J. Appl. Prob. **17**, 662–673 (1980)
Printed in Israel
0021–9002/80/030662–12 $01.45
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A MARTINGALE APPROACH TO CENTRAL LIMIT THEOREMS FOR EXCHANGEABLE RANDOM VARIABLES

N. C. WEBER,* *University of Cambridge*

Abstract

In this paper it will be shown that by an appropriate choice of σ-fields, martingale methods can be used to obtain simple proofs of many of the central limit theorems known for triangular arrays of exchangeable random variables.

EXCHANGEABLE RANDOM VARIABLES; MARTINGALES; CENTRAL LIMIT THEOREM; INVARIANCE PRINCIPLE; REVERSED MARTINGALES; TRIANGULAR ARRAYS

1. Introduction

A sequence of random variables $\{X_n\}$, possibly finite, is said to be *exchangeable* if every subset of k distinct elements of the sequence, $\{X_{n_1}, X_{n_2}, \cdots, X_{n_k}\}$ say, has the same joint distribution depending only on k. In particular, if $\{X_n\}$ is a sequence of independent and identically distributed random variables then $\{X_n\}$ is exchangeable. The concept of exchangeability has existed in the literature for some time, being first introduced by De Finetti in 1930. The behaviour of an infinite sequence of exchangeable random variables is well known in that, conditional on the tail σ-field, it is a sequence of independent and identically distributed random variables. There are, however, sets of exchangeable random variables which cannot be embedded into an infinite sequence and these sets are called finitely exchangeable. An example of such a situation is a set of ranks where $\{R_1, R_2, \cdots, R_k\}$ is a random permutation of the integers $\{1, 2, \cdots, k\}$. Chernoff and Teicher (1958) proved that if $\{X_1, X_2, \cdots, X_m\}$ was a finitely exchangeable set of zero-mean random variables with unit variance then $E(X_1X_2) \geqq -(m-1)^{-1}$, and so if variables in an exchangeable set have a negative covariance then there is a definite upper bound to the size of any exchangeable set into which the given set may be embedded.

In this paper we are concerned with central limit theorems and invariance principles for triangular arrays of exchangeable random variables. That is, in the

Received 26 October 1978; revision received 30 July 1979.

* Present address: Department of Mathematical Statistics, The University of Sydney, Sydney NSW 2006, Australia.

https://doi.org/10.2307/3212960 Published online by Cambridge University Press

notation which we shall use from now on, if $\{X_{ni}: 1 \leqq i \leqq m,\ m > n \geqq 1\}$ is an array of random variables such that for each n, $\{X_{ni}\}_{i=1}^{m}$ is an exchangeable set of random variables with $EX_{n1} = 0$ and $EX_{n1}^2 = 1$, then we are interested in the conditions under which $S_n = \Sigma_{i=1}^{n} X_{ni}$, suitably normalised, converges to a normal limit.

Blum et al. (1958) proved a central limit theorem, under third-moment conditions, for the particular case where each row is embeddable into an infinite sequence of exchangeable random variables. Their proof uses the conditional independence of such a sequence. The proof given here uses martingale and reversed martingale techniques and relies only on the symmetry of the sequence rather than the conditional independence, and as such these methods can be used to obtain results for sequences of weakly exchangeable arrays. A weakly exchangeable array is an array of random variables indexed by a multi-dimensional parameter which has an exchangeability structure which reduces to an exchangeable sequence when the indexing parameter is a singleton. For details about weakly exchangeable arrays see Eagleson and Weber (1978).

Also, the results given in this paper for the special case where each row of the triangular array has a non-random sum include those given by Chernoff and Teicher (1958).

2. General results

Define the σ-fields

$$\mathscr{F}_{nj} = \sigma\left\{ X_{n1}, X_{n2}, \cdots, X_{nj}, \sum_{i=j+1}^{m} X_{ni}, \sum_{i=j+1}^{m} X_{ni}^2 \right\}, \quad j = 1, 2, \cdots, m.$$

Clearly $\mathscr{F}_{nj} \supseteq \mathscr{F}_{n,j-1}$, where $\mathscr{F}_{n0}$ can be taken to be the trivial σ-field, and X_{nj} is measurable with respect to $\mathscr{F}_{nj}$.

