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Hyperbolicity versus weak periodic orbits inside homoclinic classes

XIAODONG WANG

School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, 200240, China
School of Mathematical Sciences, Peking University, Beijing, 100871, China
Laboratoire de Mathématiques d'Orsay, Université Paris-Sud 11, Orsay, 91405, France
(e-mail: xdwang1987@sjtu.edu.cn, xdwang1987@gmail.com)

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Abstract. We prove that, for C^1-generic diffeomorphisms, if the periodic orbits contained in a homoclinic class $H(p)$ have all their Lyapunov exponents bounded away from zero, then $H(p)$ must be (uniformly) hyperbolic. This is in the spirit of the works on the stability conjecture, but with a significant difference that the homoclinic class $H(p)$ is not known isolated in advance, hence the 'weak' periodic orbits created by perturbations near the homoclinic class have to be guaranteed strictly inside the homoclinic class. In this sense the problem is of an 'intrinsic' nature, and the classical proof of the stability conjecture does not work. In particular, we construct in the proof several perturbations which are not simple applications of the connecting lemmas.

1. *Introduction*

1.1. *Background and main results.* We study in this paper a problem that is in the spirit of the works on the stability conjecture but with an 'intrinsic' nature. Let M be a compact manifold without boundary, and $\mathrm{Diff}^1(M)$ be the space of C^1-diffeomorphisms of M. Recall that the stability conjecture formulated by Palis and Smale claims that if a diffeomorphism f is structurally stable, then it is hyperbolic. Here a diffeomorphism f is called *hyperbolic* if the chain recurrent set $R(f)$ of f (see Definition 2.9) is hyperbolic. A stronger version of the conjecture is to claim that if f is Ω-stable, then it is hyperbolic. These two remarkable conjectures are solved by Mañé [**37**] and Palis [**38**], respectively.

During the long history of study of the stability conjectures, attention has increasingly been concentrated on periodic orbits of the (unperturbed) diffeomorphism f as well as its perturbations g. Liao [**35**] and Mañé [**36**] raised independently a conjecture (more precisely, a problem without a tentative answer), known as the star conjecture, stating

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that if f has no, robustly, non-hyperbolic periodic orbits, then it is hyperbolic. Being an assumption, the star condition is clearly weaker than the Ω-stability. Hence, the star conjecture is regarded as another (strong) version of the stability conjecture. It was solved by Aoki [**3**] and Hayashi [**32**]. To compare more precisely with our Main Theorem below, we state their results in a generic version. Recall that if p is a periodic point with period τ of a diffeomorphism f, and if $\lambda_1, \lambda_2, \ldots, \lambda_d$ are the eigenvalues of Df^τ (counted by multiplicity), then the d numbers $\chi_i = (1/\tau)\log|\lambda_i|$, $i = 1, \ldots, d$, are called the *Lyapunov exponents* of $\mathrm{Orb}(p)$.

THEOREM 1.1. (Aoki [**3**] and Hayashi [**32**]) *For a C^1-generic $f \in \mathrm{Diff}^1(M)$, if f is not hyperbolic, then there is a periodic orbit of f that has a Lyapunov exponent arbitrarily close to zero.*

Now we state our main result. Recall that two hyperbolic periodic points are *homoclinically related* if $W^s(\mathrm{Orb}(p))$ has non-empty transverse intersection with $W^u(\mathrm{Orb}(q))$ and $W^u(\mathrm{Orb}(p))$ has non-empty transverse intersection with $W^s(\mathrm{Orb}(q))$. To be homoclinically related is an equivalent relation, and the *homoclinic class* $H(p)$ of a hyperbolic periodic point p is the closure of the union of periodic orbits that are homoclinically related to p. Two different homoclinic classes may intersect. Nevertheless by the result of [**8**], for C^1-generic diffeomorphisms, every homoclinic class is a maximal invariant compact set that is *chain transitive* (see Definition 2.8), hence they are pairwise disjoint. Homoclinic classes are generally infinite in number, even for generic diffeomorphisms.

MAIN THEOREM. *For a C^1-generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ of f is not hyperbolic, then there is a periodic orbit of f that is homoclinically related to* $\mathrm{Orb}(p)$ *and has a Lyapunov exponent arbitrarily close to zero.*

Note that here the 'weak' periodic orbit (the one with a Lyapunov exponent arbitrarily close to zero) is homoclinically related to $\mathrm{Orb}(p)$, that is, is 'inside' the homoclinic class $H(p)$. This is the main point of this paper. In fact, under the assumptions of the Main Theorem, it is straightforward to prove (following the classical proof of the stability conjecture) that there must be a weak periodic orbit arbitrarily near $H(p)$. In contrast, here the Main Theorem claims there must be a weak periodic orbit not only near, but actually inside $H(p)$. Of course, if the homoclinic class $H(p)$ is assumed to be isolated, then being 'near' will be equivalent to being 'inside'. The point is that here $H(p)$ is not known to be isolated, hence, at each step, the periodic orbits created by perturbations have to be guaranteed to lie strictly inside the homoclinic class. It is in this sense that we say the problem is of an 'intrinsic' nature, and the classical proof of the stability conjecture does not work.

There are some other results that are concerned with whether the newly created periodic orbits are inside or outside the homoclinic class, for instance [**9**, **14**]. The lack of domination of the homoclinic class yields periodic orbits having multiple Lyapunov exponents and weak hyperbolicity [**9**] or allows the production of sinks or sources [**14**]. The difference is that [**9**] applies the Franks–Gourmelon lemma [**30**] (see also Lemma 3.4 in §3) to guarantee that the newly created periodic orbits are still inside the homoclinic

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class, while [**14**] claims the newly created sinks or sources are outside the initial class by producing an attracting or repelling region. In our paper, we build new periodic orbits by mixing a hyperbolic periodic orbit and a *weak set*, which has very weak hyperbolicity. The new periodic orbits have weak hyperbolicity because they spend a long time close to the weak set. However, by the property of hyperbolic periodic orbit, the amount of time that the new periodic orbit spends close to the initial periodic orbit (which is chosen after that close to the weak set has been fixed) can be any large number. Hence, by choosing the amount of time properly, the hyperbolicity of the new periodic orbits is still uniform in some sense. This is the reason that the new periodic orbits are homoclinically related to the initial one. We have to connect first the initial periodic orbit and the weak set by heteroclinic orbits through several perturbations. The difficulty is that the connection between the new periodic orbit and the homoclinic class may be destroyed in any step of the process. To avoid this, we have to use the generic properties to guarantee that perturbations lie inside the homoclinic class.

There are other conjectures aimed at giving a dichotomy of global dynamics. Recall that a *homoclinic tangency* of a hyperbolic periodic point p is a non-transverse intersection between $W^u(p)$ and $W^s(p)$. A diffeomorphism is with a *heterodimensional cycle* if there are two hyperbolic periodic points p and q with different stable dimensions such that $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^s(q) \cap W^u(p) \neq \emptyset$. It is obvious that any diffeomorphism with either a tangency or a heterodimensional cycle is not hyperbolic. Palis conjectured that these two phenomena are the only obstacles for hyperbolicity. More precisely, the union of hyperbolic diffeomorphisms and diffeomorphisms with tangencies or heterodimensional cycles are dense in the space of diffeomorphisms, see [**39**]. Based on the results afterwards, Bonatti and Díaz conjectured that the union of diffeomorphisms that are hyperbolic and those with heterodimensional cycles are dense in the space of diffeomorphisms, see [**7**, **11**]. There are many works related to this subject, such as [**19**, **22**, **23**, **43**]. Pujals and Sambarino [**43**] solved this conjecture for dimension two, and for higher dimension, some progress has been made in [**19**, **22**, **23**] that far from homoclinic bifurcations, the system has some weak hyperbolicity (partially hyperbolic or essentially hyperbolic).

By the Franks lemma [**26**, **30**], we can perturb weak periodic orbits to get periodic orbits with different stable dimensions. But it is not clear whether these periodic orbits are still contained in the non-hyperbolic homoclinic class after perturbation. Thus, we have the following conjecture, which is an intrinsic version of Palis conjecture for homoclinic classes.

CONJECTURE 1. (Bonatti *et al* [**7**, **9**]) *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$, such that for all $f \in \mathcal{R}$, if a homoclinic class $H(p)$ is not hyperbolic, then there is a periodic point $q \in H(p)$, whose stable dimension is different from that of p.*

By [**16**], one can decompose the dynamics into pieces, and each piece is called a *chain recurrence class* (see Definition 2.10). By [**8**], for C^1-generic diffeomorphism, a chain recurrence class is either a homoclinic class or contains no periodic point. We call a chain recurrence class without periodic point an *aperiodic class*. Recall that a *dominated splitting* $E \oplus F$ on an invariant compact Λ set is an invariant splitting of $T_\Lambda M$ and the norm of Df along E is controlled by that along F, and Λ is *partially hyperbolic* if $T_\Lambda M$

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splits into three bundles, which is a dominated splitting such that the extremal bundles are hyperbolic and the center bundle is neutral (see Definition 2.6). By [**19**, **23**], for C^1-generic diffeomorphisms far from homoclinic bifurcations (or just homoclinic tangencies), an aperiodic class is partially hyperbolic with center bundle of dimension one. In [**10**], they proved that if $\dim(M) \geq 3$, then there are an open set $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ and a residual subset $\mathcal{V}$ of $\mathcal{U}$ such that any $g \in \mathcal{V}$ has infinitely many aperiodic classes, and each of them has no non-trivial dominated splitting. We state here a conjecture by Crovisier for aperiodic classes, which implies the non-existence of aperiodic classes for C^1-generic diffeomorphisms far from homoclinic bifurcations.

CONJECTURE 2. (Crovisier [**21**]) *Let Λ be an aperiodic class for a C^1-generic diffeomorphism f and let $E^s \oplus E^c \oplus E^u$ be the dominated splitting such that E^s (respectively E^u) is the maximal uniformly contracting (respectively expanding) subbundle. Then E^c has dimension larger than or equal to two and does not admit a finer dominated splitting.*

1.2. *Main Theorem restated.* We give here a more general result rather than the Main Theorem. For a hyperbolic periodic point p, denote by $\mathrm{Ind}(p)$ its stable dimension.

THEOREM A. *For C^1-generic $f \in \mathrm{Diff}^1(M)$, assume that p is a hyperbolic periodic point of f. If the homoclinic class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, with $\dim E \leq \mathrm{Ind}(p)$, such that the bundle E is not contracting, then there are periodic orbits in $H(p)$ with index $\dim(E)$ that have the maximal Lyapunov exponents along E arbitrarily close to zero.*

Remark 1.2. If, in the assumption of Theorem A, $\dim E = \mathrm{Ind}(p)$, then the periodic orbits $\mathcal{O}_k = \mathrm{Orb}(q_k)$ obtained have the same index as p. Thus, by the genericity assumption and Lemma 2.38(4), these periodic orbits $\mathcal{O}_k$ are homoclinically related with $\mathrm{Orb}(p)$.

We give an explanation why Theorem A implies the main theorem. We assume that all Lyapunov exponents of periodic orbits that are homoclinically related to $\mathrm{Orb}(p)$ are uniformly away from zero. Then by the genericity assumption, $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, with $\dim E = \mathrm{Ind}(p)$ (see [**28**] and [**9**, Proposition 4.8]). By the conclusion of Theorem A and the assumption of no existence of weak periodic orbits homoclinically related to $\mathrm{Orb}(p)$, we get that the bundle E is contracting. With the same argument for f^{-1}, we get that the bundle F is expanding for f. Hence, $T_{H(p)}M = E \oplus F$ is a hyperbolic splitting which is a contradiction to the assumption of the main theorem.

In [**37**], Mañé introduced a very useful lemma (Theorem II.1) to get weak periodic orbits under certain hypotheses. The statement is very technical and the original proof of Mañé is difficult, thus we will not state it here. Based on a modification of the proof of Mañé, Bonatti *et al* have a result for homoclinic classes, see [**13**].

Here we point out that, different from [**37**, Theorem II.1] and the result of [**13**], there is a genericity assumption in the main theorem and Theorem A. That is to say, the conclusion of the main theorem is a perturbation result and may not be valid for all diffeomorphisms. Thus, one asks the following question naturally, whether the genericity assumption is essential in the main theorem.

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Question 1. Is there a homoclinic class $H(p)$ for a diffeomorphism f satisfying that all of the Lyapunov exponents of all periodic orbits homoclinically related to $\mathrm{Orb}(p)$ are uniformly away from zero but $H(p)$ is not hyperbolic?

In some cases, we can give a positive answer to this question. In [**45**], Rios proved that there is a diffeomorphism on the boundary of the set of hyperbolic diffeomorphisms on surface, with a homoclinic class containing a tangency inside. Hence, it is not hyperbolic (it does not have a dominated splitting because of the existence of tangency). In [**15**], they proved that for this homoclinic class, all of the Lyapunov exponents of all periodic orbits contained in the class are uniformly away from zero. In fact, they proved more in that all of the Lyapunov exponents of all ergodic measures are uniformly away from zero. Examples of non-hyperbolic homoclinic classes with a dominated splitting can be found in [**17**, **24**, **25**, **44**], but the homoclinic classes in these examples contain weak periodic orbits. For C^2 diffeomorphisms on surfaces, by the conclusions of [**44**], one cannot give a non-hyperbolic homoclinic class with domination and without weak periodic orbits, which is unknown in the C^1 dynamics. Hence, we have the following question which is a stronger version of Question 1.

Question 2. Is there a non-hyperbolic homoclinic class $H(p)$ with a non-trivial dominated splitting for a diffeomorphism f satisfying that all of the Lyapunov exponents of all periodic orbits homoclinically related to $\mathrm{Orb}(p)$ are uniformly away from zero?

1.3. *Some applications of the main theorem.* In this section, we give some applications whose proof will be given later.

1.3.1. *Structural stability and hyperbolicity.* Recall that a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *structurally stable* if there is a C^1 neighborhood $\mathcal{U}$ of f such that, for any $g \in \mathcal{U}$, there is a homeomorphism $\phi : M \to M$, satisfying $\phi \circ f = g \circ \phi$. The orbital structure of a structurally stable diffeomorphism remains unchanged under perturbations. Mañé proved that the chain recurrent set of a structurally stable diffeomorphism is hyperbolic, see [**37**]. Here we give a local version of this result.

It is known that a hyperbolic periodic point has a continuation. More precisely, for a hyperbolic periodic point p of a diffeomorphism f with period τ, there is a neighborhood U of $\mathrm{Orb}(p)$ and a C^1 neighborhood $\mathcal{U}$ of f, such that, for any $g \in \mathcal{U}$, the maximal invariant compact set of g in U is a unique periodic orbit with period τ and with the same index as p. We denote this *continuation* of p by p_g for such a diffeomorphism g, and denote the homoclinic class of p_g by $H(p_g)$. Thus, we say that a homoclinic class $H(p)$ of a diffeomorphism f is *structurally stable* if there is a C^1 neighborhood $\mathcal{U}$ of f, such that, for any $g \in \mathcal{U}$, there is a homeomorphism $\phi : H(p) \to H(p_g)$, satisfying $\phi \circ f|_{H(p)} = g \circ \phi|_{H(p)}$, where p_g is the continuation of p.

Question 3. Assume that p is a hyperbolic periodic point for a diffeomorphism: if $H(p)$ is structurally stable, then is it hyperbolic?

There are many works related to this question, see for example [**28**, **46**, **51**, **53**]. In [**51**, **53**], they prove that structural stability implies hyperbolicity for the chain recurrence class and the homoclinic class, respectively, of a hyperbolic periodic point,

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under the hypothesis that the diffeomorphism is far away from tangency, or that the stable or the unstable dimension of this periodic point is one. With the conclusion of the Main Theorem, we can give a complete answer to Question 3.

COROLLARY 1.3. *Assume that f is a diffeomorphism in* $\mathrm{Diff}^1(M)$ *and p is a hyperbolic periodic point of f. If the homoclinic class $H(p)$ is structurally stable, then $H(p)$ is hyperbolic.*

1.3.2. *Partial hyperbolicity.* The next result is that for a homoclinic class with a dominated splitting of a C^1-generic diffeomorphism, if the dimensions of the two bundles in the splitting satisfy certain hypotheses, then the splitting is a partially hyperbolic splitting (at least one bundle is hyperbolic).

COROLLARY 1.4. *For C^1-generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, such that* $\dim(E)$ *is smaller than the smallest index of periodic orbits contained in $H(p)$, then the bundle E is contracting. Symmetrically, if* $\dim(E)$ *is larger than the largest index of periodic orbits contained in $H(p)$, then the bundle F is expanding.*

As another consequence of the Main Theorem, we can give a proof of Theorem 1.1(2) in [**23**] with a different argument. More precisely, we can prove that for a C^1-generic diffeomorphism far from tangency, a homoclinic class has a partially hyperbolic splitting whose center bundle splits into one-dimensional sub-bundles, and the Lyapunov exponents of the periodic orbits along each the center sub-bundle can be arbitrarily close to zero. Denote by $\mathcal{HT}$ the set of diffeomorphisms of $\mathrm{Diff}^1(M)$ that exhibit a tangency.

COROLLARY 1.5. (Crovisier *et al* [**23**]) *For C^1-generic $f \in \mathrm{Diff}^1(M)\backslash\overline{\mathcal{HT}}$, a homoclinic class $H(p)$ has a partially hyperbolic splitting $T_{H(p)}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u$ such that each of the center sub-bundles E^c_i is neither contracting nor expanding and* $\dim(E^c_i) = 1$, *for all $i = 1, \ldots, k$. Moreover, the minimal index of periodic points contained in $H(p)$ is* $\dim(E^s)$ *or* $\dim(E^s) + 1$, *and symmetrically, the maximal index of periodic points contained in $H(p)$ is $d - \dim(E^u)$ or $d - \dim(E^u) - 1$. For each $i = 1, \ldots, k$, there exist periodic orbits contained in $H(p)$ with arbitrarily long periods with a Lyapunov exponent along E^c_i arbitrarily close to zero.*

1.3.3. *Lyapunov stable homoclinic classes.* Recall that an invariant compact set $\Lambda \subset M$ is *Lyapunov stable for f* if for any neighborhood U of Λ, there is another neighborhood V of Λ, such that $f^n(V) \subset U$ for all $n \geq 0$. We say that Λ is *bi-Lyapunov stable*, if Λ is both Lyapunov stable for f and for f^{-1}.

The following results are about C^1-generic Lyapunov stable homoclinic classes. First, for C^1-generic Lyapunov stable homoclinic classes, we can get a similar conclusion to Corollary 1.4 under a weaker hypothesis.

COROLLARY 1.6. *For C^1-generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ is Lyapunov stable and has a dominated splitting $T_{H(p)}M = E \oplus F$ such that* $\dim(E)$ *is larger than or equal to the largest index of periodic orbits contained in $H(p)$, then the bundle F is expanding.*

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With the conclusion of Corollary 1.6, we can give a positive answer to Conjecture 1 for bi-Lyapunov stable homoclinic classes.

COROLLARY 1.7. *For C^1-generic $f \in \mathrm{Diff}^1(M)$, where M is connected, if a homoclinic class $H(p)$ is bi-Lyapunov stable, then we have either:*

(i) *$H(p)$ is hyperbolic, hence $H(p) = M$ and f is Anosov; or*

(ii) *f can be C^1 approximated by diffeomorphisms that have a heterodimensional cycle.*

From [**23**] (or Corollary 1.5), we know that for C^1-generic diffeomorphisms far away from tangencies, a homoclinic class has a partially hyperbolic splitting with all central bundles of dimension one. We have the following result about the index of periodic orbits for Lyapunov stable homoclinic classes. It is a direct corollary of Corollary 1.6 and we omit the proof.

COROLLARY 1.8. *For C^1-generic $f \in \mathrm{Diff}^1(M)\backslash\overline{\mathcal{HT}}$, if a homoclinic class $H(p)$ is Lyapunov stable and assume $T_{H(p)}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u$ is the partially hyperbolic splitting, then the largest index of periodic points contained in $H(p)$ equals $d - \dim(E^s)$.*

1.4. *Propositions for the proof of Theorem A.* To prove Theorem A, we will use the following three propositions. Proposition 1 states that for any hyperbolic periodic orbit $\mathrm{Orb}(p)$ and any invariant compact set K of a diffeomorphism f linked by heteroclinic orbits, we can get a periodic orbit that spends a given proportion of time close to $\mathrm{Orb}(p)$ and K by arbitrarily C^1 small perturbation. In the whole paper, we denote by $\mathrm{Orb}^+(x, f)$ (respectively, $\mathrm{Orb}^-(x, f)$) the positive (respectively, negative) f-orbit of the point x, i.e. $\mathrm{Orb}^+(x, f) = \{x, f(x), f^2(x), \ldots\}$ (respectively, $\mathrm{Orb}^-(x, f) = \{x, f^{-1}(x), f^{-2}(x), \ldots\}$). If there is no misunderstanding, we just denote it by $\mathrm{Orb}^+(x)$ (respectively, $\mathrm{Orb}^-(x)$) for simplicity.

PROPOSITION 1. *Let $f \in \mathrm{Diff}^1(M)$. Consider a hyperbolic periodic point p and an invariant compact set K, satisfying that all periodic points contained in K are hyperbolic and $p \notin K$. Assume, moreover, that there are two points x and y, satisfying that:*

(i) *$x \in W^u(p)$ and $\omega(x) \cap K \neq \emptyset$;*

(ii) *$y \in W^s(p)$ and $\alpha(y) = K$.*

Then for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, any neighborhood U_p of $\mathrm{Orb}(p)$, and any neighborhood U_K of K, there are two integers l and n_0, such that, for any integer T_K, we have the following result.

