks, each of length $2a$ and with their centres at distance d ($> 2a$) apart, situated along the real axis, the region being otherwise unbounded. We assume that each line crack is opened by equal and opposite normal pressures on each side of the crack which are the same for each crack. The stresses and displacements are then periodic functions of x with period d. On $y = 0$ the conditions are

$$\left.\begin{aligned}\sigma_{yy} &= -f(x) - g(x) \quad (y = 0,\ |x - nd| \leqslant a),\\ \sigma_{xy} &= 0 \quad (y = 0, \text{ all } x),\\ u_y &= 0 \quad (y = 0,\ a \leqslant |x - nd| \leqslant d - a),\end{aligned}\right\} \tag{4·1}$$

for $n = 0, \pm 1, \pm 2, \ldots$. The functions $f(x)$, $g(x)$ are respectively even and odd functions of x, sectionally continuous in $0 \leqslant x \leqslant a$. We also assume that stresses and rotations vanish for large $|y|$. We again use (2·5) with $\psi(z) \equiv 0$ and now we take†

$$\Omega(z) = \int_0^a \frac{F(\eta)\tan(\pi\eta/d)\cos(\pi z/d) + G(\eta)\cot(\pi\eta/d)\sin(\pi z/d)}{\sqrt{[\cos(2\pi\eta/d) - \cos(2\pi z/d)]}}\,d\eta, \tag{4·2}$$

where $F(\eta)$, $G(\eta)$ are real continuous functions of η. A real constant may be added to the form (4·2) to allow for a rigid body translation along the x-axis, but this may be ignored for our purpose. To avoid undue length we proceed formally and omit detailed discussion of the validity of (4·2) for our problem. This discussion is similar in many respects to that of § 3 but in any given problem we may verify, if necessary, that the solution obtained satisfies all the required conditions. We note, however, that $\Omega(z)$ is a periodic function with period d. Also, on examining the square root, we have

$$M = \sqrt{[\cos(2\pi\eta/d) - \cos(2\pi z/d)]} = \xi e^{i\tau} \quad (-\pi \leqslant \tau \leqslant \pi), \tag{4·3}$$

where

$$\left.\begin{aligned}\xi^2\cos(2\tau) &= \cos(2\pi\eta/d) - \cos(2\pi x/d)\cosh(2\pi y/d),\\ \xi^2\sin(2\tau) &= \sin(2\pi x/d)\sinh(2\pi y/d),\end{aligned}\right\} \tag{4·4}$$

and

$$\left.\begin{aligned}M &= \sqrt{[\cos(2\pi\eta/d) - \cos(2\pi x/d)]} && (\tfrac{1}{2}d > x > \eta,\ y \to \pm 0),\\ M &= -\sqrt{[\cos(2\pi\eta/d) - \cos(2\pi x/d)]} && (\tfrac{1}{2}d > |x| > \eta,\ x < 0,\ y \to \pm 0),\\ M &= i\sqrt{[\cos(2\pi x/d) - \cos(2\pi\eta/d)]} && (|x| < \eta,\ y \to +0),\\ M &= -i\sqrt{[\cos(2\pi x/d) - \cos(2\pi\eta/d)]} && (|x| < \eta,\ y \to -0).\end{aligned}\right\} \tag{4·5}$$

Finally, $\Omega(z) = O(1)$ for large $|y|$.

† A somewhat similar technique has been employed in some parts of the book by Woods ((8), p. 505).

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In view of (4·5) the third condition in (4·1) is satisfied over all sections of the line $y = 0$ outside of the cracks. The first condition in (4·1) on the crack $|x| \leqslant a$ gives the two equations

$$4\frac{d}{dx}\int_0^x \frac{F(\eta)\tan(\pi\eta/d)\cos(\pi x/d)\,d\eta}{\sqrt{[\cos(2\pi\eta/d)-\cos(2\pi x/d)]}} = -f(x) \quad (0 \leqslant x \leqslant a), \tag{4·6}$$

$$4\frac{d}{dx}\int_0^x \frac{G(\eta)\cot(\pi\eta/d)\sin(\pi x/d)\,d\eta}{\sqrt{[\cos(2\pi\eta/d)-\cos(2\pi x/d)]}} = -g(x) \quad (0 \leqslant x \leqslant a), \tag{4·7}$$

and then the conditions on the remaining cracks are automatically satisfied. The solution of the integral equations is straightforward. From (4·6) we have

$$4\int_0^x \frac{F(u)\tan(\pi u/d)\cos(\pi x/d)\,du}{\sqrt{[\cos(2\pi u/d)-\cos(2\pi x/d)]}} = -\int_0^x f(u)\,du.$$

