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Cocycles associated with normalizers of topological dynamical systems

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Abstract. We investigate normalizers of full groups of a topological dynamical system by means of cocycles. Some of the investigations are conducted in connection with index maps, the centralizer of a topological dynamical system and groups of automorphisms of a C^*-crossed product.

1. *Introduction*

In the present paper, we extend some results obtained by [**3**] from the dynamical viewpoint which concern structures of the normalizer of the topological full group of a Cantor minimal system to the class of topologically free dynamical systems and, furthermore, we report some progress in investigations of the normalizer along the lines of [**3, 9**] from the C^*-algebraic viewpoint. The investigations are conducted by means of cocycles.

By turning his attention to the continuity property of orbit cocycles, Boyle [**1**] showed that if an orbit equivalence of topologically transitive homeomorphisms on a compact metric space has a continuous orbit cocycle, then the orbit equivalence implies the flip conjugacy of the homeomorphisms. Using this result, Giordano *et al* [**3**] showed that the normalizer $N[[\varphi]]$ of the topological full group $[[\varphi]]$ of a minimal homeomorphism φ on a Cantor set X is generated by $[[\varphi]]$ and the group $C^{\epsilon}(\varphi)$ of homeomorphisms on X which commute with φ or φ^{-1} and, furthermore, the semi-direct product $[[\varphi]] \rtimes C^{\epsilon}(\varphi)$ of $[[\varphi]]$ by $C^{\epsilon}(\varphi)$ is an extension of $N[[\varphi]]$ by $\mathbb{Z}$.

Boyle and Tomiyama [**2**] extended Boyle's result [**1**] to the class of topologically free homeomorphisms on a compact Hausdorff space. By using this result, in Lemma 2.3 and Theorem 2.1 of the present paper, we generalize the previously mentioned results [**3**] to the class of topologically free dynamical systems. Actually, for a topologically free dynamical system (X, φ), we single out a subgroup G of the group $\mathrm{Homeo}(X)$ of homeomorphisms on X so that $N[[\varphi]]$ is generated by the groups $[[\varphi]]$ and G and, furthermore, the semi-direct product $[[\varphi]] \rtimes G$ is an extension of $N[[\varphi]]$ by I, where I is the subgroup of elements

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of $[[\varphi]]$ with the φ-invariant, continuous orbit cocycle. The groups G and I coincide with $C^\epsilon(\varphi)$ and $\mathbb{Z}$ for a minimal homeomorphism φ on a Cantor set X, respectively, and G will be denoted by the notation $C^\epsilon(\varphi)$ in the present paper.

From §3 to the end of the present paper, we deal with the centralizer $C(\varphi)$ of a topologically free homeomorphism φ on a compact Hausdorff space X in connection with the normalizer $N[[\varphi]]$. We take up the group $N[[\varphi]]_1$ consisting of $\gamma \in N[[\varphi]]$ for which $\gamma\varphi\gamma^{-1}$ has a continuous orbit cocycle which is cohomologous to 1. As for structures of $N[[\varphi]]_1$, in Lemma 3.1 and Theorem 3.1, we obtain similar results to the ones mentioned in the preceding paragraph: $N[[\varphi]]_1$ is generated by the groups $[[\varphi]]$ and $C(\varphi)$; and $[[\varphi]] \rtimes C(\varphi)$ is an extension of $N[[\varphi]]_1$ by I. By means of index maps which were introduced in [**3**, Proposition 3.1], Proposition 3.1 establishes the fact that $N[[\varphi]]$ is realized as the normalizer $N(\Gamma)$ in $\mathrm{Homeo}(X)$ of some subset Γ of the topological full group $[[\varphi]]$. In addition to these results, we also show in Theorems 2.2 and 3.2 that for a topologically free dynamical system (X, φ), the subgroup I of $[[\varphi]]$ implements the relationships that $N[[\varphi]]/[[\varphi]] \cong G/I$ and $N[[\varphi]]_1/[[\varphi]] \cong C(\varphi)/I$. For a minimal homeomorphism φ on a compact Hausdorff space X, we also prove the fact, which seems to be intuitively natural, that $N[[\varphi]] \cap [\varphi] = [[\varphi]]$, where $[\varphi]$ denotes the full group of (X, φ).

In connection with a C^*-crossed product $C^*(X, \varphi)$ associated with a topologically free dynamical system (X, φ), Tomiyama [**9**] proved that the group $\mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ of automorphisms which globally fix the subalgebra $C(X)$ of continuous functions on X is isomorphic to the semi-direct product of $N[[\varphi]]$ by the group of unitary elements of $C(X)$, which is a generalization of a result obtained by [**3**] for a minimal homeomorphism on a Cantor set. Along this line, in Theorem 4.1, we single out a subgroup $\mathrm{Aut}_{C(X)}(C^*(X, \varphi))_1$ of $\mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ which is isomorphic to the semi-direct product of some group by $N[[\varphi]]_1$.

2. *Normalizers of full groups*

Let us recall some basic facts for our arguments. By a *topological dynamical system*, we shall mean the pair (X, φ) of a compact Hausdorff space X and a homeomorphism φ on X. The following intrinsic property of a topological dynamical system was introduced by Tomiyama, which we deal extensively with in the present paper:

Definition. We say that a topological dynamical system (X, φ) is *topologically free* if the set of aperiodic points of φ is dense in X. If this happens, then we also say that the homeomorphism φ is topologically free.

Let us recall definitions of full groups of a topological dynamical system.

Definition. The *full group* $[\varphi]$ of a topological dynamical system (X, φ) is the subgroup of homeomorphisms $\psi \in \mathrm{Homeo}(X)$ with a function $g : X \to \mathbb{Z}$ such that

$$\psi(x) = \varphi^{g(x)}(x) \quad \text{for any } x \in X.$$

We call such a function g an *orbit cocycle* of ψ. The *topological full group* $[[\varphi]]$ of (X, φ) is the subgroup of homeomorphisms ψ in $[\varphi]$ which have continuous orbit cocycles.

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We must mention here that these full groups are complete invariants for orbit structures in a couple of classes of topological dynamical systems, see [**3, 4**]. The notation '$[[\varphi]]$' is adopted from [**6**]. Note that if a topological dynamical system (X, φ) is topologically free, then a continuous orbit cocycle of any element of the topological full group $[[\varphi]]$ is uniquely determined.

