Ergod. Th. & Dynam. Sys. (2004), **24**, 945–953
DOI: 10.1017/S014338570300035X
© *2004 Cambridge University Press*
Printed in the United Kingdom

Commuting holomorphic maps of the unit disc

ROBERTO TAURASO

Dipartimento di Matematica, Università di Roma 'Tor Vergata',
Via della Ricerca Scientifica, 00133 Roma, Italy
(e-mail: tauraso@mat.uniroma2.it)

(*Received* 18 *December* 2002 *and accepted* 16 *June* 2003)

Abstract. Let f and g be two commuting holomorphic self-maps of the open unit disc $\mathbb{D}$ in the complex plane with a common Wolff point $\tau \in \partial\mathbb{D}$: if these two maps agree at τ up to the third order, then $f \equiv g$.

1. *Introduction*

The purpose of this paper is to show a connection between iteration theory and the study of commuting holomorphic maps of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

The dynamical properties of a map $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ are well known (see the survey article by Burckel [**6**]): if f is not the identity map, then it has at most one fixed point in $\mathbb{D}$ which is attracting provided f is not an elliptic automorphism (a rotation). However, if f is fixed-point-free, then there is still an attracting point τ, called the *Wolff point* of f, but it is located on the boundary $\partial\mathbb{D}$ and the sequence of iterates f^n converges to τ uniformly on compact subsets of $\mathbb{D}$.

Now take another map $g \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ and assume that it commutes with f:

$$f \circ g = g \circ f.$$

If f has a fixed point $z_0 \in \mathbb{D}$, then

$$f(g(z_0)) = g(f(z_0)) = g(z_0)$$

and, by uniqueness, $g(z_0) = z_0$, that is z_0 is also the fixed point for the map g. If we compare the expansions of f and g at z_0 then, by the commuting property, it is not hard to show that it suffices to check only a finite number of derivatives to prove that f and g coincide: if $f^{(k)}(z_0) = g^{(k)}(z_0) = 0$ for $1 \le k < n$ and $f^{(n)}(z_0) = g^{(n)}(z_0) \neq 0$ for some $n \ge 1$, then $f \equiv g$ (see [**4**]). Note that the number n can be arbitrary large: z^p and z^q commute and their expansions at 0 agree up to the $(\min(p, q) - 1)$-order.

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

When f is fixed-point-free, f and g have the same Wolff point $\tau \in \partial\mathbb{D}$ unless they are two hyperbolic automorphisms with the same set of fixed points (see [**2, 9**]). The common Wolff point τ contains a lot of interesting information about the two maps. To extract this information, we need the following regularity notion: we say that $f \in C^r_K(\tau)$ if it has an expansion at τ of the form

$$f(z) = \tau + f'(\tau)(z-\tau) + \cdots + \frac{1}{r!} f^{(r)}(\tau)(z-\tau)^r + o_K(|z-\tau|^r),$$

where o_K means that the limit is taken non-tangentially (i.e. within an angular region with vertex at τ):

$$\operatorname*{K-lim}_{z\to\tau} \frac{o_K(|z-\tau|^r)}{|z-\tau|^r} = 0.$$

Moreover, we say that $f \in C^r(\tau)$ if $o_K(|z-\tau|^r)$ can be replaced by $o(|z-\tau|^r)$ that is the limit is taken in the full disc. The Julia–Wolff–Carathéodory theorem states that some regularity at τ is expected although τ belongs to the boundary $\partial\mathbb{D}$: $f \in C^1_K(\tau)$ and $0 < f'(\tau) \le 1$. If $f'(\tau) < 1$, then f is called *hyperbolic*, whereas if $f'(\tau) = 1$, then f is called *parabolic*. If $f \in C^2(\tau)$ is parabolic, then $\operatorname{Re}(f''(\tau)\tau) \ge 0$ (see [**3**, p. 52]).

What happens if we compare the expansions of f and g at the common Wolff point τ? As we will see, to establish that $f \equiv g$ it suffices that the two maps agree at τ up to the third order. The case when one of these two maps is the identity is due to Burns and Krantz [**5**] and the general case has been discussed in [**4**].

More precisely, here are the boundary versions of this kind of 'identity principle' for commuting holomorphic maps. For the 'extreme' cases, that is when f is hyperbolic or the identity, we have the following theorem (see [**4**]).

THEOREM 1.1. *If one of the following conditions holds, then* $f \equiv g$.