(1) *There is $h \in \mathcal{U}$ such that:*
   (a) *h coincides with f outside U_K;*
   (b) *the point y is on the positive orbit of x under h;*
   (c) *$\#(\mathrm{Orb}(x, h) \cap U_K) \geq T_K$ and $\#((\mathrm{Orb}(x, h)\backslash(U_K \cup U_p)) \leq n_0$.*

(2) *For any $m \in \mathbb{N}$, there is $h_m \in \mathcal{U}$ such that:*
   (a) *h_m coincides with h on $\mathrm{Orb}(p)$ and outside U_p;*
   (b) *h_m has a periodic orbit O, satisfying $O\backslash U_p = (\mathrm{Orb}(x, h))\backslash U_p$, and $\#(O \cap U_p) \in \{l + m\tau, l + m\tau + 1, \ldots, l + (m + 1)\tau - 1\}$.*

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Remark 1.9. We make the following remarks.

(1) It is easy to see that if we take U_K small enough, then h coincides with f on $\mathrm{Orb}(p) \cup \mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)$.
(2) For the diffeomorphism h, the point x (also y) is a homoclinic point of the periodic orbit $\mathrm{Orb}(p)$.
(3) It is obvious that, in the settings of the proposition, if we change '$\omega(x) \cap K \neq \emptyset$ and $\alpha(y) = K$' to '$\alpha(y) \cap K \neq \emptyset$ and $\omega(x) = K$', the conclusion still holds.

Propositions 2 and 3 are in some sense performing an asymptotic connecting process from a point to an invariant compact set. Proposition 2 states that if the closure of the unstable manifold $\overline{W^u(p)}$ of a hyperbolic periodic point p intersects an invariant compact set K, then by an arbitrarily small perturbation, one can obtain a point which belongs to the unstable manifold of p and whose positive limit set is contained in K. Moreover, the perturbation will not change certain pieces of orbit. In fact, the first property can be obtained from the proof of Proposition 10 in [**18**], but the second property is not a direct consequence.

PROPOSITION 2. (A modified case of [**18**, Proposition 10]) *Let $f \in \mathrm{Diff}^1(M)$. Consider an invariant compact set K which contains no non-hyperbolic periodic point and a point $x \in M$ with $\alpha(x) \subset K$. Assume that p is a hyperbolic periodic point satisfying that $p \notin K$ and $\overline{W^u(p)} \cap K \neq \emptyset$.*

Then for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there is a diffeomorphism $g \in \mathcal{U}$, satisfying the following properties:

(1) *g coincides with f on $\mathrm{Orb}(p) \cup K$;*
(2) *g and Dg coincides with f and Df respectively on $\mathrm{Orb}^-(x)$, hence $\alpha(x, g) \subset K$;*
(3) *there is a point $y \in W^u(p, g)$, such that $\omega(y, g) \subset K$.*

In the assumptions of the above two propositions, the point and invariant compact sets are linked by true orbits. However, Proposition 3 deals with the case that they are linked by pseudo-orbits which is more complicated. We use the techniques of [**8**, **18**].

PROPOSITION 3. *Assume that f_0 is a diffeomorphism in $\mathrm{Diff}^1(M)$. For any neighborhood $\mathcal{U}$ of f_0 in $\mathrm{Diff}^1(M)$, there are a smaller neighborhood $\mathcal{U}'$ of f_0 with $\overline{\mathcal{U}'} \subset \mathcal{U}$ and an integer T, with the following properties.*

For any diffeomorphism $f \in \mathcal{U}'$, considering an invariant compact set K, a positively invariant compact set X and a point $z \in X$, suppose that the following conditions are satisfied:

(i) *all periodic orbits contained in K are hyperbolic;*
(ii) *all periodic orbits contained in X with period less than T are hyperbolic;*
(iii) *for any $\varepsilon > 0$, there is an ε-pseudo-orbit contained in X connecting z to K;*

then for any neighborhood U of $X \backslash K$ and for any $\gamma > 0$, there is a diffeomorphism $g \in \mathcal{U}$, such that $g|_{M \backslash U} = f|_{M \backslash U}$ and $\omega(z, g) \subset K$. Moreover, the C^0 distance between g and f is smaller than γ.

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Remark 1.10. We make the following remarks.

(1) Proposition 2 is not a direct corollary of Proposition 3, because we wish to keep the negative orbit of a point that accumulates to the invariant compact set unchanged after perturbation in Proposition 2.

(2) In Proposition 3, we can see that $X \cap K \neq \emptyset$. Thus, $X \backslash K$ is not a compact set and we have that $\overline{U} \cap K \neq \emptyset$, where U is the neighborhood of $X \backslash K$.

1.5. *Organization of the paper.* In §2, we give some basic definitions and well-known results that we will use in the proofs. In §3, we give a slightly different version (Theorem B) of Theorem A, and we prove Theorem A using Theorem B. Later, we give the proof of Theorem B from Propositions 1–3 in §4. The proofs of Propositions 1–3 will be given in §§5, 6 and 7, respectively. Finally, we give the proofs of the applications of the Main Theorem in §8.

2. *Preliminary*

In this section, we give some definitions and some well-known results. Denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms of M.

2.1. *Hyperbolicity and dominated splitting.*

Definition 2.1. Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$, Λ is an invariant compact set of f and E is a Df-invariant sub-bundle of $T_\Lambda M$. We say that the bundle E is (C, λ)-*contracting* if there are constants $C > 0$ and $\lambda \in (0, 1)$, such that

$$\|Df^n|_{E(x)}\| < C\lambda^n,$$

for all $x \in \Lambda$ and all $n \geq 1$. We say that E is (C, λ)-*expanding* if it is (C, λ)-contracting with respect to f^{-1}. If the tangent bundle of Λ has an invariant splitting $T_\Lambda M = E^s \oplus E^u$, such that E^s is (C, λ)-contracting and E^u is (C, λ)-expanding for some constants $C > 0$ and $\lambda \in (0, 1)$, then we call Λ a *hyperbolic set* and $\dim(E^s)$ the *index* of the hyperbolic splitting. If a periodic orbit $\mathrm{Orb}(p)$ is a hyperbolic set, then we call p a hyperbolic periodic point, and the dimension of the contracting bundle E^s in the hyperbolic splitting is called the *index* of p, denoted by $\mathrm{Ind}(p)$.

Definition 2.2. For any point $x \in M$ and any number $\delta > 0$, we define the *local stable set* and *local unstable set* of x of size δ, respectively, as follows:

$$W^s_\delta(x) = \Big\{y : \forall n \geq 0,\, d(f^n(x),\, f^n(y)) \leq \delta \text{ and } \lim_{n\to+\infty} d(f^n(x),\, f^n(y)) = 0\Big\};$$

$$W^u_\delta(x) = \Big\{y : \forall n \geq 0,\, d(f^{-n}(x),\, f^{-n}(y)) \leq \delta \quad \text{and}$$

$$\lim_{n\to+\infty} d(f^{-n}(x),\, f^{-n}(y)) = 0\Big\}.$$

We define the *stable set* and *unstable set* of x respectively as follows:

$$W^s(x) = \Big\{y : \lim_{n\to+\infty} d(f^n(x),\, f^n(y)) = 0\Big\};$$

$$W^u(x) = \Big\{y : \lim_{n\to+\infty} d(f^{-n}(x),\, f^{-n}(y)) = 0\Big\}.$$

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Remark 2.3. We make the following remarks.

(1) It is obvious that, for any $\delta > 0$, we have

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^s_\delta(f^n(x)))$$

and

$$W^u(x) = \bigcup_{n\geq 0} f^{n}(W^u_\delta(f^{-n}(x))).$$

(2) To belong to the same stable set is an equivalent relation, thus two stable sets either coincide or are disjoint with each other. The same holds for the unstable set.

For hyperbolic sets, the (local) stable (respectively unstable) set has the following properties, see for example [**33**].

LEMMA 2.4. *Assume that Λ is an invariant compact set of f. If Λ is hyperbolic and $T_\Lambda M = E^s \oplus E^u$ is the hyperbolic splitting, then there is a number $\delta > 0$, such that for any $x \in \Lambda$, the local stable (respectively unstable) set $W^s_\delta(x)$ (respectively $W^u_\delta(x)$) is an embedded disk with dimension* $\dim(E^s)$ *(respectively* $\dim(E^u)$*) and is tangent to E^s (respectively E^u) at x. Moreover, the stable (respectively unstable) set $W^s(x)$ (respectively $W^u(x)$) of x is an immersed sub-manifold of M.*

Definition 2.5. Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$ and $p, q \in M$ are two hyperbolic periodic points of f. We say p and q are *homoclinically related* and denote the relation by $p \sim q$, if $W^u(\mathrm{Orb}(p))$ has non-empty transverse intersections with $W^s(\mathrm{Orb}(q))$, and $W^s(\mathrm{Orb}(p))$ has non-empty transverse intersections with $W^u(\mathrm{Orb}(q))$, denoted by $W^u(\mathrm{Orb}(p)) \pitchfork W^s(\mathrm{Orb}(q)) \neq \emptyset$ and $W^s(\mathrm{Orb}(p)) \pitchfork W^u(\mathrm{Orb}(q)) \neq \emptyset$. We call the closure of the set of periodic orbits homoclinically related to $\mathrm{Orb}(p)$ the *homoclinic class* of p and denote it by $H(p, f)$ or $H(p)$ for simplicity.

Definition 2.6. Let $f \in \mathrm{Diff}^1(M)$. An invariant compact set Λ of M is said to have an (m, λ)-*dominated splitting*, if the tangent bundle has a Df-invariant splitting $T_\Lambda M = E \oplus F$ and there are an integer m and a constant $\lambda \in (0, 1)$ such that

$$\|Df^m|_{E(x)}\| \cdot \|Df^{-m}|_{F(f^m x)}\| < \lambda.$$

We call $\dim(E)$ the *index* of the dominated splitting. Moreover, we say Λ has a *partially hyperbolic splitting*, if the tangent bundle has an invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$, such that the two splittings $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ are both dominated splittings and, moreover, the bundle E^s is contracting, the bundle E^u expanding and the central bundle E^c is neither contracting nor expanding.

Remark 2.7. We point out here that if an invariant compact set Λ has two dominated splittings $T_\Lambda M = E_1 \oplus F_1 = E_2 \oplus F_2$ such that $\dim(E_1) \leq \dim(E_2)$, then we have $E_1 \subset E_2$. Hence, two dominated splittings on an invariant compact set with the same index would coincide.

By [**29**], there is always an *adapted metric* for a dominated splitting, that is to say, an (m, λ)-dominated splitting is a $(1, \lambda)$-dominated splitting by considering a metric equivalent to the original one. Also, it is obvious that an (m, λ)-dominated splitting is always an (mN, λ)-dominated splitting for any positive integer N.

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2.2. *Recurrence.* We give some definitions of recurrence.

Definition 2.8. For a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a number $\varepsilon > 0$, we call a sequence of points $\{x_i\}_{i=a}^b$ of M an *ε-pseudo-orbit of f* if $d(f(x_i), x_{i+1}) < \varepsilon$ for any $i = a, a+1, \ldots, b-1$, where $-\infty \le a < b \le \infty$. An invariant compact set K is called a *chain transitive set*, if for any $\varepsilon > 0$, there is a periodic ε-pseudo-orbit contained in K and ε-dense in K.

Definition 2.9. Assume that $f \in \mathrm{Diff}^1(M)$. We say that a point y is *chain attainable* from x if for any number $\varepsilon > 0$, there is an ε-pseudo-orbit of f $(x_0, x_1, \ldots, x_n)$ such that $x_0 = x$ and $x_n = y$, and we denote it by $x \dashv y$. The *chain recurrent set* of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, denoted by $R(f)$, is the set of the points x such that x is chain attainable from itself.

It is well-known that the chain recurrent set $R(f)$ of f can be decomposed into a disjoint union of invariant compact 'undecomposable' sets. More precisely, we give the definition as the following.

Definition 2.10. Assume that $f \in \mathrm{Diff}^1(M)$. For any two points $x, y \in M$, we use the notation $x \sim y$ if x is chain attainable from y and y is chain attainable from x. Obviously $\sim$ is an equivalent relation on $R(f)$, and an equivalent class of $\sim$ is called a *chain recurrence class*.

Definition 2.11. Assume that $f \in \mathrm{Diff}^1(M)$ and Λ is an invariant compact set of f. We say that Λ is *shadowable*, if for any $\varepsilon > 0$, there is $\delta > 0$, such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of f, where $-\infty \le a < b \le \infty$, there is a point $y \in M$, such that $d(f^i(y), x_i) < \varepsilon$ for all $a \le i \le b$.

Now we give another definition of a relation, which is denoted by $\prec$.

Definition 2.12. Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$ and W is an open set of M. For any two points $x, y \in M$, we use the notation $x \prec y$ if for any neighborhood U of x and any neighborhood V of y, there are a point $z \in M$ and an integer $n \ge 1$ such that $z \in U$ and $f^n(z) \in V$. We denote $x \prec_W y$ if for any neighborhood U of x and any neighborhood V of y, there is a piece of orbit $(z, f(z), \ldots, f^n(z))$ contained in W such that $z \in U$ and $f^n(z) \in V$. Moreover, let K be a compact set of M, then we we use the notation $x \prec K$ (respectively, $x \prec_W K$) if there is a point $y \in K$ such that $x \prec y$ (respectively, $x \prec_W y$).

For the relation $\prec$, we have the following result, whose proof is similar to the proof of Lemma 6 in [**18**].

LEMMA 2.13. *Assume that K is an invariant compact set. Then for any two neighborhoods $U_2 \subset U_1$ of K and any point $y \in U_1$ satisfying $y \prec_{U_1} K$, there is a point $y' \in U_2$ such that $y \prec_{U_1} y' \prec_{U_2} K$ and the positive orbit of y' is contained in U_2.*

It is obvious that $x \prec y$ implies $x \dashv y$, but the two relations are not equivalent. In [**8**], they have proved that for generic diffeomorphisms, the two relations are equivalent.

LEMMA 2.14. (Bonatti and Crovisier [**8**]) *For generic diffeomorphism $f \in \mathrm{Diff}^1(M)$, if $x \dashv y$, then $x \prec y$.*

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2.3. *Pliss points and weak sets.*

Definition 2.15. Assume that Λ is an invariant compact set of a diffeomorphism f in $\mathrm{Diff}^1(M)$ and E is an invariant sub-bundle of $T_\Lambda M$. For a constant $\lambda \in (0, 1)$, we call $x \in \Lambda$ an (m, λ)-*E-Pliss point*, if for any integer $n > 0$, we have

$$\prod_{i=0}^{n-1} \|Df^{im}|_{E(f^{im}(x))}\| \le \lambda^n.$$

In particular, if $m = 1$, we call x a λ-*E-Pliss point* for short.

Remark 2.16. It is not difficult to see that, if $\{x_i\}$ is a sequence of λ-E-Pliss points, then any limit point y of the sequence is also a λ-E-Pliss point.

Definition 2.17. Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$, an invariant compact set K of f, an invariant sub-bundle E of $T_K M$, an integer m and a constant $\lambda \in (0, 1)$. We say that K is an (m, λ)-*E-weak set*, if for any point $x \in K$, there is an integer n_x, such that

$$\prod_{i=0}^{n_x-1} \|Df^m|_{E(f^{im}(x))}\| > \lambda^{n_x}.$$

We denote N_x the smallest integer that satisfies the above inequality. In particular, if $m = 1$, we call K a λ-*E-weak set* for short.

Remark 2.18. If K is an (m, λ)-E-weak set, by the compactness of K, we can see that N_x is bounded by an integer N_K for all $x \in K$. Also from the definition, we can see that an invariant compact set K is an (m, λ)-E-weak set if and only if K does not contain any (m, λ)-E-Pliss point.

One can obtain Pliss points by the following lemma given by Pliss, see [**40**, **43**].

LEMMA 2.19. (Pliss lemma) *Assume that Λ is an invariant compact set of a diffeomorphism f in $\mathrm{Diff}^1(M)$ and E is an invariant sub-bundle of $T_\Lambda M$. For any two numbers $0 < \lambda_1 < \lambda_2 < 1$, we have the following results.*

(1) *There are a positive integer $N = N(\lambda_1, \lambda_2, f)$ and a number $c = c(\lambda_1, \lambda_2, f)$ such that for any $x \in \Lambda$ and any number $n \ge N$, if*

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i x)}\| \le {\lambda_1}^n,$$

then there are $0 \le n_1 < n_2 < \cdots < n_l \le n$ such that $l \ge cn$, and, for any $j = 1, \ldots, l$ and any $k = n_j + 1, \ldots, n$,

$$\prod_{i=n_j}^{k-1} \|Df|_{E(f^i x)}\| \le {\lambda_2}^{k-n_j}.$$

(2) *For any point $x \in \Lambda$, and any integer m, if for all $n \ge m$,*

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i x)}\| \le {\lambda_1}^n,$$

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then there is an infinite sequence $0 \le n_1 < n_2 < \cdots$, *such that*

$$\prod_{i=n_j}^{k-1} \|Df|_{E(f^i x)}\| \le {\lambda_2}^{k-n_j},$$

for all $k > n_j$ *and all* $j = 1, 2, \ldots$.

COROLLARY 2.20. *For a diffeomorphism* $f \in \mathrm{Diff}^1(M)$ *and an* f*-invariant continuous bundle* $E \subset T_\Lambda M$ *of an invariant compact set* Λ, *we have, for any* $x \in \Lambda$, *the following results.*

(1) *If* x *is an* (m, λ)*-*E*-Pliss point, then there is a point* $y \in \omega(x)$, *such that* y *is also a* (m, λ)*-*E*-Pliss point.*

(2) *If for any* $y \in \omega(x)$, *there is an integer* $n_y \in \mathbb{N}$, *such that*

$$\prod_{i=0}^{n_y-1} \|Df^m|_{E(f^{im}(y))}\| \le \lambda^{n_y},$$

then for any $\lambda' \in (\lambda, 1)$, *there are infinitely many* (m, λ')*-*E*-Pliss points on* $\mathrm{Orb}^+(x)$.

Proof. By considering the diffeomorphism f^m instead of f, we can assume that $m = 1$. The proof of the general case is similar.

(1) By (2) of the Pliss lemma, for any $\lambda' \in (\lambda, 1)$, there are infinitely many λ'-E-Pliss points on $\mathrm{Orb}^+(x)$. Take a limit point of these λ'-E-Pliss points, denote it by $y_{\lambda'}$, then $y_{\lambda'} \in \omega(x)$ is a λ'-E-Pliss point. We take a sequence of numbers $(\lambda_n)_{n\ge1}$ such that $\lambda_n \in (\lambda, 1)$ and $\lambda_n \to \lambda$ when n goes to infinity. Then for any $n \ge 1$, there is a λ_n-E-Pliss point $y_{\lambda_n} \in \omega(x)$. Taking a subsequence if necessary, we assume that $(y_{\lambda_n})_{n\ge1}$ converges to a point $y \in \omega(x)$. Then y is a λ_n-E-Pliss point for any $n \ge 1$. Since $\lambda_n \to \lambda$, the point y is a λ-E-Pliss point.

(2) By the compactness of $\omega(x)$, there is an integer N, such that $n_y \le N$ for any $y \in \omega(x)$. There is a constant $C > 0$, such that, for any $y \in \omega(x)$, we have, for all $n > 0$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(y))}\| < C\lambda^n.$$

Take a constant $\lambda' \in (\lambda, 1)$. Take three constants $\lambda_1 < \lambda_2 < \lambda_3$ contained in (λ, λ'). There is $N \in \mathbb{N}$, such that $C\lambda^n < \lambda_1^n$ for any $n \ge N$. There is $\varepsilon > 0$, such that, for any two points $x_1, x_2 \in \Lambda$, if $d(f(x_1), f(x_2)) < \varepsilon$, then $\|Df|_{E(f^i(x_1))}\| / \|Df|_{E(f^i(x_2))}\| < \lambda_2/\lambda_1$, for all $i = 0, 1, \ldots, N$. By considering an iterate of x instead of x, we can assume that $d_H(\mathrm{Orb}^+(x), \omega(x)) < \varepsilon$, where $d_H(\cdot, \cdot)$ is the Hausdorff distance. Then, for any $n \ge 1$, we have

$$\prod_{i=0}^{nN} \|Df|_{E(f^i(x))}\| < (C\lambda^N)^n \left(\frac{\lambda_2}{\lambda_1}\right)^{nN} < \lambda_2^{nN}.$$

There is $T > 0$, such that, for any $k \ge T$, we have $\lambda_2^{kN}\|Df\|^j < \lambda_3^{kN+j}$ for all $j = 0, 1, \ldots, N-1$. Then for any $n > TN$, assume $n = kN + j$, where $0 \le j < N$, we have

$$\prod_{i=0}^{n} \|Df|_{E(f^i(x))}\| \le \left(\prod_{i=0}^{kN} \|Df|_{E(f^i(x))}\|\right)\|Df\|^j < \lambda_2^{kN}\|Df\|^j < \lambda_3^n.$$

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Then by item (2) of the Pliss lemma, there are infinitely many $(1, \lambda')$-E-Pliss points on $\mathrm{Orb}^+(x)$. □

Definition 2.21. Assume that Λ is an invariant compact set of a diffeomorphism f in $\mathrm{Diff}^1(M)$ and E is an invariant sub-bundle of $T_\Lambda M$. We call two (m, λ)-E-Pliss points $(f^n(x), f^l(x))$ on a single orbit *consecutive* (m, λ)-E-Pliss points, if $n < l$ and for all $n < k < l$, $f^k(x)$ is not an (m, λ)-E-Pliss point. Furthermore, if there is a dominated splitting $T_\Lambda M = E \oplus F$ on Λ, we call $x \in \Lambda$ an (m, λ)-*bi-Pliss point*, if it is an (m, λ)-E-Pliss point for f and an (m, λ)-F-Pliss point for f^{-1}.