Definition. We denote by $N[\varphi]$ the normalizer of the full group $[\varphi]$ of a topological dynamical system (X, φ) in the group $\mathrm{Homeo}(X)$, i.e.

$$N[\varphi] = \{\gamma \in \mathrm{Homeo}(X);\ \gamma[\varphi]\gamma^{-1} = [\varphi]\}.$$

We also denote by $N[[\varphi]]$ the normalizer of the topological full group $[[\varphi]]$ in the group $\mathrm{Homeo}(X)$, i.e.

$$N[[\varphi]] = \{\gamma \in \mathrm{Homeo}(X);\ \gamma[[\varphi]]\gamma^{-1} = [[\varphi]]\}.$$

As shown in [**3**, Proposition 1.4], it follows that $\gamma \in \mathrm{Homeo}(X)$ belongs to $N[\varphi]$ if and only if $\gamma(\mathrm{Orb}_\varphi(x)) = \mathrm{Orb}_\varphi(\gamma(x))$ for any $x \in X$. By using this characterization, we can see that $N[[\varphi]] \subset N[\varphi]$, and $\gamma\varphi\gamma^{-1}$ has the same orbits as φ for any $\gamma \in N[\varphi]$. Also from the characterization, we may think of an element of $N[\varphi]$ as an automorphism of φ up to 'orbit equivalence'.

In our arguments, we exploit the following result [**2**, Theorem 3.1] of Boyle and Tomiyama which is a generalization of Boyle's result [**1**].

LEMMA 2.1. *Suppose that φ and ψ are topologically free homeomorphisms on a compact Hausdorff space X with the same orbits. If there exists a continuous function $f : X \to \mathbb{Z}$ such that $\varphi(x) = \psi^{f(x)}(x)$ for any $x \in X$, then there exist φ-invariant, disjoint clopen (closed and open) subsets A and B of X whose union is X so that*

$$f = \chi_A - \chi_B + g - g \circ \varphi$$

for some continuous function $g : X \to \mathbb{Z}$. Furthermore, the homeomorphism $\xi \in [[\psi]]$ defined for $x \in X$ by

$$\xi(x) = \psi^{g(x)}(x)$$

yields that

$$\xi\varphi\xi^{-1}(x) = \psi(x) \quad \text{for any } x \in A \quad \text{and} \quad \xi\varphi\xi^{-1}(x) = \psi^{-1}(x) \quad \text{for any } x \in B.$$

For a topological dynamical system (X, φ) and a function $f : X \to \mathbb{Z}$, let us denote by f_φ the cocycle for φ which is defined for $(k, x) \in \mathbb{Z} \times X$ by

$$f_\varphi(k, x) = \begin{cases} \displaystyle\sum_{j=0}^{k-1} f(\varphi^j x) & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ \displaystyle -\sum_{j=1}^{|k|} f(\varphi^{-j} x) & \text{if } k < 0. \end{cases}$$

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Let φ and ψ be homeomorphisms on a compact Hausdorff space X. Note that if there exists a function $n : X \to \mathbb{Z}$ such that $\varphi(x) = \psi^{n(x)}(x)$ for any $x \in X$, then it is easily verified that

$$\varphi^k(x) = \psi^{n_\varphi(k,x)}(x) \quad \text{for any } (k, x) \in \mathbb{Z} \times X.$$

We use the following result [**2**, Proposition 2.2] of Boyle and Tomiyama in order to show Proposition 2.1, which gives a characterization when a $\gamma \in N[\varphi]$ belongs to $N[[\varphi]]$ for a topologically free φ.

LEMMA 2.2. *Suppose that φ and ψ are topologically free homeomorphisms with the same orbits acting on a compact Hausdorff space X and, moreover, there exists a continuous function $n : X \to \mathbb{Z}$ such that $\varphi(x) = \psi^{n(x)}(x)$ for any $x \in X$. Then, the map $k \mapsto n_\varphi(k, x)$ is a permutation on $\mathbb{Z}$ for any $x \in X$.*

For a topological dynamical system (X, φ), let us put

$$N[\varphi]_c = \{\gamma \in N[\varphi]; \gamma\varphi\gamma^{-1} \text{ has a continuous orbit cocycle}\}.$$

Obviously, it follows that

$$N[\varphi]_c = \{\gamma \in N[\varphi]; \gamma[[\varphi]]\gamma^{-1} \subset [[\varphi]]\}. \tag{2.1}$$

PROPOSITION 2.1. *Suppose that a topological dynamical system (X, φ) is topologically free. Then, the set $N[\varphi]_c$ is a subgroup of $N[\varphi]$. Hence, we obtain that $N[\varphi]_c = N[[\varphi]]$.*

Proof. It is clear from (2.1) that $\gamma_1\gamma_2 \in N[\varphi]_c$ for any $\gamma_1, \gamma_2 \in N[\varphi]_c$. To show that $N[\varphi]_c$ is also closed under taking the inverse, suppose that $\gamma \in N[\varphi]_c$. There exists a continuous function $g : X \to \mathbb{Z}$ such that

$$\gamma\varphi\gamma^{-1}(x) = \varphi^{g(x)}(x) \quad \text{for any } x \in X.$$

We have that

$$\gamma\varphi^k\gamma^{-1}(x) = \varphi^{g_{\gamma\varphi\gamma^{-1}}(k,x)}(x)$$
$$\varphi^k(x) = \gamma^{-1}\varphi^{g_{\gamma\varphi\gamma^{-1}}(k,\gamma(x))}\gamma(x) \quad \text{for any } (k, x) \in \mathbb{Z} \times X.$$

From Lemma 2.2, for an arbitrary $x \in X$, there exists a unique integer $k(x)$ such that $g_{\gamma\varphi\gamma^{-1}}(k(x), \gamma(x)) = 1$. Let us see that the function $k : X \to \mathbb{Z}$ is continuous. Fix an arbitrary $x_0 \in X$. It is immediate that $g_{\gamma\varphi\gamma^{-1}}(k(x_0), \gamma(x_0)) = 1$. Since the function $g_{\gamma\varphi\gamma^{-1}}(l, \cdot) : X \to \mathbb{Z}$ is continuous for each $l \in \mathbb{Z}$ and γ is also continuous, there exists an open neighborhood U of x_0 such that $g_{\gamma\varphi\gamma^{-1}}(k(x_0), \gamma(x)) = 1$ for any $x \in U$. Therefore, we obtain that $k(x) = 1$ for any $x \in U$.

