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# Heat transport by laminar boundary layer flow with polymers

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Motivated by recent experimental observations, we consider a steady-state boundary layer flow with polymers in forced convection above a heated plate and study how the heat transport might be affected by the polymers. We discuss how a set of equations can be derived for the problem and how these equations can be solved numerically by an iterative scheme. By carrying out such a scheme, we find that the effect of the polymers is equivalent to producing a space-dependent effective viscosity that first increases from the zero-shear value at the plate then decreases rapidly back to the zero-shear value far from the plate. We further show that such an effective viscosity leads to a decrease in the streamwise velocity near the plate, which in turn leads to a reduction in heat transport.

**Key words:** boundary layers, polymers

## 1. Introduction

It has been known for more than 60 years that adding polymers into turbulent wall-bounded flows can reduce the friction drag significantly (see, for example, Sreenivasan & White 2000; Procaccia, L'vov & Benzi 2008 and references therein). This phenomenon of drag reduction, or equivalently mass transport enhancement, by polymers has been studied extensively during these 60 or so years. On the other hand, the possible effects of polymers additives on other transport properties like heat transport are much less studied. Recently, an experimental study reported (Ahlers & Nikolaenko 2010) that for turbulent Rayleigh–Bénard (RB) convection of water, confined within a cylindrical cell heated from below and cooled on top, adding polymers to the flow reduces the heat transport. In turbulent RB convection within a container of given shape, the flow state is characterized by two dimensionless parameters: the Rayleigh number ($Ra$) and the Prandtl number ($Pr$), where $Ra$ measures the size of the thermal forcing and $Pr$ is the ratio of the kinematic viscosity to the thermal diffusivity of the fluid. Moreover, there is an exact balance (Siggia 1994) between the heat transport and the energy and thermal dissipation rates. The energy and thermal dissipation rates have contributions from the bulk of the flow as

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well as from the boundary layers near the top and bottom plates (Grossmann & Lohse 2000). The experimental study mentioned (Ahlers & Nikolaenko 2010) was conducted in water at $Ra$ of the order of $10^{10}$. At such a moderate $Ra$, contributions to the energy and thermal dissipation rates from the boundary layers are significant (Grossmann & Lohse 2000). This suggests that the observed reduction in heat transport is likely to be an effect of the polymers interacting with the boundary layer flow in turbulent RB convection. A similar conclusion that the polymers cannot have a large influence on the bulk has also been drawn (Ahlers & Nikolaenko 2010) based on a comparison of the Zimm relaxation time of the polymers and the typical eddy turnover time of the flow. The classical Prandtl–Blasius theory, which was developed for a steady-state fluid flow above a semi-infinite flat plate and also extended to the case of forced convection above a heated plate (Landau & Lifshitz 1987; Schlichting & Gersten 2004), has been applied to the fluctuating boundary layers in turbulent RB convection within a container of finite lateral extent with considerable success (Ahlers, Grossmann & Lohse 2009). In particular, experimental measurements (Zhou *et al.* 2010) indicated that the dynamically time-averaged velocity and temperature boundary layer profiles in turbulent RB convection at moderate $Ra$ could be described by the profiles in the steady-state Prandtl–Blasius boundary layer flow.

Motivated by these experimental observations, we consider a steady-state boundary layer flow with polymers in forced convection above a heated plate and study the possible effect of polymer additives on the heat transport. Physically, we think of this steady-state boundary layer flow as an approximation of the boundary layer flow near the bottom plate of the convection cell in turbulent RB convection in a polymer solution.

This paper is organized as follows. In § 2, we formulate the problem and set up the equations of motion for the system. We discuss how the set of equations can be solved numerically by an iterative scheme in § 3. After solving the problem using such a scheme, our results show that the effect of the polymers is to produce a space-dependent effective viscosity that first increases from the zero-shear value at the plate then decreases rapidly back to the zero-shear value far from the plate. We further show that such an effective viscosity, which is the result of the stretching of the polymers, would lead to a reduction in heat transport. We present and discuss our results in § 4. Finally, we summarize and conclude in § 5.

## 2. The problem

For the Prandtl–Blasius boundary layer flow above a large flat plate, the velocity equation is:

$$v_x \partial_x v_x + v_y \partial_y v_x = \nu \partial_{yy}^2 v_x. \tag{2.1}$$

Here $x$ denotes the direction along the plate, $y$ denotes the direction away from the plate, and $\nu$ is the kinematic viscosity of the fluid. Both $v_x$ and $v_y$ vanish at the plate and $v_x$ merges with the uniform main-stream velocity $U$ far away from the plate. The crucial point about (2.1) is that the viscous term is balanced by the nonlinear advection term and that the flow changes more rapidly away from the plate than along the plate such that $\partial_y \gg \partial_x$. The latter condition is satisfied in flows with large Reynolds number ($Re$).

We first introduce the dimensionless variable

$$\xi \equiv \sqrt{\frac{U}{\nu x}} y \tag{2.2}$$

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and the stream function

$$\Psi(x, y) \equiv \sqrt{\nu x U}\phi(\xi), \tag{2.3}$$

in terms of $\phi(\xi)$, which is a function to be determined, such that

$$v_x = \partial_y \Psi = U\phi'(\xi), \quad v_y = -\partial_x \Psi = \frac{1}{2}\sqrt{\frac{\nu U}{x}}(\xi\phi_\xi - \phi). \tag{2.4}$$