Theorem 1. Given $\{X_{nj}, \mathscr{F}_{nj}\}$ as above, with

(i) $EX_{n1}X_{n2} \to 0$ as $n \to \infty$,

(ii) $\max_{j \leqq m} |X_{nj}|/\sqrt{n} \xrightarrow{p} 0$ as $n \to \infty$,

and

(iii) $n^{-1}\Sigma_{j=1}^{n} X_{nj}^2 \xrightarrow{p} 1$ as $n \to \infty$,

then

$$W_n(t) = n^{-1/2}\left[\sum_{j=1}^{[nt]} (X_{nj} - E(X_{nj} \mid \mathscr{F}_{n,j-1}) \right] \Rightarrow W(t) \text{ as } n \to \infty,$$

where $W(t)$ is standard Brownian motion on $D[0, 1]$.

In particular,

$$\sqrt{n}\left[n^{-1}S_n - n^{-1} \sum_{j=1}^{n} E(X_{nj} \mid \mathscr{F}_{n,j-1}) \right] \xrightarrow{\mathscr{L}} N(0, 1) \quad \text{as} \quad n \to \infty.$$

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Proof. Let $Y_{nj} = X_{nj} - E(X_{nj} \mid \mathcal{F}_{n,j-1})$. Since $\{X_{ni}\}_{i=1}^{m}$ is an exchangeable set,

$$E(X_{nj} \mid \mathcal{F}_{n,j-1}) = (m-j+1)^{-1} \sum_{i=j}^{m} X_{ni}. \tag{1}$$

The result will follow from Theorem 1 of Scott (1973), slightly modified by replacing $ES_n^2 = 1$ with $ES_n^2 \to 1$, once we show that

(a) $n^{-1} \sum_{j=1}^{n} EY_{nj}^2 \to 1$ as $n \to \infty$,

(b) $\max_{j \leqq n} | Y_{nj} | / \sqrt{n} \xrightarrow{P} 0$ as $n \to \infty$,

and

(c) $n^{-1} \sum_{j=1}^{n} Y_{nj}^2 \xrightarrow{P} 1$ as $n \to \infty$.

First,

$$\begin{aligned} E\left[n^{-1} \sum_{j=1}^{n} E^2(X_{nj} \mid \mathcal{F}_{n,j-1}) \right] &= n^{-1} \sum_{j=1}^{n} E[X_{nj} E(X_{nj} \mid \mathcal{F}_{n,j-1})] \\ &\leqq n^{-1} \log[m/(m-n)] + n^{-1} | E(X_{n1} X_{n2}) | \sum_{j=1}^{n} (m-j)/(m-j+1) \\ &\leqq n^{-1} \log(n+1) + | E(X_{n1} X_{n2}) | \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned} \tag{2}$$

by (i). Thus

$$\begin{aligned} n^{-1} \sum_{j=1}^{n} EY_{nj}^2 &= n^{-1} \sum_{j=1}^{n} EX_{nj}^2 - E\left[n^{-1} \sum_{j=1}^{n} E^2(X_{nj} \mid \mathcal{F}_{n,j-1}) \right] \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Now

$$\begin{aligned} \max_{j \leqq n} | Y_{nj} | / \sqrt{n} &\leqq \max_{j \leqq n} | X_{nj} | / \sqrt{n} + \max_{j \leqq n} \left| (m-j+1)^{-1} \sum_{i=j}^{m} X_{ni} \right| \Big/ \sqrt{n} \\ &\leqq 2 \max_{j \leqq m} | X_{nj} | / \sqrt{n} \end{aligned}$$

and so (b) follows from (ii). Next,

$$n^{-1} \sum_{j=1}^{n} Y_{nj}^2 = n^{-1} \sum_{j=1}^{n} [X_{nj}^2 - 2X_{nj} E(X_{nj} \mid \mathcal{F}_{n,j-1}) + E^2(X_{nj} \mid \mathcal{F}_{n,j-1})]$$

and by (2)

$$n^{-1} \sum_{j=1}^{n} E^2(X_{nj} \mid \mathcal{F}_{n,j-1}) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

Also,

$$\begin{aligned} \left| n^{-1} \sum_{j=1}^{n} X_{nj} E(X_{nj} \mid \mathcal{F}_{n,j-1}) \right| &\leqq \left(n^{-1} \sum_{j=1}^{n} X_{nj}^2 \right)^{1/2} \left(n^{-1} \sum_{j=1}^{n} E^2(X_{nj} \mid \mathcal{F}_{n,j-1}) \right)^{1/2} \\ &\xrightarrow{P} 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

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Therefore $n^{-1}\Sigma_{j=1}^{n} Y_{nj}^2 \xrightarrow{P} 1$ as $n \to \infty$, by (iii), and the theorem is proved.