As $N[\varphi]_c$ is closed under taking the inverse, it follows that $\gamma[[\varphi]]\gamma^{-1} = [[\varphi]]$ for any $\gamma \in N[\varphi]_c$, which completes the proof. □

Next, we shall go into structures of the normalizer $N[[\varphi]]$ for a topologically free dynamical system (X, φ). The first of such structures which we will see is a generalization of [**3**, Proposition 5.11].

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Let us denote by $C^\epsilon(\varphi)$ the group of homeomorphisms $\eta \in \mathrm{Homeo}(X)$ for which there exist φ-invariant, disjoint clopen subsets A and B of X whose union is X such that

$$\eta\varphi\eta^{-1}(x) = \varphi(x) \quad \text{for any } x \in A \quad \text{and} \quad \eta\varphi\eta^{-1}(x) = \varphi^{-1}(x) \quad \text{for any } x \in B, \quad (2.2)$$

where one of the sets A and B is possibly empty. This condition is equivalent to the statement that there exist φ-invariant, disjoint clopen sets $A, B \subset X$ whose union is X such that the continuous function $\chi_A - \chi_B$ is an orbit cocycle of the homeomorphism $\eta\varphi\eta^{-1}$. It is clear that $C^\epsilon(\varphi) \subset N[[\varphi]]$. This implies, in particular, that both the clopen sets $\eta(A)$ and $\eta^{-1}(A)$ are φ-invariant for any $\eta \in C^\epsilon(\varphi)$ whenever a clopen set $A \subset X$ is φ-invariant, which guarantees that $C^\epsilon(\varphi)$ is indeed a group. The notation '$C^\epsilon(\varphi)$' is adopted from [**3**]. Actually, for a Cantor minimal system (X, φ), the group $C^\epsilon(\varphi)$ defined earlier coincides with the group $C^\epsilon(\varphi)$ defined in [**3**].

To achieve the desired generalization of [**3**, Proposition 5.11], we use the following lemmas.

LEMMA 2.3. *Suppose that a topological dynamical system (X, φ) is topologically free. Then, each $\gamma \in N[[\varphi]]$ is decomposed into a product $\psi\eta$ for $\psi \in [[\varphi]]$ and $\eta \in C^\epsilon(\varphi)$. So, the normalizer $N[[\varphi]]$ is generated by the groups $[[\varphi]]$ and $C^\epsilon(\varphi)$.*

Proof. Suppose that $\gamma \in N[[\varphi]]$. There exists an integer-valued continuous function $g : X \to \mathbb{Z}$ such that $\gamma\varphi\gamma^{-1}(x) = \varphi^{g(x)}(x)$ for any $x \in X$. Lemma 2.1 implies that there exist φ-invariant, disjoint clopen subsets A and B of X whose union is X, and a $\xi \in [[\varphi]]$ such that

$$\xi\gamma\varphi\gamma^{-1}\xi^{-1}(x) = \varphi(x) \quad \text{for any } x \in A;$$
$$\xi\gamma\varphi\gamma^{-1}\xi^{-1}(x) = \varphi^{-1}(x) \quad \text{for any } x \in B.$$

Thus, we obtain that $\xi\gamma \in C^\epsilon(\varphi)$. It follows finally that $\gamma = \xi^{-1}(\xi\gamma)$. □

For a topological dynamical system (X, φ), we shall define the subgroup I of the topological full group $[[\varphi]]$ by

$$I = \{\psi \in [[\varphi]];\ \psi \text{ has a } \varphi\text{-invariant, continuous orbit cocycle}\}.$$

Of course, a function f on X is said to be φ-invariant if $f \circ \varphi = f$. The group I is a commutative normal subgroup of each of $C^\epsilon(\varphi)$ and the *centralizer*

$$C(\varphi) := \{\eta \in \mathrm{Homeo}(X);\ \eta\varphi = \varphi\eta\}$$

of φ. To see this, suppose that $\psi \in [[\varphi]]$ has a φ-invariant, continuous orbit cocycle. It is readily verified that both $\varphi\psi$ and $\psi\varphi$ have an orbit cocycle $g + 1$, i.e. $\psi \in C(\varphi)$. For an $\eta \in C^\epsilon(\varphi)$ with condition (2.2), it is verified that $\eta\psi\eta^{-1}$ has a φ-invariant, continuous orbit cocycle $(g \circ \eta^{-1})\chi_A - (g \circ \eta^{-1})\chi_B$. This implies that I is a normal subgroup of $C^\epsilon(\varphi)$. If we suppose that the clopen set B is empty, then we conclude that I is a normal subgroup of $C(\varphi)$. The centralizer of I in $[[\varphi]]$ is I itself. Clearly, the space X does not have any non-trivial, φ-invariant clopen subsets if and only if $I = \{\varphi^n;\ n \in \mathbb{Z}\}$.

LEMMA 2.4. *Suppose that a topological dynamical system (X, φ) is topologically free. Then, it follows that $C^\epsilon(\varphi) \cap [[\varphi]] = C(\varphi) \cap [[\varphi]] = I$.*

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Proof. To show the first equality, suppose that $\gamma \in C^\epsilon(\varphi) \cap [[\varphi]]$. There exists a continuous function $g : X \to \mathbb{Z}$ such that $\gamma(x) = \varphi^{g(x)}(x)$ for any $x \in X$. From this, we obtain that

$$\gamma\varphi\gamma^{-1}(x) = \varphi^{g(\varphi\gamma^{-1}(x)) - g(\gamma^{-1}(x)) + 1}(x) \quad \text{for any } x \in X.$$

However, from the assumption that $\gamma \in C^\epsilon(\varphi)$, there exist φ-invariant, disjoint clopen sets $A, B \subset X$ whose union is X such that $\chi_A - \chi_B$ is an orbit cocycle of $\gamma\varphi\gamma^{-1}$. The topological freeness of φ implies, therefore, that

$$g(\varphi\gamma^{-1}(x)) - g(\gamma^{-1}(x)) + 1 = \chi_A(x) - \chi_B(x) \quad \text{for any } x \in X.$$

By integrating both sides of this equation with respect to any φ-invariant Borel probability measure μ, we obtain that $1 = \mu(A) - \mu(B)$. This forces that B is empty. We obtain that $\gamma \in C(\varphi) \cap [[\varphi]]$. Then, as

$$\varphi^{g(\varphi(x))+1}(x) = \gamma\varphi(x) = \varphi\gamma(x) = \varphi^{g(x)+1}(x) \quad \text{for any } x \in X,$$

the topological freeness of φ implies that $g(\varphi(x)) = g(x)$ for any $x \in X$. This implies finally that $\gamma \in I$. □

The group $C^\epsilon(\varphi)$ acts on $[[\varphi]]$ by conjugation. Let us denote by $[[\varphi]] \rtimes C^\epsilon(\varphi)$ the semi-direct product of $[[\varphi]]$ by $C^\epsilon(\varphi)$. The following is the desired generalization of [**3**, Proposition 5.11].