For Pliss-points, we have the following lemma. The techniques of the proof can be found in many papers, for example [**43**].

LEMMA 2.22. *Assume that Λ is an invariant compact set of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ with an (m, λ^2)-dominated splitting $T_\Lambda M = E \oplus F$. Let $x \in \Lambda$ and $\{x_i\}_{i\geq 0}$ be a sequence of points contained in Λ. We have, for any $\lambda' \in (\lambda, 1)$, the following results.*

(1) *If a sequence of consecutive (m, λ')-E-Pliss points $(f^{n_i}(x_i), f^{l_i}(x_i))_{i\geq 0}$ satisfies that $l_i - n_i \to +\infty$, then any limit point y of the sequence $(f^{l_i}(x_i))$ is a (m, λ')-bi-Pliss point.*

(2) *If there are both (m, λ')-E-Pliss points for f on $\mathrm{Orb}^+(x)$ and (m, λ')-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(x)$, then there is at least one (m, λ')-bi-Pliss point on $\mathrm{Orb}(x)$.*

(3) *If $x \in \Lambda$ is an (m, λ')-E-Pliss point and there are no other (m, λ')-E-Pliss points on $\mathrm{Orb}^-(x)$, then x is also an (m, λ)-F-Pliss point for f^{-1}. Thus, x is an (m, λ')-bi-Pliss point.*

We have the following selecting lemma of Liao to get weak periodic orbits (see [**34**, **49**]).

LEMMA 2.23. (Liao's selecting lemma) *Assume that $f \in \mathrm{Diff}^1(M)$. Consider an invariant compact set Λ with a non-trivial (m, λ)-dominated splitting $T_\Lambda M = E \oplus F$, and $\lambda_0 \in (\lambda, 1)$, suppose that the following two conditions are satisfied.*

(i) *There is a point $b \in \Lambda$, such that, for all $n \geq 1$, we have*

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^{im}(b))}\| \geq 1.$$

(ii) *For any invariant compact subset $K \subsetneq \Lambda$, there is an (m, λ_0)-E-Pliss point $x \in K$. Then for any neighborhood U of Λ, for any $\lambda_1 < \lambda_2$ contained in $(\lambda_0, 1)$, there is a periodic orbit $\mathrm{Orb}(q) \subset U$ with period $\tau(q)$ a multiple of m, such that, for all $n = 1, \ldots, \tau(q)/m$, the following two inequalities are satisfied:*

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^{im}(q))}\| \leq {\lambda_2}^n,$$

and

$$\prod_{i=n-1}^{\tau(q)/m-1} \|Df^m|_{E(f^{im}(q))}\| \geq {\lambda_1}^{\tau(q)/m-n+1}.$$

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In particular, one can find a sequence of periodic points that are homoclinic related with each other and converges to a point in Λ. *Similar assertions for* F *hold with respect to* f^{-1}.

2.4. *Perturbation techniques.* Consider a diffeomorphism f and a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$. For a perturbation $f_1 \in \mathcal{U}$ of f, if there is an open set $U \subset M$ such that $f_1|_{M\setminus U} = f|_{M\setminus U}$, then U is called the *perturbation neighborhood* of f_1. Consider two perturbations f_1 and f_2 of f with disjoint perturbation neighborhoods U_1 and U_2, respectively. In general, the diffeomorphism g, where $g|_{M\setminus(U_1\cup U_2)} = f|_{M\setminus(U_1\cup U_2)}$ and $g|_{U_i} = f_i|_{U_i}$ for $i = 1, 2$, is not contained in $\mathcal{U}$ any more. However, there is a basis of neighborhoods $\mathcal{U}$ of f, such that if the element of $\mathcal{U}$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}$, where $\mathcal{V}$ is a C^1 neighborhood of Id, then $\mathcal{V}$ satisfies the following property (F), see [**42**, §2].

Definition 2.24. (Property (F)) Assume that $\mathcal{V}$ is a C^1 neighborhood of Id. We say that $\mathcal{V}$ satisfies the *property (F)*, if for any perturbations ϕ and ϕ' of Id in $\mathcal{V}$ with disjoint perturbation neighborhoods, the composed perturbation $\phi \circ \phi'$ is still in $\mathcal{V}$.

We give some tools for C^1-perturbation. The first is the famous Hayashi's connecting lemma, see [**31**, **50**].

THEOREM 2.25. (Connecting lemma) *Assume that* f *is a diffeomorphism in* $\mathrm{Diff}^1(M)$. *For any* C^1 *neighborhood* $\mathcal{U}$ *of* f *in* $\mathrm{Diff}^1(M)$, *there is an integer* $N \in \mathbb{N}$, *satisfying the following properties.*

Assume that $z \in M$ *is not a periodic point of period less than or equal to* N. *Then for any neighborhood* U_z *of* z, *there is a smaller neighborhood* $V_z \subset U_z$ *of* z, *such that, for any two points* x *and* y *that are outside* $\Delta = \bigcup_{0\le i\le N-1} f^i(U_z)$, *if there are two positive integer* n_x *and* n_y, *such that* $f^{n_x}(x) \in V_z$ *and* $f^{-n_y}(y) \in V_z$, *then there are a diffeomorphism* $g \in \mathcal{U}$ *and a positive integer* m *such that* $g^m(x) = y$ *and* $g = f$ *outside* Δ. *Moreover, the piece of orbit* $\{x, g(x), \ldots, g^m(x) = y\}$ *is contained in the set* $\{x, f(x), \ldots, f^{n_x}(x)\} \cup \Delta \cup \{y, f^{-1}(y), \ldots, f^{-n_y}(y)\}$ *and the number* m *is no more than* $n_x + n_y$.

Theorem 2.25 deals with a single diffeomorphism and a given neighborhood. Below we give a uniform version that is valid in a neighborhood of a diffeomorphism, see [**48**]. We point out that Theorem 2.25 is a corollary of Theorem 2.26. We put the two theorems here because in the proof of Propositions 1 and 2, we only need to apply Theorem 2.25, and the notation is more simple. Theorem 2.26 is applied in the proof of Theorem B in §4 and in the proof of Proposition 3 in §7.

THEOREM 2.26. (A uniform connecting lemma, [**48**, Theorem A]) *Assume that* f *is a diffeomorphism in* $\mathrm{Diff}^1(M)$. *For any* C^1 *neighborhood* $\mathcal{U}$ *of* f *in* $\mathrm{Diff}^1(M)$, *there are three numbers* $\rho > 1$, $\delta_0 > 0$ *and* $N \in \mathbb{N}$, *together with a* C^1 *neighborhood* $\mathcal{U}_1 \subset \mathcal{U}$ *of* f *in* $\mathrm{Diff}^1(M)$, *that satisfy the following property.*

For any $f_1 \in \mathcal{U}_1$, *any point* $z \in M$ *and any number* $0 < \delta < \delta_0$, *as long as the* N *balls* $(f_1^i(B(z, \delta)))_{0\le i\le N-1}$ *are pairwise disjoint and each is of size smaller than* δ_0 *(that is to say,* $f_1^i(B(z, \delta)) \subset B(f_1^i(z), \delta_0)$*), then for any two points* x *and* y *that are outside*

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the set $\Delta = \bigcup_{0\le i\le N-1} f_1^i(B(z,\delta))$*, if there are two positive integers* n_x *and* n_y *such that* $f_1^{n_x}(x) \in B(z, \delta/\rho)$ *and* $f_1^{-n_y}(y) \in B(z, \delta/\rho)$*, then there are a diffeomorphism* $g \in \mathcal{U}$ *and a positive integer* m *such that* $g^m(x) = y$ *and* $g = f_1$ *outside* Δ*. Moreover, the piece of orbit* $\{x, g(x), \dots, g^m(x) = y\}$ *is contained in the set* $\{x, f_1(x), \dots, f_1^{n_x}(x)\} \cup \Delta \cup \{y, f_1^{-1}(y), \dots, f_1^{-n_y}(y)\}$ *and the number* m *is no more than* $n_x + n_y$*.*

To control the perturbing neighborhood when connecting two points that are close, we have the following lemma, see [**5**].

LEMMA 2.27. (Basic perturbation lemma) *For any neighborhood* $\mathcal{U}$ *of a diffeomorphism* $f \in \mathrm{Diff}^1(M)$*, there are two numbers* $\theta > 1$ *and* $r_0 > 0$ *satisfying: for any two points* $x, y \in M$ *contained in a ball* $B(z, r)$*, where* $r \le r_0$*, there is a diffeomorphism* $g \in \mathcal{U}$*, such that* $g(x) = f(y)$*, and* g *coincides with* f *outside the ball* $B(z, \theta \cdot r)$*.*

Definition 2.28. For a chart $\varphi : V \to \mathbb{R}^d$ of M, we call a set $C \subset V$ a *cube* of φ if $\varphi(C)$ is the image of $[-a, a]^d$ by a translation of $\mathbb{R}^d$. Here a is called the *radius* of the cube. If a cube $C' \subset V$ is of radius $(1+\varepsilon)a$ and $\varphi(C')$ is of the same center of $\varphi(C)$, we also use the notation $C' = (1+\varepsilon)C$.

Definition 2.29. Consider a chart $\varphi : V \to \mathbb{R}^d$. A *tiled domain* according to the chart of φ is an open set $U \subset V$ and a family $\mathcal{C}$ of cubes of φ (called *tiles* of domain), such that:

(1) the interior of the tiles are pairwise disjoint;
(2) the union of all tiles of $\mathcal{C}$ is equal to U;
(3) the geometry of the tiling is bounded, i.e.
   (i) the number of tiles around each point is uniformly bounded (by 2^d), that is to say, there is a neighborhood for each point that meets at most 2^d tiles;
   (ii) for any two pairs (C, C') of intersecting tiles, the rate of their diameters is uniformly bounded (by 2).

By a standard construction, any open set $U \subset V$ can be tiled according to the coordinates of φ (see, e.g., [**8**, **20**]).

Definition 2.30. Assume that $f \in \mathrm{Diff}^1(M)$. Consider a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ and a number N. A tiled domain $(U, \mathcal{C})$ is called a *perturbation domain* of order N of $(f, \mathcal{U})$, if the following properties are satisfied.

(1) The set U is disjoint from its N first iterates of f.
(2) For any finitely many sequence of pairs of points $\{(x_i, y_i)\}_{1\le i\le l}$ in U, such that for any $i = 1, 2, \dots, l$, the points x_i and y_i are contained in the same tile of $\mathcal{C}$, then there exist:
   (i) a diffeomorphism $g \in \mathcal{U}$, that coincides with f outside $\bigcup_{0\le i\le N-1} f^i(U)$;
   (ii) a strictly increasing sequence $1 = n_0 < n_1 < \cdots < n_k \le l$, such that $g^N(x_{n_i}) = f^N(y_{n_{i+1}-1})$ for any $i \ne k$, and $g^N(x_{n_k}) = f^N(y_l)$.

The union $\bigcup_{0\le i\le N-1} f^i(U)$ is called the *support* of the perturbation domain $(U, \mathcal{C})$ and denoted by $\mathrm{supp}(U)$.

Definition 2.31. A pseudo-orbit $(x_0, x_1, \dots, x_l)$ is said to *keep the tiles* of a perturbation domain $(U, \mathcal{C})$ of order N of $(f, \mathcal{U})$, if the intersection of the pseudo-orbit and $\mathrm{supp}(U)$

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is a union of segments $x_{n_i}, x_{n_i+1}, \ldots, x_{n_i+N-1}$ of the form that $x_{n_i} \in U$ and for any $j = 1, 2, \ldots, N-1$, $x_{n_i+j} = f^j(y_{n_i})$, where y_{n_i} is a point contained in the same tile of $\mathcal{C}$ as x_{n_i}. A pseudo-orbit $(x_0, x_1, \ldots, x_k)$ is said to *have jumps only in tiles* of a perturbation domain $(U, \mathcal{C})$ of order N of $(f, \mathcal{U})$, if it keeps the tiles and for any $x_i \notin \mathrm{supp}(U)$, we have $x_{i+1} = f(x_i)$. For a family of perturbation domains $(U_k, \mathcal{C}_k)_{k\geq 0}$ of order N_k of $(f, \mathcal{U}_k)$ with disjoint support, we say that a pseudo-orbit $(x_0, x_1, \ldots, x_l)$ has *jumps only in tiles* of the perturbation domains $(U_k, \mathcal{C}_k)_{k\geq 0}$, if it keeps the tiles of the perturbation domains and for any $x_i \notin \bigcup_k \mathrm{supp}(U_k)$, we have $x_{i+1} = f(x_i)$.

By the proof of connecting lemma in [**6**], the perturbation domain always exists (see also [**8**, Théorème 2.1] and [**20**, Théorème 3.3]).

THEOREM 2.32. (Another statement of the connecting lemma) *For any neighborhood $\mathcal{U}$ of f, there is an integer $N \geq 1$, and for all point $p \in M$, there is a chart $\varphi : V \to \mathbb{R}^d$ such that any tiled domain $(U, \mathcal{C})$ according to φ disjoint from its N first iterates is a perturbation domain of order N for $(f, \mathcal{U})$.*

From the definitions above, we can get the following lemma easily.

LEMMA 2.33. (Bonatti and Crovisier [**8**, Lemme 2.3]) *For a family of disjoint perturbation domains $(U_k, \mathcal{C}_k)$ of order N_k of $(f, \mathcal{U}_k)$ with disjoint support, if there is a pseudo-orbit $(p = p_0, p_1, \ldots, p_m = q)$ that has only jumps in the tiles of $(U_k, \mathcal{C}_k)_{k\geq 0}$ and $p_0, p_m \notin U_k \cup \cdots \cup f^{N_k-1}(U_k)$ for all $k \geq 0$, then for any i, there is $g_i \in \mathcal{U}_i$ and a new pseudo-orbit $(p = p'_0, \ldots, p'_{m'} = q)$ of g_i that has only jumps in the tiles of domains $(U_k, \mathcal{C}_k)_{k\geq 0, k\neq i}$. Moreover, $g_i = f$ outside $U_i \cup \cdots \cup f^{N_i-1}(U_i)$ and $\{p'_0, \ldots, p'_{m'}\} \backslash (U_i \cup \cdots \cup f^{N_i-1}(U_i)) \subset \{p_0, p_1, \ldots, p_m\}$, and $m' \leq m$.*

2.5. *Topological towers.* In this subsection, we introduce two lemmas of [**8**] that are useful to get a true orbit by perturbing a pseudo-orbit. These two lemmas are the key tools in the proof of Proposition 3. First we give the following lemma to choose perturbation neighborhoods. In fact, it is a generalized result of [**8**, Lemme 3.7], but one can get the conclusion from the proof in [**8**].

LEMMA 2.34. *There is a constant $\kappa_d > 0$ (which only depends on the dimension d of M) satisfying the following property.*

Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and an integer $T > 0$. Assume that W' and V' are two compact d-dimensional sub-manifolds with boundary, satisfying V' is disjoint from its $\kappa_d T$ first iterates. Then for any neighborhood U_1 of W' and any neighborhood U_2 of V', there is an open set S, such that the following hold.

(1) *We have $V' \subset \bigcup_{i=0}^{\kappa_d T} f^{-i}(S)$.*

(2) *We have $S = W \cup V$ where W and V satisfy the following:*

  (i) *$\overline{W}$ and $\overline{V}$ are two compact d-dimensional sub-manifolds with boundary;*

  (ii) *$W' \subset W \subset \overline{W} \subset U_1$;*

  (iii) *$\overline{V}$ is contained in $U_2 \cup f(U_2) \cup \cdots \cup f^{\kappa_d T}(U_2)$ and disjoint from its T first iterates.*

  (iv) *$\overline{W} \cap (\bigcup_{i=-T}^{T} f^i(\overline{V})) = \emptyset$.*

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Remark 2.35. We make the following remarks.

(1) We point out here that, the two sets W' and V' in Lemma 2.34 correspond to U and V, respectively, in [**8**, Lemme 3.7]. Lemme 3.7 of [**8**] assumes further that W' is disjoint from its T first iterates, and in the conclusion the set $W \cup V$ is disjoint from its T first iterates. Moreover, the statement of [**8**, Lemme 3.7] does not involve the two neighborhoods U_1 and U_2. But the proof of Lemme 3.7 of [**8**] gives all of the information stated in Lemma 2.34, see [**8**, pp. 61–62].

(2) In Lemma 2.34, if we further assume that W' is disjoint with its first T' iterates where $T' \leq T$, then by taking U_1 small enough, we can obtain that W is disjoint from its T' first iterates, and the union $W \cup V$ is also disjoint from its T' first iterates.

Then, we give a lemma of [**8**] for the construction of what they called the *topological tower* (see [**8**, Théorème 3.1 and Corollaire 3.1]). Denote by $\mathrm{Per}_T(f)$ the set of periodic orbits with period less than T.

LEMMA 2.36. (Topological tower) *There is a constant $\kappa_d > 0$ (which only depends on the dimension d of M), such that, for any $T \in \mathbb{N}$, any constant $\delta > 0$, any compact set K of $f \in \mathrm{Diff}^1(M)$ that does not contain any non-hyperbolic periodic orbits with periods less than $\kappa_d T$ and any neighborhood U of K, there exist an open set V and a compact set $D \subset V$, satisfying the following properties.*

(1) *The closure of V is a compact d-dimensional sub-manifold with boundary.*

(2) *For any point $x \in K$ with $x \notin \bigcup_{p \in \mathrm{Per}_T(f)} W^s_\delta(p)$, there is $n > 0$, such that $f^n(x) \in \mathrm{Int}(D)$.*

(3) *The sets $\overline{V}$, $f(\overline{V}), \ldots, f^T(\overline{V})$ are pairwise disjoint.*

(4) *The set $\overline{V}$ is contained in $U \cup f(U) \cup \cdots \cup f^{\kappa_d T}(U)$.*

Moreover, the diameter of all connected components of V can be arbitrarily small.

Remark 2.37. We make the following remarks.

(1) In [**8**], Théorème 3.1 is stated for an invariant compact set K, and Lemma 2.36(1) and (4) are not stated. But from the proof (see [**8**, pp. 62–63]), we can see that the conclusion is also true for non-invariant compact sets and also Lemma 2.36(1) and (4) are true.

(2) We give a sketch of the proof of Lemma 2.36. Take κ_d to be the constant in Lemma 2.34. First, one can take a compact sub-manifold U_0 of M with boundary that is disjoint from its first T iterates, such that, any point in a small neighborhood O of $\mathrm{Per}_T(f)$ that is not on the local stable manifold of $\mathrm{Per}_T(f)$ has a positive iterate in the interior of U_0. Then one can take a finite cover of the compact set $K \setminus O$ by open sets $\{V_i\}_{0 \leq i \leq r}$ that are disjoint from their first $\kappa_d T$ iterates (they are not disjoint from each other in general). Moreover, for each $0 \leq i \leq r$ the closure of V_i is a compact d-dimensional sub-manifold with boundary. Then one can apply Lemma 2.34 inductively with respect to i to the triple (T, U_i, V_i), where the triple (T, U_i, V_i) corresponds to (T, W', V') in Lemma 2.34, and obtain U_{i+1} as S in Lemma 2.34. Moreover, since, in this setting, U_0 is disjoint from its first T iterates,

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one can obtain that U_i is disjoint from its first T iterates for each $1 \le i \le r+1$. Finally, one can take V to be the interior of U_{r+1}. For more details, the reader should refer to [**8**, pp. 62–63].

2.6. *Generic properties.* A set R of a topological Baire space X is called a *residual* set, if R contains a dense G_δ set of X. We say a property is a *generic* property of X, if there is a residual set $R \subset X$, such that each element contained in R satisfies the property. We give some well-known C^1-generic properties of diffeomorphisms in the following lemma. These results can be found in many papers, such as [**1**, **12**, **18**, **41**].

LEMMA 2.38. *There is a residual set $\mathcal{R}$ in* $\mathrm{Diff}^1(M)$ *of diffeomorphisms, such that any $f \in \mathcal{R}$ satisfies the following properties.*

(1) *The diffeomorphism f is Kupka-Smale: all periodic points of f are hyperbolic and the stable and unstable manifolds of periodic orbits intersect transversely.*
(2) *The periodic points are dense in the chain recurrent set and any chain recurrence class is either a homoclinic class or contains no periodic point.*
(3) *For every periodic point p of f, there exists a C^1 neighborhood $\mathcal{U}_1$ of f, such that every $g \in \mathcal{U}_1 \cap \mathcal{R}$ is a continuity point for the map $g \mapsto H(p_g, g)$ where p_g is the continuation of p for g, where the continuity is with respect to the Hausdorff distance between compact subsets of M.*
(4) *If $H(p)$ is a homoclinic class of f, then there exists an interval $[\alpha, \beta]$ of natural numbers and a C^1 neighborhood $\mathcal{U}_2$ of f, such that for every $g \in \mathcal{U}_2$, the set of indices of hyperbolic periodic points contained in $H(p_g, g)$ is $[\alpha, \beta]$. Also, all periodic points of the same index contained in $H(p)$ are homoclinically related.*
(5) *If a homoclinic class $H(p)$ contains periodic orbits with different indices, then f can be C^1 approximated by diffeomorphisms having a heterodimensional cycle.*
(6) *If a homoclinic class $H(p)$ is Lyapunov stable, then there is a C^1 neighborhood $\mathcal{U}_3$ of f, such that for any $g \in \mathcal{U}_3 \cap \mathcal{R}$, the homoclinic class $H(p_g, g)$ is also Lyapunov stable.*
(7) *For any two points $x, y \in M$, if $x \dashv y$, then $x \prec y$.*

3. *Norm of products and product of norms: reduction of the proof of Theorem A*
Theorem A essentially follows from the theorem below.