In practice we are only interested in the partial sums and their asymptotic distribution, and so we have the following corollary.

Corollary 1. Given $\{X_{nj}\}$ as above, with

(i) $nEX_{n1}X_{n2} \to 0$ as $n \to \infty$,

(ii) $\max_{j \leqq m} |X_{nj}|/\sqrt{n} \xrightarrow{P} 0$ as $n \to \infty$,

(iii) $n^{-1}\Sigma_{j=1}^{n} X_{nj}^2 \xrightarrow{P} 1$ as $n \to \infty$, and

(iv) $n/m \to 0$ as $n \to \infty$,

then $n^{-1/2}\Sigma_{j=1}^{[nt]} X_{nj} \Rightarrow W(t)$ as $n \to \infty$, and $n^{-1/2}S_n \xrightarrow{\mathcal{D}} N(0,1)$ as $n \to \infty$.

Proof. Consider

$$
\begin{aligned}
&E\left(n^{-1/2} \sum_{j=1}^{n} E(X_{nj} \mid \mathcal{F}_{n,j-1}) \right)^2 \\
&\quad = E\left[n^{-1} \sum_{j=1}^{n} E^2(X_{nj} \mid \mathcal{F}_{n,j-1}) \right] + 2n^{-1} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} E(X_{nk} E(X_{nj} \mid \mathcal{F}_{n,j-1})) \\
&\quad = n^{-1} \sum_{j=1}^{n} E[E^2(X_{nj} \mid \mathcal{F}_{n,j-1})] \\
&\qquad + 2n^{-1} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (m-j+1)^{-1}[1 + (m-j)E(X_{n1}X_{n2})] \\
&\quad \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}
$$

by (i), (iv) and (2). Thus $n^{-1/2}\Sigma_{j=1}^{n} E(X_{nj} \mid \mathcal{F}_{n,j-1}) \xrightarrow{P} 0$ as $n \to \infty$.

Remarks. First it should be noted that the conditional mean term in Theorem 1 can be written as

$$
n^{-1} \sum_{j=1}^{n} E(X_{nj} \mid \mathcal{F}_{n,j-1}) = n^{-1} \sum_{i=1}^{m} X_{ni} \left(\sum_{j=1}^{i \wedge n} (m-j+1)^{-1} \right). \tag{3}
$$

Often Conditions (ii) and (iii) above are difficult to work with directly and so it is useful to notice that if $nm^{-1} \to \alpha > 0$ as $n \to \infty$, (ii) is implied by

$$
\forall \varepsilon > 0, \quad EX_{n1}^2 I(|X_{n1}| > \varepsilon \sqrt{n}) \to 0 \quad \text{as} \quad n \to \infty, \tag{4}
$$

and can in any case be replaced by

$$
E|X_{n1}|^{2+\delta} = o(n^{\delta/2}) \quad \text{for some } \delta > 0. \tag{5}
$$

That (4) implies (ii) follows from the fact that (ii) is equivalent to

$$
n^{-1} \sum_{j=1}^{m} X_{nj}^2 I(|X_{nj}| > \varepsilon \sqrt{n}) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty, \quad \text{for all } \varepsilon > 0.
$$

If X_{n1} has finite fourth moments and if $n/m \to \alpha$, $0 \leqq \alpha < 1$, then (ii) and (iii) can be replaced in the above results by

$$
EX_{n1}^4 = o(n), \tag{6}
$$

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and

$$EX_{n1}^2 X_{n2}^2 \to 1 \quad \text{as} \quad n \to \infty. \tag{7}$$

Condition (6) is just Condition (5) with $\delta = 2$, and so implies (ii). To see that (6) and (7) jointly can replace (iii), note that (iii) is only used to imply $n^{-1}\Sigma_{j=1}^n Y_{nj}^2 \xrightarrow{P} 1$ as $n \to \infty$, and by Scott's lemma, this condition is equivalent to $n^{-1}\Sigma_{j=1}^n E(Y_{nj}^2 \mid \mathscr{F}_{n,j-1}) \xrightarrow{P} 1$ as $n \to \infty$. As

$$n^{-1}\sum_{j=1}^n E(Y_{nj}^2 \mid \mathscr{F}_{n,j-1}) = n^{-1}\sum_{j=1}^n [E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) - E^2(X_{nj} \mid \mathscr{F}_{n,j-1})]$$

it is sufficient to show that

$$E\left| n^{-1}\sum_{j=1}^n E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) - 1 \right| \to 0 \quad \text{as} \quad n \to \infty,$$