THEOREM 2.1. *Suppose that a topological dynamical system (X, φ) is topologically free. Then, we obtain the following short exact sequence*:

$$0 \to I \xrightarrow{\iota} [[\varphi]] \rtimes C^\epsilon(\varphi) \xrightarrow{\tau} N[[\varphi]] \to 1,$$

where $\iota(\psi) = (\psi, \psi^{-1})$ for $\psi \in I$, and $\tau(\psi, \eta) = \psi\eta$ for $(\psi, \eta) \in [[\varphi]] \rtimes C^\epsilon(\varphi)$.

Proof. It was already proved by Lemma 2.3 that τ is surjective. Clearly, it follows that $\mathrm{Im}(\iota) \subset \ker(\tau)$. Conversely, suppose that $\tau(\psi, \eta) = \psi\eta = 1$ for $\psi \in [[\varphi]]$ and $\eta \in C^\epsilon(\varphi)$. We obtain that $\psi = \eta^{-1} \in [[\varphi]] \cap C^\epsilon(\varphi)$ and so $\psi \in I$ in virtue of Lemma 2.4, which completes the proof. □

To see another relationship among all the groups which appear in the previous theorem, let us define a homomorphism $\pi : N[[\varphi]] \to C^\epsilon(\varphi)/I$ as follows. Let $\gamma \in N[[\varphi]]$ be arbitrary. Suppose that $\gamma = \eta_1\psi_1 = \eta_2\psi_2$ for $\eta_i \in C^\epsilon(\varphi)$ and $\psi_i \in [[\varphi]]$, $i = 1, 2$. Then, it follows from Lemma 2.4 that $\eta_2^{-1}\eta_1 = \psi_2\psi_1^{-1} \in C^\epsilon(\varphi) \cap [[\varphi]] = I$. Therefore, the cosets $\eta_1 I$ and $\eta_2 I$ in $C^\epsilon(\varphi)$ coincide. Finally, we can define a map $\pi : N[[\varphi]] \to C^\epsilon(\varphi)/I$ by $\pi(\gamma) = \eta I$, where η is any element of $C^\epsilon(\varphi)$ satisfying that $\gamma = \eta\psi$ for some $\psi \in [[\varphi]]$. It follows from the fact that $C^\epsilon(\varphi)$ acts by conjugation on $[[\varphi]]$ that π is indeed a homomorphism.

THEOREM 2.2. *Suppose that a topological dynamical system (X, φ) is topologically free. Then, we obtain the following short exact sequence*:

$$1 \to [[\varphi]] \xrightarrow{i} N[[\varphi]] \xrightarrow{\pi} C^\epsilon(\varphi)/I \to 1,$$

where i is the inclusion map.

Proof. As $\pi(\eta) = \eta I$ for any $\eta \in C^\epsilon(\varphi)$, π is surjective. It is straightforward to see that $\mathrm{Im}(i) \subset \ker(\pi)$. Suppose that $\gamma \in \ker(\pi)$ and $\gamma = \eta\psi$ for $\eta \in C^\epsilon(\varphi)$ and $\psi \in [[\varphi]]$. Then, $\eta \in I$, which implies that $\gamma \in [[\varphi]]$. □

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3. *Centralizers*

For a topological dynamical system (X, φ), we say that continuous integer-valued functions f and g on X are *cohomologous* if $f - g$ is a *coboundary* for φ, i.e. a function of the form $h \circ \varphi - h$ for some continuous function $h : X \to \mathbb{Z}$. We shall denote by $N[[\varphi]]_1$ the set of those $\gamma \in N[[\varphi]]$ for which $\gamma\varphi\gamma^{-1}$ has a continuous orbit cocycle which is cohomologous to 1. This condition is equivalent to the statement that the clopen set B in Lemma 2.1 is empty which is determined for the homeomorphisms $\gamma\varphi\gamma^{-1}$ and φ in place of φ and ψ in Lemma 2.1, respectively. This is verified by similar arguments using φ-invariant Borel probability measures to the proof of Lemma 2.4.

Remark. For each $\gamma \in N[[\varphi]]$, $\gamma\varphi\gamma^{-1}$ has a continuous orbit cocycle which is cohomologous to 1 if and only if there exists a continuous function $f : X \to \mathbb{Z}$ which is cohomologous to 1 such that $\gamma\varphi(x) = \varphi^{f(x)}\gamma(x)$ for any $x \in X$. For, if there exists a continuous function $g : X \to \mathbb{Z}$ such that

$$\gamma\varphi\gamma^{-1}(x) = \varphi^{g(\varphi(x))-g(x)+1}(x) \quad \text{for any } x \in X,$$

then $\gamma\varphi(x) = \varphi^{g(\varphi\gamma(x))-g(\gamma(x))+1}\gamma(x)$ for any $x \in X$. Since $\gamma^{-1} \in N[[\varphi]]$, there exists a continuous function $h : X \to \mathbb{Z}$ such that $\varphi\gamma(x) = \gamma\varphi^{h(x)}(x)$ for any $x \in X$. If we put $X_k = \{x \in X; h(x) = k\}$ for $k \in \mathbb{Z}$, then we see that

$$g \circ \varphi \circ \gamma - g \circ \gamma = \sum_{k\in\mathbb{Z}} \{(g \circ \gamma \circ \varphi^k) \cdot \chi_{X_k} - (g \circ \gamma) \cdot (\chi_{X_k} \circ \varphi^{-k})\},$$

which is a coboundary for φ. The converse implication can be checked in the same way.