THEOREM B. *For C^1-generic $f \in \mathrm{Diff}^1(M)$, assume that p is a hyperbolic periodic point of f and that the homoclinic class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$, with* $\dim E \le \mathrm{Ind}(p)$, *such that the bundle E is not contracting. Then there are a constant $\lambda_0 \in (0, 1)$, an integer $m_0 \in \mathbb{N}$, satisfying, for any $m \in \mathbb{N}$ with $m \ge m_0$, any constants $\lambda_1, \lambda_2 \in (\lambda_0, 1)$ with $\lambda_1 < \lambda_2$, there is a sequence of different periodic orbits $\mathcal{O}_k = \mathrm{Orb}(q_k)$ with period $\tau(q_k)$ contained in $H(P)$, such that*

$$\lambda_1{}^{\tau(q_k)} < \prod_{0 \le i < \tau(q_k)/m} \|Df^m|_{E(f^{im}(q_k))}\| < \lambda_2{}^{\tau(q_k)}.$$

From Theorem B, we can get periodic orbits that have certain controls of the product of norms along the bundle E. To control Lyapunov exponents of the periodic orbits, we have

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to control the norm of products along the bundle E. We use the following two lemmas. The first is a perturbation lemma for matrices to control exponents, see [**20**, **40**] (also see [**34**, **36**]).

LEMMA 3.1. *For any integer $d \geq 1$, $K \geq 1$, any constant $\varepsilon > 0$ and $\lambda > 0$, there are two integers N and τ_0, such that for any $A_1, \ldots, A_\tau$ in $\mathrm{GL}(d, \mathbb{R})$ with $\tau \geq \tau_0$, and $\max_{1 \leq i \leq \tau}\{\|A_i\|, \|A_i^{-1}\|\} \leq K$, if*

$$\prod_{0 \leq i < \tau/N} \|A_{(i+1)N} \cdots A_{iN+2} A_{iN+1}\| \geq \lambda^\tau,$$

then there are $B_1, \ldots, B_\tau$ in $\mathrm{GL}(d, \mathbb{R})$, with $\|B_i - A_i\| < \varepsilon$ and $\|B_i^{-1} - A_i^{-1}\| < \varepsilon$, for all $i = 1, \ldots, \tau$, such that the maximal absolute value of eigenvalues of $B_\tau \circ \cdots \circ B_2 \circ B_1$ is bigger than λ.

Remark 3.2. In [**20**], it is presented for the constant $\lambda = 1$. If $\lambda \neq 1$, then by considering $A_i' = \lambda^{-1}\mathrm{Id} \circ A_i$ and applying the special case for the constant 1, we can get the general statement as above.

COROLLARY 3.3. *For any integer $d \geq 1$, $K \geq 1$, any constant $\varepsilon > 0$ and $\lambda_1 < \lambda_2$, there are two integers N and τ_0, such that for any $A_1, \ldots, A_\tau$ in $\mathrm{GL}(d, \mathbb{R})$ with $\tau \geq \tau_0$, and $\max_{1 \leq i \leq \tau}\{\|A_i\|, \|A_i^{-1}\|\} \leq K$, if*

$$\lambda_1^\tau < \prod_{0 \leq i < \tau/N} \|A_{(i+1)N} \cdots A_{iN+2} A_{iN+1}\| < \lambda_2^\tau,$$

then there are $B_1, \ldots, B_\tau$ in $\mathrm{GL}(d, \mathbb{R})$, with $\|B_i - A_i\| < \varepsilon$ and $\|B_i^{-1} - A_i^{-1}\| < \varepsilon$, for all $i = 1, \ldots, \tau$, such that the maximal norm of eigenvalue of $B_\tau \circ \cdots \circ B_2 \circ B_1$ is in the interval (λ_1, λ_2).

Proof. We take ε small enough such that, for any $A \in \mathrm{GL}(d, \mathbb{R})$, if $\|A^{-1}\| \leq K$, then $B(A, \varepsilon) \in \mathrm{GL}(d, \mathbb{R})$, where $B(A, \varepsilon)$ is the ε ball of A. By the assumption of A_i, we have that the maximal norm of eigenvalue of $A_\tau \circ \cdots \circ A_2 \circ A_1$ is smaller than λ_2. By Lemma 3.1, we can get $B_1^0, \ldots, B_\tau^0$ in $\mathrm{GL}(d, \mathbb{R})$ that satisfies the conclusion for the number λ_1. We take a path $A_{i,t}|_{0 \leq t \leq 1}$ contained in $B(A_i, \varepsilon)$ that connects A_i to B_i^0. We have that the maximal norm of eigenvalue of $B_\tau^0 \circ \cdots \circ B_2^0 \circ B_1^0$ is bigger than λ_1. Then there must be a time $0 < t < 1$, such that the maximal norm of eigenvalue of $A_{\tau,t} \circ \cdots \circ A_{2,t} \circ A_{1,t}$ is in the interval (λ_1, λ_2). We take $B_i = A_{i,t}$ and get the conclusion. □

The next lemma is a generalized Frank's lemma by Gourmelon that preserves some pieces of invariant manifolds of hyperbolic period orbits, see [**30**].

LEMMA 3.4. *Consider a constant $\varepsilon > 0$, a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a hyperbolic periodic orbit $\mathcal{O} = \mathrm{Orb}(q)$ of f with period τ. Assume there is a one-parameter family of linear maps $(A_{n,t})_{n=0,1,\ldots,\tau-1; t \in [0,1]}$ in $\mathrm{GL}(d, \mathbb{R})$, satisfying:*

(1) $A_{n,0} = Df(f^n(q))$;
(2) *for all $n = 0, 1, \ldots, \tau - 1$ and $t \in [0, 1]$, we have $\|Df(f^n(q)) - A_{n,t}\| < \varepsilon$ and $\|Df^{-1}(f^{n+1}(q)) - A_{n,t}^{-1}\| < \varepsilon$;*
(3) *$A_{\tau-1,t} \circ \cdots \circ A_{0,t}$ is hyperbolic for all $t \in [0, 1]$.*

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Then, for any neighborhood V of $\mathcal{O}$, any $\delta > 0$, and any pair of compact sets $K^s \subset W^s_\delta(\mathcal{O}, f)$ and $K^u \subset W^u_\delta(\mathcal{O}, f)$ disjoint from V, there is a diffeomorphism $g \in \mathrm{Diff}^1(M)$ that is ε-C^1 close to f, such that:

(a) *g coincides with f on $\mathcal{O}$ and outside V;*
(b) *$K^s \subset W^s_\delta(\mathcal{O}, g)$ and $K^u \subset W^u_\delta(\mathcal{O}, g)$;*
(c) *$Dg(g^n(q)) = Dg(f^n(q)) = A_{n,1}$ for all $n = 0, \ldots, \tau - 1$.*

Now we give the proof of Theorem A from Theorem B.

Proof. By Theorem B, we get two constants $\lambda_0 \in (0, 1)$ and $m_0 \in \mathbb{N}$. We prove that for any $\lambda_0 < \lambda_1 < \lambda_2 < 1$ and any $\varepsilon > 0$, there is a diffeomorphism g that is C^1-ε close to f having a periodic orbit $\mathrm{Orb}(q)$ homoclinic related to p_g such that the largest Lyapunov exponent along E of $\mathrm{Orb}(q)$ is in the interval $(\log \lambda_1, \log \lambda_2)$. Then by the genericity of f and [**28**, Lemma 2.1], f itself has such periodic orbits. Since λ_1 can be taken arbitrarily close to 1, we get the conclusion of Theorem A.

Take $d = \dim(M)$ and $K = \max\{\|Df\|, \|Df^{-1}\|\}$. Now we fix the constants $\varepsilon > 0$ and $\lambda_1 < \lambda_2$ in $(\lambda_0, 1)$. Since $E \oplus F$ is a dominated splitting, the two bundles E and F are transverse with each other, thus the angle between E and F has a lower bound. As a result, the perturbation of f along the periodic orbit $\mathrm{Orb}(q)$ can be realized by the perturbation restricted to the derivative of f along the two bundles E and F. That is to say, for the constant $\varepsilon > 0$, there is $\varepsilon' > 0$, such that any ε' perturbation of Df on the bundles E and F independently gives an ε perturbation of f. For $\varepsilon' > 0$, we get two integers N and τ_0 by Corollary 3.3.

By Theorem B, there is a periodic orbit $\mathrm{Orb}(q)$ of f with period $\tau > \tau_0$ that is homoclinically related to $\mathrm{Orb}(p)$, such that

$$\lambda_1{}^\tau < \prod_{0 \le i < \tau/m} \|Df^m|_{E(f^{im}(q))}\| < \lambda_2{}^\tau,$$

where $m > m_0$ is a multiple of N. We use the notation $A_i = Df|_{f^i(q)}$ for $i = 0, \ldots, \tau - 1$. By Corollary 3.3, there are $B_0, \ldots, B_{\tau-1}$ in $\mathrm{GL}(d, \mathbb{R})$, with $\|(B_i - A_i)|_E\| < \varepsilon'$ and $\|(B_i^{-1} - A_i^{-1})|_E\| < \varepsilon'$, for all $i = 0, \ldots, \tau - 1$, and B_i coincides with A_i along the bundle F such that the maximal norm of eigenvalue of $B_{\tau-1} \circ \cdots \circ B_1 \circ B_0$ along the bundle E is in the interval $(\lambda_1^\tau, \lambda_2^\tau)$. By the choice of ε', we have that $\|B_i - A_i\| < \varepsilon$ and $\|B_i^{-1} - A_i^{-1}\| < \varepsilon$. We take a path $A_{i,t}|_{0 \le t \le 1}$ contained in $B(A_i, \varepsilon)$ that connects A_i to B_i such that $A_{i,t}$ coincides with A_i along the bundle F for all $i = 0, \ldots, \tau - 1$ and all $t \in (0, 1)$. If there is a time $t \in (0, 1)$ such that $A_{\tau-1,t} \circ \cdots A_{0,t}$ is not hyperbolic, then there must be a time $t_0 < t$ such that $A_{\tau-1,s} \circ \cdots A_{0,s}$ is hyperbolic for all $0 \le s \le t_0$, and the maximal norm of eigenvalue of $A_{\tau-1,t_0} \circ \cdots A_{0,t_0}$ along the bundle E is in the interval $(\lambda_1^\tau, \lambda_2^\tau)$. Otherwise, we can take $t_0 = 1$.

Take a small number $\delta > 0$, since $\mathrm{Orb}(q)$ is homoclinically related to $\mathrm{Orb}(p)$, there exist two points $x \in W^s_\delta(\mathrm{Orb}(q)) \pitchfork W^u(\mathrm{Orb}(p))$ and $y \in W^u_\delta(\mathrm{Orb}(q)) \pitchfork W^s(\mathrm{Orb}(p))$. Consider the two compact sets $K^s = \{x\}$ and $K^u = \{y\}$. We take a neighborhood V of $\mathrm{Orb}(q)$ such that $V \cap (K^s \cup K^u) = \emptyset$ and $V \cap (\mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)) = \emptyset$. By Lemma 3.4, considering the one-parameter family of linear maps $(A_{i,t})_{i=0,\ldots,\tau-1;t\in[0,t_0]}$, there is a

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diffeomorphism g that is C^1-ε close to f, such that:
(a) g coincides with f on $\mathrm{Orb}(q)$ and outside V;
(b) $K^s \subset W^s_\delta(\mathrm{Orb}(q), g)$ and $K^u \subset W^u_\delta(\mathrm{Orb}(q), g)$;
(c) $Dg(g^i(q)) = Df(f^i(q)) = A_{i,t_0}$ for all $i = 0, \ldots, \tau - 1$.

By item (a) and the choice of V, we have that g coincides with f on $\mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)$, hence we have that $x \in W^u(\mathrm{Orb}(p), g)$ and $y \in W^s(\mathrm{Orb}(p), g)$. Then by item (b), we can see that $x \in W^s_\delta(\mathrm{Orb}(q), g) \cap W^u(\mathrm{Orb}(p), g)$ and $y \in W^u_\delta(\mathrm{Orb}(q), g) \cap W^s(\mathrm{Orb}(p), g)$. By another small perturbation if necessary, we can assume that the two intersections are transverse. Then the two periodic orbits $\mathrm{Orb}(q)$ and $\mathrm{Orb}(p)$ of g are still homoclinically related with each other, and the largest Lyapunov exponent of $\mathrm{Orb}(q)$ along the bundle E under the diffeomorphism g is in the interval $(\log \lambda_1, \log \lambda_2)$. This finishes the proof of Theorem A. □

4. *Non-hyperbolicity implies the existence of weak periodic orbits: proof of Theorem B*

In this section we give the proof of Theorem B. We assume that $\mathcal{R}$ is the residual set of $\mathrm{Diff}^1(M)$ stated in Lemma 2.38 and $f \in \mathcal{R}$ is a diffeomorphism that satisfies the hypotheses of Theorem B. Later we will assume also that f belongs to another two residual subsets $\mathcal{R}_0$ and $\mathcal{R}_1$ defined below.

Since $E \oplus F$ is a dominated splitting and $\dim E \le \mathrm{Ind}(p)$, we have that there are $\lambda_0 \in (0, 1)$ and $m_0 \in \mathbb{N}$ such that, for any $m \ge m_0$, the splitting $E \oplus F$ is (m, λ_0^2)-dominated and, for the hyperbolic periodic orbit $\mathrm{Orb}(p)$,

$$\|Df^{\tau(p)}|_{E(p)}\| < \lambda_0^{\tau(p)},$$

where $\tau(p)$ is the period of $\mathrm{Orb}(p)$. In the following, we fix the constant λ_0 and the integer $m \ge m_0$. In order to simplify the notation, we will assume that $m = 1$ and that p is a fixed point of f, but the general case is identical.

4.1. *Existence of weak sets.*

LEMMA 4.1. *For any $\lambda \in (\lambda_0, 1)$, there is a λ-E-weak set contained in $H(p)$.*

Proof. Since E is not contracting, by a compactness argument, there is a point $b \in H(p)$ such that, for any $n \ge 1$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(b))}\| \ge 1.$$

Then the first assumption for the bundle E in Lemma 2.23 is satisfied.

Assume by contradiction that there is a constant $\lambda \in (\lambda_0, 1)$, such that there is no λ-E-weak set contained in $H(p)$. Thus, the second assumption in Lemma 2.23 is satisfied for the bundle E and the constant λ. Hence, for any $\lambda_1, \lambda_2 \in (\lambda, 1)$ with $\lambda_1 < \lambda_2$, there is a sequence of periodic orbits $\mathrm{Orb}(q_k)$ with period $\tau(q_k)$ that are homoclinically related with each other and that converges to a subset of $H(p)$ such that for any $k \ge 0$, the following properties are satisfied:

$$\lambda_1{}^{\tau(q_k)} \le \prod_{0 \le i < \tau(q_k)} \|Df|_{E(f^i(q_k))}\| \le \lambda_2{}^{\tau(q_k)}.$$

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Then $H(p) = H(q_k)$ by Lemma 2.38(2), hence $q_k \in H(p)$. It is obvious that $\mathrm{Orb}(q_k)$ is a λ_1-E-weak set contained in $H(p)$, thus is also a λ-E-weak set. This contradicts the assumption that there is no λ-E-weak set contained in $H(p)$. □

4.2. *Existence of a bi-Pliss point accumulating backward to an E-weak set.* From now on, we fix any two numbers $\lambda_1 < \lambda_2$ in $(\lambda_0, 1)$. Then there is a λ_2-E-weak set contained in $H(p)$. By the domination, any λ_2-E-weak set K is (C, λ_0, F)-expanding for some constant $C > 0$ depending on K. By [**33**], any point $x \in K$ has a uniform local unstable manifold $W^u_{\mathrm{loc}}(x)$ with uniform size depending on K. Recall Remark 2.16, any limit point of a sequence of λ-E-Pliss points is still a λ-E-Pliss point.

We extend the dominated splitting $E \oplus F$ to the maximal invariant compact set of a small neighborhood U of $H(p)$ and denote it still by $E \oplus F$. We take a constant $\lambda_3 \in (\lambda_2, 1)$.

LEMMA 4.2. *There are a λ_2-E-weak set K and a λ_3-bi-Pliss point $x \in H(p)\backslash K$ satisfying $\alpha(x) = K$.*

It is obvious that any compact invariant subset of a λ_2-E-weak set is still a λ_2-E-weak set. So we only have to prove that: *there are a λ_2-E-weak set K and a λ_3-bi-Pliss point $x \in H(p)\backslash K$ satisfying $\alpha(x) \subset K$.*

Proof. By Lemma 4.1, there exists a λ_2-E-weak set in $H(p)$. To prove this lemma, we consider two cases: either all of the λ_2-E-weak sets are uniformly E-weak or not. More precisely, if we take the closure of the union of all λ_2-E-weak sets contained in $H(p)$, and denote it by $\hat{K}$, then there are two cases: either $\hat{K}$ is still a λ_2-E-weak set or not.

4.2.1. *The uniform case: $\hat{K}$ is a λ_2-E-weak set.* In this case, $\hat{K}$ is the maximal λ_2-E-weak set in $H(p)$ and we will take $K = \hat{K}$.

CLAIM 4.3. *The set K is locally maximal in $H(p)$, and there exists a point $z \in H(p)\backslash K$, such that $\alpha(z) \subset K$.*

Proof. We prove by contradiction. Assume that K is not locally maximal in $H(p)$. Take a decreasing sequence of neighborhoods $(U_n)_{n\geq 0}$ of K, such that $\bigcap_n U_n = K$. Then for any $n \geq 0$, there is a compact invariant set $K_n \subset U_n \cap H(p)$ such that $K \subsetneq K_n$. Since K is the maximal λ_2-E-weak set in $H(p)$, we have that K_n is not a λ_2-E-weak set, thus there is a λ_2-E-Pliss point $y_n \in K_n$. Take a converging subsequence of (y_n), and assume that y is the limit point. Then we have that $y \in K$ and y is a λ_2-E-Pliss point. This contradicts the fact that K is a λ_2-E-weak set.

Since K is locally maximal in $H(p)$, there is a neighborhood U of K such that K is the maximal compact invariant set contained in $\overline{U} \cap H(p)$. Now we consider the diffeomorphism f^{-1}. Since $K \subset H(p)$, we have that $p \prec K$ under f^{-1}. By Lemma 2.13, there exists a point $z \in U\backslash K$, such that $p \prec_M z \prec_U K$ under f^{-1} and $\mathrm{Orb}^+(z, f^{-1}) \subset U$. Since $H(p)$ is a chain recurrence class both for f and f^{-1} and $K \subset H(p)$, the fact $p \prec z \prec K$ under f^{-1} implies that $z \in H(p)$. Since $\mathrm{Orb}^+(z, f^{-1}) \subset U$ we have that $\alpha(z, f) = \omega(z, f^{-1})$ is an f-invariant compact set contained in $\overline{U} \cap H(p)$. By the fact

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that K is the maximal compact invariant set contained in $\overline{U} \cap H(p)$, one has $\alpha(z) \subset K$. This ends the proof of Claim 4.3. □

Now we take the point $z \in H(p)\backslash K$ satisfying $\alpha(z) \subset K$ from Claim 4.3.

CLAIM 4.4. *There exists at least one λ_2-E-Pliss point contained in $\omega(z)$.*

Proof. We prove this claim by contradiction. If $\omega(z)$ contains no λ_2-E-Pliss points, by item (1) of Corollary 2.20, $\mathrm{Orb}(z) \cup \omega(z)$ contains no λ_2-E-Pliss points. Then $K \cup \mathrm{Orb}(z) \cup \omega(z)$ is a λ_2-E-weak set, which contradicts the maximality of λ_2-E-weak set K since $z \notin K$. Thus $\omega(z)$ contains at least one λ_2-E-Pliss point. □

Since K is a λ_2-E-weak set, by the domination, for any point $w \in K$, there is an integer n_w, such that $\prod_{i=0}^{n_w-1} \|Df^{-1}|_{F(f^{-i}(w))}\| \le (\lambda_0^2/\lambda_2)^{n_w} < {\lambda_0}^{n_w}$. By Corollary 2.20(2), considering the bundle F, there are infinitely many λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(z)$. We take all of the λ_1-F-Pliss points $\{f^{n_i}(z)\}$ with $n_{i+1} > n_i$ on $\mathrm{Orb}(z)$. Note that the index $i \in \mathbb{Z}$ but not $i \ge 0$, since there are infinitely many λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(z)$. We consider the following two cases:

(a) either the sequence (n_i) has an upper bound or $(n_{i+1} - n_i)$ can be arbitrarily large;
(b) or the sequence (n_i) has no upper bounds and $(n_{i+1} - n_i)$ is bounded.