(1) f *is hyperbolic with a Wolff point at* τ *and* $f'(\tau) = g'(\tau)$.

(2) $f = Id$, $g \in C^3_K(\tau)$ *and* $f(\tau) = g(\tau) = \tau$, $f'(\tau) = g'(\tau) = 1$, $f''(\tau) = g''(\tau) = 0$, $f'''(\tau) = g'''(\tau) = 0$.

In this paper, by using the linear model of Baker and Pommerenke in a more extensive way, we simplify the proof of this theorem and we establish a new result for the 'middle case', that is when f is parabolic. In particular, we improve the corresponding statement appeared in [**4**] by achieving a more 'natural' smoothness conditions:

THEOREM 1.2. *If* f *is parabolic with a Wolff point at* τ *and one of the following conditions holds, then* $f \equiv g$.

(1) $f \in C^2(\tau)$, $g \in C^2_K(\tau)$, $f''(\tau) = g''(\tau) \neq 0$ *and* $\operatorname{Re}(f''(\tau)\tau) > 0$.

(2) $f, g \in C^2(\tau)$, $f''(\tau) = g''(\tau) \neq 0$ *and* $\operatorname{Re}(f''(\tau)\tau) = 0$.

(3) $f \in C^3(\tau)$, $g \in C^3_K(\tau)$, $f''(\tau) = g''(\tau) = 0$ *and* $f'''(\tau) = g'''(\tau)$.

2. *The linear model in* $\mathbb{H}$

The main tool that we are going to use is the construction of a *linear model* for our maps: a 'change of coordinates' in a neighbourhood of the Wolff point which transforms f into an automorphism of the upper half-plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ or of the entire plane $\mathbb{C}$. To simplify notation, from now on we will work in the upper half-plane

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

which is biholomorphically equivalent to the unit disc $\mathbb{D}$ by the *Cayley transformation* $C(z) = i(\tau + z)/(\tau - z)$ that maps τ to ∞. Then, by the Julia–Wolff–Carathéodory Theorem,

$$\operatorname*{K-lim}_{w\to\infty} \frac{F(w)}{w} = \alpha$$

where

$$\alpha := \inf\left\{\frac{\operatorname{Im} F(w)}{\operatorname{Im} w} : w \in \mathbb{H}\right\} \in [0, +\infty).$$

Therefore, if $\alpha > 0$, then $F \in C_K^1(\infty)$, $\alpha = 1/f'(\tau)$ and

$$F(w) = \alpha w + \Gamma(w) \quad \text{with} \quad \Gamma(w) = o_K(|w|).$$

If ∞ is the Wolff point of F, then $\alpha \geq 1$. When $\alpha = 1$ and F is not the identity, then $\Gamma(w) \in \operatorname{Hol}(\mathbb{H}, \mathbb{H})$ and

$$\Gamma(w) = \beta + o_K(1) \qquad \text{if } f \in C_K^2(\tau),$$
$$\Gamma(w) = \beta + \frac{\gamma}{w} + o_K\left(\frac{1}{|w|}\right) \quad \text{if } f \in C_K^3(\tau),$$

where $\beta = if''(\tau)\tau$, $\gamma = 2S_f(\tau)\tau^2/3$ and $S_f(\tau) = f'''(\tau) - \frac{3}{2}(f''(\tau))^2$ (the Schwarzian derivative of f at τ). As we remarked before, if $f \in C^2(\tau)$ is parabolic, then $\operatorname{Im}\beta \geq 0$.

The following result, due to Cowen [**7**], gives some precious information about the orbits behaviour.

THEOREM 2.1. (Cowen) *Let $F \in \operatorname{Hol}(\mathbb{H}, \mathbb{H})$ with a Wolff point at ∞. Then there is an open connected, simply connected set V, called the* fundamental set *for F, such that:*

(1) *V is F-invariant, that is $F(V) \subset V$;*
(2) *for all compact set K of $\mathbb{H}$, the sequence $F^n(K)$ is eventually contained in V;*
(3) *F is univalent in V.*

The pseudohyperbolic distance in $\mathbb{H}$ is defined by

$$d(w, w') = \tanh^{-1}\left|\frac{w - w'}{w - \overline{w'}}\right| \quad \forall w, w' \in \mathbb{H}.$$