LEMMA 3.1. *Let (X, φ) be a topological dynamical system. Then, each $\gamma \in N[[\varphi]]_1$ is decomposed into a product $\eta\psi$ for $\eta \in C(\varphi)$ and $\psi \in [[\varphi]]$. So, the set $N[[\varphi]]_1$ is a group generated by the groups $[[\varphi]]$ and $C(\varphi)$.*

Proof. Suppose that $\gamma \in N[[\varphi]]_1$. There exists a continuous function $g : X \to \mathbb{Z}$ such that

$$\gamma\varphi(x) = \varphi^{g(\varphi(x))-g(x)+1}\gamma(x) \quad \text{for any } x \in X.$$

Therefore, we obtain that

$$\varphi^{-g(\varphi(x))}\gamma\varphi(x) = \varphi\varphi^{-g(x)}\gamma(x) \quad \text{for any } x \in X. \tag{3.1}$$

If we define a continuous map $\eta : X \to X$ by

$$\eta(x) = \varphi^{-g(x)}\gamma(x) \quad \text{for } x \in X, \tag{3.2}$$

then it follows from (3.1) that

$$\eta\varphi(x) = \varphi\eta(x) \quad \text{for any } x \in X. \tag{3.3}$$

If we define a continuous map $\psi : X \to X$ by

$$\psi(x) = \varphi^{g(x)}(x) \quad \text{for } x \in X,$$

then from (3.2) and the fact that $\eta\varphi = \varphi\eta$, it follows that

$$\gamma(x) = \eta\psi(x) \quad \text{for any } x \in X. \tag{3.4}$$

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As γ is injective, ψ is also injective. Assume that there exists a point $x_0 \in X$ such that $\psi(\mathrm{Orb}_\varphi(x_0)) \subsetneq \mathrm{Orb}_\varphi(x_0)$, where $\mathrm{Orb}_\varphi(x_0)$ is the orbit of x_0 under φ. We shall put $\psi(\mathrm{Orb}_\varphi(x_0)) = \{\varphi^{n_i}(x_0); i \geq 1\}$, which is possibly finite. Then, it follows from (3.3) and (3.4) that

$$\gamma(\mathrm{Orb}_\varphi(x_0)) = \eta\psi(\mathrm{Orb}_\varphi(x_0)) = \{\varphi^{n_i}\eta(x_0); i \geq 1\} \subsetneq \mathrm{Orb}_\varphi(\eta(x_0)),$$

which is a contradiction to the fact that γ maps one φ-orbit to another φ-orbit in a bijective way. We have shown that $\psi(\mathrm{Orb}_\varphi(x)) = \mathrm{Orb}_\varphi(x)$ for any $x \in X$ and so ψ is surjective or bijective. Finally we obtain that $\psi \in [[\varphi]]$.

To show the second statement of the lemma, it is enough to note the following facts. The centralizer $C(\varphi)$ acts on the topological full group $[[\varphi]]$ by conjugation. For any $\eta \in C(\varphi)$ and $\psi \in [[\varphi]]$, the product $\eta\psi$ belongs to the set $N[[\varphi]]_1$ because

$$\eta\psi\varphi(x) = \varphi^{g(\varphi(x))-g(x)+1}\eta\psi(x) \quad \text{for any } x \in X,$$

where $g : X \to \mathbb{Z}$ is a continuous orbit cocycle of ψ. □

Remark. For a topologically free dynamical system (X, φ), the centralizer $C(\varphi)$ is not a normal subgroup of $N[[\varphi]]_1$ in general. To see this, assume that $\gamma\varphi\gamma^{-1} \in C(\varphi)$ for $\gamma \in N[[\varphi]]_1$. As $\gamma \in N[[\varphi]]_1$, there exists a continuous function $g : X \to \mathbb{Z}$ such that

$$\gamma\varphi\gamma^{-1}(x) = \varphi^{g(\varphi(x))-g(x)+1}(x) \quad \text{for any } x \in X. \tag{3.5}$$

From this, we obtain that

$$\gamma\varphi\gamma^{-1}\varphi(x) = \varphi^{(g\circ\varphi-g)\circ\varphi(x)-(g\circ\varphi-g)(x)+1}\gamma\varphi\gamma^{-1}(x) \quad \text{for any } x \in X. \tag{3.6}$$

As $\gamma\varphi\gamma^{-1} \in C(\varphi)$ and φ is topologically free, (3.6) implies that the function $g \circ \varphi - g$ is φ-invariant. Put $X_k = g^{-1}\{k\}$ for $k \in \mathbb{Z}$, which is clopen. Arrange the set $S := \{k \in \mathbb{Z}; X_k \neq \emptyset\}$ so that $S = \{k_1 < k_2 < \cdots < k_n\}$. To show that each set X_{k_i}, $1 \leq i \leq n$, is φ-invariant, let us show first that the clopen set X_{k_n} is φ-invariant. Fix an $x \in X_{k_n}$ arbitrarily. It is clear that $g(\varphi(x)) - g(x) \leq 0$. Let us assume that $g(\varphi(x)) - g(x) < 0$. This implies that $g(x) - g(\varphi^{-1}(x)) = g(\varphi(x)) - g(x) < 0$, which contradicts the fact that $x \in X_{k_n}$. Hence, we obtain that $g(\varphi(x)) = g(x) = g(\varphi^{-1}(x))$ and so $\varphi(X_{k_n}) = X_{k_n}$. By applying the same argument as before to each set X_{k_i}, we can conclude that each set X_{k_i} is φ-invariant and so the function g is φ-invariant. From (3.5), we obtain that $\gamma \in C(\varphi)$.

By using Lemmas 2.3 and 3.1, one can see that the group $N[[\varphi]]_1$ is a normal subgroup of $N[[\varphi]]$ if and only if $C(\varphi)$ is a normal subgroup of $C^\epsilon(\varphi)$. The centralizer $C(\varphi)$ becomes a normal subgroup of $C^\epsilon(\varphi)$ and so $N[[\varphi]]_1$ becomes a normal subgroup of $N[[\varphi]]$ if a topologically free dynamical system (X, φ) has no non-trivial, φ-invariant clopen subset of X. The latter condition is satisfied if (X, φ) is a minimal system, i.e. φ has no non-trivial, invariant closed subset of X.