CLAIM 4.5. *In case (a), there exists a λ_2-E-Pliss point $y \in H(p)$ such that, for any $\delta > 0$, there is $n_i \in \mathbb{Z}$, satisfying $d(y, f^{n_i}(z)) < \delta$. Thus, by taking δ small enough, we can take $x \in W^u(f^{n_i}(z)) \cap W^s(y)$ such that x is a λ_3-bi-Pliss point.*

Proof. If the sequence $\{n_i\}$ has an upper bound, we take the maximal n_i. That is to say, $f^{n_i}(z)$ is a λ_1-F-Pliss point for f^{-1} and there is no λ_1-F-Pliss point for f^{-1} on $\mathrm{Orb}^+(f^{n_i}(z))$. By Lemma 2.22(3), we have that $f^{n_i}(z)$ is also a λ_1-E-Pliss point, thus $f^{n_i}(z)$ is a λ_1-bi-Pliss point. We take $x = y = f^{n_i}(z)$ in this case.

Otherwise, the sequence $\{n_i\}$ has no upper bounds but $(n_{i+1} - n_i)$ can be arbitrarily large. By Lemma 2.22(1), we can take a subsequence of $\{n_i\}$ such that $f^{n_i}(z)$ converges to a λ_1-bi-Pliss point $y \in \omega(z)$. Then, for any $\delta > 0$, we can take n_i large enough, such that $d(y, f^{n_i}(z)) < \delta$ and, moreover, we can take $x \in W^u(f^{n_i}z) \cap W^s(y)$ such that $d(f^j(x), f^j(y)) < \delta$ and $d(f^{-j}(x), f^{-j}(f^{n_i}z)) < \delta$, for all $j \ge 0$. Thus, by taking δ small enough, x is a λ_3-bi-Pliss point. □

CLAIM 4.6. *In case (b), there is a λ_2-E-Pliss point $y \in \omega(z)$ such that there is $n \in \mathbb{N}$ satisfying $W^u(f^n(z)) \cap W^s(y) \neq \emptyset$. Thus, we can take a point $\bar{x} \in W^u(f^n(z)) \cap W^s(y)$, such that* $\mathrm{Orb}(\bar{x})$ *contains some λ_3-bi-Pliss point.*

Proof. In this case, there are infinitely many λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^+(z)$, and the time between any consecutive λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^+(z)$ is bounded. Then for any point $w \in \overline{\mathrm{Orb}^+(z)}$, there is an integer $n_w \in \mathbb{N}$ such that $\prod_{i=0}^{n_w-1} \|Df^{-1}|_{F(f^{-i}(w))}\| \le {\lambda_1}^{n_w}$. Hence, $\overline{\mathrm{Orb}^+(z)}$ is a positively invariant F-expanding compact set, and any point $w \in \overline{\mathrm{Orb}^+(z)}$ has a uniform unstable manifold. By Claim 4.4, there is a λ_2-E-Pliss point $y \in \omega(z)$. For any $\delta > 0$, there is $n \in \mathbb{N}$ such that $d(y, f^n(z)) < \delta$ and $W^u(f^n(z)) \cap W^s(y) \neq \emptyset$. We take $\bar{x} \in W^s(y) \cap W^u(f^n(z))$. Then $\alpha(\bar{x}) = \alpha(z)$ and

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by Corollary 2.20(2), there are λ_3-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(\bar{x})$. Also by taking δ small enough, $d(f^i(\bar{x}), f^i(y))$ can be small for all $i \geq 0$. Since y is a λ_2-E-Pliss point, we can take $\bar{x}$ to be a λ_3-E-Pliss point. Then, by Lemma 2.22(2) there exists a λ_3-bi-Pliss point x on $\mathrm{Orb}(\bar{x})$. Then we have that $\alpha(x) = \alpha(\bar{x}) = \alpha(z) \subset K$. □

From the above two claims, we get a λ_3-bi-Pliss point $x \in H(p)$, such that $\alpha(x) \subset K$. We have to show that $x \notin K$. Note that in the two cases, we both have $\omega(x) = \omega(y)$ where y is a λ_2-E-Pliss point. By Corollary 2.20(1), $\omega(x)$ contains some λ_2-E-Pliss point. Since K contains no λ_2-E-Pliss point, we have that $x \notin K$. This ends the proof of the uniform case.

4.2.2. *The non-uniform case: $\hat{K}$ is not a λ_2-E-weak set.* Recall that for any point z contained in a λ_2-E-weak set, there is an integer n_z such that

$$\prod_{i=0}^{n_z-1} \|Df|_{E(f^i(z))}\| > {\lambda_2}^{n_z}.$$

We denote N_z the smallest integer that satisfies the above inequality.

CLAIM 4.7. *In this case, for any number $L > 0$ there are a λ_2-E-weak set K_L and a point $z \in K_L$ such that z is a λ_1-F-Pliss point for f^{-1} and $N_z > L$.*

Proof. Since $\hat{K}$ is not a λ_2-E-weak set, for any number $L > 0$ there is a λ_2-E-weak set K_L and a point $z \in K_L$ such that $N_z > L$, that is to say, for $1 \leq n \leq N_z$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(z))}\| \leq \lambda_2^n,$$

and

$$\prod_{i=0}^{N_z-1} \|Df|_{E(f^i(z))}\| > \lambda_2^{N_z}.$$

We only have to show that we can choose z to be a λ_1-F-Pliss point for f^{-1}. Since K_L is a λ_2-E-weak set, similarly to the arguments above, by Corollary 2.20(2), there are λ_1-F-Pliss points for f^{-1} on $\mathrm{Orb}^-(z)$. If z is not a λ_1-F-Pliss point for f^{-1}, we can take the minimal number $l \in \mathbb{N}$ such that $w = f^{-l}(z)$ is a λ_1-F-Pliss point for f^{-1}. We will prove that $N_w \geq N_z + l > L$. Hence, by replacing z by w, we get the conclusion of the claim. To prove this, we only have to show that for any $1 \leq n \leq l$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(w))}\| \leq \lambda_2^n.$$

We prove this by contradiction. If the above statement is not true, then there is an integer $k \in \{1, 2, \ldots, l\}$ such that

$$\prod_{i=0}^{k-1} \|Df|_{E(f^i(w))}\| > \lambda_2^k,$$

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and, for any $1 \le n < k$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(w))}\| \le \lambda_2^n.$$

Thus, we have, for any $1 \le n \le k$,

$$\prod_{i=1}^{n} \|Df|_{E(f^{k-i}(w))}\| = \left(\prod_{i=0}^{k-1} \|Df|_{E(f^i(w))}\|\right) \Big/ \left(\prod_{i=0}^{k-n-1} \|Df|_{E(f^i(w))}\|\right) > \lambda_2^n.$$

By the domination of $E \oplus F$, we have, for all $1 \le n \le k$,

$$\prod_{i=0}^{n-1} \|Df^{-1}|_{F(f^{k-i}(w))}\| \le \left(\frac{\lambda_0^2}{\lambda_2}\right)^n \le \lambda_1^n.$$

Moreover, since w is a λ_1-F-Pliss point for f^{-1}, we will have, for any $n \ge 1$,

$$\prod_{i=0}^{n-1} \|Df^{-1}|_{F(f^{k-i}(w))}\| \le \lambda_1^n.$$

Thus, $f^k(w) = f^{-l+k}(z)$ is a λ_1-F-Pliss point, which contradicts the choice of w. This finishes the proof of Claim 4.7. □

By taking L large enough, the point z in Claim 4.7 is close to a λ_2-E-Pliss point $y \notin K_L$. Since z has a uniform local unstable manifold and y has a uniform local stable manifold, we have that $W^s(y) \cap W^u(z) \neq \emptyset$ if we take these two points close enough. We take the proper L, z and K_L, satisfying this property. Let $K = K_L$. We explain that the λ_2-E-weak set K satisfies Lemma 4.2.

By similar arguments as in the proof of case (b) in the uniform case, we can take a point $\bar{x} \in W^s(y) \cap W^u(z)$ satisfying that $\mathrm{Orb}(\bar{x})$ contains a λ_3-bi-Pliss point $x \in H(p)$. Then we have that $\alpha(x) = \alpha(z) \subset K$. Moreover, since y is a λ_2-E-Pliss point, we know that $\omega(y)$ contains λ_2-E-Pliss points by Corollary 2.20(1). Hence, $\omega(x)$ contains λ_2-E-Pliss points because $\omega(x) = \omega(y)$. This implies $x \notin K$ since K is a λ_2-E-weak set. To sum up, we have obtained a λ_2-E-weak set K and a λ_3-bi-Pliss point $x \in H(p) \backslash K$, satisfying that $\alpha(x) \subset K$. This finishes the proof of Lemma 4.2. □

4.3. *Continuation of Pliss points.* Denote by $\mathcal{M}$ the space of all compact subsets of M, associated with the Hausdorff topology. Denote by $\mathcal{S}$ the space of all finite subsets of $M \times \mathcal{M}$ associated with the Hausdorff topology. For any positive integer $N \in \mathbb{N}$, and a diffeomorphism $g \in \mathrm{Diff}^1(M)$, denote by $\mathrm{Per}_N(g)$ the set of periodic points of g with period less than or equal to N, and denote by $\mathcal{C}(q, g)$ the chain recurrence class of a periodic point q of g. It is well-known that for any $N \ge 1$, there is a dense and open subset $\mathcal{U}_N \subset \mathrm{Diff}^1(M)$, such that, for any $g \in \mathcal{U}_N$, the set $\mathrm{Per}_N(g)$ is a finite set and any point $q \in \mathrm{Per}_N(g)$ is a hyperbolic periodic point.

We define a map $\Phi_N : \mathcal{U}_N \mapsto \mathcal{S}$, sending a diffeomorphism g to the set of pairs $(q, P_{\lambda_3}(q, g))$, where $q \in \mathrm{Per}_N(g)$, and $P_{\lambda_3}(q, g)$ is a compact set contained in $\mathcal{C}(q, g)$ defined as follows.

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(i) If $\mathcal{C}(q, g)$ has a λ_0^2-dominated splitting $E \oplus F$ such that $\dim(E) = \mathrm{Ind}(q)$, then the set $P_{\lambda_3}(q, g)$ is the set of λ_3-E-Pliss points contained in $\mathcal{C}(q, g)$.
(ii) Otherwise, $P_{\lambda_3}(q, g) = \emptyset$.

LEMMA 4.8. *For each positive integer $N \in \mathbb{N}$, the set of continuity points of Φ_N, denoted by $\mathcal{B}_N$, is a residual subset of* $\mathrm{Diff}^1(M)$.

Proof. Assume that $g \in \mathrm{Diff}^1(M)$ and p_g is a hyperbolic periodic point of g. There is a C^1 neighborhood $\mathcal{U}$ of g such that, for any $h \in \mathcal{U}$, the point p_g has a continuation p_h. For any neighborhood V of $\mathcal{C}(q, g)$, there is a C^1 neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of g, such that $\mathcal{C}(q, h) \subset V$ for any $h \in \mathcal{U}_1$.

If $\mathcal{C}(q, g)$ has a λ_0^2-dominated splitting, then it is a robust λ_0^2-dominated splitting. More precisely, there is a C^1 neighborhood $\mathcal{U}_2 \subset \mathcal{U}$ of g, such that $\mathcal{C}(q_h, h)$ has a λ_0^2-dominated splitting for any $h \in \mathcal{U}$. Hence, by the choice of $\mathcal{U}_N$, there is an open and dense subset $\mathcal{U}'_N \subset \mathcal{U}_N$ such that, for any $g \in \mathcal{U}'_N$ and any $q \in \mathrm{Per}_N(g)$, the chain recurrence class $\mathcal{C}(q, g)$ either has a robust λ_0^2-dominated splitting or has no λ_0^2-dominated splitting robustly. Moreover, if there is a sequence of diffeomorphisms $\{g_n\}_{n\geq 0}$ such that g_n converges to g and g_n has a λ_3-E-Pliss point $x_n \in \mathcal{C}(q_h, h)$, then any limit point x of the sequence $\{x_n\}$ is a λ_3-E-Pliss point of g.

By the above arguments, we can see that Φ_N is an upper-semi-continuous map restricted to $\mathcal{U}'_N$. It is known that the set of continuity points of a semi-continuous map is a residual subset. Then $\mathcal{B}_N$ contained a residual subset of $\mathcal{U}'_N$. Since $\mathcal{U}'_N$ is open and dense in $\mathcal{U}_N$, we know that $\mathcal{B}_N$ is a residual subset of $\mathcal{U}_N$. Hence, $\mathcal{B}_N$ is a residual subset of $\mathrm{Diff}^1(M)$, since $\mathcal{U}_N$ is open and dense in $\mathrm{Diff}^1(M)$. □

Denote by $\mathcal{R}_0 = \bigcap_{N\geq 1} \mathcal{B}_N$, then $\mathcal{R}_0$ is a residual subset of $\mathrm{Diff}^1(M)$. In the following we take $f \in \mathcal{R}_0 \cap \mathcal{R}$.

4.4. *The perturbation to make $W^u(p)$ accumulate to K.* We take the λ_2-E-weak set $K \subset H(p)$ of f obtained by Lemma 4.2. By Proposition 2, one can obtain a heteroclinic orbit connecting p to K by a C^1 perturbation, since $K \subset H(p)$. Hence, the set K is still a λ_2-E-weak set if the perturbation is C^1 small. Moreover, using the continuation of Pliss points (§§4.2 and 4.3), we can guarantee that the set K is contained in the chain recurrence class of p after the perturbation.

LEMMA 4.9. *Assume that $f \in \mathcal{R}_0 \cap \mathcal{R}$, then for any neighborhood $\mathcal{U}$ of f in* $\mathrm{Diff}^1(M)$, *there are a diffeomorphism $g_1 \in \mathcal{U}$ and a point $y \in M$, such that:*
(1) *g_1 coincides with f on the set $K \cup \mathrm{Orb}(p)$, and $y \in W^u(p, g_1)$;*
(2) *$\omega(y, g_1) \subset K$;*
(3) *K is contained in $\mathcal{C}(p, g_1)$.*

Proof. By Lemma 4.2, we obtain that, for the diffeomorphism f, there is a λ_3-bi-Pliss point $x \in H(p)\backslash K$ satisfying $\alpha(x) = K$. Since $K \subset H(p)$, we have that $K \subset \overline{W^u(p)}$. By Proposition 2, for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there are a point $y \in W^u(p, f)$ and a diffeomorphism $g_1 \in \mathcal{U}$ such that $\omega(y, g_1) \subset K$ and $y \in W^u(p, g_1)$. Moreover, the

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diffeomorphism g_1 coincides with f on the set $\mathrm{Orb}^-(x) \cup K \cup \mathrm{Orb}(p)$ and Dg_1 coincides with Df on $\mathrm{Orb}^-(x)$. Thus, items (1) and (2) are satisfied, and x is a λ_3-F-Pliss point for g_1^{-1}.

Since p is a hyperbolic fixed point and $x \in P_{\lambda_3}(p, f)$, by Lemma 4.8 and the fact that f is a continuity point of Φ_1, if we choose g_1 close enough to f (by taking the neighborhood $\mathcal{U}$ small), then there is a λ_3-E-Pliss x' close to x, such that $x' \in \mathcal{C}(p, g_1)$. Moreover, if x' is close enough to x (by taking g_1 close to f), then $W^u(x, g_1) \cap W^s(x', g_1) \neq \emptyset$.

Now we prove that $K \subset \mathcal{C}(p, g_1)$. Take any constant $\varepsilon > 0$. Since $y \in W^u(p, g_1)$ and $\omega(y, g_1) \subset K$, there is an ε-pseudo-orbit connecting from p to a point contained in K. In fact the pseudo-orbit can be taken as an orbit segment of $\mathrm{Orb}(y, g_1)$. On the other hand, since $\alpha(x, g_1) = K$ and $W^u(x, g_1) \cap W^s(x', g_1) \neq \emptyset$, there is an $\varepsilon/2$-pseudo-orbit connecting from K to x'. By the fact that $x' \in \mathcal{C}(p, g_1)$, there is an $\varepsilon/2$-pseudo-orbit connecting from x' to p. The composition of the two $\varepsilon/2$-pseudo-orbits is an ε-pseudo-orbit connecting from K to p. Finally by the fact that ε can be arbitrarily small and the fact that $K = \alpha(x, g_1)$ is a chain transitive set, we have that $K \subset \mathcal{C}(p, g_1)$. This finishes the proof of Lemma 4.9. □

4.5. *The perturbations to connect p and K by true orbits.* In this section, we prove that we can get heteroclinic connections between the hyperbolic fixed point p and the weak set K for a diffeomorphism C^1 close to f. In the former section, we have obtained a diffeomorphism g_1 that is C^1 close to f, and an orbit $\mathrm{Orb}(y)$ that connects p to K. Moreover, K is still contained in the chain recurrence class of p for g_1. We take two steps to get heteroclinic connections between p and K. First, since $K \subset \mathcal{C}(p, g_1)$, by Proposition 3, we can connect K by a true orbit to any neighborhood of p by a C^1 small perturbation. Then, by the hyperbolicity of p, we use the uniform connecting lemma to 'push' this orbit onto the stable manifold of p. We will see that in these two steps, the orbit $\mathrm{Orb}(y)$ that connects p to K is not changed.

LEMMA 4.10. *Assume $f \in \mathcal{R}_0 \cap \mathcal{R}$, then for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there are a diffeomorphism $g_2 \in \mathcal{U}$ and two points $y, y' \in M$, such that:*

(1) $y \in W^u(p, g_2)$ *and* $\omega(y, g_2) \subset K$;
(2) $y' \in W^s(p, g_2)$ *and* $\alpha(y', g_2) \subset \omega(y, g_2)$;
(3) g_2 *coincides with* f *on the set* $\omega(y, g_2) \cup \mathrm{Orb}(p)$.

Proof. We take several steps to prove the lemma.

Choice of neighborhoods. For any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there are a neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ and three numbers $\rho > 1$, $\delta_0 > 0$ and $N \in \mathbb{N}$ that satisfy the uniform connecting lemma (Theorem 2.26). We can also assume that the fixed point p has a continuation for any $g \in \mathcal{U}_1$. There are a smaller neighborhood $\mathcal{U}' \subset \mathcal{U}_1$ of f and an integer T satisfying the conclusions of Proposition 3. By the hyperbolicity of periodic orbits of f, for the integer T, there is a neighborhood $\mathcal{U}_2 \subset \mathrm{Diff}^1(M)$ of f such that, for any diffeomorphism $h \in \mathcal{U}_2$, any periodic point of h with period less than or equal to T is hyperbolic. Take a neighborhood $\mathcal{U}_3$ of f in $\mathrm{Diff}^1(M)$, such that $\overline{\mathcal{U}_3} \subset \mathcal{U}_2 \cap \mathcal{U}'$.

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The connection from K to a neighborhood of p by pseudo-orbits. By Lemma 4.9, there are a diffeomorphism $g_1 \in \mathcal{U}_3$ and a point $y \in M$, such that:

(i) g_1 coincides with f on the set $K \cup \mathrm{Orb}(p) \cup \mathrm{Orb}^-(y)$;

(ii) $y \in W^u(p, g_1)$ and $\omega(y, g_1) \subset K \subset \mathcal{C}(p, g_1)$.

Denote $K_0 = \omega(y, g_1)$.

CLAIM 4.11. *For any neighborhood V of p, there are a negatively g_1-invariant compact set X and a point $z \in V \cap X$, satisfying that:*

(i) *the point $p \notin X$;*

(ii) *for any $\varepsilon > 0$, there is a g_1-ε-pseudo-orbit $Y_\varepsilon = (y_0, \ldots, y_m)$ contained in X such that $y_0 \in K_0$ and $y_m = z$.*

Proof. For any neighborhood V of p, take a smaller neighborhood V_0 of p, such that $\overline{V_0} \subset V$. For any $k \geq 1$, there is a g_1-$1/k$-pseudo-orbit $X_k = \{x_0^k, x_1^k, \ldots, x_{m_k}^k\}$ such that $X_k \cap K_0 = \{x_0^k\}$ and $X_k \cap V_0 = \{x_{m_k}^k\}$. Take a subsequence of $\{X_k\}_{k\geq 1}$ if necessary, we assume that X_k converges to a compact set X and $x_{m_k}^k$ converges to a point $z \in \overline{V_0} \subset V$ as k goes to $+\infty$. Obviously, X is a negatively g_1-invariant set, $p \notin X$ and $X \cap K_0 \neq \emptyset$.