We say that F is of *automorphic type* ($F \in \mathcal{A}$) if all orbits are separated in the pseudohyperbolic distance:

$$\lim_{n\to\infty} d(w_{n+1}, w_n) > 0 \quad \forall w \in \mathbb{H}$$

where $w_n = x_n + iy_n = F^n(w)$. This limit exists because F is a d-contraction. If F is hyperbolic, then $F \in \mathcal{A}$; however, if F is parabolic, then it can be of automorphic type or not. Furthermore, $F \notin \mathcal{A}$ if and only if for some $w^0 \in \mathbb{H}$ (and thus for all $w \in \mathbb{H}$)

$$q_n^0 := \frac{w_{n+1}^0 - w_n^0}{w_{n+1}^0 - \overline{w_n^0}} \to 0.$$

Here is the construction of the *linear model* for F due to Baker and Pommerenke (see [**1, 8**]).

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

THEOREM 2.2. (Baker–Pommerenke) *Let $w^0 \in \mathbb{H}$ and*

$$\sigma_n(w) := \frac{w_n - x_n^0}{y_n^0},$$

then the limit

$$\sigma := \lim_{n\to\infty} \sigma_n$$

exists locally uniformly in $\mathbb{H}$ and satisfies $\sigma(w^0) = i$. Moreover, there is an automorphism Φ of $\mathbb{H}$ which fixes ∞ such that

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{F} & \mathbb{H} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathbb{H} & \xrightarrow{\Phi} & \mathbb{H} \end{array}$$

The map $\sigma \not\equiv i$ if and only if $F \in \mathcal{A}$.

If $F \notin \mathcal{A}$, let

$$\rho_n(w) := \frac{\sigma_n(w) - i}{q_n^0},$$

then the limit

$$\rho := \lim_{n\to\infty} \rho_n$$

exists locally uniformly in $\mathbb{H}$ and satisfies $\rho(w^0) = 0$. Moreover,

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{F} & \mathbb{H} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathbb{C} & \xrightarrow{w+2i} & \mathbb{C} \end{array}$$

From now on, the transformations σ and ρ are considered to be always related to the map F. The following key lemma will be very useful later.

LEMMA 2.1. *Let $F, G \in \mathrm{Hol}(\mathbb{H}, \mathbb{H})$ be two commuting maps with a Wolff point at ∞. If one of the following conditions holds, then $F \equiv G$.*

(1) *$F \in \mathcal{A}$ and $\sigma \circ F = \sigma \circ G$.*

(2) *$F \notin \mathcal{A}$ and $\rho \circ F = \rho \circ G$.*

Proof. Let V be a fundamental set for F. Since F is univalent in V and the set V is F-invariant, then also F^n, σ_n, ρ_n are univalent in V for any $n \in \mathbb{N}$. If $F \in \mathcal{A}$, then the limit σ is not constant and, by Hurwitz' theorem, it is univalent in V. In the same way, if $F \notin \mathcal{A}$ then the limit ρ (which is not constant) is univalent in V too. Moreover, if G commutes with F and K is a non-empty compact subset of $G(V)$ then $F^n(K) \subset V$ for some $n \in \mathbb{N}$ and

$$F^n(K) \subset F^n(G(V)) = G(F^n(V)) \subset G(V).$$

Therefore, the open set $G(V) \cap V$ is non-empty because it contains $F^n(K)$ and it is possible to find a non-empty open set $W \subset V$ such that $G(W) \subset V$. Since σ is injective in V and both sets $F(W)$ and $G(W)$ are contained in V, then the condition

$$\sigma(F(w)) = \sigma(G(w)) \quad \forall w \in W$$

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

implies that $F \equiv G$ in $\mathbb{H}$. In a similar way,

$$\rho(F(w)) = \rho(G(w)) \quad \forall w \in W$$

implies that $F \equiv G$ in $\mathbb{H}$. □

As a first application of the previous results, we discuss the cases when f is hyperbolic or f is the identity map.

Proof of Theorem 1.1.