for $n^{-1}\Sigma_{j=1}^n E^2(X_{nj} \mid \mathscr{F}_{n,j-1}) \xrightarrow{P} 0$ under Condition (i). Now

$$\begin{aligned} E\left| n^{-1}\sum_{j=1}^n E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) - 1 \right| &\leqq n^{-1}\sum_{j=1}^n E^{1/2}(E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) - 1)^2 \\ &= n^{-1}\sum_{j=1}^n E^{1/2}(E^2(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) - 1), \end{aligned}$$

and

$$\begin{aligned} E(E^2(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) - 1) &= (m-j+1)^{-2} E\left(\sum_{i=j}^m X_{ni}^2\right)^2 - 1 \\ &= (m-j+1)^{-1}[EX_{n1}^4 + (m-j)EX_{n1}^2 X_{n2}^2] - 1, \end{aligned}$$

and this tends to zero under Conditions (6) and (7).

Another way of obtaining a central limit theorem for these triangular arrays is to use a reversed martingale approach. Define the σ-fields

$$\mathscr{H}_{nj} = \sigma\{(X_{n1}^{(1)}, \cdots, X_{nj}^{(j)}), X_{n,j+1}, X_{n,j+2}, \cdots, X_{n,m}\}; \quad 1 \leqq j \leqq m,$$

where $(X_{n1}^{(1)}, \cdots, X_{nj}^{(j)})$ denotes the order statistics of the first j elements of the nth row. Clearly, $\mathscr{H}_{nj} \supseteq \mathscr{H}_{n,j+1}$ and $E(X_{n1} \mid \mathscr{H}_{nj}) = j^{-1}\Sigma_{i=1}^j X_{ni} = j^{-1}S_{nj}$. Thus $\{j^{-1}S_{nj}, \mathscr{H}_{nj}\}$ is a reversed martingale for each n.

Theorem 2. Given $\{X_{nj}\}$ as above, with

(i) $n/m \to \alpha$ as $n \to \infty$, $0 < \alpha < 1$,

(ii) $EX_{n1}X_{n2} \to 0$ as $n \to \infty$,

(iii) $\max_{i \leqq m} |X_{ni}|/\sqrt{n} \xrightarrow{P} 0$ as $n \to \infty$,

and

(iv) $n^{-1}\Sigma_{i=1}^n X_{ni}^2 \xrightarrow{P} 1$ as $n \to \infty$,

then

$$\sqrt{n}[n^{-1}S_n - E(X_{n1} \mid \mathscr{H}_{n,m})] \xrightarrow{\mathscr{D}} N(0, 1-\alpha) \quad \text{as} \quad n \to \infty.$$

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Proof. Let

$$Z_{ni} = \sqrt{n}[E(X_{n1} \mid \mathscr{H}_{ni}) - E(X_{ni} \mid \mathscr{H}_{n,i+1})]$$

$$= \sqrt{n}\, i^{-1}[E(X_{n1} \mid \mathscr{H}_{n,i+1}) - X_{n,i+1}], \tag{8}$$

so that

$$\sqrt{n}[n^{-1}S_n - E(X_{n1} \mid \mathscr{H}_{n,m})] = \sum_{i=n}^{m-1} Z_{ni}.$$

Noting that $\{Z_{n,(m-j+1)}, \mathscr{H}_{n,m-j+1}\}_{j=1}^{(m-n+1)}$ is a martingale difference array, the result will follow from Theorem 1 of Scott (1973) once we show

(a) $\sum_{i=n}^{m-1} EZ_{ni}^2 \to 1-\alpha$ as $n \to \infty$,

(b) $\sum_{i=n}^{m-1} Z_{ni}^2 \xrightarrow{P} 1-\alpha$ as $n \to \infty$,

and

(c) $\max_{n \le i \le m-1} |Z_{ni}| \xrightarrow{P} 0$ as $n \to \infty$.