We multiply both sides of this equation by

$$\frac{\sin(\pi x/d)}{\sqrt{[\cos(2\pi x/d)-\cos(2\pi\eta/d)]}}$$

and integrate with respect to x from $x = 0$ to $x = \eta$. Then, after changing the orders of integration, we find that

$$F(\eta) = -\frac{1}{d}\int_0^\eta \frac{f(u)\cos(\pi u/d)\,du}{\sqrt{[\cos(2\pi u/d)-\cos(2\pi\eta/d)]}}. \tag{4·8}$$

If $f(u)$ is continuous in $0 \leqslant u \leqslant a$ then $F(\eta)$ reduces to the form

$$F(\eta) = -\frac{1}{\pi\sqrt{2}}\int_0^{\frac{1}{2}\pi} f^*(\phi)\,d\phi, \tag{4·9}$$

where
$$f^*(\phi) = f(u), \quad \sin(\pi u/d) = \sin(\pi\eta/d)\sin\phi. \tag{4·10}$$

Similarly, from (4·7),

$$G(\eta) = -\frac{1}{d}\int_0^\eta \frac{g(u)\sin(\pi u/d)\,du}{\sqrt{[\cos(2\pi u/d)-\cos(2\pi\eta/d)]}}, \tag{4·11}$$

and when $g(u)$ is continuous in $0 \leqslant u \leqslant a$,

$$G(\eta) = -\frac{1}{\pi\sqrt{2}}\int_0^{\frac{1}{2}\pi} g^*(\phi)\,d\phi, \tag{4·12}$$

where
$$g(u)\tan(\pi u/d) = g^*(\phi), \quad \sin(\pi u/d) = \sin(\pi\eta/d)\sin\phi. \tag{4·13}$$

Particular examples of this theory are omitted but we observe that Koiter's results (4) for cracks opened by constant pressure may be obtained at once by putting $g(x) \equiv 0, f(x) = p$. A similar analysis could be developed to allow for applied shear stresses at the cracks.

5. *Infinite row of punches on a half-space.* The problem of an infinite row of identical smooth punches acting on the surface $y = 0$ of the half-space $y \geqslant 0$, and separated by equal distances d, can be solved by a method similar to that used in § 4 for the row of cracks.

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The boundary conditions are

$$\left.\begin{aligned} \sigma_{yy} &= 0 && (y=0,\ a \leqslant |x-nd| \leqslant d-a),\\ \sigma_{xy} &= 0 && (y=0,\ \text{all } x),\\ u_y &= f(x)+g(x) && (y=0,\ |x-nd| \leqslant a), \end{aligned}\right\} \tag{5·1}$$

for $n = 0, \pm 1, \pm 2, \ldots$, where $2a$ is the length of the region of contact between a punch and the half-space, and $f(x)$, $g(x)$ are even and odd functions of x respectively. We also assume that stresses and rotation vanish for large $|y|$.

We again use equations (2·5) with $\psi \equiv 0$ and now consider

$$\Omega'(z) = i\int_0^a \frac{F(\eta)\cos(\pi z/d)+G(\eta)\sin(\pi z/d)}{\sqrt{[\cos(2\pi\eta/d)-\cos(2\pi z/d)]}}\,d\eta + i\frac{A\cos(\pi z/d)+B\sin(\pi z/d)}{\sqrt{[\cos(2\pi a/d)-\cos(2\pi z/d)]}} + iC, \tag{5·2}$$

where A, B, C are real constants and $F(\eta)$, $G(\eta)$ are real continuous functions.