Case 1. We know $F(w) = \alpha_F w + \Gamma_F(w)$, $G(w) = \alpha_G w + \Gamma_G(w)$. Since F and G commute, then

$$\frac{F^n(G(w)) - x_n^0}{y_n^0} = \frac{G(w_n) - x_n^0}{y_n^0} = \alpha_G \frac{w_n - x_n^0}{y_n^0} + (\alpha_G - 1)\frac{x_n^0}{y_n^0} + \frac{\Gamma_G(w_n)}{w_n}\left(\frac{w_n - x_n^0}{y_n^0} + \frac{x_n^0}{y_n^0}\right).$$

By a result due to Cowen (see Lemma 2.2 in [**7**]), w_n tends to ∞ non-tangentially and, therefore, the sequence x_n^0/y_n^0 is bounded and, up to a subsequence, we can assume that it converges to the real number M. Moreover,

$$\lim_{n\to\infty} \frac{\Gamma_G(w_n)}{w_n} = 0.$$

Thus, by Theorem 2.2, taking the limit, we find that

$$\sigma(G(w)) = \alpha_G \sigma(w) + (\alpha_G - 1)M.$$

Similarly,

$$\sigma(F(w)) = \alpha_F \sigma(w) + (\alpha_F - 1)M.$$

If $f'(\tau) = g'(\tau)$, then $\alpha_F = \alpha_G$ and, therefore, $\sigma \circ F \equiv \sigma \circ G$. Since $F \in \mathcal{A}$, by Lemma 2.1, we find that $F \equiv G$.

Case 2. By hypothesis,

$$G(w) = w + \Gamma(w) \quad \text{with } \underset{w\to\infty}{\text{K-lim}}\, w\Gamma(w) = 0.$$

If G is not the identity map then, by the maximum principle, $T(\mathbb{H}) \subset \mathbb{H}$ where $T(w) := -1/\Gamma(w)$. Furthermore,

$$\underset{w\to\infty}{\text{K-lim}} \frac{T(w)}{w} = \underset{w\to\infty}{\text{K-lim}} \frac{-1}{w\Gamma(w)} = \infty$$

and this is a contradiction because, by the Julia–Wolff–Carathéodory theorem applied to the map T, this limit has to be finite. □

3. *The parabolic case*

First we establish a result about the kind of convergence of the orbits of F to ∞. The first case has been proved by Bourdon and Shapiro [**3**].

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

PROPOSITION 3.1. *Let $F \in \mathrm{Hol}(\mathbb{H}, \mathbb{H})$ be parabolic with a Wolff point at ∞.*

(1) *If $F \in C^2(\infty)$ and $\beta \neq 0$, then w_n goes to ∞ non-tangentially if and only if $Im(\beta) > 0$.*

(2) *If $F \in C^3(\infty)$ and $\beta = 0$, then $\gamma < 0$ and w_n has a subsequence that tends to ∞ non-tangentially.*

Proof.

Case 1. If $F \in C^2(\infty)$, then

$$\frac{w_n}{n} = \frac{w}{n} + \beta + \frac{1}{n}\sum_{j=0}^{n-1} \Gamma(w_j).$$

Therefore, since $\Gamma(w) = o(1)$,

$$\lim_{n\to\infty} \frac{w_n}{n} = \beta.$$

Hence,

$$\lim_{n\to\infty} \frac{x_n}{n} = \mathrm{Re}\,\beta \quad \text{and} \quad \lim_{n\to\infty} \frac{y_n}{n} = \mathrm{Im}\,\beta$$

and if $\beta \neq 0$ then w_n goes to ∞ non-tangentially if and only if $\mathrm{Im}(\beta) > 0$.

Case 2. Now $\beta = 0$, $F \in C^3(\infty)$ and since F is not the identity map, then $\gamma < 0$ by Theorem 5.1 in [**4**]. Therefore,

$$w_{n+1} = w_n + \frac{\gamma}{w_n} + o\left(\frac{1}{|w_n|}\right) = w_n + \frac{\gamma\overline{w_n} + |w_n| \cdot o(1)}{|w_n|^2}.$$

Taking the real part, we find that

$$x_{n+1} = x_n\left(1 + \frac{\gamma + (|w_n|/x_n)\,\mathrm{Re}(o(1))}{|w_n|^2}\right).$$

Assume, by contradiction, that the sequence w_n has no subsequence which goes to ∞ non-tangentially. Then the sequence $|x_n|/y_n$ is bounded away from zero and the sequence

$$\frac{|w_n|}{x_n} = \frac{|x_n|}{x_n}\sqrt{1 + \left(\frac{y_n}{x_n}\right)^2}$$

is bounded.

Hence, eventually

$$c_n := 1 + \frac{\gamma + (|w_n|/x_n)\,\mathrm{Re}(o(1))}{|w_n|^2} \in (0, 1)$$

and

$$|x_{n+1}| = c_n|x_n| \leq |x_n|$$

which means that the sequence x_n is also bounded. Since $|w_n|$ goes to infinity, y_n cannot be bounded; therefore, $|x_n|/y_n$ goes to zero against our assumption. □

The following theorem establishes a necessary and sufficient condition so that two parabolic commuting maps coincide.