Then we obtain the famous Blasius equation (Landau & Lifshitz 1987)

$$2\phi_{\xi\xi\xi} + \phi\phi_{\xi\xi} = 0 \tag{2.5}$$

showing that the velocity profile is self-similar in $\xi$ at different positions $x$ along the plate. Here $\phi_\xi$ denotes $\partial_\xi \phi$. The boundary conditions are

$$\phi(0) = \phi_\xi(0) = 0, \quad \phi_\xi(\infty) = 1 \tag{2.6}$$

as $v_x \to U$ when far away from the plate. The plate is heated at a temperature $T_1$ above the ambient temperature $T_0$ far away from the plate. Writing the temperature field as

$$T(x, y) = T_0 + (T_1 - T_0)\theta(\xi) \tag{2.7}$$

(relating to turbulent RB convection, $T_0$ can be taken as the temperature at the centre of the cell and thus the total temperature difference across the cell is $2(T_1 - T_0)$), then $\theta$ satisfies the equation

$$2\theta_{\xi\xi} + \phi\theta_\xi Pr = 0 \tag{2.8}$$

where $Pr = \nu/\kappa$ and $\kappa$ is the thermal diffusivity of the fluid, and the boundary conditions are

$$\theta(0) = 1, \quad \theta(\infty) = 0. \tag{2.9}$$

We want to investigate the effect of polymers on heat transport in this laminar Prandtl–Blasius flow. The polymers produce an additional stress in the momentum equation of the fluid. This polymer stress $\mathscr{T}_{ij}$ depends on the amount of stretching of the polymers and is thus a function of the dimensionless conformation tensor $R_{ij}$ of the polymers. Let the vector $\boldsymbol{d}$ denote the end-to-end distance of the polymer and $\rho_0$ be the equilibrium value of $|\boldsymbol{d}|$ in the unstretched regime, then $R_{ij}$ represents the average over many ($N$) polymers in a small region around the point $(x, y)$ of the product $d_i d_j/\rho_0^2$, i.e. $R_{ij} = N^{-1}\sum d_i d_j/\rho_0^2$. In the simplest Oldroyd-B model of polymers (Bird *et al.* 1987),

$$\mathscr{T}_{ij} = \frac{\nu_p}{\tau}(R_{ij} - \delta_{ij}), \tag{2.10}$$

where $\tau$ is the (longest) relaxation time of the polymers, and $\nu_p$ is the polymer contribution to the viscosity of the solution at zero shear, which depends on the concentration of the polymers. Thus in the presence of polymers, the equation of motion for the velocity field is modified by an additional stress that depends on $R_{ij}$. By employing the same ideas leading to the Blasius equation (2.1) we have

$$v_x\partial_x v_x + v_y\partial_y v_x = \nu\partial_{yy}^2 v_x + \partial_y\left[\frac{\nu_p}{\tau}R_{xy}\right]. \tag{2.11}$$

It has been found (Larson *et al.* 1997) that the friction coefficient of the force exerted by the fluid on a polymer depends on the polymer configuration: the

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friction coefficient is smaller when the polymers are in the coiled configuration and larger when they are stretched. Thus the relaxation time of the polymers, which is proportional to the friction coefficient, is also increased by the stretching of the polymers. In Doyle *et al.* (1998) and Celani, Puliafito & Vincenzi (2006), the friction coefficient is approximated to increase linearly with the end-to-end distance of the polymers. This implies that $\tau$ is a linear function of $R \equiv (R_{xx} + R_{yy})^{1/2}$:

$$\tau = \tau_0 \left( \frac{1 + aR}{1 + a} \right), \tag{2.12}$$

where $\tau_0$ is the Zimm relaxation time when the polymers are not stretched and $a > 0$ is a parameter. The ratio of the relative changes in the relaxation time and the polymer end-to-end distance is given by $a/(1 + a)$. When there is no stretching, $R = 1$ and $\tau$ reduces back to $\tau_0$. When there is stretching, $R > 1$ and $\tau > \tau_0$.

In the presence of polymers, the transformation using $\xi$ does not lead to a similarity solution in that explicit appearance of $x$ remains in the equation for $\phi$. This is well-known in the literature. A similarity solution has been obtained in some special cases with some special velocity or temperature boundary conditions (Olagunju 2006; Bataller 2008) that do not have obvious physical relevance. Here, to circumvent this difficulty, we recall that the Prandtl–Blasius flow is meant to be applicable when $x$ is large (such that $\partial_y \gg \partial_x$). Thus we make the following approximations:

$$v_y \approx \frac{1}{2}\sqrt{\frac{\nu_0 U}{L}}(\xi\phi_\xi - \phi), \tag{2.13}$$

$$\partial_x \approx -\frac{\xi}{2L}\frac{\mathrm{d}}{\mathrm{d}\xi}, \quad \partial_y \approx \sqrt{\frac{U}{\nu_0 L}}\frac{\mathrm{d}}{\mathrm{d}\xi}, \tag{2.14}$$

by putting $x = L$, where $L$ is the length of the large plate. In the presence of polymers, even when they are not stretched, the viscosity of the solution increases. Specifically, the zero-shear viscosity of the polymer solution is $\nu_0 = \nu + \nu_p$. Thus, we replace $\nu$ by $\nu_0$ in $\xi$ (equation (2.2)), namely,

$$\xi = \sqrt{\frac{U}{\nu_0 x}}y \quad \text{with polymers.} \tag{2.15}$$

With these approximations, the scaling transformation of $\xi$ leads to a similarity solution. The resulting modified Blasius equation is

$$-\frac{1}{2}\phi\phi_{\xi\xi} = (1 - \gamma)\phi_{\xi\xi\xi} + \frac{\gamma}{Wi\sqrt{Re}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{(1 + a)R_{xy}}{1 + aR}\right], \tag{2.16}$$

where the Weissenberg number ($Wi$) and $Re$ are defined as

$$Wi \equiv \frac{\tau_0 U}{L}, \quad Re \equiv \frac{UL}{\nu_0}, \tag{2.17}$$

and $\gamma \equiv \nu_p/\nu_0$ is a function of the polymer concentration. As usual in the Prandtl–Blasius approximation, all terms of the order of $1/Re$ are neglected in (2.16). In the present problem, we assume that the main-stream velocity $U$ remains the same in the presence of polymers. At the end of § 4, we shall discuss how a change in $U$ by the polymers would modify our findings and show that we expect only a minor change in our results.

