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Sequences characterizing the sine function

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Recently J. Roe(1) established the striking characterization of the sine and cosine functions given in Theorem 1 (below). His proof was an ingenious application of Fourier transforms of generalized functions.

The use of the Laplace transform in place of the Fourier transform enables Theorem 1 to be proved without generalized functions. Two simple properties are needed:

(i) the Laplace transform is regular in the (open) half plane of its existence;

(ii) the function $(az+b)/(z^2+1)$ is the transform of $a\cos t + b\sin t$ $(t \geqslant 0)$.

It will be seen that the Laplace transform of the function to be specified has an analytic continuation throughout the complex plane C with the exception of singularities at $\pm i$. This fact corresponds to the Fourier transform in Roe's proof vanishing at all points of the real line other than ± 1.

Working with the Laplace transform has the further advantage of permitting a generalization of Theorem 1. To prove this result we need a lemma in which the rational function of property (ii) is replaced by one with denominator $(z^2+1)^n$. The final corollary has the same conclusion as Theorem 1, but a weaker hypothesis.

THEOREM 1. *Suppose that the real valued functions* f_n $(n = 0, \pm 1, \pm 2, \ldots)$ *on* $(-\infty, \infty)$ *satisfy the relation* $f_n' = f_{n+1}$ *for all* n. *If there exists a constant* M *such that*

$$|f_n(t)| \leqslant M$$

for all n *and* t, *then there are real numbers* a, b *such that*

$$f_0(t) = a\cos t + b\sin t$$

for $-\infty < t < \infty$.

Proof. We denote the function f_0 by f. The Laplace transform F of f, given by

$$F(z) = \int_0^\infty e^{-zt} f(t)\, dt,$$

evidently exists and is regular in the half-plane $\{z: \operatorname{Re} z > 0\}$.

The argument now falls into three parts.

I. F can be continued analytically into the region $C\backslash\{-i, i\}$.

Let $z = x+iy$ and suppose that $x > 0$. For every positive integer n, we then have, by integration by parts,

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$$F(z) = \int_0^\infty e^{-zt} f(t)\,dt$$

$$= \frac{1}{z} f_0(0) + \frac{1}{z^2} f_1(0) + \ldots + \frac{1}{z^n} f_{n-1}(0) + \frac{1}{z^n} \int_0^\infty e^{-zt} f_n(t)\,dt$$

and

$$F(z) = -f_{-1}(0) - z f_{-2}(0) - \ldots - z^{n-1} f_{-n}(0) + z^n \int_0^\infty e^{-zt} f_{-n}(t)\,dt.$$

But, for $m = 0, \pm 1, \pm 2, \ldots,$

$$\left| \int_0^\infty e^{-zt} f_m(t)\,dt \right| \leqslant M \int_0^\infty e^{-xt}\,dt = \frac{M}{x}.$$

Hence, when $x > 0$ and $|z| > 1$,

$$F(z) = \sum_{n=0}^{\infty} \frac{f_n(0)}{z^{n+1}} \tag{1}$$

and, when $x > 0$ and $|z| < 1$,

$$F(z) = -\sum_{n=0}^{\infty} f_{-n-1}(0)\, z^n. \tag{2}$$

Since the power series in (1) and (2) converge for $|z| > 1$ and for $|z| < 1$, respectively, they provide an analytic continuation of F into the region

$$C_1 = C\backslash\{z: (|z| = 1 \text{ and } \operatorname{Re} z \leqslant 0)\}.$$

We next define the functions ϕ_n, ψ_n on $(-\infty, \infty)$ by $\phi_n(t) = \frac{1}{2}\{f_n(t) + (-1)^n f_n(-t)\}$, $\psi_n(t) = \frac{1}{2}\{f_n(t) - (-1)^n f_n(-t)\}$. Then

$$f_n = \phi_n + \psi_n$$

for each n, and the sequences (ϕ_n), (ψ_n) satisfy the same conditions as (f_n). In addition, for $k = 0, \pm 1, \pm 2, \ldots, \phi_{2k}(0) = f_{2k}(0)$, $\phi_{2k+1}(0) = 0$, $\psi_{2k}(0) = 0$, $\psi_{2k+1}(0) = f_{2k+1}(0)$. Hence the Laplace transforms Φ, Ψ of $\phi = \phi_0$ and $\psi = \psi_0$, respectively, are regular in the half-plane $\{z: \operatorname{Re} z > 0\}$ and their analytic continuations into the region C_1 are given by

$$\Phi(z) = \begin{cases} -\sum_{k=1}^{\infty} f_{-2k}(0)\, z^{2k-1} & (|z| < 1), \\ \sum_{k=0}^{\infty} f_{2k}(0)/z^{2k+1} & (|z| > 1), \end{cases}$$

$$\Psi(z) = \begin{cases} -\sum_{k=0}^{\infty} f_{-2k-1}(0)\, z^{2k} & (|z| < 1), \\ \sum_{k=1}^{\infty} f_{2k-1}(0)/z^{2k} & (|z| > 1). \end{cases}$$