From equations (2·5) and (4·3), we see that the first and second boundary conditions in (5·1) are satisfied, and the third condition can be written as

$$\Omega'(z) - \bar{\Omega}'(\bar{z}) = \frac{2i\mu}{1+\kappa}\{f'(x)+g'(x)\}, \tag{5·3}$$

so that

$$\int_0^x \frac{G(\eta)\sin(\pi x/d)\,d\eta}{\sqrt{[\cos(2\pi\eta/d)-\cos(2\pi x/d)]}} = \frac{\mu}{1+\kappa}g'(x) - C \quad (0 \leqslant x \leqslant a), \tag{5·4}$$

$$\int_0^x \frac{F(\eta)\cos(\pi x/d)\,d\eta}{\sqrt{[\cos(2\pi\eta/d)-\cos(2\pi x/d)]}} = \frac{\mu}{1+\kappa}f'(x) \quad (0 \leqslant x \leqslant a). \tag{5·5}$$

Since the left-hand side of (5·4) tends to zero as $x \to 0$ we must choose

$$C = \frac{\mu}{1+\kappa}g'(0) \tag{5·6}$$

and the equations (5·4), (5·5) may then be solved in a manner similar to that in § 4, giving

$$\left.\begin{aligned} F(\eta) &= \frac{4\mu}{(1+\kappa)d}\frac{d}{d\eta}\int_0^\eta \frac{f'(x)\sin(\pi x/d)\,dx}{\sqrt{[\cos(2\pi x/d)-\cos(2\pi\eta/d)]}},\\ G(\eta) &= \frac{4\mu}{(1+\kappa)d}\frac{d}{d\eta}\int_0^\eta \frac{\{g'(x)-g'(0)\}\cos(\pi x/d)\,dx}{\sqrt{[\cos(2\pi x/d)-\cos(2\pi\eta/d)]}}, \end{aligned}\right\} \tag{5·7}$$

where $f'(x)$, $g'(x)$ are assumed to be continuous.

The normal stress under a typical punch is found to be

$$\sigma_{yy} = 4\int_{|x|}^a \frac{F(\eta)\cos(\pi x/d)+G(\eta)\sin(\pi x/d)}{\sqrt{[\cos(2\pi x/d)-\cos(2\pi\eta/d)]}}\,d\eta + 4\,\frac{A\cos(\pi x/d)+B\sin(\pi x/d)}{\sqrt{[\cos(2\pi x/d)-\cos(2\pi a/d)]}}, \tag{5·8}$$

using (2·5), (4·3) and (5·2), and hence the resultant pressure P on a punch is given by

$$\begin{aligned} P &= -\int_{-a}^a \sigma_{yy}\,dx,\\ &= -8\int_0^a \left\{\int_x^a \frac{F(\eta)\cos(\pi x/d)\,d\eta}{\sqrt{[\cos(2\pi x/d)-\cos(2\pi\eta/d)]}} + \frac{A\cos(\pi x/d)}{\sqrt{[\cos(2\pi x/d)-\cos(2\pi a/d)]}}\right\} dx,\\ &= -2d\sqrt{2}\left\{\int_0^a F(\eta)\,d\eta + A\right\}, \end{aligned} \tag{5·9}$$

on changing the order of integration.

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From (5·2) the condition that the rotation is zero at infinity implies that

$$\int_0^a G(\eta)\,d\eta + B + C\sqrt{2} = 0, \tag{5·10}$$

and the constant B can now be determined.

In the case of complete penetration by a punch when its whole end-face is in contact with the body, the length $2a$ is known and (5·6), (5·9), (5·10) determine the constants. For incomplete penetration, however, when there is at most one corner of a punch in contact with the half-space, we must use (5·8) to ensure that the normal stress is finite at a point where the medium leaves the punch smoothly. In this case the length $2a$ of the region of contact and the position of its mid-point are unknown but can be determined from (5·9) and (5·10), completing the solution.

As a particular example we consider a row of punches with inclined bases, so that in (5·1)

$$f(x) \equiv 0, \quad g(x) = \epsilon x, \tag{5·11}$$

where ϵ is a constant. Then, from (5·6), (5·7), (5·10) and (5·2) we find that

$$B = -C\sqrt{2} = -\frac{\mu\epsilon\sqrt{2}}{1+\kappa}, \quad A = -\frac{P}{2d\sqrt{2}}, \tag{5·12}$$

and

$$\Omega'(z) = -\frac{iP}{4d}\frac{\cos(\pi z/d)}{\sqrt{[\cos^2(\pi a/d) - \cos^2(\pi z/d)]}} - \frac{i\mu\epsilon}{1+\kappa}\frac{\sin(\pi z/d)}{\sqrt{[\cos^2(\pi a/d) - \cos^2(\pi z/d)]}} + \frac{i\mu\epsilon}{1+\kappa}. \tag{5·13}$$