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It is common to measure the heat flux $Q$ in terms of the dimensionless Nusselt number ($Nu$), which is the ratio of $Q$ to its value when there is only conduction. In turbulent RB convection, $Nu$ is defined by

$$Nu = \frac{Q}{2k(T_1 - T_0)/H} = \frac{\left\langle -\left.\frac{\partial T}{\partial y}\right|_{y=0} \right\rangle_A}{2(T_1 - T_0)/H}, \tag{2.18}$$

where $k$ is the thermal conductivity of the fluid, $H$ is the height of the convection cell, and $\langle \ldots \rangle_A$ is the average over the cross-section of the cell. For the Prandtl–Blasius flow, taking $H = L$ and dropping the numerical factor, $Nu$ can be written as

$$Nu = \sqrt{\frac{UL}{\nu_0}}[-\theta_\xi(0)]. \tag{2.19}$$

We would like to compare the value of $Nu$ in the presence of polymers to the reference value $Nu_0$ for a Newtonian fluid with the same kinematic viscosity $\nu_0$. In doing so, we focus on the possible effect due to polymer stretching, i.e. the non-Newtonian nature of the polymers, and not merely the increase of the zero-shear viscosity from $\nu$ to $\nu_0$. To get the reference value $Nu_0$, we start with the Prandtl–Blasius velocity profile for a Newtonian fluid of kinematic viscosity $\nu_0$, denoted as $\phi^{(B)}(\xi)$. This is just the solution to (2.5) with $\xi$ given by (2.15) or (2.16) with $\gamma = 0$. With this $\phi^{(B)}$, we solve (2.8) and obtain the resulting temperature profile $\theta^{(B)}(\xi)$ at the same $Pr = \nu_0/\kappa$ as the polymer solution. Then

$$Nu_0 = \sqrt{\frac{UL}{\nu_0}}[-\theta_\xi^{(B)}(0)] \tag{2.20}$$

and we are interested in the ratio

$$\frac{Nu}{Nu_0} = \frac{\theta_\xi(0)}{\theta_\xi^{(B)}(0)}. \tag{2.21}$$

To proceed, we must supplement (2.16) with specific information on $R_{xy}$. In a fluid flow of velocity $\boldsymbol{v}$, the components of the dimensionless polymer end-to-end distance $l_i = d_i/\rho_0$, $i = x, y$, obey the differential equations:

$$\frac{\mathrm{d}l_i}{\mathrm{d}t} = -\frac{1}{2\tau}(l_i - l_{0i}) + l_j \partial_j v_i, \tag{2.22}$$

where $l_{0x} = \cos\alpha$, $l_{0y} = \sin\alpha$, $\alpha$ being the angle of the equilibrium orientation, and we have neglected the thermal noise. We rewrite (2.22) for the two-dimensional Prandtl–Blasius flow as

$$-\frac{L}{U}\frac{\mathrm{d}l_x}{\mathrm{d}t} = \frac{(1+a)(l_x - l_{0x})}{2Wi(1+aR)} + \frac{\xi\phi_{\xi\xi}}{2}l_x - \sqrt{Re}\phi_{\xi\xi}l_y, \tag{2.23}$$

$$-\frac{L}{U}\frac{\mathrm{d}l_y}{\mathrm{d}t} = \frac{(1+a)(l_y - l_{0y})}{2Wi(1+aR)} + \frac{\xi^2\phi_{\xi\xi}}{4\sqrt{Re}}l_x - \frac{\xi\phi_{\xi\xi}}{2}l_y. \tag{2.24}$$

In order to obtain $l_i$ as a function of $\xi$, we assume that each polymer follows the streamlines of the flow such that $\mathrm{d}\xi/\mathrm{d}t = v_x\partial_x\xi + v_y\partial_y\xi = -(U/2L)\phi$, again using the approximations in (2.13) and (2.14). Thus $(L/U)\,\mathrm{d}/\mathrm{d}t = -(\phi/2)\,\mathrm{d}/\mathrm{d}\xi$

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and (2.23) and (2.24) become

$$\phi \frac{\mathrm{d}l_x}{\mathrm{d}\xi} = \frac{(1+a)(l_x - l_{0x})}{Wi(1+aR)} + \xi\phi_{\xi\xi} l_x - 2\sqrt{Re}\phi_{\xi\xi} l_y, \tag{2.25}$$

$$\phi \frac{\mathrm{d}l_y}{\mathrm{d}\xi} = \frac{(1+a)(l_y - l_{0y})}{Wi(1+aR)} + \frac{\xi^2\phi_{\xi\xi}}{2\sqrt{Re}} l_x - \xi\phi_{\xi\xi} l_y. \tag{2.26}$$

The quantity $R_{xy}$ is the average of $l_x l_y$ over the polymers contained in a small volume centred near the point $(x, y)$. The equilibrium magnitude of $\boldsymbol{l}$ for these all polymers is equal to 1 but each of the polymers can have a different value of the angle $\alpha$. Thus the ensemble average over the polymers is equivalent to an average over $\alpha$, which has a uniform distribution in $[0, 2\pi]$. That is, $R_{xy} = \langle l_x l_y \rangle_\alpha$. Such an average over $\alpha$ can be evaluated analytically only in the limit of small $Wi$. For small $Wi$, the left-hand sides of (2.25) and (2.26) are negligible as the stretching of the polymers due to the velocity field is balanced by the relaxation, represented by the term with the factor $1/Wi$. As a result, one can solve $l_x$ and $l_y$ in terms of $\phi$ and its derivatives, and then carry out the averaging over $\alpha$ explicitly to obtain:

$$R_{xy} = \langle l_x l_y \rangle_\alpha = Wi\sqrt{Re}\left(\frac{1+aR}{1+a}\right)\phi_{\xi\xi} + \langle (l_x - l_{0x})\,(l_y - l_{0y})\rangle_\alpha \tag{2.27}$$

and $\langle (l_x - l_{0x})\,(l_y - l_{0y})\rangle_\alpha = Wi^2\sqrt{Re}\,[(1+aR)/(1+a)]^2\,\xi\phi_{\xi\xi}^2$. Here we have further neglected terms of order $1/\sqrt{Re}$. We expect (2.27) to remain true for general $Wi$. Using (2.27), we shall see that the physical effects of the polymers is to produce a space-dependent effective viscosity. To show this, note that the first term on the right-hand side of (2.27) is just $\tau\partial_y v_x$. Thus (2.27) can be further rewritten as

$$\frac{R_{xy}}{\tau} = \left[1 + \frac{\langle (l_x - l_{0x})\,(l_y - l_{0y})\rangle_\alpha}{Wi\sqrt{Re}\phi_{\xi\xi}(1+aR)/(1+a)}\right]\partial_y v_x. \tag{2.28}$$