First, from (8) we have

$$\max_{n \le i \le m-1} |Z_{ni}| \le n^{-1/2} \max_{n \le i \le m-1} \left[|X_{n,i+1}| + \left| (i+1)^{-1} \sum_{j=1}^{i+1} X_{nj} \right| \right]$$

$$\le 2n^{-1/2} \max_{i \le m} |X_{ni}|,$$

and so (c) holds as a result of Condition (iii). Next, since

$$E(X_{n,i+1} \mid \mathscr{H}_{n,i+1}) = E(X_{n1} \mid \mathscr{H}_{n,i+1}),$$

$$\sum_{i=n}^{m-1} EZ_{ni}^2 = n \sum_{i=n}^{m-1} i^{-2} E(E(X_{n1}^2 \mid \mathscr{H}_{n,i+1}) - E^2(X_{n1} \mid \mathscr{H}_{n,i+1}))$$

$$= n(1 - E(X_{n1}X_{n2})) \sum_{i=n}^{m-1} [i(i+1)]^{-1}$$

$$= (1 - E(X_{n1}X_{n2}))(1 - (n/m)) \to 1-\alpha \quad \text{as} \quad n \to \infty,$$

by Conditions (i) and (ii). Also

$$n \sum_{i=n}^{m-1} i^{-2} E(E^2(X_{n1} \mid \mathscr{H}_{n,i+1})) = n \sum_{i=n}^{m-1} \left[i^{-2}(i+1)^{-1} + \frac{EX_{n1}X_{n2}}{i(i+1)} \right]$$

$$\le (n-1)^{-1} + |EX_{n1}X_{n2}| \to 0 \quad \text{as} \quad n \to \infty,$$

and so

$$n \sum_{i=n}^{m-1} i^{-2} E^2(X_{n1} \mid \mathscr{H}_{n,i+1}) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty. \tag{9}$$

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Now

$$\sum_{i=n}^{m-1} Z_{ni}^2 = n\sum_{i=n}^{m-1} i^{-2}[X_{n,i+1}^2 - 2X_{n,i+1}E(X_{n1}\mid \mathscr{H}_{n,i+1}) + E^2(X_{n1}\mid \mathscr{H}_{n,i+1})]$$

and

$$\left| n\sum_{i=n}^{m-1} i^{-2}X_{n,i+1}E(X_{n1}\mid \mathscr{H}_{n,i+1})\right| \leqq \left(n\sum_{i=n}^{m-1} i^{-2}X_{n,i+1}^2\right)^{1/2}\cdot\left(n\sum_{i=n}^{m-1} i^{-2}E^2(X_{n1}\mid \mathscr{H}_{n,i+1})\right)^{1/2}$$

so by (9), (b) will follow if we can show that

$$n\sum_{i=n}^{m-1} i^{-2}X_{n,i+1}^2 \xrightarrow{p} 1-\alpha \quad \text{as} \quad n\to\infty. \tag{10}$$

Note that

$$\left| n\sum_{i=n}^{m-1} i^{-2}X_{n,i+1}^2 - (1-\alpha)\right| \leqq \left| n\sum_{i=n+1}^{m}(i-1)^{-2}(X_{ni}^2-1)\right| + \left|\alpha - \frac{n}{m}\right| + O(n^{-1}),$$

so we must show that

$$\left| n\sum_{i=n+1}^{m}(i-1)^{-2}(X_{ni}^2-1)\right| \xrightarrow{p} 0 \quad \text{as} \quad n\to\infty.$$

Writing

$$n\sum_{i=n+1}^{m}(i-1)^{-2}(X_{ni}^2-1) = \sum_{j=n+1}^{m} c_{nj}\sum_{i=n+1}^{j}(X_{ni}^2-1)$$

where

$$c_{nj} = \frac{n}{(j-1)^2} - \frac{n}{j^2}, \quad n+1 \leqq j < m$$

and

$$c_{nm} = \frac{n}{(m-1)^2},$$

we see that

$$\left| n\sum_{i=n+1}^{m}(i-1)^{-2}(X_{ni}^2-1)\right| \leqq \max_{n<j\leqq m}\left|\sum_{i=n+1}^{j}(X_{ni}^2-1)\right|\cdot\sum_{j=n+1}^{m} c_{nj}$$

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$$= \max_{n<j\leqq m} \left| n^{-1} \sum_{i=n+1}^{j} (X_{ni}^2 - 1) \right|$$

$$\leqq \frac{m}{n} \max_{n\leqq j\leqq m} \left| j^{-1} \sum_{i=1}^{j} (X_{ni}^2 - 1) \right| + \left| n^{-1} \sum_{i=1}^{n} (X_{ni}^2 - 1) \right|.$$

Thus, using conditions (iv) and (i), the proof will be complete once we show that

$$\max_{n\leqq j\leqq m} \left| j^{-1} \sum_{i=1}^{j} (X_{ni}^2 - 1) \right| \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty. \tag{11}$$