For incomplete penetration it can be shown that contact is maintained over a region of length $2a$, where

$$\tan\left(\frac{\pi a}{d}\right) = \frac{P(1+\kappa)}{4\mu\epsilon d}. \tag{5·14}$$

6. *Infinite row of parallel cracks.* In this section we consider an infinite row of identical line cracks, each parallel to the x-axis, of length $2a$, and uniformly spaced at a distance d apart along the y-axis. The origin is taken at the centre of one crack and the y-axis passes through the mid-point of each crack. As in §§ 3 and 4 we suppose each crack is opened by equal and opposite normal pressures $\sigma_{yy} = -f(x) - g(x)$, where $f(x)$, $g(x)$ are sectionally continuous even and odd functions respectively. A corresponding problem in which the cracks are opened by a uniform shear stress at infinity has been considered by Koiter (5).

The lines $y = (2n+1)\,d/2$, $n = 0, \pm 1, \pm 2, \ldots$ are lines of symmetry along which $u_y = \sigma_{xy} = 0$, so we consider only the strip $-\frac{1}{2}d \leqslant x \leqslant \frac{1}{2}d$ subject to the boundary conditions

$$\left.\begin{aligned} \sigma_{yy} &= -f(x) - g(x) && (y = 0,\ |x| \leqslant a),\\ \sigma_{xy} &= 0 && (y = 0,\ \text{all } x),\\ u_y &= 0 && (y = 0,\ |x| > a),\\ \sigma_{xy} &= u_y = 0 && (y = \pm\tfrac{1}{2}d,\ \text{all } x). \end{aligned}\right\} \tag{6·1}$$

Using formulae (2·5) and guided by § 3, forms are assumed for $\Omega(z)$ and $\psi(z)$, reducing the problem to the solution of two Fredholm integral equations. These are then solved in terms of a power series in a/d. An equivalent closed form solution to that obtained

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in § 4 has not been found, but it may be remarked here that a method analogous to that used in this section may be applied to the problem of § 4 and the solution obtained in a power series form.

We assume

$$\left.\begin{aligned}\Omega(z) &= \int_0^a \frac{F(t)+zG(t)}{\sqrt{(z^2-t^2)}}\,dt + \int_0^\infty A(\xi)\cos(\xi z)\,d\xi + \int_0^\infty B(\xi)\sin(\xi z)\,d\xi,\\ \psi(z) &= \int_0^\infty C(\xi)\cos(\xi z)\,d\xi + \int_0^\infty D(\xi)\sin(\xi z)\,d\xi,\end{aligned}\right\} \tag{6·2}$$

where $F(t)$, $G(t)$, $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$ are real continuous functions.

Remembering the definition of the square root (3·7) we see that the second and third boundary conditions in (6·1) are satisfied identically. Similarly, after some manipulation, the fourth condition yields a simple set of simultaneous equations determining $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$ in terms of $F(t)$, $G(t)$. These can be solved to give

$$\left.\begin{aligned}A(\xi)\sinh(\tfrac{1}{2}\xi d) &= -e^{-\frac{1}{2}\xi d}\int_0^a tG(t)\,J_1(\xi t)\,dt,\\ B(\xi)\sinh(\tfrac{1}{2}\xi d) &= e^{-\frac{1}{2}\xi d}\int_0^a F(t)\,J_0(\xi t)\,dt,\\ C(\xi)\sinh(\tfrac{1}{2}\xi d) &= -\xi d\{1+\coth(\tfrac{1}{2}\xi d)\}\,e^{-\frac{1}{2}\xi d}\int_0^a tG(t)\,J_1(\xi t)\,dt,\\ D(\xi)\sinh(\tfrac{1}{2}\xi d) &= \xi d\{1+\coth(\tfrac{1}{2}\xi d)\}\,e^{-\frac{1}{2}\xi d}\int_0^a F(t)\,J_0(\xi t)\,dt,\end{aligned}\right\} \tag{6·3}$$

where $J_0(x)$, $J_1(x)$ are Bessel functions of the first kind. We note that if the pressure distribution is symmetric then $A(\xi)$, $C(\xi)$ are identically zero.