Hence the polymer term in (2.11) can be written in terms of a space-dependent viscosity $\nu_{eff}(\xi)$:

$$\partial_y\left[\frac{\nu_p}{\tau}R_{xy}\right] \equiv \partial_y[\nu_{eff}(\xi)\partial_y v_x] \tag{2.29}$$

and

$$\nu_{eff}(\xi) = \nu_p\left[1 + \frac{\langle (l_x - l_{0x})\,(l_y - l_{0y})\rangle_\alpha}{Wi\sqrt{Re}\phi_{\xi\xi}(1+aR)/(1+a)}\right]. \tag{2.30}$$

Moreover, we see that the space-dependent part is due to polymer stretching. We note that in strongly turbulent thermal convection, non-Boussinesq effects could be important and the temperature dependence of the viscosity could then result in a variable or space-dependent viscosity (see, for example, Cebeci 2002; Ahlers *et al.* 2006). On the other hand, in our present work, the zero-shear viscosity of the polymer solution is temperature independent and a space-dependent effective viscosity arises because of the stretching of the polymers.

For the case of general $Wi$, the averaging over $\alpha$ has to be evaluated numerically, which is very cumbersome. So we make an approximation. We see that the equations of motion are invariant under inversion and reflection, thus averaging over $0 \leqslant \alpha \leqslant 2\pi$ is equivalent to averaging over $0 \leqslant \alpha \leqslant \pi/2$. Hence, we

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approximate $\langle (l_x - l_{0x})\,(l_y - l_{0y})\rangle_\alpha$ by its value evaluated at the average value of $\alpha$ in $[0, \pi/2]$, which is $\pi/4$:

$$\langle (l_x - l_{0x})\,(l_y - l_{0y})\rangle_\alpha \approx (l_x - l_{0x})\,(l_y - l_{0y})_{\alpha=\pi/4} = (\hat{l}_x - l_0)(\hat{l}_y - l_0) \tag{2.31}$$

where $\hat{l}_x$ and $\hat{l}_y$ are the results for $\alpha = \pi/4$, i.e. they satisfy (2.25) and (2.26) with $l_{0x} = l_{0y} = l_0 = 1/\sqrt{2}$:

$$\phi \frac{\mathrm{d}\hat{l}_x}{\mathrm{d}\xi} = \frac{(1+a)(\hat{l}_x - l_0)}{Wi(1+aR)} + \xi\phi_{\xi\xi}\hat{l}_x - 2\sqrt{Re}\phi_{\xi\xi}\hat{l}_y, \tag{2.32}$$

$$\phi \frac{\mathrm{d}\hat{l}_y}{\mathrm{d}\xi} = \frac{(1+a)(\hat{l}_y - l_0)}{Wi(1+aR)} + \frac{\xi^2\phi_{\xi\xi}}{2\sqrt{Re}}\hat{l}_x - \xi\phi_{\xi\xi}\hat{l}_y. \tag{2.33}$$

This approximation is exact to the leading order in the small-$Wi$ limit. Using this approximation, we have

$$\begin{aligned} R_{xy} &\approx Wi\sqrt{Re}\left(\frac{1+aR}{1+a}\right)\phi_{\xi\xi} + (\hat{l}_x - l_0)(\hat{l}_y - l_0) \\ &= Wi\sqrt{Re}\frac{(1+aR)}{(1+a)}\phi_{\xi\xi}(1+g), \end{aligned} \tag{2.34}$$

where

$$g(\xi) \equiv \frac{[\hat{l}_x(\xi) - l_0][\hat{l}_y(\xi) - l_0]}{Wi\sqrt{Re}\phi_{\xi\xi}(\xi)[1 + aR(\xi)]/(1+a)} \tag{2.35}$$

with $R(\xi) = [\hat{l}_x^2(\xi) + \hat{l}_y^2(\xi)]^{1/2}$. Moreover, $\nu_{eff} \approx \nu_p(1+g)$ and the total effective viscosity of the polymer solution is

$$\nu_{tot}(\xi) = \nu + \nu_{eff}(\xi) \approx \nu_0 + \nu_p g(\xi). \tag{2.36}$$

Substituting (2.34) into (2.16), we have

$$2\phi_{\xi\xi\xi} + \phi\phi_{\xi\xi} + 2\gamma\frac{\mathrm{d}}{\mathrm{d}\xi}(g\phi_{\xi\xi}) = 0. \tag{2.37}$$

Solving (2.32), (2.33), (2.35) and (2.37) gives $\phi(\xi)$ for the polymer-laden flow. With this $\phi(\xi)$, (2.8) is then solved to obtain $\theta(\xi)$. $Nu$ is then found from $\theta_\xi(0)$.

## 3. Calculations

We solve (2.32), (2.33), (2.35) and (2.37) consistently by iteration at fixed values of $Re$ and $a$. We start with $\phi = \phi^{(B)}$ in (2.32) and (2.33) and solve for $\hat{l}_x(\xi)$ and $\hat{l}_y(\xi)$. Using the boundary conditions that $\hat{l}_x$ and $\hat{l}_y$ tend to $l_0$ as $\xi \to \infty$, we put $\hat{l}_x = \hat{l}_y = l_0$ at some large value of $\xi$, denoted as $\xi_\infty$ (we use $\xi_\infty = 30$), and integrate backwards until $\xi$ equals some finite $\xi_0 \approx 0.2$ close to zero. We cannot integrate forwards in $\xi$ because $\phi$ vanishes at $\xi = 0$ (see (2.6)). Then we extrapolate $\hat{l}_x$ and $\hat{l}_y$ for $0 < \xi < \xi_0$ by a quadratic fit ($\hat{l}_x(0)$ and $\hat{l}_y(0)$ can be obtained from (2.32) and (2.33)). With such calculated $\hat{l}_x$ and $\hat{l}_y$, we obtain $g(\xi)$ using (2.35). From (2.33), we see that $\hat{l}_y(0) = l_0$ and thus $g(0) = 0$. For ease of computation, we fit the resulting $g(\xi)$ by an analytical form and input this fitted form for $g$ into (2.37) to solve for an updated $\phi$. Then we use this updated $\phi$ in (2.32) and (2.33) to obtain an updated $g$. We repeat the procedure until convergence in both $g$ and $\phi$ is achieved.