Let us denote by $[[\varphi]] \rtimes C(\varphi)$ the semi-direct product of $[[\varphi]]$ by $C(\varphi)$. In a similar way to the proofs of Theorems 2.1 and 2.2, we can prove the following theorems.

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THEOREM 3.1. *Suppose that a topological dynamical system* (X, φ) *is topologically free. Then, we obtain the following short exact sequence*:

$$0 \to I \xrightarrow{\iota} [[\varphi]] \rtimes C(\varphi) \xrightarrow{\tau} N[[\varphi]]_1 \to 1,$$

where $\iota(\psi) = (\psi, \psi^{-1})$ *for* $\psi \in I$, *and* $\tau(\psi, \eta) = \psi\eta$ *for* $(\psi, \eta) \in [[\varphi]] \rtimes C(\varphi)$. *This exact sequence implements a central extension of the group* $N[[\varphi]]_1$ *if and only if the continuous orbit cocycle of any element of* $[[\varphi]]$ *is also invariant under the action of any element of the centralizer* $C(\varphi)$.

THEOREM 3.2. *Suppose that a topological dynamical system* (X, φ) *is topologically free. Then, we obtain the following exact sequence*:

$$1 \to [[\varphi]] \xrightarrow{i} N[[\varphi]]_1 \xrightarrow{\pi_1} C(\varphi)/I \to 1,$$

where i *is the inclusion map and* π_1 *is the restriction of* π *in Theorem* 2.2.

Suppose that a topological dynamical system (X, φ) is topologically free. Next, we see the fact that the group $N[[\varphi]]_1$ is the normalizer of some subset of the topological full group $[[\varphi]]$. Let us denote by M_φ the set of φ-invariant Borel probability measures. Following [**3**], we shall define the *index map* $I_\mu : [[\varphi]] \to \mathbb{R}$ for each $\mu \in M_\varphi$ by

$$I_\mu(\psi) = \int_X g \, d\mu \quad \text{for } \psi \in [[\varphi]],$$

where g is the continuous orbit cocycle of ψ. It is readily verified that the map I_μ is a homomorphism. Let us put

$$\Gamma = \{\psi \in [[\varphi]]; I_\mu(\psi) = 1 \text{ for any } \mu \in M_\varphi\};$$
$$N(\Gamma) = \{\gamma \in \mathrm{Homeo}(X); \gamma\Gamma\gamma^{-1} = \Gamma\}.$$

The group $N[[\varphi]]_1$ can be realized as the normalizer $N(\Gamma)$:

PROPOSITION 3.1. *Suppose that a topological dynamical system* (X, φ) *is topologically free. Then, it follows that* $N[[\varphi]]_1 = N(\Gamma)$.

Proof. To show the inclusion that $N(\Gamma) \subset N[[\varphi]]_1$, suppose that $\gamma \in N(\Gamma)$. As $\gamma\varphi\gamma^{-1} \in \Gamma \subset [[\varphi]]$, and $\gamma\varphi\gamma^{-1}$ and φ have the same orbits, Lemma 2.1 implies that there exist φ-invariant, disjoint clopen subsets A and B of X whose union is X such that a continuous orbit cocycle of $\gamma\varphi\gamma^{-1}$ is $\chi_A - \chi_B + f \circ \varphi - f$ for some continuous function $f : X \to \mathbb{Z}$. The fact that $\gamma\varphi\gamma^{-1} \in \Gamma$ implies that $\mu(A) - \mu(B) = 1$ for any $\mu \in M_\varphi$. This implies that B is empty, which shows that the continuous cocycle of $\gamma\varphi\gamma^{-1}$ is cohomologous to 1.

To show the converse inclusion, suppose that $\gamma \in N[[\varphi]]_1$. There exists a continuous function $g : X \to \mathbb{Z}$ such that

$$\gamma\varphi(x) = \varphi^{g(\varphi(x)) - g(x) + 1}\gamma(x) \quad \text{for any } x \in X.$$

From this, it follows that

$$\gamma\varphi^k(x) = \varphi^{g(\varphi^k(x)) - g(x) + k}\gamma(x) \quad \text{for any } (k, x) \in \mathbb{Z} \times X.$$

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Let $\psi \in \Gamma$ be arbitrary. Then, there exists a continuous function $f : X \to \mathbb{Z}$ such that $\psi(x) = \varphi^{f(x)}(x)$ for any $x \in X$, and $\int_X f\, d\mu = 1$ for any $\mu \in M_\varphi$. It is readily verified that

$$\gamma\psi\gamma^{-1}(x) = \varphi^{g(\psi\gamma^{-1}(x))-g(\gamma^{-1}(x))+f(\gamma^{-1}(x))}(x) \quad \text{for any } x \in X.$$

Hence, as

$$\begin{aligned} I_\mu(\gamma\psi\gamma^{-1}) &= \int_X \{g(\psi(x)) - g(x) + f(x)\}\, d(\mu \circ \gamma)(x) \\ &= \int_X f(x)\, d(\mu \circ \gamma)(x) = 1 \quad \text{for any } \mu \in M_\varphi, \end{aligned}$$

it follows that $\gamma \in N(\Gamma)$. This completes the proof. □

By paying attention to cocycles, we can show the following relationships between some groups dealt with so far, which may be seem to be intuitively natural.

PROPOSITION 3.2. *For a minimal dynamical system* (X, φ)*, it follows that*

$$N[[\varphi]] \cap [\varphi] = N[[\varphi]]_1 \cap [\varphi] = [[\varphi]].$$

Proof. First, we show the second equality. Suppose that $\gamma \in N[[\varphi]]_1 \cap [\varphi]$. There exists a function $f : X \to \mathbb{Z}$ such that $\gamma(x) = \varphi^{f(x)}(x)$ for any $x \in X$. From this, we obtain that

$$\gamma\varphi(x) = \varphi^{f(\varphi(x))-f(x)+1}\gamma(x) \quad \text{for any } x \in X.$$

As $\gamma \in N[[\varphi]]_1$, there exists a continuous function $g : X \to \mathbb{Z}$ such that

$$\gamma\varphi(x) = \varphi^{g(\varphi(x))-g(x)+1}\gamma(x) \quad \text{for any } x \in X.$$

Therefore, if we put $h = g - f$, then we see that $h(\varphi(x)) = h(x)$ for any $x \in X$. As $f^{-1}\{i\}$ and $g^{-1}\{i\}$ are closed subsets of X for every $i \in \mathbb{Z}$, the set

$$h^{-1}\{i\} = \bigcup_{k-l=i} (g^{-1}\{k\} \cap f^{-1}\{l\})$$

is a countable disjoint union of closed sets for every $i \in \mathbb{Z}$. Therefore, there exists a countable partition $\{X_i; i \geq 1\}$ of X into closed sets so that h is constant on each X_i. Baire's category theorem implies the existence of an integer $i_0 \geq 1$ such that X_{i_0} has a non-empty interior. Since φ is minimal, $X = \bigcup_{n\in\mathbb{Z}} \varphi^n(X_{i_0})$. Thus, there exists an integer c such that $h(x) = c$ for any $x \in X$. Finally, we obtain that $f = g - c$, which is a continuous function. This establishes the second equality.