Let

$$U_{nj} = |E(X_{n1}^2 - 1 \mid \mathcal{H}_{n,j})|,$$

then

$$\max_{n\leqq j\leqq m} \left| j^{-1} \sum_{i=1}^{j} (X_{ni}^2 - 1) \right| = \max_{n\leqq j\leqq m} U_{nj}$$

and for each n, $\{U_{n,m-j+1}, \mathcal{H}_{n,m-j+1}\}_{j=1}^{(m-n+1)}$ is a submartingale. Thus from Theorem 3.2 of Doob ((1953), p. 314), given any $\varepsilon > 0$,

$$P\left(\max_{n\leqq j\leqq m} U_{nj} > \varepsilon \right) \leqq \varepsilon^{-1} E \mid U_{nn} \mid$$

$$= \varepsilon^{-1} E \left| n^{-1} \sum_{i=1}^{n} X_{ni}^2 - 1 \right|.$$

Since $EX_{ni}^2 = 1$, Condition (iv) is equivalent to $n^{-1} \Sigma_{i=1}^{n} X_{ni}^2 \xrightarrow{L_1} 1$ and so (11) holds and the proof is completed.

Note that Condition (iii) is equivalent to

$$\max_{i\leqq m} |X_{ni}| / \sqrt{m} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty, \quad \text{since } \alpha > 0.$$

3. Triangular arrays with non-random row sums

There are several situations where a triangular array of exchangeable random variables satisfies the extra condition that the sum $\Sigma_{i=1}^{m} X_{ni}$ is non-random. Examples of such situations include arrays of ranks and arrays of percentiles as explained in Chernoff and Teicher (1958). Theorem 2 and Corollary 1 above can be used to obtain conditions under which the partial sums of the rows of such arrays have a limiting normal distribution.

Theorem 3. Given $\{X_{ni}\}$ as above with $\Sigma_{i=1}^{m} X_{ni} = 0$ for each n, then if

(i) $n/m \to \alpha$ as $n \to \infty$, $(0 \leqq \alpha < 1)$,

(ii) $\max_{i\leqq m} |X_{ni}| / \sqrt{n} \xrightarrow{P} 0$ as $n \to \infty$,

and

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(iii) $n^{-1}\Sigma_{i=1}^{n} X_{ni}^2 \xrightarrow{P} 1$ as $n \to \infty$,

$$n^{-1/2} S_n \xrightarrow{\mathcal{D}} N(0, 1-\alpha) \quad \text{as} \quad n \to \infty.$$

Proof. Since $\Sigma_{i=1}^{m} X_{ni} = 0$ and $EX_{n1}^2 = 1$, we have that $EX_{n1}X_{n2} = -(m-1)^{-1}$. Thus if $\alpha = 0$, the result follows from Corollary 1. If $0 < \alpha < 1$, then the result follows from Theorem 2 as $E(X_{n1} \mid \mathscr{H}_{n,m}) = m^{-1}\Sigma_{i=1}^{m} X_{ni} = 0$.

Theorem 3 generalises Theorem 1 of Chernoff and Teicher (1958) in that it allows $\alpha = 0$ as a possible condition. If $\alpha > 0$ then the conditions of Theorem 3 above, are equivalent to those of Chernoff and Teicher.

4. Triangular arrays with rows embeddable in infinite sequences

Suppose that for each n, $\{X_{ni}\}_{i \geq 1}$ is an infinite sequence of exchangeable random variables. A more general form of the central limit theorem for such an array is one which admits convergence to mixtures of normal variables and to allow these mixing variables to be easily identified we introduce the σ-fields

$$\mathscr{I}_{nj} = \sigma\{(X_{n1}^{(1)}, \cdots, X_{nj}^{(j)}), X_{n,j+1}, X_{n,j+2}, \cdots\}, \quad j \geq 1.$$

Let $\mathscr{I}_{n\infty} = \bigcap_{j=1}^{\infty} \mathscr{I}_{nj}$ and put $\mathscr{I} = \bigcap_{n=1}^{\infty} \mathscr{I}_{n\infty}$. Clearly $\mathscr{I}_{nj} \supseteq \mathscr{I}_{n,j+1}$ and $\{j^{-1}S_{nj}, \mathscr{I}_{nj}\}$ is a reversed martingale, for each n. By the reversed martingale convergence theorem we have, for each n,

$$j^{-1}S_{nj} = E(X_{n1} \mid \mathscr{I}_{nj}) \xrightarrow[L_2]{\text{a.s.}} E(X_{n1} \mid \mathscr{I}_{n\infty}) \quad \text{as} \quad j \to \infty. \tag{12}$$