Thus Φ is odd in the set $C\backslash\{z: |z| = 1\}$. If we now also define $\Phi(z)$ to be $-\Phi(-z)$ when $|z| = 1$ and $\operatorname{Re} z < 0$, then we finally obtain an odd regular function in the region $C\backslash\{-i, i\}$. Similarly Ψ can be defined as an even regular function in this region, and F is continued by means of the identity

$$F(z) = \Phi(z) + \Psi(z).$$

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II. $\Phi(z)(z-i)$ and $\Psi(z)(z-i)$ are bounded for $0 < |z-i| < \frac{1}{2}$; $\Phi(z)(z+i)$ and $\Psi(z)(z+i)$ are bounded for $0 < |z+i| < \frac{1}{2}$.

Let $z = x+iy$. If $x > 0$, then

$$|\Phi(z)| \leqslant \int_0^\infty e^{-xt} M\, dt = \frac{M}{x}.$$

If $x < 0$, then $-x > 0$ and, since Φ is odd,

$$|\Phi(z)| = |\Phi(-z)| \leqslant \frac{M}{-x}.$$

Thus, whenever $x \neq 0$,

$$|\Phi(z)| \leqslant \frac{M}{|x|}.$$

Now suppose that $z \neq 0, i$ and put $|z| = \rho$, $\arg z = \alpha$, $|z-i| = \sigma$, $\arg(z-i) = \beta$. If $|\beta| \leqslant \frac{1}{4}\pi$ or $\frac{3}{4}\pi \leqslant |\beta| \leqslant \pi$, then $|x| = \sigma|\cos\beta| \geqslant \sigma/\sqrt{2}$ and

$$|\Phi(z)| \leqslant \frac{M}{|x|} \leqslant \frac{\sqrt{2}\,M}{|z-i|}.$$

If $\frac{1}{4}\pi \leqslant \beta \leqslant \frac{3}{4}\pi$, then $|z| > 1$ and

$$|z| - 1 \geqslant y - 1 = \sigma \sin\beta \geqslant \sigma/\sqrt{2};$$

hence

$$|\Phi(z)| = \left| \sum_{k=0}^{\infty} \frac{f_{2k}(0)}{z^{2k+1}} \right| \leqslant \frac{M}{|z|} \sum_{k=0}^{\infty} \frac{1}{|z|^{2k}} = \frac{M|z|}{|z|^2 - 1} \leqslant \frac{\sqrt{2}\,M}{|z-i|}.$$

If $0 < |z-i| < \frac{1}{2}$ and $-\frac{3}{4}\pi \leqslant \beta \leqslant -\frac{1}{4}\pi$, then $|z| < 1$, $\frac{1}{3}\pi < \alpha < \frac{2}{3}\pi$ and

$$\rho = \rho\sin^2\alpha + \rho\cos^2\alpha \leqslant \rho\sin\alpha + \rho\cos^2\alpha = y + x\cos\alpha,$$

so that

$$1 - |z| \geqslant (1-y) - x\cos\alpha = -\sigma\sin\beta - \sigma\cos\beta\cos\alpha \geqslant \sigma\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{2}\right) = \frac{\sigma}{2\sqrt{2}};$$

hence

$$|\Phi(z)| = \left| -\sum_{k=1}^{\infty} f_{-2k}(0)\, z^{2k-1} \right| \leqslant \frac{M|z|}{1-|z|^2} \leqslant \frac{M}{1-|z|} \leqslant \frac{2\sqrt{2}M}{|z-i|}.$$

Thus

$$|\Phi(z)| \leqslant \frac{2\sqrt{2}M}{|z-i|} \quad \text{for} \quad 0 < |z-i| < \tfrac{1}{2}.$$

Similarly

$$|\Phi(z)| \leqslant \frac{2\sqrt{2}M}{|z+i|} \quad \text{for} \quad 0 < |z+i| < \tfrac{1}{2}.$$

It is proved in very much the same way that identical inequalities hold for Ψ.