Now we prove that for any $\varepsilon > 0$, there is a g_1-ε-pseudo-orbit contained in X from K_0 to z. By the continuity of g_1, for any $\varepsilon > 0$, there is $k > 3/\varepsilon$, such that for all $x, y \in M$, if $d(x, y) < 1/k$, then $d(g_1(x), g_1(y)) < \varepsilon/3$. Then we take a $1/k'$-pseudo-orbit $X_{k'} = \{x_0^{k'}, x_1^{k'}, \ldots, x_{m_k}^{k'}\}$, such that $x_0^{k'} \in K_0$ and $x_{m_{k'}}^{k'} \in V_0$ for a number $k' > k$. By choosing k' large enough, we can assume that $d_H(X_{k'}, X) < 1/k$ and there is a point $y_0 \in X \cap K_0$ such that $d(y_0, x_0^{k'}) < 1/k$ and $d(z, x_{m_{k'}}^{k'}) < 1/k$. By the assumption, for any $1 \leq i \leq m_{k'} - 1$, there is $y_i \in X$, such that $d(x_i^{k'}, y_i) < \eta$. We prove that $Y_\varepsilon := (y_0, \ldots, y_{m_{k'}} = z)$ is a ε-pseudo-orbit of g_1. In fact, for any $0 \leq i \leq m_{k'} - 1$,

$$\begin{aligned} d(g_1(y_i), y_{i+1}) &\leq d(g_1(y_i), g_1(x_i^{k'})) + d(g_1(x_i^{k'}), x_{i+1}^{k'}) + d(x_{i+1}^{k'}, y_{i+1}) \\ &< \frac{\varepsilon}{3} + \frac{1}{k'} + \frac{1}{k} < \varepsilon\,. \end{aligned}$$

Hence, $Y_\varepsilon \subset X$ is a ε-pseudo-orbit of g_1 from the set K_0 to the point z. □

The perturbation to connect K to a neighborhood of p. We take a local stable manifold $W^s_{\mathrm{loc}}(p, g_1)$ of p, and take a compact fundamental domain I_{g_1} of $W^s_{\mathrm{loc}}(p, g_1)$. Then there is a number $\delta < \delta_0$, where δ_0 is chosen in the paragraph *Choice of neighborhoods* such that, for any point $w \in I_{g_1}$, the N balls $(g_1^j(B(w, 2\delta)))_{0\leq j\leq N-1}$ are each of size smaller than δ_0, pairwise disjoint and disjoint from the set $K \cup \mathrm{Orb}(y, g_1) \cup \mathrm{Orb}(p)$. By the compactness of I_{g_1}, there are finite points $w_1, w_2, \ldots, w_L \in I_{g_1}$ such that $(B(w_i, \delta/\rho))_{1\leq i\leq L}$ is a finite open cover of I_{g_1}. Then there is a number $\eta > 0$ such that, for any diffeomorphism $h \in \mathcal{U}_1$ that is η-C^0 close to g_1, we have that:

(a) $W^s_{\mathrm{loc}}(p_h, h)$ is C^0 close to $W^s_{\mathrm{loc}}(p, g_1)$;

(b) $(B(w_i, \delta/\rho))_{1\leq i\leq L}$ is still a finite open cover of a fundamental domain I_h of $W^s_{\mathrm{loc}}(p_h, h)$;

(c) for any $1 \leq i \leq L$, the N balls $(h^j(B(w_i, 2\delta))_{0\leq j\leq N-1}$ are each of size smaller than δ_0, pairwise disjoint and disjoint with the set $K \cup \mathrm{Orb}(y, g_1) \cup \mathrm{Orb}(p, g_1)$.

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Recall that $p \notin X$ and X is negatively invariant by Claim 4.11. Since $y \in W^u(p, g_1)$, we have that $\mathrm{Orb}(y, g_1) \cap X = \emptyset$. By the choice of g_1, we have that all periodic orbits of g_1 contained in X with period less than or equal to T are hyperbolic. Under all of these hypotheses, if $(X \backslash K_0) \cap \overline{\mathrm{Orb}(y, g_1)} = \emptyset$, then there is a neighborhood U_0 of $X \backslash K_0$ such that $U_0 \cap \overline{\mathrm{Orb}(y, g_1)} = \emptyset$. By Proposition 3, there is a diffeomorphism $h \in \mathcal{U}_1$ which is η-C^0 close to g_1, such that $h = g_1 = f|_{\{p\} \cup \mathrm{Orb}(y) \cup K_0}$, and $\alpha(z, h) \subset K_0$. Thus, the above items (a), (b) and (c) are satisfied for such a diffeomorphism h.

The perturbation to get a heteroclinic connection between p and K. By the hyperbolicity of the periodic point p, if we take the neighborhood V of p small enough, then the diffeomorphism h and the point z chosen above satisfy that the negative orbit of z under h intersects with $B(w_i, \delta/\rho)$ for some $i \in \{1, 2, \ldots, L\}$. Since $\alpha(z, h) \subset K_0$ and $B(w_i, \delta/\rho) \cap K_0 = \emptyset$, there is a point $w = h^{-t}(z)$ for some integer $t > 0$, such that $\mathrm{Orb}^-(w) \cap B(w_i, \delta/\rho) = \emptyset$ and w has a positive iterate under h contained in $B(w_i, \delta/\rho)$. By the item (b), there is a point $y' \in W^s(p, h)$ such that $\mathrm{Orb}^+(y', h) \cap (\bigcup_{0 \le j \le N-1} h^i(B(w_i, \delta/\rho))) = \emptyset$ and y' has a negative iterate under h contained in $B(w_i, \delta/\rho)$. By Theorem 2.26, there is a diffeomorphism $g_2 \in \mathcal{U}$ such that y' is on the positive iterate of w under g_2. Moreover, $g_2 = g_1$ on the set $K_0 \cup \mathrm{Orb}(y) \cup \mathrm{Orb}(p) \cup \mathrm{Orb}^-(w) \cup \mathrm{Orb}^+(y')$, hence $g_2 = f$ on the set $\mathrm{Orb}(p) \cup K_0$, where $K_0 = \omega(y, g_1) = \omega(y, g_2)$. Thus, the three items of the lemma are satisfied for g_2. This finishes the proof of Lemma 4.10. □

4.6. *Last perturbation to get a weak periodic orbit.* The following lemma estimates the average contraction along the bundle E on periodic orbits. Recall that λ_0 satisfies that $E \oplus F$ is $(1, \lambda_0^2)$-dominated splitting and, for the hyperbolic fixed point p, we have $\|Df|_{E(p)}\| < \lambda_0$.

LEMMA 4.12. *Assume that $f \in \mathcal{R}_0 \cap \mathcal{R}$. Then for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, for any integer $L > 0$, any neighborhood U_p of p, there is $g \in \mathcal{U}$, which coincides with f on $\{p\}$, satisfying that, g has a periodic point $q \in U_p$ with period $\tau > L$ such that $\mathrm{Orb}(q)$ has λ_0^2-dominated splitting $E \oplus F$, and*

$$\lambda_1{}^\tau \le \prod_{0 \le i \le \tau - 1} \|Dg|_{E(g^i(q))}\| \le \lambda_2{}^\tau.$$

Proof. We take several steps to prove the lemma. We take the λ_2-E-weak set $K \subset H(p)$ of f obtained by Lemma 4.2. Take two numbers λ_1' and λ_2', such that $\lambda_1 < \lambda_1' < \lambda_2' < \lambda_2$.

Choice of neighborhoods and constants. There is a neighborhood V of $H(p)$ and a neighborhood $\mathcal{V} \subset \mathrm{Diff}^1(M)$ of f such that, for any $h \in \mathcal{V}$, the following properties are satisfied.

(i) The maximal invariant compact set of h in V has a dominated splitting which is a continuation of $E \oplus F$. To simplify the notations, we still denote this domination by $E \oplus F$.
(ii) The fixed point p has a continuation $p_h \in V$ for h, and $\|Dh|_{E(p_h)}\| < \lambda_0$.
(iii) The chain recurrence class $\mathcal{C}(p_h, h)$ of p_h is contained in V.

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Moreover, since K is a λ_2-E-weak set for f, there are a neighborhood $U_K \subset V$ of K and a number N_K satisfying the following property: consider a point z whose whole orbit is contained in V, hence $\mathrm{Orb}(z)$ has the dominated splitting $E \oplus F$ by the choice of V, if the piece of orbit $(z, f(z), \ldots, f^n(z))$ is contained in $\overline{U_K}$ with $n \geq N_K$, then we have

$$\prod_{0\leq i\leq n-1} \|Df|_{E(f^i(z))}\| > {\lambda_2}^n.$$

To simplify the proof, we just assume that $N_K = 1$, but the general case is identical.

We can take the neighborhoods $\mathcal{V}$ and U_p small, such that for any diffeomorphism $h \in \mathcal{V}$, the following additional properties are satisfied.

(i) For any point $z \in U_p$ whose orbit under h is contained in V, hence $\mathrm{Orb}(z, h)$ has the dominated splitting $E \oplus F$ by the choice of V and $\mathcal{V}$, we have that $\lambda_1/\lambda_1' < \|Dh|_{E(z)}\|/\|Df|_{E(p)}\| < \lambda_2/\lambda_2'$.

(ii) For any point $z \in U_K$ whose orbit under h is contained in V, we have that $\|Dh|_{E(z)}\| > \lambda_2$.

We can also assume that $\overline{U_K} \cap \overline{U_p} = \emptyset$ and $\overline{U_K} \cup \overline{U_p} \subset V$. Moreover, we can assume that $\overline{\mathcal{U}} \subset \mathcal{V}$.

By Lemma 4.10, there are a diffeomorphism $g_2 \in \mathcal{U}$ and two points $y, y' \in M$ satisfying that:

(i) $y \in W^u(p, g_2)$ and $\omega(y, g_2) \subset K$;

(ii) $y' \in W^s(p, g_2)$ and $\alpha(y', g_2) \subset \omega(y, g_2)$;

(iii) g_2 coincides with f on the set $\omega(y, g_2) \cup \mathrm{Orb}(p)$.

We use the notation $K_0 = \omega(y, g_2)$. Since all periodic points of f are hyperbolic and $g_2 = f|_{K_0}$, then K_0 contains no non-hyperbolic periodic point of g_2.

Choice of time. Now we fix the neighborhoods U_p and U_{K_0}. Then there are two integers l and n_0 satisfying the conclusion of Proposition 1 for g_2 and the neighborhood $\mathcal{U}$. Then we take $T_{K_0} > L$ large such that, for any $h \in \mathcal{V}$, the inequality

$$m(h)^{l+n_0}{\lambda_2}^{T_{K_0}} > (\lambda_2')^{T_{K_0}+l+n_0}$$

holds. Here $m(h)$ is the mininorm of h, i.e. $m(h) = \min_{\|v\|=1} \|Dh(v)\| = \|Dh^{-1}\|^{-1}$.

By the first item of Proposition 1, there is a diffeomorphism $h \in \mathcal{U}$, such that:

(i) h coincides with g_2 on $\mathrm{Orb}(p) \cup \mathrm{Orb}^-(y) \cup \mathrm{Orb}^+(y')$ and outside U_K;

(ii) the point y' is on the positive orbit of y under h, with $n_{K_0} = \#(\mathrm{Orb}(y, h) \cap U_{K_0}) \geq T_{K_0}$ and $n_c = \#((\mathrm{Orb}(y, h)\backslash(U_K \cup U_p)) \leq n_0$.

Hence, by the choice of T_{K_0} and the neighborhoods, we have that

$$\prod_{h^i(y)\notin U_p} \|Dh|_{E(h^i(x))}\| > (\lambda_2')^{n_{K_0}+n_c}.$$

CLAIM 4.13. *There is an integer $m > 0$ such that*

$$(\lambda_1')^{n_{K_0}+n_c+m+l} < \|Df|_{E(p)}\|^{l+m} \cdot \prod_{h^i(y)\notin U_p} \|Dh|_{E(h^i(x))}\| < (\lambda_2')^{n_{K_0}+n_c+m+l}.$$

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Proof. Let $\bar{\lambda} > 0$ be the constant such that

$$\prod_{h^i(y)\notin U_p} \|Dh|_{E(h^i(x))}\| = \bar{\lambda}^{n_{K_0}+n_c},$$

then we have $\bar{\lambda} > \lambda_2'$.

The inequality in the claim is equivalent to

$$\frac{(n_{K_0}+n_c)\log(\bar{\lambda}/\lambda_2')}{\log(\lambda_2'/\|Df|_{E(p)}\|)} < l+m < \frac{(n_{K_0}+n_c)\log(\bar{\lambda}/\lambda_1')}{\log(\lambda_1'/\|Df|_{E(p)}\|)}.$$

By the choice of T_{K_0} and $n_{K_0} \geq T_{K_0}$, we have that

$$\frac{(n_{K_0}+n_c)\log(\bar{\lambda}/\lambda_2')}{\log(\lambda_2'/\|Df|_{E(p)}\|)} > l.$$

So we only need that

$$\frac{(n_{K_0}+n_c)\log(\bar{\lambda}/\lambda_1')}{\log(\lambda_1'/\|Df|_{E(p)}\|)} - \frac{(n_{K_0}+n_c)\log(\bar{\lambda}/\lambda_2')}{\log(\lambda_2'/\|Df|_{E(p)}\|)} > 1.$$

It is equivalent to

$$(n_{K_0}+n_c)\left(\left(\frac{1}{\log(\lambda_1'/\|Df|_{E(p)}\|)} - \frac{1}{\log(\lambda_2'/\|Df|_{E(p)}\|)}\right)\log\bar{\lambda} + \frac{\log\lambda_2'}{\log(\lambda_2'/\|Df|_{E(p)}\|)} - \frac{\log\lambda_1'}{\log(\lambda_1'/\|Df|_{E(p)}\|)}\right) > 1.$$

Since $\bar{\lambda} > \lambda_2'$, and $n_{K_0} > T_{K_0}$, it is sufficient to require that

$$\frac{T_{K_0}(\log\lambda_2' - \log\lambda_1')}{\log(\lambda_1'/\|Df|_{E(p)}\|)} > 1.$$

By taking T_{K_0} large enough, the above inequality is satisfied. □

Choice of the diffeomorphism g. We take $g = h_m \in \mathcal{U}$ from Proposition 1(2), then g has a periodic orbit $O = \mathrm{Orb}(q)$ such that $O\backslash U_p = \mathrm{Orb}(y, h)\backslash U_p$ and $\#(O \cap U_p) = l+m$. Denote by $\pi(O)$ the period of O, then we have $\pi(O) = n_{K_0} + n_c + m + l$. By the choice of the neighborhood $\mathcal{U}$ and the constants, we have

$$\prod_{0\leq i\leq\pi(O)-1} \|Dg|_{E(g^i(q))}\| = \prod_{g^i(q)\in U_p} \|Dg|_{E(g^i(q))}\| \prod_{h^i(y)\notin U_p} \|Dh|_{E(h^i(x))}\|.$$

By the choice of the neighborhoods $\mathcal{V}$ and U_p, and the constants λ_1' and λ_2', we have that

$$\left(\frac{\lambda_1}{\lambda_1'}\right)^{l+m} \|Df|_{E(p)}\|^{l+m} < \prod_{g^i(q)\in U_p} \|Dg|_{E(g^i(q))}\| < \left(\frac{\lambda_2}{\lambda_2'}\right)^{l+m} \|Df|_{E(p)}\|^{l+m}.$$

Then by the estimation in Claim 4.13, we can see that

$${\lambda_1}^{\pi(O)} \leq \prod_{0\leq i\leq\pi(O)-1} \|Dg|_{E(g^i(q))}\| \leq {\lambda_2}^{\pi(O)}$$

This finishes the proof of Lemma 4.12. □

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4.7. *The genericity argument.* Lemma 4.12 is a perturbation result to get weak periodic orbits. To get the conclusion of Theorem B, we have to prove the genericity argument based on Lemma 4.12, see for instance [**27**].

Take a countable basis $(V_n)_{n\geq 1}$ of M, and take the countable family $(U_n)_{n\geq 1}$, where each U_n is a union of finitely many sets of $(V_n)_{n\geq 1}$. Take the countable pairs $(\eta_n, \gamma_n)_{n\geq 1}$ of rational numbers contained in $(\lambda_0, 1)$ with $\eta_n < \gamma_n$ for each $n \geq 1$.

Let $\mathcal{H}_{n,m}$ be the set of C^1 diffeomorphisms h such that every h_1 in a C^1 neighborhood $\mathcal{V} \subset \mathrm{Diff}^1(M)$ of h has a hyperbolic periodic point $q \in U_n$ satisfying that the hyperbolic splitting $E^s \oplus E^u$ of $\mathrm{Orb}(q, h_1)$ is a λ_0^2-dominated splitting and

$$\eta_m{}^{\tau(q)} < \prod_{0\leq i\leq \tau(q)-1} \|Dh_1|_{E^s(h_1^i)}\| < \gamma_m{}^{\tau(q)},$$

where $\tau(q)$ is the period of q. Let $\mathcal{N}_{n,m}$ be the set of C^1 diffeomorphisms h such that every h_1 in a C^1 neighborhood $\mathcal{V} \subset \mathrm{Diff}^1(M)$ of h has no hyperbolic periodic point $q \in U_n$ satisfying that the hyperbolic splitting $E^s \oplus E^u$ of $\mathrm{Orb}(q, h_1)$ is a λ_0^2-dominated splitting and

$$\eta_m{}^{\tau(q)} < \prod_{0\leq i\leq \tau(q)-1} \|Dh_1|_{E^s(h_1^i)}\| < \gamma_m{}^{\tau(q)},$$

where $\tau(q)$ is the period of q.

Note that $\mathcal{N}_{n,m} = \mathrm{Diff}^1(M)\backslash\overline{\mathcal{H}_{n,m}}$. Hence $\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}$ is C^1 open and dense in $\mathrm{Diff}^1(M)$. Let

$$\mathcal{R}_1 = \bigcap_{n\geq 1, m\geq 1} (\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}).$$

Then $\mathcal{R}_1$ is a residual subset of $\mathrm{Diff}^1(M)$, and $\mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$ is also a residual subset of $\mathrm{Diff}^1(M)$.

CLAIM 4.14. *Assume that $f \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$. Then for any two numbers $\lambda_1 < \lambda_2 \in (\lambda_0, 1)$, for any neighborhood U_p of $\mathrm{Orb}(p)$ and any integer $L > 0$, there is a periodic point $q \in U_p$ with period $\tau > L$ such that $\mathrm{Orb}(q)$ has the λ_0^2-dominated splitting $E \oplus F$, and*

$$\lambda_1{}^{\tau} \leq \prod_{0\leq i\leq \tau-1} \|Df|_{E(f^i(q))}\| \leq \lambda_2{}^{\tau}.$$

Proof. We take two rational numbers $\eta_i, \gamma_i \in (\lambda_0, 1)$ such that $\lambda_1 < \eta_i < \gamma_i < \lambda_2$, and take U_j from the countable basis of M such that $U_j \subset U_p$. Then by Lemma 4.12, there is a diffeomorphism g arbitrarily C^1 close to f such that g has a periodic point $q \in U_p$ with period $\tau > T$ such that the λ_0^2-dominated splitting $E \oplus F$ is the hyperbolic splitting on $\mathrm{Orb}(q, g)$, and

$$\eta_i{}^{\tau} \leq \prod_{0\leq i\leq \tau-1} \|Dg|_{E(g^i(q))}\| \leq \gamma_i{}^{\tau}.$$

Then $f \notin \mathcal{N}_{j,i}$, thus $f \in \mathcal{H}_{j,i}$ and f satisfies the conclusion of Claim 4.14. □

CLAIM 4.15. *Theorem B holds for any diffeomorphism in $\mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$.*

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Proof. Assume that $f \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}$ and f satisfies the assumptions of Theorem B. By Claim 4.14, we get a sequence of periodic orbits $\mathrm{Orb}(q_k)$ of f such that $q_k \to p$ with $\tau(q_k) \to \infty$, and

$$\lambda_1{}^{\tau(q_k)} \le \prod_{0\le i\le \tau(q_k)-1} \|Df|_{E(f^i(q_k))}\| \le \lambda_2{}^{\tau(q_k)}.$$

Hence, by the λ_0^2-domination of $E \oplus F$, we have that

$$\prod_{0\le i\le \tau(q_k)-1} \|Df^{-1}|_{F(f^{-i}(q_k))}\| \le \lambda_2{}^{\tau(q_k)}.$$

Then by Lemma 2.20(2) and Lemma 2.22(2), there is a λ_2-bi-Pliss point r_k on $\mathrm{Orb}(q_k)$ for each k. Taking a subsequence if necessary, we assume that (r_k) is a converging sequence. Then there is $l > 0$ such that for any $m, n \ge l$, the stable and unstable manifolds of r_m and r_n intersect, respectively, since r_k has uniform stable and unstable manifolds. Hence, $(\mathrm{Orb}(q_m))_{m\ge l}$ are homoclinically related together, thus $p \in H(q_k)$. By Lemma 2.38(2), we have that $q_k \in H(p)$. This finishes the proof of the claim. □

The proof of Theorem B is now complete.

5. *Periodic orbits around a periodic orbit and a set: proof of Proposition 1*

In this section, we give the proof of Proposition 1. We consider the periodic point p and the invariant compact set K satisfying that $p \notin K$ and K contains no non-hyperbolic periodic point. Recall that the point $x \in W^u(p)$ satisfies $\omega(x) \cap K \ne \emptyset$ and the point $y \in W^s(p)$ satisfies $\alpha(y) = K$.

Taking a smaller neighborhood if necessary, we can assume that the element of $\mathcal{U}$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}$, where $\mathcal{V}$ is a C^1 neighborhood of Id and satisfies the property (F) in Definition 2.24. By Theorem 2.25, there is an integer N associated to the neighborhood $\mathcal{U}$. By Lemma 2.27, there are two numbers $\theta > 1$ and $r_0 > 0$ associated to $\mathcal{U}$.

It is easy to see that we only need to prove the proposition for U_p and U_K small. More precisely, we assume that $U_p \cap U_K = \emptyset$ and $(\mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)) \cap U_K = \emptyset$. Moreover, by the hyperbolicity of periodic orbits in K, we assume that there is no periodic point with period less than or equal to N contained in $U_K \setminus K$.

To simplify the notation, we assume that p is a hyperbolic fixed point of f, and the proof of the general case is similar. The only difference is that, in item (2) of the conclusion, the condition $\#(O \cap U_p) \in \{l + m\tau, l + m\tau + 1, \ldots, l + (m+1)\tau - 1\}$ should be $\#(O \cap U_p) = l + m$. In the general case, the number $\#(O \cap U_p)$ cannot be chosen arbitrarily, but we can make sure it is contained in an interval whose length is the period of p.