Using (6·2) and (3·7) the first condition in (6·1) yields the integral equations

$$4\frac{d}{dx}\int_0^x \frac{F(t)\,dt}{\sqrt{(x^2-t^2)}} + 4\frac{d}{dx}\int_0^\infty \{B(\xi)+\tfrac{1}{2}D(\xi)\}\sin(\xi x)\,d\xi = -f(x) \quad (0 \leqslant x \leqslant a),$$

$$4\frac{d}{dx}\int_0^x \frac{xG(t)\,dt}{\sqrt{(x^2-t^2)}} + 4\frac{d}{dx}\int_0^\infty \{A(\xi)+\tfrac{1}{2}C(\xi)\}\cos(\xi x)\,d\xi = -g(x) \quad (0 \leqslant x \leqslant a);$$

which, on inverting, become

$$F(t) + t\int_0^\infty \xi\{B(\xi)+\tfrac{1}{2}D(\xi)\}\,J_0(\xi t)\,d\xi = -\frac{t}{2\pi}\int_0^t \frac{f(x)\,dx}{\sqrt{(t^2-x^2)}},$$

$$G(t) - \int_0^\infty \xi\{A(\xi)+\tfrac{1}{2}C(\xi)\}\,J_1(\xi t)\,d\xi = -\frac{1}{2\pi t}\int_0^t \frac{xg(x)\,dx}{\sqrt{(t^2-x^2)}}.$$

Using (6·3), these equations reduce finally to Fredholm integral equations

$$\left.\begin{aligned}F(t) + t\int_0^a F(s)\,K(s,t)\,ds &= -\frac{t}{2\pi}\int_0^t \frac{f(x)\,dx}{\sqrt{(t^2-x^2)}},\\ G(t) + \int_0^a sG(s)\,L(s,t)\,ds &= -\frac{1}{2\pi t}\int_0^t \frac{xg(x)\,dx}{\sqrt{(t^2-x^2)}},\end{aligned}\right\} \tag{6·4}$$

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where
$$\left.\begin{aligned} K(s,t) &= \int_0^\infty \frac{\xi}{\sinh(\frac{1}{2}\xi d)}[1+\tfrac{1}{2}\xi d\{1+\coth(\tfrac{1}{2}\xi d)\}]\,e^{-\frac{1}{2}\xi d}J_0(\xi s)\,J_0(\xi t)\,d\xi, \\ L(s,t) &= \int_0^\infty \frac{\xi}{\sinh(\frac{1}{2}\xi d)}[1+\tfrac{1}{2}\xi d\{1+\coth(\tfrac{1}{2}\xi d)\}]\,e^{-\frac{1}{2}\xi d}J_1(\xi s)\,J_1(\xi t)\,d\xi. \end{aligned}\right\} \tag{6·5}$$

We solve the equations (6·4) in terms of power series expansions in a/d. It is convenient first to put these equations in non-dimensional form by writing†

$$F(ax) = a\bar{F}(x), \quad f(ax) = \bar{f}(x),$$
$$G(ax) = \bar{G}(x), \quad g(ax) = \bar{g}(x).$$

Then
$$\left.\begin{aligned} \bar{F}(x) + x\int_0^1 \bar{F}(y)\,a^2K(ay,ax)\,dy &= -\frac{x}{2\pi}\int_0^x \frac{\bar{f}(y)\,dy}{\sqrt{(x^2-y^2)}}, \\ \bar{G}(x) + \int_0^1 y\bar{G}(y)\,a^2L(ay,ax)\,dy &= -\frac{1}{2\pi x}\int_0^x \frac{y\bar{g}(y)\,dy}{\sqrt{(x^2-y^2)}}, \end{aligned}\right\} \tag{6·6}$$

and
$$\left.\begin{aligned} a^2K(ay,ax) &= 4\frac{a^2}{d^2}\int_0^\infty \frac{\xi}{\sinh\xi}[1+\xi\{1+\coth\xi\}]\,e^{-\xi}J_0\left(\frac{2a\xi y}{d}\right)J_0\left(\frac{2a\xi x}{d}\right)d\xi, \\ a^2L(ay,ax) &= 4\frac{a^2}{d^2}\int_0^\infty \frac{\xi}{\sinh\xi}[1+\xi\{1+\coth\xi\}]\,e^{-\xi}J_1\left(\frac{2a\xi y}{d}\right)J_1\left(\frac{2a\xi x}{d}\right)d\xi. \end{aligned}\right\} \tag{6·7}$$