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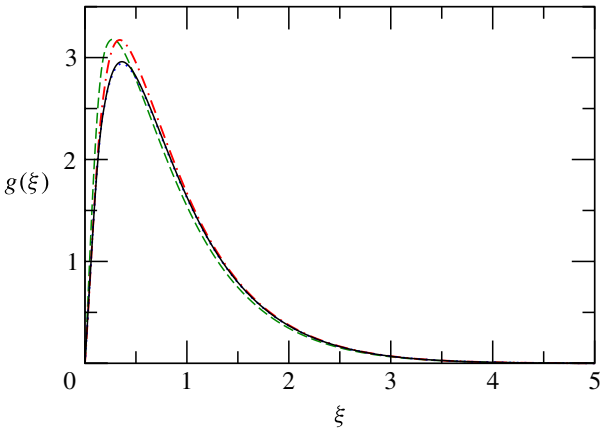



FIGURE 1. (Colour online available at journals.cambridge.org/flm) Fast convergence of the numerical iterative procedure. Result for $g(\xi)$ obtained after the first (dashed), second (dot-dashed), fifth (dotted) and tenth iteration (solid).

In figure 1, we show $g(\xi)$ obtained using this iterative procedure for $Re = 4900$, $a = 0.01$, $Wi = 2.8$ and $\gamma = 0.2$. Convergence is fast and achieved after only a few iterations. Using the converged $\phi$, we obtain $Nu$ as discussed in § 2 and study the ratio $Nu/Nu_0$ for different values of the parameters.

We note a technical point in the mathematical treatment of the problem, namely, there is a constraint on the value of $a$ imposed by the realizability of $\hat{l}_x(0)$. From (2.32) and $\hat{l}_y(0) = l_0$, we see that $\hat{l}_x(0)$ satisfies the following equation:

$$\frac{1 + 1/a}{2Wi\sqrt{Re}\phi_{\xi\xi}(0)l_0}[\hat{l}_x(0) - l_0] = \sqrt{\hat{l}_x(0)^2 + l_0^2} + \frac{1}{a}. \tag{3.1}$$

Thus the condition for (3.1) to have a finite real solution for $l_x(0)$ is

$$\frac{a}{1 + a} < \left[\sqrt{2}Wi\sqrt{Re}\phi_{\xi\xi}(0)\right]^{-1}. \tag{3.2}$$

This constraint means that for given values of $Re$ and $Wi$, the fraction $a/(1 + a)$ cannot be too large, which is physically reasonable since $a/(1 + a)$ is the ratio of the relative change in the relaxation time to the relative change in the polymer end-to-end distance. In all the cases that we have studied, we use a value of $a$ such that $a/(1 + a)$ is a few per cent, compatible with Doyle *et al.* (1998). Specifically, we have studied two values of $Re$: 100 and 4900. For $Re = 4900$, we take $a = 0.01$ and study different values of $Wi$ up to $\sim$2.8 for $\gamma = 0.2$. For $Re = 100$, we take $a = 0.07$ and study different values of $Wi$ up to 3.0 for the same value of $\gamma$. We then fix $Wi = 2.5$ and study the effect of polymer concentration by varying the value of $\gamma$ for $Re = 4900$ and $a = 0.01$. We also study a few other values of the parameters to investigate the dependence of the results on the parameters.

## 4. Results and discussions

We shall first focus on results for $Pr = 4.4$, the value used in the experiment (Ahlers & Nikolaenko 2010). Results for other values ($Pr = 1$, 10, 50 and 100) will be discussed near the end of this section. In figure 1, we see that $g(\xi)$ increases from

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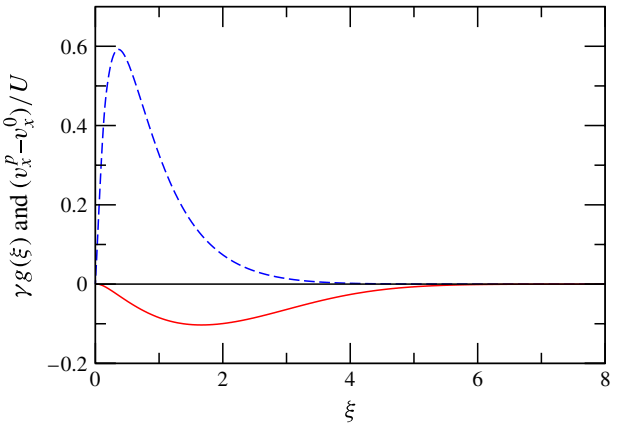



FIGURE 2. (Colour online) Effective viscosity due to the polymers as represented by $\gamma g(\xi)$ (dashed curve) and the difference in the horizontal velocity of the polymer solution from that of the pure solvent, $(v_x^p - v_x^0)/U$ (solid curve), as a function of $\xi$ for $Re = 4900$, $Pr = 4.4$, $a = 0.01$, $Wi = 0.28$ and $\gamma = 0.2$.

zero at the plate ($\xi = 0$) up to a certain maximum then decreases rather rapidly back to zero far away from the plate ($\xi \to \infty$). Whenever the polymers stretch, $g(\xi)$ is larger than zero, and the effective viscosity of the polymer solution is enhanced. We expect that such an increase in the viscosity near the plate leads to a reduction in the horizontal velocity in the region near the plate. Indeed, we find that $v_x^p = U\phi_\xi$, the horizontal velocity for the flow with polymers, is reduced compared to $v_x^0 = U\phi_\xi^{(B)}$, the horizontal velocity for the flow without polymers, near the plate (see figure 2).