Now we fix the neighborhoods $\mathcal{U}$, U_p and U_K, and the numbers N, θ and r_0.

5.1. *The choice of n_0, the point z_1 and the perturbation domain at z_1.*

Recall that the point $x \in W^u(p)$ satisfies $\omega(x) \cap K \ne \emptyset$.

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LEMMA 5.1. *There is a point $z_1 \in U_K \backslash K$, such that:*
(i) *for any neighborhood V_{z_1} of z_1, there is $n \geq 1$ such that $f^n(x) \in V_{z_1}$;*
(ii) $z_1 \prec_{U_K} K$ *and* $\mathrm{Orb}^+(z_1) \subset U_K$.

Proof. The proof is similar to that of Lemma 2.13. We take a smaller open neighborhood V of K such that $\overline{V} \subset U_K$. Since $\omega(x) \cap K \neq \emptyset$, for any $k \geq 1$, there is $n_k \geq 1$, such that $f^{n_k}(x) \in B(K, 1/k)$. Take the smallest integer m_k, such that the piece of orbit $(f^{m_k}(x), f^{m_k+1}(x), \ldots, f^{n_k}(x))$ is contained in V. Taking a convergent subsequence if necessary, we assume that the sequence $\{f^{m_k}(x)\}_{k\geq 1}$ converges to a point $z_1 \in \overline{V} \subset U_K$ and the sequence $\{f^{n_k}(x)\}_{k\geq 1}$ converges to a point $z_2 \in K$. Then we have that $z_1 \prec_{U_K} z_2$, and the pieces of orbit that connects the neighborhoods of z_1 and z_2 are $(f^{m_k}(x), f^{m_k+1}(x), \ldots, f^{n_k}(x))_{k\geq 1}$. Since $z_2 \in K$, we have that $z_1 \prec_{U_K} K$. By the choice of m_k, we have that $f^{m_k-1}(x) \in M \backslash V$. Since $M \backslash V$ is compact, and $f^{-1}(z_1)$ is a limit point of the sequence $\{f^{m_k-1}(x)\}_{k\geq 1}$, we have that $f^{-1}(z_1) \in M \backslash V$. By the invariance of K, we have that $z_1 \notin K$ and $n_k - m_k$ goes to $+\infty$. Since $(f^{m_k}(x), f^{m_k+1}(x), \ldots, f^{n_k}(x))$ is contained in U_K and $f^{m_k}(x)$ converges to z_1, we have that $\mathrm{Orb}^+(z_1) \subset \overline{V} \subset U_K$. Thus, the second item is satisfied. The first item is a trivial fact by the choice of z_1. □

By the assumption on U_K, we have that z_1 is not a periodic point with period less than or equal to N. Also, since $y \in W^s(p)$ and $\mathrm{Orb}^+(z_1) \subset U_K$, we have $z_1 \notin \mathrm{Orb}(y)$. Moreover, by the fact that $\alpha(y) = K$, we know $z_1 \notin \overline{\mathrm{Orb}(y)}$. Then there are two neighborhoods $V_{z_1} \subset U_{z_1}$ of z_1 satisfying the conclusion of Theorem 2.25 for the triple $(f, \mathcal{U}, N)$, and also satisfying the following conditions:
(i) $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1}) \subset U_K \backslash K$;
(ii) $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cap \mathrm{Orb}(y) = \emptyset$.

By Lemma 5.1, there is $n_1 \in \mathbb{N}$ such that $f^{n_1}(x) \in V_{z_1}$. Moreover, since $\alpha(y) = K$, there is $n_2 \in \mathbb{N}$ such that, for any $n \geq n_2$, we have $f^{-n}(y) \in U_K$. Let $n_0 = n_1 + n_2$.

5.2. *The choices of points and perturbation domains in K and to get h.* Take any integer T_K. By Lemma 5.1, we have that $z_1 \prec_{U_K} K$, that is to say, there is a point $z_2 \in K$ such that $z_1 \prec_{U_K} z_2$. Now we consider two cases, depending on whether there is such a point z_2 that is not a periodic point with period less than or equal to N.

5.2.1. *The non-periodic case.* Assume that there is a point $z_2 \in K$ which is not a periodic point with period less than or equal to N, and $z_1 \prec_{U_K} z_2$. Then there are two neighborhoods $V_{z_2} \subset U_{z_2}$ of z_2 satisfying the conclusion of Theorem 2.25 for the triple $(f, \mathcal{U}, N)$ and also satisfying the following conditions:
(i) $U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2}) \subset U_K$;
(ii) $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cap (U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})) = \emptyset$;
(iii) $f^n(x) \notin U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})$, for any $-\infty < n \leq n_1$;
(iv) $f^{-n}(y) \notin U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})$, for any $n \leq n_2 + T_K$.
Then there is $n_3 > n_2 + T_K$, such that $f^{-n_3}(y) \in V_{z_2}$. Since we have the fact that $z_1 \prec_{U_K} z_2$, there is a piece of orbit $(w, f(w), \ldots, f^k(w))$ contained in U_K, such that $w \in V_{z_1}$

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and $f^k(w) \in V_{z_2}$. Moreover, since $U_{z_1} \cap \mathrm{Orb}(y) = \emptyset$ and $w \in V_{z_1} \subset U_{z_1}$, we have that $w \notin \mathrm{Orb}(y)$.

Perturbations to get h in the non-periodic case. Now we do the perturbations step by step to get the conclusion.

Step 1. From the choice of points and neighborhoods above, we can see that the point x has a positive iterate $f^{n_1}(x) \in V_{z_1}$ and the point $f^k(w)$ has a negative iterate $w \in V_{z_1}$. Then by Theorem 2.25, there is a diffeomorphism $f_1 \in \mathcal{U}$ such that f_1 coincides with f outside $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$ and $f^k(w)$ is on the positive orbit of x under f_1.

Step 2. For the diffeomorphism f_1, the point x has a positive iterate $f^k(w) \in V_{z_2}$, and the point y has a negative iterate $f^{-n_3}(y) \in V_{z_2}$. Since f_1 coincides with f outside $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$ and $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cap (U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})) = \emptyset$, then by Theorem 2.25, there is a diffeomorphism $h \in \mathcal{U}$, such that y is on the positive orbit of x under h and h coincides with f_1 outside $U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2})$.

By the constructions above, h coincides with f outside $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cup (U_{z_2} \cup f(U_{z_2}) \cup \cdots \cup f^N(U_{z_2}))$. Hence, h coincides with f on $\mathrm{Orb}(p) \cup \mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)$ and outside U_K. Moreover, $\#(\mathrm{Orb}(x, h) \cap U_K) \geq n_3 - n_2 \geq T_K$ and $\#(\mathrm{Orb}(x, h) \backslash (U_K \cup U_p)) \leq n_1 + n_2 \leq n_0$.

5.2.2. *The periodic case.* Assume that any point $z_2 \in K$ satisfying $z_1 \prec_{U_K} z_2$ is a periodic point with period less than or equal to N. We take such a point $q \in K$. In this case, we cannot use Theorem 2.25 at the point q since its period is small but we can do perturbations at the stable and unstable manifolds of q since it is hyperbolic. To simplify the proof, we assume that q is a hyperbolic fixed point of f, but the general case is identical. We take a neighborhood U_q of q such that $\overline{U_q} \subset U_K \backslash (U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1}))$, and such that for any point w satisfying $\mathrm{Orb}^+(w) \subset U_q$ (respectively, $\mathrm{Orb}^-(w) \subset U_q$), we have $w \in W^s(q)$ (respectively, $w \in W^u(q)$).

Since $z_1 \prec_{U_K} q$, by a similar argument as in Lemma 5.1, we can get that there is a point $x' \in U_q$ such that $z_1 \prec_{U_K} x'$ and $\mathrm{Orb}^+(x) \subset U_q$. By the choice of U_q, we have that $x' \in W^s(q)$ and $x' \notin U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$. Note that since x' is not a periodic point and $z_1 \prec_{U_K} x'$, we have $x' \notin K = \alpha(y)$. Hence, $x' \notin \overline{\mathrm{Orb}(y)}$.

Since $q \in \alpha(y)$, there is $y' \in W^u(q) \cap U_q$, such that for any neighborhood U of y', there is an integer $n \geq 1$, such that $f^{-n}(y) \in U$. (In fact, if $\alpha(y) = \{q\}$, we can choose y' to be a negative iterate of y. If $\{q\} \subsetneqq \alpha(y)$, we can choose y' to be contained in $\alpha(y) \cap W^u(q)$.) Moreover, we have $y' \notin \mathrm{Orb}(x)$.

By the λ-lemma, there are two neighborhoods $W_{x'}$ and $W_{y'}$ of x' and y', respectively, such that, for any two smaller neighborhoods $W'_{x'} \subset W_{x'}$ and $W'_{y'} \subset W_{y'}$ of x' and y', respectively, there is a piece of orbit $(z', f(z'), \ldots, f^t(z'))$ contained in U_q, such that $z' \in W'_{x'}$, $f^t(z') \in f^{-1}(W'_{y'})$, $f^i(z') \notin W'_{x'} \cup W'_{y'}$ for any $i \in \{1, 2, \ldots, t\}$ and $t \geq T_K$.

Now we construct the perturbation domains at the points x' and y', respectively. Recall that $\theta > 1$ and $r_0 > 0$ are the two constants obtained by Lemma 2.27 associated to $\mathcal{U}$ and f.

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Perturbation domain at x'. We can take two neighborhoods $V_{x'} \subset U_{x'}$ of x' that satisfy the conclusions of Theorem 2.25 for the triple $(f, \mathcal{U}, N)$, and also satisfy that:

(i) $U_{x'} \subset W_{x'}$;
(ii) $U_{x'} \cup f(U_{x'}) \cup \cdots, \cup f^N(U_{x'}) \subset U_q$;
(iii) $f^n(x) \notin U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})$, for any $-\infty < n \le n_1$;
(iv) $U_{x'} \cap \mathrm{Orb}(y) = \emptyset$ and $q \notin U_{x'}$.

Then, there is a piece of orbit $(w', \ldots, f^{k'}(w'))$ contained in U_K such that $w' \in V_{z_1}$ and $f^{k'}(w') \in V_{x'}$. Moreover, $w' \notin \mathrm{Orb}(y)$.

Perturbation domain at y'. We take a number $r' < r_0$ small enough, such that if we take the neighborhood $U_{y'} = f(B(f^{-1}(y'), \theta r'))$ of y', then the following properties are satisfied:

(i) $U_{y'} \subset W_{y'}$;
(ii) $U_{y'} \cup f^{-1}(U_{y'}) \subset U_q \backslash (\{q\} \cup U_{x'} \cup f(U_{x'}) \cup \cdots, \cup f^N(U_{x'}))$;
(iii) $U_{y'} \cap f^{-1}(U_{y'}) = \emptyset$;
(iv) $f^n(x) \notin U_{y'} \cup f^{-1}(U_{y'})$, for any $-\infty < n \le n_1$;
(v) $\{w', \ldots, f^{k'}(w')\} \cap (U_{y'} \cup f^{-1}(U_{y'})) = \emptyset$.

Then by the choice of $U_{x'}$ and $U_{y'}$, there is a piece of orbit $(z', f(z'), \ldots, f^{n_4}(z'))$ contained in U_q, such that $z' \in V_{x'}$, $f^{n_4}(z') \in B(f^{-1}(y'), r')$, $f^i(z') \notin U_{x'}$ for any $i \in \{1, 2, \ldots, n_4\}$ and $n_4 \ge T_K$. By the choice of y', there is a negative iterate $f^{-n_5}(y)$ of y contained in $B(f^{-1}(y'), r')$.

Perturbations to get h in the periodic case. From the above constructions, we can see that the perturbation domains are pairwise disjoint and contained in U_K, and the pieces of orbits that connect two perturbation domains are pairwise disjoint and disjoint from the other perturbation domains. Then we can perform the perturbations step by step as in *Case 1*.

Step 1. By Lemma 2.27, there is $f_1 \in \mathcal{U}$ such that f_1 coincides with f outside $f^{-1}(U_{y'})$ and $f_1(f^{n_4}(z')) = f^{-n_5+1}(y)$. Since $f^n(x) \notin U_{y'} \cup f^{-1}(U_{y'})$, for any $-\infty < n \le n_1$, we have that f_1 coincides with f on $\{f^n(x)\}_{-\infty<n\le n_1}$.

Step 2. For the diffeomorphism f_1, the point y has a negative iterate $z' \in V_{x'}$ and the point w' has a positive iterate $f^{k'}(w') \in V_{x'}$. Then by Theorem 2.25, there is $f_2 \in \mathcal{U}$ such that f_2 coincides with f_1 outside $U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})$ and w' is on the negative orbit of y under f_2. Since $f^n(x) \notin U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})$, for any $-\infty < n \le n_1$, we have that f_2 coincides with f on $\{f^n(x)\}_{-\infty<n\le n_1}$.

Step 3. For the diffeomorphism f_2, the point y has a negative iterate w' in V_{z_1} and the point x has a positive iterate $f^{n_1}(x) \in V_{z_1}$. By Theorem 2.25, there is $h \in \mathcal{U}$ such that h coincides with f_2 outside $U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})$ and y is on the positive orbit of x under h.

By the constructions above, the diffeomorphism h coincides with f outside $(U_{z_1} \cup f(U_{z_1}) \cup \cdots \cup f^N(U_{z_1})) \cup (U_{x'} \cup f(U_{x'}) \cup \cdots \cup f^N(U_{x'})) \cup f^{-1}(U_{y'})$. Hence, h coincides with f on $\mathrm{Orb}(p) \cup \mathrm{Orb}^-(x) \cup \mathrm{Orb}^+(y)$ and outside U_K. Moreover, $\#(\mathrm{Orb}(x, h) \cap U_K) \ge n_4 \ge T_K$ and $\#(\mathrm{Orb}(x, h) \backslash (U_K \cup U_p)) \le n_1 + n_2 \le n_0$.

Remark 5.2. We point out that, in the construction, each of the perturbation supports are pairwise disjoint with each other. Since $\mathcal{U} = f \circ \mathcal{V}$, where $\mathcal{V}$ is a C^1 neighborhood of Id

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and satisfies the property (F) in Definition 2.24, the perturbations f_1, f_2 and h are still contained in $\mathcal{U}$.

5.3. *The choice of l and the perturbation domains at x and y, and to get h_m.* From the constructions in §5.2, we get the diffeomorphism h that satisfies the first item of Proposition 1. In this section, we perform the perturbations to get the diffeomorphism h_m.

Assume $h^t(x) = y$. By replacing x and y to a negative or positive iteration, we assume that $x, y \in U_p$ and $\mathrm{Orb}^-(x, h) \cup \mathrm{Orb}^+(y, h) \subset U_p$. Assume that $\#(\{h^n(x)\}_{1\le n\le t} \cap U_p) = m_0$. We take a number $r < r_0$ small enough such that if we take the neighborhood $U_x = h(B(h^{-1}(x), \theta r))$ of x and the neighborhood $U_y = B(y, \theta r)$ of y, then the four sets U_x, $h^{-1}(U_x)$, U_y and $h(U_y)$ are contained in U_p and pairwise disjoint from each other and disjoint with $\{h^n(x)\}_{1\le n\le t}$. By the λ-lemma, there is $l_0 \in \mathbb{N}$ such that, for any $m \ge 1$, there is a piece of orbit $(h(z), h^2(z), \ldots, h^{l_0+m-1}(z))$ contained in U_p such that $h(z) \in B(y, r)$, $h^{l_0+m-1}(z) \in B(h^{-1}(x), r)$ and $h^i(z) \notin U_y \cup h^{-1}(U_x)$ for any $i = 2, 3, \ldots, l_0 + m - 2$. Let $l = l_0 + m_0$.

By Lemma 2.27 and the disjointness of U_y, $f^{-1}(U_x)$ and U_K, there is $h_m \in \mathcal{U}$ such that h_m coincides with h outside $U_y \cup f^{-1}(U_x)$ and $h_m(y) = h^2(z)$, $h_m(f^{l_0+m-1}(z)) = h_m^{l_0+m-1}(y) = x$. Hence, h_m coincides with h on $\mathrm{Orb}(p)$ and outside U_p. Moreover, the point x is a periodic point of h_m, and putting $O = \mathrm{Orb}(x, h_m)$, we have that $O \backslash U_p = \mathrm{Orb}(x, h) \backslash U_p$, and $\#(O \cap U_p) = l_0 + m_0 + m = l + m$.

This finishes the proof of Proposition 1.

Remark 5.3. We point out here that, in the periodic case, we cannot do the same perturbations at x' to connect $f^{k'}(w')$ to $f(z')$ just as at the points y' to connect $f^{n_4}(z')$ and $f^{-n_5+1}(y)$, because the piece of orbit $(w', \ldots, f^{k'}(w'))$ that connects the neighborhoods V_{z_1} and $V_{x'}$ of z_1 and x', respectively, may enter into the neighborhood $U_{x'}$ many times before $f^{k'}(w')$. Thus, if we use the basic perturbation lemma to connect $f^{k'}(w')$ to $f(z')$, the piece of orbit $(w', \ldots, f^{k'}(w'))$ may be modified and it is not clear whether the negative orbit of y can intersect V_{z_1} after such perturbation.

6. *Asymptotic approximation for true orbits: proof of Proposition 2*

In this section, we give the proof of Proposition 2. In fact, the proof is almost the same as the proof of Proposition 10 in [**18**]. Recall that K is an invariant compact set and $\alpha(x) \subset K$. We consider the hyperbolic periodic point $p \notin K$ with $\overline{W^u(p)} \cap K \ne \emptyset$. We assume that for any point $y \in W^u(p, f)$, we have $\omega(y) \backslash K \ne \emptyset$, otherwise there is nothing needed to prove. Also we assume that $x \notin K$ because the other case can be obtained directly if the proposition is true under the assumption $x \notin K$. We take two steps to reach our goal:

(i) we choose a sequence of non-periodic points $(z_n)_{n\ge 0}$ such that

$$z_0 \prec z_1 \prec \cdots \prec K, \quad z_0 \in \overline{W^u(p)} \quad \text{and} \quad z_n \notin \overline{\mathrm{Orb}^-(x)} \quad \text{for any } n \ge 0;$$

(ii) then we perturb at every z_n to connect all of the points together and avoid $\mathrm{Orb}^-(x)$.

In order to prove Proposition 2, we take a decreasing sequence of C^1 neighborhoods $(\mathcal{U}_n)$ of f that satisfies the following properties:

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(i) $\overline{\mathcal{U}_0} \subset \mathcal{U}$;

(ii) the element of $\mathcal{U}_n$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}_n$, where $(\mathcal{V}_n)$ is a decreasing sequence of C^1 neighborhoods of Id that satisfy the property (F) in Definition 2.24, and $\bigcap_n \mathcal{V}_n = \{\mathrm{Id}\}$.

Then we have that $\bigcap_n \mathcal{U}_n = \{f\}$. Theorem 2.25 associates to each pair $(f, \mathcal{U}_k)$ a number N_k.

Recall that $\alpha(x) \subset K$ and we have assumed that $x \notin K$. We need the following three lemmas for the proof of Proposition 2.

LEMMA 6.1. *For any neighborhood W of K, there is a point $z \in (W \cap \overline{W^u(p)})\backslash K$ such that $z \prec_W K$ and $\mathrm{Orb}^+(z) \subset W$. Moreover, $z \notin \overline{\mathrm{Orb}^-(x)}$.*

Proof. The proof is similar to the proof of Lemma 5.1. We assume that W is a small neighborhood of K such that $x \notin \overline{W}$ and $p \notin W$. Take an open neighborhood $V \subset W$ of K such that $\overline{V} \subset W$. Since $\overline{W^u(p)} \cap K \neq \emptyset$, for any $k \geq 1$, there is a point $x_k \in W^u(p)$ and a positive integer $n(k)$ such that $f^{n(k)}(x_k) \in B(K, 1/k)$. For k large, the set $B(K, 1/k)$ is contained in V. We consider the smallest integer $m(k)$ such that the piece of orbit $(f^{m(k)}(x_k), \ldots, f^{n(k)}(x_k))$ is contained in V. By the assumption that $\omega(x_k)\backslash K \neq \emptyset$ for any $k \geq 1$, we can see that $n(k) - m(k)$ goes to infinity as k goes to infinity.