To show the first equality, suppose that $\gamma \in N[[\varphi]] \cap [\varphi]$. There exists a continuous function $g : X \to \mathbb{Z}$ such that $\gamma\varphi(x) = \varphi^{g(x)}\gamma(x)$ for any $x \in X$. It follows from Lemma 2.1 together with the fact that φ has no non-trivial, invariant clopen subset of X that g is cohomologous to 1 or -1. As $\gamma \in [\varphi]$, there exists a function $f : X \to \mathbb{Z}$ such that $\gamma(x) = \varphi^{f(x)}(x)$ for any $x \in X$, and so $\gamma\varphi(x) = \varphi^{f(\varphi(x))-f(x)+1}\gamma(x)$ for any $x \in X$. Therefore, we obtain that $g(x) = f(\varphi(x)) - f(x) + 1$ for any $x \in X$, and so $\int_X g\, d\mu = 1$ for any $\mu \in M_\varphi$. This implies that g is cohomologous to 1. □

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4. *Automorphisms of a C^*-crossed product*

The aim of this section is to show Theorem 4.1, where we single out a subgroup of automorphisms of a C^*-crossed product associated with a topologically free dynamical system (X, φ), which is affiliated with the group which we have so far denoted by $N[[\varphi]]_1$, and see that the previously mentioned subgroup of automorphisms is isomorphic to a semi-direct product of some group by $N[[\varphi]]_1$.

For the moment, in order to show Theorem 4.1, let us review Tomiyama's results seen in [**7–9**] which give the relationships among a topological full group, its normalizer and automorphisms of a C^*-crossed product. They are generalizations of some results of [**3, 5**]. For a topological dynamical system (X, φ), let us denote by $C^*(X, \varphi)$ the C^*-crossed product which is generated by the commutative C^*-algebra $C(X)$ of complex-valued continuous functions on X and a unitary element u with

$$ufu^* = f \circ \varphi^{-1} \quad \text{for any } f \in C(X).$$

Let us denote by $\mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ the group of automorphisms of $C^*(X, \varphi)$ which fix the subalgebra $C(X)$ globally and define $\mathrm{Aut}_{C(X)}(C^*(X, \varphi))_1$ to be the subgroup of automorphisms $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ with the property that

$$\alpha(u) = wuw^* \quad \text{for some } w \in UN(C(X), C^*(X, \varphi)),$$

where $UN(C(X), C^*(X, \varphi))$ is the group of unitary elements $w \in C^*(X, \varphi)$ with the property that $wC(X)w^* = C(X)$. It is readily verified that $\mathrm{Aut}_{C(X)}(C^*(X, \varphi))_1$ is actually a group.

We will use the following characterization [**7**, Theorem 5.4] of the topological freeness from the C^*-algebraic viewpoint.

LEMMA 4.1. *For a topological dynamical system (X, φ), the following statements are equivalent*:

(i) *(X, φ) is topologically free*;

(ii) *$C(X)$ is a maximal Abelian subalgebra of $C^*(X, \varphi)$.*

Let us denote by $U(C(X))$ the group of unitary elements of $C(X)$. The following is seen in [**8**].

LEMMA 4.2. *Suppose that a topological dynamical system (X, φ) is topologically free. For each $v \in UN(C(X), C^*(X, \varphi))$ there exist unique $f \in U(C(X))$ and projections $\{p_n\}_{n \geq 1}$ in $C(X)$, only finitely many number of which are non-zero, such that*

$$\sum_{n \in \mathbb{Z}} p_n = \sum_{n \in \mathbb{Z}} p_n \circ \varphi^{-n} = 1 \quad \textit{and} \quad v = f \sum_{n \in \mathbb{Z}} u^n p_n.$$

Then, we define a group homomorphism $\Phi : UN(C(X), C^*(X, \varphi)) \to [[\varphi]]$, which is described in [**3**, §2] and designated by η in [**5**, Theorem 5.2], by

$$\Phi(v)(x) = \varphi^n(x) \quad \text{if } x \in \mathrm{supp}(p_n),$$

where $\mathrm{supp}(p_n)$ denotes the clopen subset of X whose characteristic function is p_n.

The following is due to [**8**, Theorem 1].

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LEMMA 4.3. *We obtain the following short exact sequence of groups*:

$$1 \to U(C(X)) \xrightarrow{i} UN(C(X), C^*(X, \varphi)) \xrightarrow{\Phi} [[\varphi]] \to 1,$$

where i is the inclusion map. This short exact sequence has a splitting v defined by

$$v_\psi = \sum_{n\in\mathbb{Z}} u^n \chi_{X_n} \quad \textit{for } \psi \in [[\varphi]],$$

where $X_n = \{x \in X;\ \psi(x) = \varphi^n(x)\}$. Moreover, it follows that

$$wgw^* = g \circ \Phi(w)^{-1} \quad \textit{for any } g \in C(X) \quad \textit{and} \quad w \in UN(C(X), C^*(X, \varphi)).$$

In [**9**, Theorem 3], it was shown that $N[[\varphi]]$ acts on the C^*-crossed product $C^*(X, \varphi)$. In fact, $N[[\varphi]]$ will be a subgroup of $\mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ by considering the split short exact sequence of groups obtained in the following lemma [**9**, Theorem 3].