This decrease in the horizontal velocity implies a reduction in the heat transport. To see this, we note that upon double integration of (2.8) by $\xi$, we obtain:

$$-\theta_\xi(0) = \frac{1}{\displaystyle\int_0^\infty \mathrm{d}t \exp\left[-\frac{Pr}{2}\int_0^t \phi(s)\,\mathrm{d}s\right]} \tag{4.1}$$

which tells us that $Nu$ is a functional of $\phi$. Defining $\Phi(\xi) \equiv \int_0^\xi \phi(s)\,\mathrm{d}s$, we can calculate the change in $Nu$, $\delta Nu = Nu - Nu_0$, due to a change in $\Phi$ caused by the polymers:

$$\delta Nu = \frac{Pr}{2\sqrt{Re}} Nu_0^2 \int_0^\infty \exp[-Pr\Phi(s)/2]\delta\Phi(s)\,\mathrm{d}s. \tag{4.2}$$

Now $\int_0^\xi v_x/U\,\mathrm{d}\xi = \phi(\xi)$. A decrease in the streamwise velocity, therefore, implies a decrease in $\phi$ and thus $\delta\Phi < 0$. Hence (4.2) shows that a decrease in the streamwise velocity implies $\delta Nu < 0$ and thus $Nu/Nu_0 < 1$. This reduction in $Nu$ is observed, as shown in figure 3.

A decrease in the streamwise velocity near the plate also indicates a reduction in the mass throughput near the plate, which one expects to be accompanied by an enhancement in friction drag, measured by the drag coefficient $C$ (Landau & Lifshitz 1987):

$$C = \frac{\nu_0 \partial_y v_x|_{y=0}}{U^2/2} = \frac{1}{2\sqrt{Re}}\phi_{\xi\xi}(0). \tag{4.3}$$

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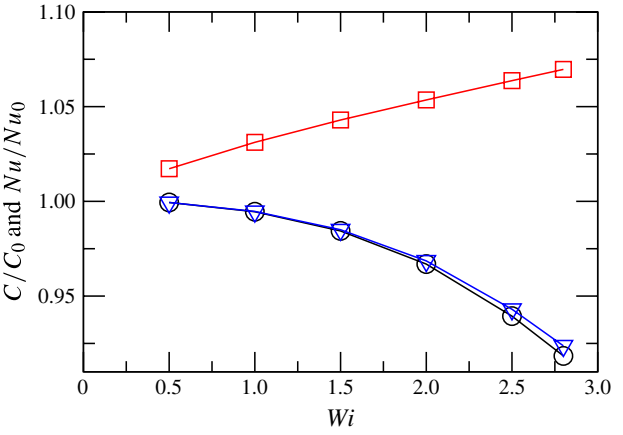



FIGURE 3. (Colour online) $C/C_0$ (squares) and $Nu/Nu_0$ (circles) as a function of $Wi$ for $Re = 4900$, $Pr = 4.4$, $a = 0.01$ and $\gamma = 0.2$. The triangles are the maximum possible values for $Nu/Nu_0$ when the main-stream velocity can be increased in the presence of polymers (see discussion near the end of § 4.)

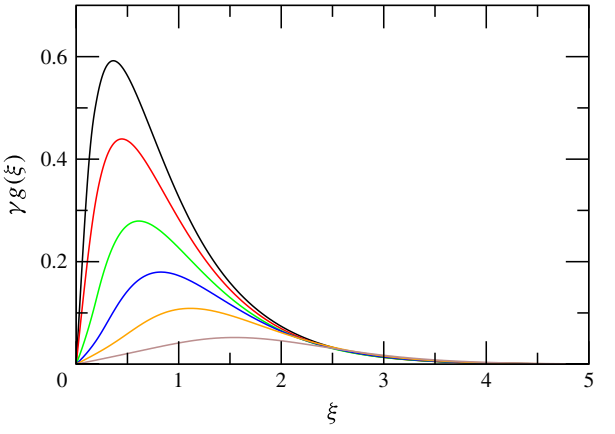



FIGURE 4. (Colour online) Dependence of the effective viscosity as measured by $\gamma g(\xi)$ on $Wi$ at fixed polymer concentration for $Re = 4900$, $a = 0.01$ and $\gamma = 0.2$. From bottom to top, $Wi$ increases from 0.5, 1.0, 1.5, 2.0, 2.5 to 2.8.

We are interested in the ratio

$$\frac{C}{C_0} = \frac{\phi_{\xi\xi}(0)}{\phi_{\xi\xi}^{(B)}(0)}. \tag{4.4}$$

It is indeed found that $C/C_0 > 1$ (see also figure 3).

As shown in figure 3, the amount of heat reduction (HR) and drag enhancement increases with $Wi$. This is understood to be the result of an increase in the effective viscosity with $Wi$. In figure 4, we show the dependence of $\gamma g(\xi)$ on $Wi$. It can be seen that the effective viscosity increases with $Wi$ while the region in which the polymers are stretched such that $g(\xi) > 0$ is approximately independent of $Wi$.

In figures 5 and 6, we show the dependence of the amount of HR, $1 - Nu/Nu_0$, and drag enhancement (DE), $C/C_0 - 1$, on $Wi$ at a fixed polymer concentration for the two

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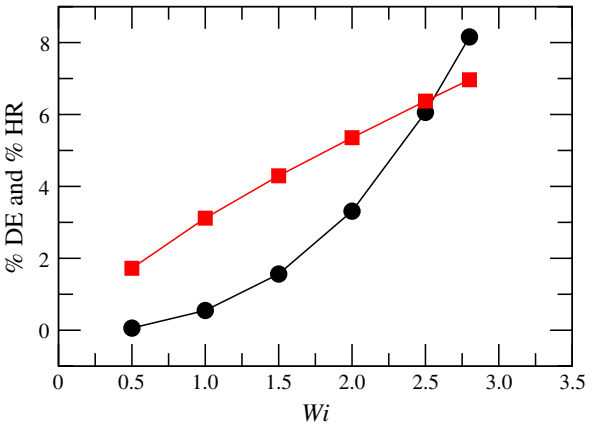



FIGURE 5. (Colour online) Percentage of drag enhancement (% DE) (squares) and heat reduction (% HR) (circles) as a function of $Wi$ at fixed $\gamma = 0.2$ for $Re = 4900$, $Pr = 4.4$ and $a = 0.01$.