III. $f(t) = a\cos t + b\sin t$ for $-\infty < t < \infty$.

If $0 < |z-i| < \frac{1}{2}$,

$$|\Phi(z)(z^2+1)| \leqslant 2\sqrt{2}M|z+i| \leqslant 5\sqrt{2}M.$$

Since $\Phi(z)(z^2+1)$ is regular in the punctured disc $\{z: 0 < |z-i| < \frac{1}{2}\}$, the function has

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a removable singularity at i. Similarly $\Phi(z)(z^2+1)$ has a removable singularity at $-i$. Let $P(z)$ be the regular continuation of $\Phi(z)(z^2+1)$ into C. Then, for $|z| \geqslant 2$,

$$|P(z)| \leqslant \frac{M|z|}{|z|^2-1}(|z|^2+1) \leqslant \tfrac{5}{3}M|z|.$$

Hence P is a polynomial of degree not exceeding 1. Since P is also odd, $P(z)$ is a constant multiple of z, so that

$$\Phi(z) = \frac{az}{z^2+1} \quad (z \neq \pm i).$$

Thus

$$\phi(t) = a\cos t$$

for $t \geqslant 0$. Since ϕ is even, the same identity holds for $t \leqslant 0$.

A similar argument shows that

$$\Psi(z) = \frac{b}{z^2+1} \quad (z \neq \pm i)$$

and that therefore

$$\psi(t) = b\sin t$$

for $-\infty < t < \infty$. This completes the proof of Theorem 1.

Next we dispose of the lemma mentioned in the introduction.

LEMMA. *Let $P_1(z)$, $P_2(z)$ be even and odd polynomials, respectively, of degrees less than $2n$ $(n \geqslant 1)$. Then*

$$\frac{P_1(z)}{(z^2+1)^n} \quad \textit{and} \quad \frac{P_2(z)}{(z^2+1)^n}$$

are the Laplace transforms, respectively, of functions

$$\pi_1(t)\cos t + \pi_2(t)\sin t \quad \textit{and} \quad \pi_3(t)\cos t + \pi_4(t)\sin t$$

on $[0,\infty)$, where π_1, π_2, π_3, π_4 are polynomials of degrees less than n and π_1, π_4 are odd, while π_2, π_3 are even.

Proof. It is convenient to proceed in stages.

(i) For $n \geqslant 1$,

$$\frac{1}{(z^2+1)^n} = \sum_{k=1}^{n} A_{nk} i^k \left\{\frac{1}{(z-i)^k} + \frac{(-1)^k}{(z+i)^k}\right\} \tag{3}$$

and, for $n \geqslant 2$,

$$\frac{z}{(z^2+1)^n} = \sum_{k=2}^{n} B_{nk} i^{k-1} \left\{\frac{1}{(z-i)^k} + \frac{(-1)^{k-1}}{(z+i)^k}\right\}, \tag{4}$$

where the A_{nk} and the B_{nk} are real constants.

The relation (3) is obvious for $n = 1$ and easily checked for $n = 2$. For $n > 2$, it follows, by use of induction, from the identity

$$\frac{4n(n-1)}{(z^2+1)^{n+1}} = \frac{(2n-1)(2n-2)}{(z^2+1)^n} - \frac{d^2}{dz^2}\frac{1}{(z^2+1)^{n-1}} \quad (n \geqslant 2).$$

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The identity

$$\frac{2(n-1)z}{(z^2+1)^n} = -\frac{d}{dz}\frac{1}{(z^2+1)^{n-1}}$$

then immediately yields (4).

(ii) The lemma holds when $P_1(z) \equiv 1$ and when $P_2(z) \equiv z$.

Both results are familiar in the case $n = 1$. When $n > 1$, they follow from (i) and the fact that, if $k \geqslant 1$ and α is real,

$$\frac{(k-1)!}{(z-i\alpha)^k}$$

is the Laplace transform of $t^{k-1}e^{i\alpha t}$.

(iii) To obtain the general form of the lemma we note that there are constants $c_0, \ldots, c_{n-1}$ and $d_0, \ldots, d_{n-1}$ such that

$$\frac{P_1(z)}{(z^2+1)^n} = \frac{1}{(z^2+1)^n}\sum_{m=0}^{n-1} c_m(z^2+1)^m = \sum_{m=0}^{n-1}\frac{c_m}{(z^2+1)^{n-m}}$$

and

$$\frac{P_2(z)}{(z^2+1)^n} = \sum_{m=0}^{n-1}\frac{d_m z}{(z^2+1)^{n-m}}.$$

The lemma now follows at once from (ii). ∎

In the theorem below we use the notation $\lfloor x \rfloor$ for the greatest integer less than or equal to the real number x and $\lceil x \rceil$ for the least integer greater than or equal to x.