LEMMA 4.4. *Suppose that a topological dynamical system (X, φ) is topologically free. Then, we obtain the following split short exact sequence*:

$$1 \to U(C(X)) \xrightarrow{\iota} \mathrm{Aut}_{C(X)}(C^*(X, \varphi)) \xrightarrow{\pi} N[[\varphi]] \to 1,$$

where ι is defined for each $g \in U(C(X))$ by

$$\iota(g)f = f \quad \textit{for } f \in C(X) \quad \textit{and} \quad \iota(g)u = ug,$$

and $\pi(\alpha)$ is defined for each $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ to be the homeomorphism on X satisfying that

$$\alpha(g) = g \circ \pi(\alpha)^{-1} \quad \textit{for any } g \in C(X).$$

A splitting $s : N[[\varphi]] \to \mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ of the above short exact sequence is defined for each $\gamma \in N[[\varphi]]$ so that

$$s(\gamma)(f) = f \circ \gamma^{-1} \quad \textit{for any } f \in C(X) \quad \textit{and} \quad s(\gamma)(u) = v_{\gamma\varphi\gamma^{-1}}.$$

To prove Theorem 4.1, we use the following lemma which gives a detailed structure of the short exact sequence in Lemma 4.4 in terms of the group $N[[\varphi]]_1$.

LEMMA 4.5. *Suppose that a topological dynamical system (X, φ) is topologically free. Then, the following are valid*:

(i) $\pi(\alpha) \in N[[\varphi]]_1$ *for any* $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X, \varphi))_1$;

(ii) $s(\gamma) \in \mathrm{Aut}_{C(X)}(C^*(X, \varphi))_1$ *for any* $\gamma \in N[[\varphi]]_1$.

Proof. (i) Let $\beta \in \mathrm{Aut}_{C(X)}(C^*(X, \varphi))$ be arbitrary. Then, it is readily checked that $\beta(u) \in UN(C(X), C^*(X, \varphi))$. From Lemma 4.3, it follows that

$$\beta(u)g\beta(u)^* = g \circ \Phi(\beta(u))^{-1} \quad \text{for any } g \in C(X).$$

However, a straightforward computation shows that

$$g \circ (\pi(\beta)\varphi\pi(\beta)^{-1})^{-1} = \beta(u)g\beta(u)^* \quad \text{for any } g \in C(X).$$

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Hence, we obtain that

$$\Phi(\beta(u)) = \pi(\beta)\varphi\pi(\beta)^{-1} \quad \text{for any } \beta \in \mathrm{Aut}_{C(X)}(C^*(X,\varphi)). \tag{4.1}$$

If $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X,\varphi))_1$, then there exists a $w \in UN(C(X), C^*(X,\varphi))$ such that $\alpha(u) = wuw^*$. From this together with (4.1), we obtain that

$$\pi(\alpha)\varphi\pi(\alpha)^{-1} = \Phi(w)\varphi\Phi(w)^{-1}.$$

Hence, it follows that $\Phi(w)^{-1}\pi(\alpha) \in C(\varphi)$ and so

$$\pi(\alpha) = \Phi(w)\Phi(w)^{-1}\pi(\alpha) \in N[[\varphi]]_1.$$

(ii) Suppose that $\gamma \in N[[\varphi]]_1$. It follows from Lemma 3.1 that there exist a $\psi \in [[\varphi]]$ and an $\eta \in C(\varphi)$ such that $\gamma = \psi\eta$. As $\gamma\varphi\gamma^{-1} = \psi\varphi\psi^{-1}$, we see that $s(\gamma)(u) = v_\psi u {v_\psi}^{-1}$, which shows that $s(\gamma) \in \mathrm{Aut}_{C(X)}(C^*(X,\varphi))_1$. □

Let us keep the notation defined in this section and put, as in [**3**],

$$U_\varphi = \{f \in U(C(X));\ f = (g\circ\varphi)g^* \quad \text{for some } g \in U(C(X))\}.$$

THEOREM 4.1. *Suppose that a topological dynamical system (X,φ) is topologically free. Then, we obtain the following exact sequence with splitting s:*

$$1 \to U_\varphi \xrightarrow{\iota} \mathrm{Aut}_{C(X)}(C^*(X,\varphi))_1 \xrightarrow{\pi} N[[\varphi]]_1 \to 1.$$

Proof. It is trivial from Lemma 4.4 that $\mathrm{Im}(\iota) \subset \ker(\pi)$. To show the converse inclusion, suppose that $\alpha \in \ker(\pi)$. As $\alpha(f) = f$ for any $f \in C(X)$, it follows that

$$\alpha(u)f\alpha(u)^* = \alpha(ufu^*) = ufu^*$$

and so

$$u^*\alpha(u)f = fu^*\alpha(u) \quad \text{for any } f \in C(X).$$

Since $C(X)$ is a maximal Abelian subalgebra of $C^*(X,\varphi)$ in virtue of Lemma 4.1, it follows that $u^*\alpha(u)$ is a unitary element in $C(X)$. Put $g_\alpha = u^*\alpha(u)$. However, as $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X,\varphi))_1$, there exists a $w \in UN(C(X), C^*(X,\varphi))$ such that $\alpha(u) = wuw^*$. There exists a unique decomposition $f\sum_n u^n p_n$ of w as in Lemma 4.2. Then, since

$$ug_\alpha = \alpha(u) = wuw^* = f\sum_{n,k\in\mathbb{Z}} u^{n-k+1}(p_n\circ\varphi^{-k+1})(p_k\circ\varphi^{-k})f^* \tag{4.2}$$

and thus

$$\sum_{n,k\in\mathbb{Z}} u^{n-k+1}(p_n\circ\varphi^{-k+1})(p_k\circ\varphi^{-k}) = u(f^*\circ\varphi)g_\alpha f,$$

we see that $(p_n\circ\varphi)p_k = 0$ whenever $n \neq k$. This implies that $p_n\circ\varphi = p_n$ for every $n \in \mathbb{Z}$. Hence, it follows from (4.2) that

$$\alpha(u) = wuw^* = fuf^*. \tag{4.3}$$

However, Lemma 4.4 implies that

$$\alpha(u) = ug \quad \text{for some } g \in U(C(X)). \tag{4.4}$$

From (4.3) and (4.4), we obtain that $g = (f\circ\varphi)f^*$. This shows the desired inclusion $\ker(\pi) \subset \mathrm{Im}(\iota)$.

What remains to be proved is readily verified. □

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