Theorem 4. Let $\{X_{ni}\}_{i \geq 1}$ be an array of exchangeable random variables with $EX_{n1} = 0$ and $EX_{n1}^2 = 1$. If there exists an $\mathscr{I}$-measurable, almost surely finite, strictly positive random variable η such that

(a) $E(X_{n1}^2 \mid \mathscr{I}_{n\infty}) - E^2(X_{n1} \mid \mathscr{I}_{n\infty}) = \eta_n^2 \xrightarrow{P} \eta^2$ as $n \to \infty$,

(b) $\forall \varepsilon > 0$, $E(X_{n1}^2 I(|X_{n1}| > \varepsilon \sqrt{n})) \to 0$ as $n \to \infty$,

then

$$\sqrt{n}(n^{-1}S_n - E(X_{n1} \mid \mathscr{I}_{n\infty})) \to \eta' N(0,1),$$

where η' has the same distribution as η and is independent of $N(0,1)$.

Proof. For the purposes of the proof we can suppose that the σ-fields $\mathscr{F}_{nj}$ are extended so that $\mathscr{I}_{n\infty} \subseteq \mathscr{F}_{n1}$ for all n. The result will now follow from a corollary to Theorem 3 of Eagleson (1975) once we show, in the notation of Theorem 1,

(i) $n^{-1}\Sigma_{j=1}^{n} E(Y_{nj}^2 \mid \mathscr{F}_{n,j-1}) \xrightarrow{P} \eta^2$ as $n \xrightarrow{P} \infty$,

(ii) $\sqrt{n}[n^{-1}\Sigma_{j=1}^{n} E(X_{nj} \mid \mathscr{F}_{n,j-1}) - E(X_{n1} \mid \mathscr{I}_{n\infty})] \xrightarrow{P} 0$ as $n \to \infty$,

(iii) $\forall \varepsilon > 0$, $n^{-1}\Sigma_{j=1}^{n} E(Y_{nj}^2 I(|Y_{nj}| > \varepsilon \sqrt{n}) \mid \mathscr{F}_{n,j-1}) \xrightarrow{P} 0$ as $n \to \infty$.

It should be remembered that $\mathscr{F}_{nj}$ depends on m, although the m is suppressed in the notation, where m, depending on n, is to be chosen later.

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Note that by the reversed martingale convergence theorem

$$E(X_{nj}^2 \mid \mathcal{F}_{n,j-1}) = (m-j+1)^{-1} \sum_{i=j}^{m} X_{ni}^2$$

$$= \frac{m}{(m-j+1)} E(X_{n1}^2 \mid \mathcal{I}_{nm}) - \left(\frac{j-1}{m-j+1}\right) E(X_{n1}^2 \mid \mathcal{I}_{n,j-1})$$

$$\xrightarrow[L_1]{\text{a.s.}} E(X_{n1}^2 \mid \mathcal{I}_{n\infty}) \quad \text{as} \quad m \to \infty, \text{ for fixed } n \text{ and } j.$$

Thus for n fixed,

$$n^{-1} \sum_{j=1}^{n} E(X_{nj}^2 \mid \mathcal{F}_{n,j-1}) \xrightarrow{L_1} E(X_{n1}^2 \mid \mathcal{I}_{n\infty}) \quad \text{as} \quad m \to \infty.$$

Also,

$$E[E(X_{nj} \mid \mathcal{F}_{n,j-1}) - E(X_{n1} \mid \mathcal{I}_{n\infty})]^2 = E[E^2(X_{nj} \mid \mathcal{F}_{n,j-1}) - E^2(X_{n1} \mid \mathcal{I}_{n\infty})]$$

and so again by the reversed martingale convergence theorem, for fixed n,

$$E\left| n^{-1} \sum_{j=1}^{n} E^2(X_{nj} \mid \mathcal{F}_{n,j-1}) - E^2(X_{n1} \mid \mathcal{I}_{n\infty}) \right| \to 0 \quad \text{as} \quad m \to \infty.$$

Let

$$A_{n,m} = n^{-1} \sum_{j=1}^{n} E(Y_{nj}^2 \mid \mathcal{F}_{n,j-1}) - E(X_{n1}^2 \mid \mathcal{I}_{n\infty}) + E^2(X_{n1} \mid \mathcal{I}_{n\infty}),$$

and let

$$B_{n,m} = \sqrt{n}\left[n^{-1} \sum_{j=1}^{n} E(X_{nj} \mid \mathcal{F}_{n,j-1}) - E(X_{n1} \mid \mathcal{I}_{n\infty}) \right].$$

For each n, choose $m = m(n) > n$ such that $n/m \to 0$, $E|A_{n,m}| < n^{-1}$ and $E|B_{n,m}| < n^{-1}$. Then for this sequence $\{m(n)\}$ we have that $A_{n,m} \xrightarrow{p} 0$ and $B_{n,m} \xrightarrow{p} 0$ as $n \to \infty$ and so combining these results with Condition (a) we have that (i) and (ii) hold.