THEOREM 2. *Suppose that the real valued functions f_n $(n = 0, \pm 1, \pm 2, \ldots)$ on $(-\infty, \infty)$ are such that $f_n' = f_{n+1}$ for all n. If there exist non-negative real numbers M, λ, μ such that*

$$|f_n(t)| \leqslant M(|n|+1)^\lambda (|t|+1)^\mu \tag{5}$$

for all n and t, then

$$f_0(t) = \Pi_1(t)\cos t + \Pi_2(t)\sin t \quad (-\infty < t < \infty),$$

where Π_1 and Π_2 are polynomials of degrees less than or equal to $\min(\lfloor\lambda\rfloor, \lfloor\mu\rfloor)$.

Proof. Let $\lceil\lambda\rceil = p, \lceil\mu\rceil = q$, so that the condition (5) implies that

$$|f_n(t)| \leqslant M(|n|+1)^p (|t|+1)^q$$

for all n and t.

The method of proof is largely the same as in the case of Theorem 1 and we shall only point out where changes of detail are needed. The sections I–III below correspond to the same sections in the proof of Theorem 1.

I. If $z = x + iy$ and $x > 0$, we use integration by parts to show that, for $m = 0, \pm 1, \pm 2, \ldots$,

$$\begin{aligned}\left|\int_0^\infty e^{-zt} f_m(t)\,dt\right| &\leqslant M(|m|+1)^p \int_0^\infty e^{-xt}(t+1)^q\,dt \\ &= M(|m|+1)^p\left(\frac{1}{x}+\frac{q}{x^2}+\frac{q(q-1)}{x^3}+\ldots+\frac{q!}{x^{q+1}}\right) \\ &\leqslant M(|m|+1)^p (q+1)!\max\left(\frac{1}{x}, \frac{1}{x^{q+1}}\right). \end{aligned} \tag{6}$$

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Thus, if $x > 0$ and $|z| > 1$,

$$\frac{1}{z^n}\int_0^\infty e^{-zt}f_n(t)\,dt \to 0 \quad \text{as} \quad n \to \infty,$$

while if $x > 0$ and $|z| < 1$,

$$z^n\int_0^\infty e^{-zt}f_{-n}(t)\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore the expansions (1) and (2) continue to hold; and no other modification of section I is needed.

II. $\Phi(z)\,(z-i)^{r+1}$ and $\Psi(z)\,(z-i)^{r+1}$ are bounded for $0 < |z-i| < \frac{1}{2}$, and $\Phi(z)\,(z+i)^{r+1}$, $\Psi(z)\,(z+i)^{r+1}$ are bounded for $0 < |z+i| < \frac{1}{2}$, where $r = \max(p, q)$.

If $|\arg(z-i)| \leqslant \frac{1}{4}\pi$ and $0 < x < 1$, then $x \geqslant |z-i|/\sqrt{2}$ and, by (6) (with f_m replaced by ϕ_m),

$$|\Phi(z)| = \left|\int_0^\infty e^{-zt}\phi_0(t)\,dt\right| \leqslant \frac{M(q+1)!}{x^{q+1}} \leqslant \frac{M(q+1)!\,2^{(q+1)/2}}{|z-i|^{q+1}}.$$

The same inequality holds when $\frac{3}{4}\pi \leqslant |\arg(z-i)| \leqslant \pi$ and $-1 < x < 0$.

If $\frac{1}{4}\pi \leqslant \arg(z-i) \leqslant \frac{3}{4}\pi$ and $1 < |z| < \frac{3}{2}$, then $|z|-1 \geqslant |z-i|/\sqrt{2}$ and

$$\begin{aligned}|\Phi(z)| &= \left|\sum_{k=0}^{\infty}\frac{f_{2k}(0)}{z^{2k+1}}\right| \leqslant \frac{M}{|z|}\sum_{k=0}^{\infty}\frac{(2k+1)^p}{|z|^{2k}}\\ &\leqslant \frac{2^p p!M}{|z|}\sum_{k=0}^{\infty}\frac{(k+1)\ldots(k+p)}{p!}\frac{1}{|z|^{2k}} = 2^p p!M\,\frac{|z|^{2p+1}}{(|z|^2-1)^{p+1}}\\ &\leqslant \frac{2^p p!M|z|^p}{(|z|-1)^{p+1}} \leqslant \frac{3^p 2^{(p+1)/2}p!M}{|z-i|^{p+1}}. \end{aligned} \tag{7}$$