By expressing the Bessel Functions in (6·7) as power series, the integrals can be expanded to give

$$\left.\begin{aligned} a^2K(ay,ax) &= \frac{a^2}{d^2}K_0 + \frac{a^4}{d^4}K_1(x^2+y^2) + \frac{a^6}{d^6}K_2(x^4+4x^2y^2+y^4) + O\left\{\left(\frac{a}{d}\right)^8\right\}, \\ a^2L(ay,ax) &= \frac{a^4}{d^4}L_1xy + \frac{a^6}{d^6}L_2xy(x^2+y^2) + O\left\{\left(\frac{a}{d}\right)^8\right\}, \end{aligned}\right\} \tag{6·8}$$

where the constants K_i, L_i are defined by

$$\left.\begin{aligned} K_0 &= 4\int_0^\infty \frac{\xi}{\sinh\xi}[1+\xi\{1+\coth\xi\}]\,e^{-\xi}\,d\xi, \\ K_1 &= -4\int_0^\infty \frac{\xi^3}{\sinh\xi}[1+\xi\{1+\coth\xi\}]\,e^{-\xi}\,d\xi, \\ K_2 &= \int_0^\infty \frac{\xi^5}{\sinh\xi}[1+\xi\{1+\coth\xi\}]\,e^{-\xi}\,d\xi, \end{aligned}\right\} \tag{6·9}$$

and
$$L_1 = -K_1, \quad L_2 = -2K_2.$$

However,
$$\int_0^\infty \frac{\xi^{2k+1}}{\sinh\xi}[1+\xi\{1+\coth\xi\}]\,e^{-\xi}\,d\xi = (2k+3)\int_0^\infty \frac{\xi^{2k+1}e^{-\xi}\,d\xi}{\sinh\xi},$$
$$= \frac{(2k+3)\,(2k+1)!}{2^{2k+1}}\,\zeta(2k+2), \tag{6·10}$$

where $\zeta(z)$ is the Riemann Zeta Function ((7), p. 32), and hence,

$$K_0 = \pi^2, \quad K_1 = -\tfrac{1}{6}\pi^4, \quad K_2 = \tfrac{1}{3\,6}\pi^6. \tag{6·11}$$

† No confusion should arise over this notation since $F(t)$, $G(t)$, $f(x)$, $g(x)$ are real functions.

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The expansions in (6·8) could be extended to higher orders in (a/d) and the corresponding coefficients found from (6·10) in terms of the Riemann Zeta Function.

Now, writing

$$\left.\begin{aligned}\overline{F}(x) &= \overline{F}_0(x)+\frac{a^2}{d^2}\overline{F}_1(x)+\frac{a^4}{d^4}\overline{F}_2(x)+\frac{a^6}{d^6}\overline{F}_3(x)+O\left\{\left(\frac{a}{d}\right)^8\right\},\\ \overline{G}(x) &= \overline{G}_0(x)+\frac{a^2}{d^2}\overline{G}_1(x)+\frac{a^4}{d^4}\overline{G}_2(x)+\frac{a^6}{d^6}\overline{G}_3(x)+O\left\{\left(\frac{a}{d}\right)^8\right\},\end{aligned}\right\} \tag{6·12}$$

and substituting (6·12) and (6·8) in (6·4), we find

$$\left.\begin{aligned}\overline{F}_0(x) &= -\frac{x}{2\pi}\int_0^x \frac{\bar{f}(y)\,dy}{\sqrt{(x^2-y^2)}},\\ \overline{F}_1(x) &= -xK_0\int_0^1 \overline{F}_0(y)\,dy,\\ \overline{F}_2(x) &= -x\int_0^1 \{K_0\overline{F}_1(y)+K_1(x^2+y^2)\,\overline{F}_0(y)\}\,dy,\\ \overline{F}_3(x) &= -x\int_0^1 \{K_0\overline{F}_2(y)+K_1(x^2+y^2)\,\overline{F}_1(y)+K_2(x^4+4x^2y^2+y^4)\,\overline{F}_0(y)\}\,dy,\end{aligned}\right\} \tag{6·13}$$

and

$$\left.\begin{aligned}\overline{G}_0(x) &= -\frac{1}{2\pi x}\int_0^x \frac{y\bar{g}(y)\,dy}{\sqrt{(x^2-y^2)}},\\ \overline{G}_1(x) &= 0,\\ \overline{G}_2(x) &= -xL_1\int_0^1 y^2\overline{G}_0(y)\,dy,\\ \overline{G}_3(x) &= -xL_2\int_0^1 y^2\overline{G}_0(y)\,(x^2+y^2)\,dy.\end{aligned}\right\} \tag{6·14}$$