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

THEOREM 3.1. *Let $F, G \in \mathrm{Hol}(\mathbb{H}, \mathbb{H})$ be two parabolic commuting maps with a Wolff point at ∞. Then the following limit exists locally uniformly in $\mathbb{H}$:*

$$H(w) := \lim_{n\to\infty} \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)}.$$

Moreover, $F \equiv G$ if and only if $H \equiv 1$.

Proof. First assume that $F \in \mathcal{A}$. Thus, by Theorem 2.2, taking the limit of

$$\sigma_n(F(w)) = \sigma_n(w) + \frac{\Gamma_F(w_n)}{y_n^0}$$

we find that

$$\sigma(F(w)) = \sigma(w) + a_F$$

where

$$a_F = \lim_{n\to\infty} \frac{\Gamma_F(w_n)}{y_n^0} \in \mathbb{R} \setminus \{0\}.$$

It cannot be zero otherwise

$$q_n^0 = \frac{1}{2i\dfrac{y_n^0}{\Gamma_F(w_n^0)} + 1} \to 0$$

contradicting the assumption that $F \in \mathcal{A}$.

Moreover, since F and G commute,

$$\sigma_n(G(w)) = \sigma_n(w) + \frac{\Gamma_F(w_n)}{y_n^0} \cdot \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)}$$

and taking the limit we find that

$$H(w) = \frac{\sigma(G(w)) - \sigma(w)}{a_F} = \frac{\sigma(G(w)) - \sigma(w)}{\sigma(F(w)) - \sigma(w)}.$$

Hence, $H \equiv 1$ if and only if $\sigma \circ F \equiv \sigma \circ G$ and, by Lemma 2.1, this is equivalent to $F \equiv G$.

Now we consider the case when $F \notin \mathcal{A}$. Thus, by Theorem 2.2, taking the limit of

$$\rho_n(F(w)) = \rho_n(w) + \frac{\Gamma_F(w_n)}{y_n^0 q_n^0}$$

we find that

$$\rho(F(w)) = \rho(w) + 2i$$

and

$$\lim_{n\to\infty} \frac{\Gamma_F(w_n)}{y_n^0 q_n^0} = 2i.$$

Moreover,

$$\rho_n(G(w)) = \rho_n(w) + \frac{\Gamma_F(w_n)}{y_n^0 q_n^0} \cdot \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)}$$

and taking the limit

$$H(w) = \frac{\rho(G(w)) - \rho(w)}{2i} = \frac{\rho(G(w)) - \rho(w)}{\rho(F(w)) - \rho(w)}.$$

Hence, $H \equiv 1$ if and only if $\rho \circ F \equiv \rho \circ G$ and, by Lemma 2.1, this is equivalent to $F \equiv G$. □

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

Note that, in the previous theorem, the condition that F and G commute is necessary. For example, taking

$$F(w) = w + i \quad \text{and} \quad G(w) = w + i - \frac{1}{(w+i)^N}$$

with $N \geq 1$, it is simple to verify that these two maps coincide up to the $(N+1)$th order at ∞ and $H \equiv 1$ even if they are not equal.

Now we are ready to prove our main result.

Proof of Theorem 1.2. First, we prove that the smoothness conditions for F and G at τ and Proposition 3.1 imply that H is identically constant.

Case 1. Since $\beta_F = if''(\tau)\tau \neq 0$ and $\operatorname{Im}\beta_F = \operatorname{Re}(f''(\tau)\tau) > 0$, then w_n goes to ∞ non-tangentially and

$$H(w) = \lim_{n\to\infty} \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)} = \lim_{w_n\to\infty} \frac{\beta_G + o_K(1)}{\beta_F + o_K(1)} = \frac{\beta_G}{\beta_F}.$$

Case 2. Since $\beta_F \neq 0$ and $\operatorname{Im}\beta_F = 0$, then w_n goes to ∞ tangentially and

$$H(w) = \lim_{n\to\infty} \frac{\Gamma_G(w_n)}{\Gamma_F(w_n)} = \lim_{w_n\to\infty} \frac{\beta_G + o(1)}{\beta_F + o(1)} = \frac{\beta_G}{\beta_F}.$$