If $-\frac{3}{4}\pi \leqslant \arg(z-i) \leqslant -\frac{1}{4}\pi$ and $0 < |z-i| < \frac{1}{2}$, then $|z| < 1$, $1-|z| \geqslant |z-i|/2\sqrt{2}$ and

$$\begin{aligned}|\Phi(z)| &= \left|-\sum_{k=1}^{\infty}f_{-2k}(0)\,z^{2k-1}\right| \leqslant M|z|\sum_{k=0}^{\infty}(2k+3)^p\,|z|^{2k}\\ &\leqslant 3^p p!M|z|\sum_{k=0}^{\infty}\frac{(k+1)\ldots(k+p)}{p!}\,|z|^{2k} = \frac{3^p p!M|z|}{(1-|z|^2)^{p+1}}\\ &\leqslant \frac{3^p 2^{3(p+1)/2}p!M}{|z-i|^{p+1}}.\end{aligned}$$

Hence

$$|\Phi(z)| \leqslant \frac{K}{|z-i|^{r+1}} \quad \text{for} \quad 0 < |z-i| < \tfrac{1}{2}.$$

There is the corresponding inequality at $-i$, and both inequalities hold also for Ψ.

III. For $0 < |z-i| < \frac{1}{2}$,

$$|\Phi(z)\,(z^2+1)^{r+1}| \leqslant K|z+i|^{r+1} \leqslant (\tfrac{3}{2})^{r+1}K.$$

Therefore $\Phi(z)\,(z^2+1)^{r+1}$ has a removable singularity at i. The singularity at $-i$ is of the same type. Let $P(z)$ be the regular continuation of $\Phi(z)\,(z^2+1)^{r+1}$ into C.

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The last but one inequality in (7) shows that, when $|z| \geqslant 2$,

$$|P(z)| \leqslant \frac{2^p p! M|z|^p}{(|z|-1)^{p+1}}(|z|^2+1)^{r+1} \leqslant 2^{2p-2r-1}5^{r+1}p!M|z|^{2r+1}.$$

Hence $P(z)$ is an odd polynomial of degree less than or equal to $2r+1$ and

$$\Phi(z) = \frac{P(z)}{(z^2+1)^{r+1}} \quad (z \neq \pm i).$$

It follows from the lemma that, for $t \geqslant 0$,

$$\phi(t) = \pi_1(t)\cos t + \pi_2(t)\sin t, \tag{8}$$

where π_1 and π_2 are even and odd polynomials, respectively, of degrees less than or equal to r; and since ϕ is even, (8) also holds for $t \leqslant 0$.

It is shown in the same way that

$$\Psi(z) = \frac{Q(z)}{(z^2+1)^{r+1}} \quad (z \neq \pm i),$$

where Q is an even polynomial and $\deg Q \leqslant 2r$. Then

$$\psi(t) = \pi_3(t)\cos t + \pi_4(t)\sin t$$

for $-\infty < t < \infty$, where π_3, π_4 are odd and even polynomials, respectively, of degrees less than or equal to r.

Hence

$$f(t) = \Pi_1(t)\cos t + \Pi_2(t)\sin t$$

for $-\infty < t < \infty$, where $\deg \Pi_1, \deg \Pi_2 \leqslant r$.

The condition

$$|f(t)| \leqslant M(|t|+1)^\lambda$$

shows that Π_1 and Π_2 must, in fact, be of degree less than or equal to $\lfloor\lambda\rfloor$. Also, if $g(t) = t^r \cos t$ and $h(t) = t^r \sin t$, then $g^{(n)}(0) = n(n-1)\ldots(n-r+1)\cos\frac{1}{2}(n-r)\pi$, $h^{(n)}(0) = n(n-1)\ldots(n-r+1)\sin\frac{1}{2}(n-r)\pi$. Therefore the condition

$$|f_n(0)| \leqslant M(n+1)^\mu \quad (n \geqslant 0)$$

shows that $\deg \Pi_1$, $\deg \Pi_2 \leqslant \lfloor\mu\rfloor$. So the degrees of Π_1 and Π_2 do not exceed $\min(\lfloor\lambda\rfloor, \lfloor\mu\rfloor)$. ∎

COROLLARY. *If, in Theorem* 2, $\min(\lambda, \mu) < 1$, *then there are real numbers* a, b *such that*

$$f_0(t) = a\cos t + b\sin t \quad (-\infty < t < \infty).$$

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