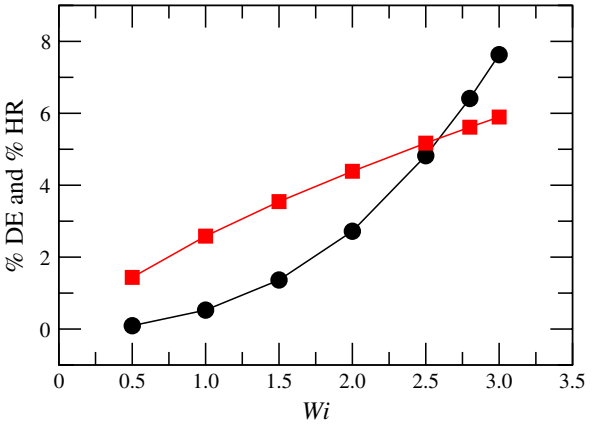



FIGURE 6. (Colour online) % DE (squares) and % HR (circles) as a function of $Wi$ at fixed $\gamma = 0.2$ for $Re = 100$, $Pr = 4.4$ and $a = 0.07$.

different sets of values of $Re$ and $a$ studied. The effect is generally larger when $Re$ is larger. Both % HR and % DE increase as $Wi$ increases.

In figure 7, we fix $Re = 4900$, $Pr = 4.4$, $a = 0.01$, $Wi = 2.5$ and show the extent of HR and DE as a function of polymer concentration specified by $\gamma$. We see that the effect increases with $\gamma$ as expected. In figure 8, we show the dependence of $\gamma g(\xi)$ on $\gamma$. Again we see that $\gamma g$ increases with $\gamma$. It is this increase of the effective viscosity of the polymers with polymer concentration that leads to the increase in the extent of HR and DE when polymer concentration increases.

We summarize the results for % HR and % DE obtained for different values of the parameters for $Pr = 4.4$ in table 1. We see that the effect generally increases with $a$, $Re$, $Wi$ and $\gamma$ while the other parameters are kept fixed. We note that HR and DE are found even for $a = 0$ although the effect is relatively modest.

Next, we study the possible dependence of the polymer effects on $Pr$. The amount of DE depends only on the ratio $\phi_{\xi\xi}(0)/\phi^{(B)}_{\xi\xi}(0)$ and % DE is thus independent of $Pr$.

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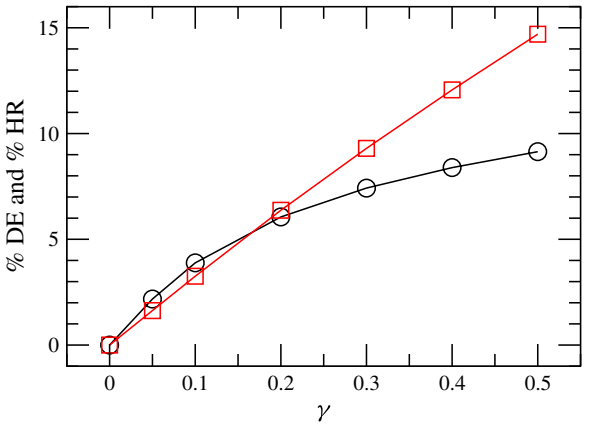



FIGURE 7. (Colour online) Dependence of the effect on polymer concentration: % DE (squares) and % HR (circles) as a function of $\gamma$ at fixed $Wi = 2.5$ with $Re = 4900$, $Pr = 4.4$ and $a = 0.01$.

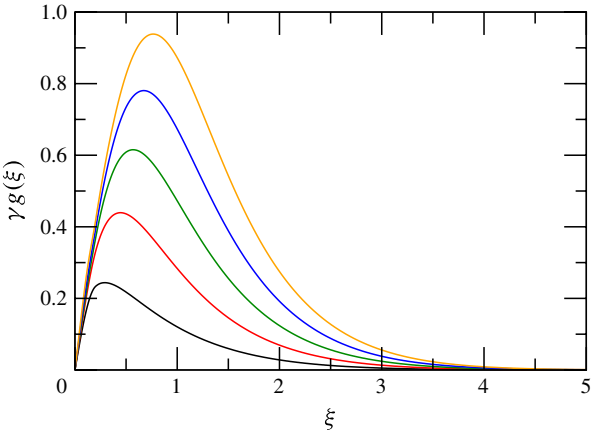



FIGURE 8. (Colour online) Dependence of $\gamma g(\xi)$ on $\gamma$ at fixed $Wi = 2.5$. From bottom to top, $\gamma$ increases from 0.1 to 0.5 in steps of 0.1.

Regarding the amount of HR, from (4.2), we see that the factor in front of the integral increases as $Pr$ while the integrand decreases exponentially with $Pr$. Thus we expect there would be a maximum amount of HR at a certain value $Pr_m$, and % HR decreases for both $Pr < Pr_m$ and $Pr > Pr_m$. This is indeed observed as shown in figure 9 and $Pr_m$ is found to be $\approx 10$.

Finally, we discuss how a change in the main-stream velocity by the polymers might modify our findings. Denote the main-stream velocity in the presence of polymers by $U_p$ and define $C_p$ and $Nu_p$ as the corresponding drag coefficient and $Nu$ for such a flow with main-stream velocity $U_p$. In our discussions so far, we have assumed $U_p = U$ and the drag coefficient and change in $Nu$ obtained under this assumption are $C$ and $\delta Nu$ respectively. Using (4.3) and (4.2), we have

$$\frac{C}{C_p} = \sqrt{\frac{U_p}{U}}, \quad \delta Nu_p = (Nu_p - Nu_0) = \delta Nu \sqrt{\frac{U}{U_p}}. \tag{4.5}$$