Case 3. Since $\beta_F = 0$ then $\gamma_F = 2/3 f'''(\tau)\tau^2 < 0$, w_n has a subsequence that goes to ∞ non-tangentially and

$$H(w) = \lim_{j\to\infty} \frac{\Gamma_G(w_{n_j})}{\Gamma_F(w_{n_j})} = \lim_{w_{n_j}\to\infty} \frac{\gamma_G + w_{n_j} o_K(1/|w_{n_j}|)}{\gamma_F + w_{n_j} o_K(1/|w_{n_j}|)} = \frac{\gamma_G}{\gamma_F}.$$

By the conditions on the derivatives, we obtain that $H \equiv 1$ and, therefore, $F \equiv G$ by the previous theorem. □

Note that if $F(w^0) = G(w^0)$ for some $w^0 \in \mathbb{H}$ and $F \circ G = G \circ F$, then $F^n(w^0) = G^n(w^0)$ for all $n \in \mathbb{N}$. We have already seen that the smoothness conditions for F and G imply that the map H is identically constant. Again this constant is just 1 and, therefore, $F \equiv G$: if $F \in \mathcal{A}$, then

$$H(w^0) = \frac{\sigma(G(w^0)) - \sigma(w^0)}{\sigma(F(w^0)) - \sigma(w^0)} = 1,$$

if $F \notin \mathcal{A}$, then

$$H(w^0) = \frac{\rho(G(w^0)) - \rho(w^0)}{\rho(F(w^0)) - \rho(w^0)} = 1.$$

This remark is non-trivial provided the sequence w_n^0 is not a Blaschke sequence, that is

$$\sum_{j=0}^{\infty} \operatorname{Im}\left(\frac{-1}{F^j(w^0)}\right) < \infty.$$

This happens for the parabolic map of automorphic type

$$F(w) = w + 1 - \frac{1}{w}.$$

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press

In fact, we have that

$$\lim_{n\to\infty} \frac{w_n}{n} = 1$$

and, therefore, $|w_n| \approx n$. Moreover,

$$\begin{aligned}
\operatorname{Im}(w_n) &= \operatorname{Im} w + \sum_{j=0}^{n-1} \operatorname{Im}\left(\frac{-1}{w_j}\right) \\
&\leq \operatorname{Im} w + \sum_{j=0}^{n-1} \frac{1}{|w_j|} \leq C \log n \\
\operatorname{Im}(w_n) &= \operatorname{Im} w + \sum_{j=0}^{n-1} \frac{\operatorname{Im}(w_j)}{|w_j|^2} \\
&\leq \operatorname{Im} w + C \sum_{j=0}^{n-1} \frac{\log j}{j^2} < +\infty.
\end{aligned}$$

Hence,

$$\sum_{j=0}^{\infty} \operatorname{Im}\left(\frac{-1}{w_j}\right) = \lim_{n\to\infty} \operatorname{Im}(w_n) - \operatorname{Im} w < +\infty.$$

REFERENCES

[1] I. N. Baker and Ch. Pommerenke. On the iteration of analytic functions in a half-plane II. *J. London Math. Soc.* **20** (1979), 255–258.

[2] D. F. Behan. Commuting analytic functions without fixed points. *Proc. Amer. Math. Soc.* **37** (1973), 114–120.

[3] P. S. Bourdon and J. H. Shapiro. *Cyclic Phenomena for Composition Operators (Memoirs American Mathematical Society, 596)*. American Mathematical Society, Providence RI, 1997.

[4] F. Bracci, R. Tauraso and F. Vlacci. Identity principles for commuting holomorphic self-maps of the unit disc. *J. Math. Anal. Appl.* **270** (2002), 451–473.

[5] D. M. Burns and S. G. Krantz. Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary. *J. Amer. Math. Soc.* **7** (1994), 661–676.

[6] R. B. Burckel. Iterating analytic self-maps of discs. *Amer. Math. Monthly* **88** (1981), 396–407.

[7] C. C. Cowen. Iteration and the solution of functional equations for functions analytic in the unit disk. *Trans. Amer. Math. Soc.* **265** (1981), 69–95.

[8] Ch. Pommerenke. On the iteration of analytic functions in a half-plane I. *J. London Math. Soc.* **19** (1979), 439–447.

[9] A. L. Shields. On fixed points of commuting analytic functions. *Proc. Amer. Math. Soc.* **15** (1964), 703–706.

https://doi.org/10.1017/S014338570300035X Published online by Cambridge University Press