Finally, using Lemma 3.3 of Dvoretzky (1972)

$$n^{-1} \sum_{j=1}^{n} E(Y_{nj}^2 I(|Y_{nj}| > \varepsilon \sqrt{n}))$$

$$= n^{-1} \sum_{j=1}^{n} E[(X_{nj} - E(X_{nj} \mid \mathcal{F}_{n,j-1}))^2 I(|X_{nj} - E(X_{nj} \mid \mathcal{F}_{n,j-1})| > \varepsilon \sqrt{n})]$$

$$\leqq 4n^{-1} \sum_{j=1}^{n} E(X_{nj}^2 I(|X_{ni}| > \tfrac{1}{2}\varepsilon \sqrt{n}))$$

$$= 4E(X_{n1}^2 I(|X_{n1}| > \tfrac{1}{2}\varepsilon \sqrt{n})).$$

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Therefore Condition (iii) is implied by (b).

Corollary 2. Given $\{X_{ni}\}_{i\geq 1}$ as above with $EX_{n1}^4 < \infty$, and
(a) $EX_{n1}^2 X_{n2}^2 \to 1$ as $n \to \infty$,
(b) $nEX_{n1}X_{n2} \to 0$ as $n \to \infty$,
and

(c) $\forall \varepsilon > 0$, $E(X_{n1}^2 I(|X_{n1}| > \varepsilon \sqrt{n})) \to 0$ as $n \to \infty$,
then $n^{-1/2} S_n \xrightarrow{\mathcal{D}} N(0,1)$ as $n \to \infty$.

Proof. The result follows from Theorem 4 once we establish that $\sqrt{n}E(X_{n1} \mid \mathcal{I}_{n\infty}) \xrightarrow{p} 0$ as $n \to \infty$ and $E(X_{n1}^2 \mid \mathcal{I}_{n\infty}) \xrightarrow{p} 1$ as $n \to \infty$. Notice that

$$\begin{aligned} E(\sqrt{n}E(X_{n1} \mid \mathcal{I}_{n\infty}))^2 &= nE\left(\lim_{j\to\infty} E^2(X_{n1} \mid \mathcal{I}_{nj})\right) \\ &= n \lim_{j\to\infty} E(X_{n1}E(X_{n1} \mid \mathcal{I}_{nj})) \\ &= n \lim_{j\to\infty} E\left(X_{n1}\left(j^{-1}\sum_{i=1}^{j} X_{ni}\right)\right) \\ &= nEX_{n1}X_{n2} \to 0 \quad \text{as} \quad n \to \infty, \quad \text{by (b).} \end{aligned}$$

Also,

$$\begin{aligned} E(E(X_{n1}^2 \mid \mathcal{I}_{n\infty}) - 1)^2 &= E(E^2(X_{n1}^2 \mid \mathcal{I}_{n\infty})) - 1 \\ &= \lim_{j\to\infty} E(X_{n1}^2 E(X_{n1}^2 \mid \mathcal{I}_{nj})) - 1 \\ &= E(X_{n1}^2 X_{n2}^2) - 1, \quad \text{if } EX_{n1}^4 < \infty. \end{aligned}$$

Remark. All of the above results on arrays of exchangeable random variables have been obtained without using the conditional independence of infinite exchangeable sets. The cost of such proofs is seen in Corollary 2 where we required finite fourth moments as compared with third moments in Theorem 2 of Blum et al. (1958). If we use the conditional independence then the conditions of Corollary 2 are sufficient without requiring $EX_{n1}^4 < \infty$, for conditional on $\mathcal{I}_{n\infty}$, X_{n1} and X_{n2} are independent random variables and so

$$\begin{aligned} E(X_{n1}^2 X_{n2}^2) - 1 &= E(E(X_{n1}^2 \mid \mathcal{I}_{n\infty})E(X_{n2}^2 \mid \mathcal{I}_{n\infty})) - 1 \\ &= E(E^2(X_{n1}^2 \mid \mathcal{I}_{n\infty})) - 1. \end{aligned}$$

Acknowledgements

The author would like to acknowledge many helpful comments made by G. K. Eagleson and J. Robinson.

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