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| $Re$ | $a$ | $Wi$ | $\gamma$ | % HR | % DE |
|---|---|---|---|---|---|
| 100 | 0.07 | 1.0 | 0.2 | 0.5 | 2.6 |
| 100 | 0.07 | 1.5 | 0.2 | 1.4 | 3.5 |
| 100 | 0.07 | 2.0 | 0.2 | 2.7 | 4.4 |
| 100 | 0.07 | 2.5 | 0.2 | 4.8 | 5.2 |
| 100 | 0.07 | 2.8 | 0.2 | 6.4 | 5.6 |
| 100 | 0.07 | 3 | 0.2 | 7.6 | 5.9 |
| 4900 | 0.0 | 2.5 | 0.5 | 3.0 | 11.2 |
| 4900 | 0.0 | 2.5 | 0.2 | 1.5 | 4.7 |
| 4900 | 0.0 | 2.8 | 0.2 | 1.8 | 5.0 |
| 4900 | 0.1 | 0.3 | 0.2 | 0.7 | 2.2 |
| 4900 | 0.05 | 0.6 | 0.2 | 1.7 | 3.4 |
| 4900 | 0.005 | 3 | 0.2 | 4.2 | 6.1 |
| 4900 | 0.005 | 4 | 0.2 | 7.8 | 7.3 |
| 4900 | 0.005 | 5 | 0.2 | 12.1 | 8.4 |
| 4900 | 0.01 | 0.5 | 0.2 | 0.06 | 1.7 |
| 4900 | 0.01 | 1.0 | 0.2 | 0.6 | 3.1 |
| 4900 | 0.01 | 1.5 | 0.2 | 1.6 | 4.3 |
| 4900 | 0.01 | 2.0 | 0.2 | 3.3 | 5.4 |
| 4900 | 0.01 | 2.5 | 0.2 | 6.1 | 6.3 |
| 4900 | 0.01 | 2.8 | 0.2 | 8.2 | 7.0 |
| 4900 | 0.01 | 2.5 | 0.05 | 2.2 | 1.6 |
| 4900 | 0.01 | 2.5 | 0.1 | 3.9 | 3.2 |
| 4900 | 0.01 | 2.5 | 0.3 | 7.4 | 9.3 |
| 4900 | 0.01 | 2.5 | 0.4 | 8.4 | 12.1 |
| 4900 | 0.01 | 2.5 | 0.5 | 9.1 | 14.7 |

TABLE 1. Percentages of HR and DE for different values of parameters for $Pr = 4.4$.

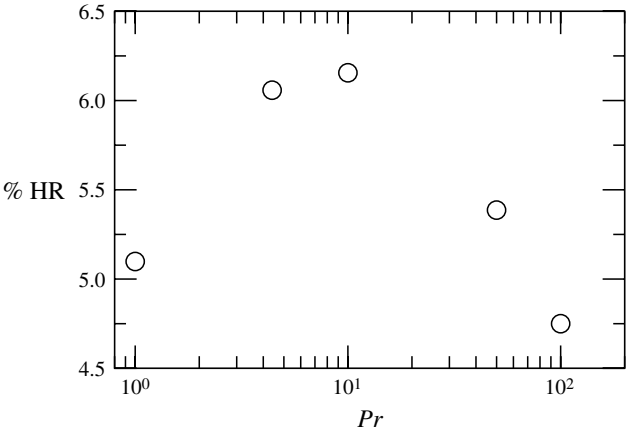



FIGURE 9. Dependence of % HR on $Pr$ at $Re = 4900$, $Wi = 2.5$, $a = 0.01$ and $\gamma = 0.2$.

If the main-stream velocity is to be changed by the polymers, it is likely that $U_p < U$ since the polymers increase the effective viscosity near the plate. This implies that $C_p > C$ and $|\delta Nu_p| > |\delta Nu|$. That is, the reduction of heat transport would be more than that calculated above by assuming $U_p = U$. On the other hand, suppose $U_p > U$, though it is unlikely from a physical point of view. In this case, $C_p < C$. However, as

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polymers always enhance drag in a laminar boundary layer flow, there exists a lower bound for $C_p$: namely, $C_p \geqslant C_0$. It follows from (4.5) that:

$$|\delta Nu_p| = |\delta Nu| \frac{C_p}{C} \geqslant |\delta Nu| \frac{C_0}{C} \equiv |\delta Nu|_{min}. \tag{4.6}$$

Hence, even taking into account the possible variation of the main-stream velocity by the polymers, we find that there is reduction in heat transport by the polymers, and the amount of HR has to be at least the minimum value given by $|\delta Nu|_{min}$. Using our results for $\delta Nu$ and $C_0/C$, we evaluate $|\delta Nu|_{min}$ and the corresponding maximum values of $Nu/Nu_0$ are shown in figure 3. One can see that the effect of variation in the main-stream velocity due to polymers is at most a few per cent of our results obtained by assuming the main-stream velocity to remain the same.

## 5. Summary and conclusions

In this paper, we have studied the problem of heat transport in steady-state boundary layer flow with polymers in forced convection above a heated plate. We have shown how a set of equations can be written for the problem and how this set of equations can be solved numerically by an iterative procedure. Our results demonstrate that the physical effect of the polymers due to their stretching is equivalent to producing a space-dependent effective viscosity, which increases near the plate. Because of this increase in the effective viscosity, the horizontal velocity near the plate is reduced. We have shown that a decrease in the horizontal velocity leads to a reduction in heat transport. One also expects to see an enhancement of drag when the horizontal velocity is decreased, and this is observed. We have obtained results for HR and DE by assuming the main-stream velocity to remain the same in the presence of polymers. We have further shown that the possible variation in the main-stream velocity by the polymers causes a minor change of at most a few per cent of our results. Measurements of heat transfer and drag were carried out for laminar flow of dilute polymer solutions around heated cylinders and an enhancement in drag and a reduction in heat transfer compared to the Newtonian flow has been reported (James & Acosta 1970). Our present work thus provides a theoretical understanding of these experimental observations even though the details of the boundary layer flows above a flat plate and around a cylinder are not the same.

Our present work may also explain the recent experimental observation of a reduction in heat transport in turbulent RB convection with polymer additives (Ahlers & Nikolaenko 2010). As discussed in § 1, this experiment was conducted in water at $Ra \sim 10^{10}$. At such moderate $Ra$, contributions to the heat transport from the boundary layers are significant and the dynamically time-averaged velocity and temperature boundary layer profiles could be described by the steady-state Prandtl–Blasius profiles. In particular, it would be interesting to check our prediction of the $Pr$-dependence of the amount of HR (see figure 9) experimentally.

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