The normal displacement over the surfaces of a crack may be obtained from (6·2), (2·5) and (3·7) giving

$$\mu u_y = -(1+\kappa)\int_{|x|}^a \frac{F(t)+xG(t)}{\sqrt{(t^2-x^2)}}\,dt \quad (|x|\leqslant a). \tag{6·15}$$

Particular problems can now be examined, for example, following § 3, if $f(x)=\frac{1}{2}p$, $g(x)=\frac{1}{2}p$, so that a crack is opened by uniform pressure over one-half $(0\leqslant x\leqslant a)$, and zero pressure over the remainder of the crack $(-a\leqslant x\leqslant 0)$, we have

$$\left.\begin{aligned}&\overline{F}_0(x) = -px/8, \quad \overline{F}_1(x) = K_0px/16,\\ &\overline{F}_2(x) = -K_0^2px/32+K_1px(1+2x^2)/32,\\ &\overline{G}_0(x) = -p/4\pi, \quad \overline{G}_1(x) = 0, \quad \overline{G}_2(x) = L_1px/12\pi.\end{aligned}\right\} \tag{6·16}$$

The corresponding displacement is, from (6·15),

$$\begin{aligned}\frac{\mu u_y}{(1+\kappa)} &= \frac{p}{8}\sqrt{(a^2-x^2)}+\frac{px}{4\pi}\cosh^{-1}\left(\frac{a}{|x|}\right)-\frac{pK_0}{16}\frac{a^2}{d^2}\sqrt{(a^2-x^2)}\\ &\quad+\frac{pK_0^2}{32}\frac{a^4}{d^4}\sqrt{(a^2-x^2)}-\frac{pK_1}{32}\frac{a^4}{d^4}\left\{3\sqrt{(a^2-x^2)}-\frac{4}{3a^2}(a^2-x^2)^{\frac{3}{2}}\right\}\\ &\quad-\frac{L_1p}{12\pi a}\frac{a^4}{d^4}x\sqrt{(a^2-x^2)}+O\{(a/d)^6\} \quad (|x|\leqslant a).\end{aligned} \tag{6·17}$$

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If we consider the case of a crack opened by a uniform pressure p, $\bar{G}(t) \equiv 0$, and the $\bar{F}(t)$ terms take twice their values in (6·16). It is of interest in this case to evaluate the change in internal energy per crack due to the opening of the cracks. After a simple calculation this is found to be equal to

$$\frac{\pi p^2(1+\kappa)\,a^2}{16\mu}\left[1-\frac{\pi^2}{2}\frac{a^2}{d^2}+\frac{\pi^4}{3}\frac{a^4}{d^4}+O\left(\frac{a^6}{d^6}\right)\right]. \tag{6·18}$$

Note added in proof. Barenblatt, G. I. and Cherepanov, G. P., *Appl. Math. Mech.* **25** (1961), 1654, have studied certain crack problems in a medium under shearing stress. This work extends that given in references (4) and (5) but does not overlap the present paper. The problems examined in (4) were first considered by Westergaard, H. M., *J. Appl. Mech.* **6** (1939), A 49.

REFERENCES

(1) MUSKHELISHVILI, N. I. *Some basic problems in the mathematical theory of elasticity*, 3rd. edn. (Noordhoff; Groningen, 1953).
(2) GREEN, A. E. and ZERNA, W. *Theoretical elasticity* (Oxford, 1954).
(3) COLLINS, W. D. *Proc. Roy. Soc. London, Ser. A*, **266** (1962), 359.
(4) KOITER, W. T. *Ing.-Arch.* **28** (1959), 168.
(5) KOITER, W. T. *Problems of continuum mechanics*, pp. 246–259 (Soc. Industr. Appl. Math.; Philadelphia, 1961).
(6) SHTAERMAN, I. YA. *The contact problems of the theory of elasticity* (in Russian) (Moscow-Leningrad, 1949).
(7) ERDÉLYI, A. *et al. Higher transcendental functions*, vol. 1 (McGraw-Hill; New York, 1953).
(8) WOODS, L. C. *The theory of subsonic plane flow* (Cambridge, 1961).

Further references to the general complex variable methods for two-dimensional problems are given in (1) and (2), p. 184.

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