By taking a convergent subsequence if necessary, assume the sequence $\{f^{m(k)}(x_k)\}$ converges to a point $z \in \overline{V}$ and $\{f^{n(k)}(x_k)\}$ converges to a point $z' \in K$. It can be obtained directly that $z \in \overline{W^u(p)}$. By a similar argument as in the proof of Lemma 5.1, we can obtain the fact that $z \notin K$, and $z \prec_W z'$. Then we have $z \prec_W K$ since $z' \in K$. Since $f^{m(k)}(x_k), \ldots, f^{n(k)}(x_k)$ is contained in V and $n(k) - m(k)$ goes to infinity, we have that $\mathrm{Orb}^+(z) \subset W$. Then by the assumption $x \notin \overline{W}$, we have $z \notin \mathrm{Orb}^-(x)$. Hence, $z \notin \overline{\mathrm{Orb}^-(x)}$ because $\overline{\mathrm{Orb}^-(x)} \subset \mathrm{Orb}^-(x) \cup K$. □

LEMMA 6.2. *There are a point $y \in W^u(p)$, a sequence of points $(z_k)_{k\geq 1}$, three sequences of neighborhoods $(U_k)_{k\geq 1}$, $(V_k)_{k\geq 1}$, $(W_k)_{k\geq 0}$ and a sequence of finite segment of orbits $Y_k = (y_k, f(y_k), \ldots, f^{m(k)}(y_k))_{k\geq 0}$, such that:*

(1) *$W_{k+1} \subset W_k$ and $\bigcap_k W_k = K$;*

(2) *Theorem 2.25 can be applied to $z_k \in V_k \subset U_k$ for the triple $(f, \mathcal{U}_k, N_k)$, and $\overline{f^n(U_k)} \subset W_k\backslash W_{k+1}$ for all $0 \leq n \leq N_k$;*

(3) *$\overline{U_k} \cap \mathrm{Orb}^-(x) = \emptyset$, for any $k \geq 1$;*

(4) *$z_k \prec_{W_k} z_{k+1}$ and $z_k \prec_{W_k} K$, for any $k \geq 1$;*

(5) *the points $f^{m(k)}(y_k)$ and y_{k+1} are contained in V_{k+1} for all $k \geq 0$ where $y_0 = y$, and $\mathrm{Orb}^-(y) \cap W_1 = \emptyset$;*

(6) *$Y_k \subset W_k\backslash W_{k+2}$ for all $k \geq 0$ and $Y_k \cap \mathrm{Orb}^-(x) = \emptyset$, for all $k \geq 0$.*

Proof. We build all of the sequences by induction. Set $W_0 = M$. We first choose W_1, z_1, V_1, U_1 and Y_0.

Since all periodic orbits in K are hyperbolic, there is a neighborhood W_1 of K such that there is no periodic points with period less than or equal to N_1 contained in $W_1\backslash K$. Also we can assume that $p \notin \overline{W_1}$. By Lemma 6.1, there is $z_1 \in W_1\backslash K$ such that $z_1 \prec_{W_1} K$, $z_1 \notin \overline{\mathrm{Orb}^-(x)}$ and $z_1 \in \overline{W^u(p)}$. By the choice of W_1 and by the fact $z_1 \in W_1\backslash K$, there is

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a neighborhood U_1 of z_1 that is disjoint from its N_1 first iterates and $\overline{f^n(U_1)} \subset W_1 \backslash K$ for any $0 \le n \le N_1$. Moreover, because $z_1 \notin \overline{\mathrm{Orb}^-(x)}$, we can assume that $\overline{U_1} \cap (\mathrm{Orb}^-(x) \cup K) = \emptyset$. By Theorem 2.25, there is $V_1 \subset U_1$ associated to $(f, \mathcal{U}_1, N_1)$. Then there are a point $y \in W^u(p) \backslash W_1$ and a positive integer $m(0)$ such that $f^{m(0)}(y) \in V_1$. Moreover, by considering a negative iterate of y if necessary, we can assume that $\mathrm{Orb}^-(y) \cap W_1 = \emptyset$. We take $y_0 = y$ and $Y_0 = (y, f(y), \ldots, f^{m(0)}(y))$. To sum up, we have obtained W_1, z_1, V_1, U_1 and Y_0.

Now we construct the sequences by induction on k. After W_k, z_k, V_k, U_k and Y_{k-1} have been built, there is $W_{k+1} \subset W_k$ such that:

(i) there is no periodic point with period less than or equal to N_{k+1} contained in $\overline{W_{k+1} \backslash K}$;
(ii) $\overline{f^n(U_k)} \cap W_{k+1} = \emptyset$, for all $1 \le n \le N_k$;
(iii) $W_{k+1} \cap Y_{k-1} = \emptyset$;
(iv) W_{k+1} is contained in $B(K, 1/k)$.

By Lemma 2.13, there is $z_{k+1} \in W_{k+1} \backslash K$, such that $z_k \prec_{W_k} z_{k+1} \prec_{W_{k+1}} K$ and $\mathrm{Orb}^+(z_{k+1}) \subset W_{k+1}$. Since $x \notin W_{k+1}$, we have that $z_{k+1} \notin \overline{\mathrm{Orb}^-(x)}$. Moreover, by the fact that $z_{k+1} \notin K$ and $\alpha(x) \subset K$, we have that $z_{k+1} \notin \mathrm{Orb}^-(x)$. By Theorem 2.25, there are neighborhoods $V_{k+1} \subset U_{k+1}$ of z_{k+1} associated to $(f, \mathcal{U}_{k+1}, N_{k+1})$, such that:

(i) $\overline{U_{k+1}} \cap (\mathrm{Orb}^-(x) \cup K) = \emptyset$;
(ii) $\overline{f^n(U_{k+1})} \subset W_{k+1}$ for all $0 \le n \le N_{k+1}$.

Then there is $Y_k = (y_k, f(y_k), \ldots, f^{m(k)}(y_k))$ such that $y_k \in V_k$ and $f^{m(k)}(y_k) \in V_{k+1}$. Since $\overline{U_k} \cap \mathrm{Orb}^-(x) = \emptyset$, we have that $y_k \notin \mathrm{Orb}^-(x)$, and hence Y_k is disjoint from $\mathrm{Orb}^-(x)$.

Then we finish the proof of Lemma 6.2. □

Remark 6.3. We point out here that, in Lemma 6.2, there is an open set V containing $\mathrm{Orb}^-(x)$ such that $V \cap U_k = \emptyset$ for all $k \ge 1$. In fact, since $x \notin K$, for any integer $n \in \mathbb{N}$, there is n_k such that $f^{-n}(x) \notin W_{n_k}$. By the fact that $U_k \subset W_k$ and $\overline{U_k} \cap \mathrm{Orb}^-(x) = \emptyset$, for any $n \ge 0$, there is an open neighborhood B_n of $f^{-n}(x)$ such that $B_n \cap U_k = \emptyset$ for any $k \ge 1$. Then we take $V = \bigcup_{n \ge 0} B_n$.

Now we fix the point $y \in W^u(p)$ and the sequences $(z_k)_{k \ge 1}$, $(U_k)_{k \ge 1}$, $(V_k)_{k \ge 1}$, $(W_k)_{k \ge 0}$ and $(Y_k)_{k \ge 0}$ as in Lemma 6.2. We have the following lemma.

LEMMA 6.4. *There are a sequence of perturbations $(g_k)_{k \ge 0}$ of f and a strictly increasing sequence of integers $(n_k)_{k \ge 0}$, such that:*

(1) *$g_0 = f$ and $n_0 = 0$;*
(2) *there is $\phi_k \in \mathcal{V}_k$, such that $\phi_k = \mathrm{Id}$ outside $U_k \cup \cdots \cup f^{N_k - 1}(U_k)$ and $g_k = g_{k-1} \circ \phi_k$, for $k \ge 1$;*
(3) *for any $l = \{0, 1, \ldots, k-1\}$, the piece of orbit $(g_k^{n_l}(y), g_k^{n_l+1}(y), \ldots, g_k^{n_{l+1}}(y))$ is contained in $W_l \backslash W_{l+2}$.*

Proof. We build inductively the sequences (g_k) and (n_k) and another sequence of integers $(m_k)_{k \ge 0}$ which satisfy the conclusions and also the following properties:

(i) $m_k > n_k$ and $g_k^{m_k}(y) \in V_{k+1}$;
(ii) the piece of orbit $(g_k^{n_k}(y), g_k^{n_k+1}(y), \ldots, g_k^{m_k}(y))$ is contained in $W_k \backslash W_{k+2}$.

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First, we take $g_0 = f$ and $n_0 = 0$. By Lemma 6.2, there is $m_0 > 0$ such that $g_0^{m_0}(y) \in V_1$ and the piece of orbit $(y = g_0^{n_0}(y), g_0(y), \ldots, g_0^{m_0}(y))$ is contained in $W_0 \backslash W_2$.

Now assume that g_k, n_k and m_k have been built, we explain how to get g_{k+1}, n_{k+1} and m_{k+1}.

The forward orbit of $g_k^{n_k}(y)$ has a positive iterate $g_k^{m_k}(y) \in V_{k+1}$, and the backward orbit of $f^{m(k+1)}(y_{k+1})$ has a negative iterate $y_{k+1} \in V_{k+1}$. Moreover, these segments of orbit are contained in $W_k \backslash W_{k+3}$. Since g_k coincides with f on the set $U_{k+1} \cup f(U_{k+1}) \cup \cdots \cup f^{N_{k+1}}(U_{k+1})$, one can apply Theorem 2.25 to $(g_k, \mathcal{U}_{k+1}, V_{k+1}, U_{k+1})$ and get a diffeomorphism g_{k+1}. The new diffeomorphism g_{k+1} is of the form $g_k \circ \phi_{k+1}$, where $\phi_{k+1} = \mathrm{Id}$ outside $U_{k+1} \cup \cdots \cup f^{N_{k+1}-1}(U_{k+1})$ and $f \circ \phi_{k+1} \in \mathcal{U}_{k+1}$, thus $\phi_{k+1} \in \mathcal{V}_{k+1}$.

Since $g_{k+1} = g_k$ outside $U_{k+1} \cup \cdots \cup f^{N_{k+1}-1}(U_{k+1})$, the piece of orbit $(y, g_k(y), \ldots, g_k^{n_k}(y))$ under g_k coincides with the piece of orbit $(y, g_{k+1}(y), \ldots, g_{k+1}^{n_k}(y))$. By the new diffeomorphism g_{k+1}, the forward orbit of $g_{k+1}^{n_k}(y)$ has an iterate $f^{m(k+1)}(y_{k+1})$ under g_{k+1} contained in V_{k+2}. That is to say, there is an integer $m_{k+1} > n_k$, such that $f^{m(k+1)}(y_{k+1}) = g_{k+1}^{m_{k+1}}(y)$. Moreover, there exists an integer n_{k+1} with $n_k < n_{k+1} < m_{k+1}$, such that:

(i) the piece of orbit $(g_{k+1}^{n_k}(y), \ldots, g_{k+1}^{n_{k+1}}(y))$ is contained in the union of $\{g_k^{n_k}(y), \ldots, g_k^{m_k}(y)\}$ and $U_{k+1} \cup \cdots \cup f^{N_{k+1}-1}(U_{k+1})$, hence it is contained in $W_k \backslash W_{k+2}$;

(ii) the piece of orbit $(g_{k+1}^{n_{k+1}}(y), \ldots, g_{k+1}^{m_{k+1}}(y))$ is contained in the union of $U_{k+1} \cup \cdots \cup f^{N_{k+1}-1}(U_{k+1})$ and $\{y_{k+1}, \ldots, f^{m(k+1)}(y_{k+1})\}$, hence it is contained in $W_{k+1} \backslash W_{k+3}$.

Then the conclusions are satisfied for $k+1$. This ends the proof of Lemma 6.4. □

End of the proof of Proposition 2. Recall that the supports $U_i \cup \cdots \cup f^{N_i-1}(U_i)$ and $U_j \cup \cdots \cup f^{N_j-1}(U_j)$ of the perturbations ϕ_i and ϕ_j are disjoint for any $i \neq j$, and $(\mathcal{V}_n)$ satisfy the property (F). Then the sequence $g_k = f \circ \phi_1 \circ \cdots \circ \phi_k$ converges to a diffeomorphism $g \in \mathcal{U}_0 \subset \mathcal{U}$. By the constructions, g coincides with $f \circ \phi_k$ in the set $U_k \cup \cdots \cup f^{N_k-1}(U_k)$ and with f elsewhere. We take V to be the neighborhood of $\mathrm{Orb}^-(x)$ in Remark 6.3, then it holds that g coincides with f on the set $\mathrm{Orb}(p) \cup K \cup V \cup \mathrm{Orb}^-(y)$ and $\omega(y, g) \subset K$. Since $\mathrm{Orb}^-(x) \subset V$, we have that Dg coincides with Df on $\mathrm{Orb}^-(x)$. Moreover, since g is the limit of the sequence (g_k), by Lemma 6.4, for any $n > n_k$, $g^n(y) \in W_k$. Then we have that $\omega(y, g) \subset K$. This finishes the proof of Proposition 2. □

7. *Asymptotic approximation for pseudo-orbits: proof of Proposition 3*

To prove Proposition 3, we use the techniques of [**8**, **18**] to get true orbits by perturbing a pseudo-orbit. Similarly to the proof of Proposition 2, we have to perturb infinitely many times in a special neighborhood to keep some part of the initial dynamic unchanged. The proof refers a lot to [**8**] and [**20**, §3.2].

We take several steps. First, we choose an open set that covers all positive orbits of X that are not on the local stable manifold of periodic orbits with small periods. Actually, we choose a special topological tower of X. Second, we construct a sequence of disjoint perturbation domains containing in their interior the special topological tower. Then, we

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choose an infinitely long pseudo-orbit in X that goes from z to K, has jumps only in the tiles of the perturbation domains and accumulates to K in the future. Finally, we perturb in the perturbation domains to construct a true orbit which goes from z to K and accumulates to K in the future.

We take a C^1 neighborhood $\mathcal{U}_0$ of the diffeomorphism f_0 with $\overline{\mathcal{U}_0} \subset \mathcal{U}$ such that the element of $\mathcal{U}_0$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}_0$, where $\mathcal{V}_0$ is a C^1 neighborhood of Id that satisfies the property (F) in Definition 2.24. Then there is a smaller C^1 neighborhood $\mathcal{U}' \subset \mathcal{U}_0$ of f_0 and an integer N_0 associated to $(f_0, \mathcal{U}_0)$ by Theorem 2.26. Take the integer $T = 10\kappa_d d N_0$ where the integer $\kappa_d \geq 1$ is the number given by Lemma 2.36.

From now on, we fix the C^1 neighborhoods $\mathcal{U}' \subset \mathcal{U}_0$ of f_0 and the integer T. Consider a diffeomorphism $f \in \mathcal{U}'$, an invariant compact set K, a positive invariant compact set X and a point z, satisfying the following properties:

(i) all periodic points contained in K are hyperbolic;
(ii) all periodic points contained in X with period less than or equal to T are hyperbolic;
(iii) for any $\varepsilon > 0$, there is an ε-pseudo-orbit contained in X that connects from z to K.

Also we fix a neighborhood U of $X \backslash K$.

We take a decreasing sequence of C^1 neighborhoods $(\mathcal{U}_n)_{n\geq 1}$ of f such that $\mathcal{U}_1 \subset \mathcal{U}_0$ and $\bigcap_n \mathcal{U}_n = \{f\}$. Moreover, the element of $\mathcal{U}_n$ is of the form $f \circ \phi$ with $\phi \in \mathcal{V}_n$, where $(\mathcal{V}_n)$ is a decreasing sequence of C^1 neighborhoods of Id that satisfy the property (F) in Definition 2.24. For any $k \geq 1$, Theorem 2.25 associates to each pair $(f, \mathcal{U}_k)$ an integer N_k. We can assume that $(N_k)_{k\geq 0}$ is an increasing sequence. We assume also that $z \notin K$, otherwise there is nothing to prove. For an integer N, denote by $\mathrm{Per}_N(f)$ the set of periodic points of f whose period is no more than N. Fix a small number $\gamma > 0$ (to determine the C^0 distance between the new created diffeomorphism and f).

7.1. *Choice of topological towers.* In this section, we construct a family of special topological towers for the set X with the properties stated in Lemma 7.1.

LEMMA 7.1. *For any $\delta > 0$, for any decreasing sequence of positive constants $(\gamma_n)_{n\geq 0}$, and for any increasing sequence of integers $(L_k)_{k\geq 0}$ where $L_0 = 10dN_0$, there are a decreasing sequence of neighborhoods $(U_k)_{k\geq 0}$ of K, a sequence of open sets $(W_k)_{k\geq 0}$, and a sequence of compact sets $(D_k)_{k\geq 0}$, such that, putting $X_k = X \cap \overline{(U_k \backslash U_{k+1})}$ for all $k \geq 0$, the following properties are satisfied.*

(i) *We have $U_0 = M$, $z \notin \overline{U_1}$, and $\bigcap_{k\geq 0} U_k = K$.*
(ii) *For any $k \geq 0$:*
  (1) *there is no periodic orbit with period less than $\kappa_d^2 L_{k+1}$ contained in $\overline{U_{k+1}} \backslash K$;*
  (2) *$f^i(\overline{U_{k+1}}) \subset U_k$, for all $-4\kappa_d^2 L_{k+1} \leq i \leq 4\kappa_d^2 L_{k+1}$;*
  (3) *the L_k sets $\overline{W_k}$, $f(\overline{W_k}), \ldots, f^{L_k-1}(\overline{W_k})$ are pairwise disjoint, contained in $U \backslash K$, and also contained in $U_{k-1} \backslash \overline{U_{k+2}}$, where we put $U_{-1} = U_0 = M$;*
  (4) *for any $-4\kappa_d^2 L_{k+2} \leq i \leq 4\kappa_d^2 L_{k+2}$, we have that $f^i(\overline{U_{k+2}}) \cap \overline{(W_0 \cup \cdots \cup W_k)} = \emptyset$;*
  (5) *for any $l < k$, any $0 \leq i \leq L_l$ and any $0 \leq j \leq L_k$, we have that $\overline{f^i(W_l)} \cap \overline{f^j(W_k)} = \emptyset$;*

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(6) *the set D_k is contained in W_k such that any point in $X_0\backslash(\bigcup_{p\in \mathrm{Per}_{L_0}(f)} W^s_\delta(p))$ has a positive iterate in* $\mathrm{Int}(D_0)$*, and for $k \geq 1$, any point in X_k has a positive iterate in* $\mathrm{Int}(D_k \cup D_{k-1})$*;*

(7) *the diameter of every connected component of W_k is smaller than γ_k.*

Remark 7.2. The set $W_0 \cup \cdots \cup W_k$ can be seen as a special topological tower for $X_0 \cup \cdots \cup X_k$, from the items (3), (5) and (6).

Proof. We build inductively the sequences $(U_k)_{k\geq 0}$, $(W_k)_{k\geq 0}$ and $(D_k)_{k\geq 0}$ from a sequence of open sets $(W'_k)_{k\geq 0}$ and a sequence of compact sets $(D'_k)_{k\geq 0}$, which satisfy the following additional properties, for any $k \geq 0$:

(1′) the set W_k is contained in a small neighborhood of W'_k, and $W'_k \subset W_k$;

(2′) the sets $\overline{W'_k}$, $f(\overline{W'_k}), \ldots, f^{L_k-1}(\overline{W'_k})$ are pairwise disjoint, contained in $U\backslash K$, and also contained in $U_{k-1}\backslash\overline{U_{k+2}}$;

(3′) for all $-4\kappa_d^2 L_{k+2} \leq i \leq 4\kappa_d^2 L_{k+2}$, we have $f^i(\overline{U_{k+2}}) \cap \overline{(W'_0 \cup \cdots \cup W'_k)} = \emptyset$;

(4′) for any $l < k$, any $0 \leq i \leq L_l$ and any $0 \leq j \leq L_k$, we have $\overline{f^i(W_l)} \cap \overline{f^j(W'_k)} = \emptyset$;

(5′) $D_k = (D'_k \cup D'_{k+1}) \cap W_k$, where $D'_0 \subset W'_0$ and $D'_{k+1} \subset W_k \cup W'_{k+1}$;

(6′) any point in $X_0\backslash(\bigcup_{p\in \mathrm{Per}_{L_0}(f)} W^s_\delta(p))$ has a positive iterate in $\mathrm{Int}(D'_0)$ and any point in X_k has a positive iterate contained in $\mathrm{Int}(D'_k)$ for any $k \geq 1$;

(7′) the diameter of every connected component of W'_k is smaller than $\gamma_k/2$.

Put $U_{-1} = U_0 = M$. We construct inductively the sets U_{k+1}, W'_k, D'_k, W_{k-1} and D_{k-1}.

The sets U_1, W'_0 and D'_0: the case where $k = 0$. By the assumption of hyperbolicity of periodic orbits in K, we can take a neighborhood $U_1 \subset B(K, 1)$ of K such that $z \notin \overline{U_1}$ and there is no periodic orbit with period less than $\kappa_d^2 L_1$ in $\overline{U_1}\backslash K$. Note that $U_0 = M$. The properties (1) and (2) are satisfied.

Recall that X contains no non-hyperbolic periodic orbit with period less than T, where $T = 10\kappa_d d N_0 = \kappa_d L_0$. Hence, X_0 contains no non-hyperbolic periodic orbit with period less than $\kappa_d L_0$. By Lemma 2.36, there are an open set $W'_0 \subset U$ whose closure $\overline{W'_0}$ is a compact d-dimensional sub-manifold with boundary and a compact set $D'_0 \subset W'_0$, such that $\overline{W'_0}$ is disjoint from its L_0 iterates, which is the first property of item (2′). Moreover, any point in $X_0\backslash(\bigcup_{p\in \mathrm{Per}_{L_0}(f)} W^s_\delta(p))$ has a positive iterate contained in $\mathrm{Int}(D'_0)$ and, hence, the item (6′) is satisfied. By Lemma 2.36(4), the set $\overline{W'_0}$ is contained in a small neighborhood of $X_0 \cup f(X_0) \cup \cdots \cup f^{\kappa_d L_0}(X_0)$. Moreover, we can choose W'_0 such that the diameter of every connected component of W'_0 is smaller than $\gamma_0/2$, which is the item (7′). Hence, we can assume that $\bigcup_{i=0}^{L_0} f^i(\overline{W'_0}) \subset U\backslash K$, since X is positively invariant. To make the sequences complete, we could put $D'_{-1} = D_{-1} = W_{-1} = \emptyset$. Note that we do not need to check other items for the case $k = 0$.

The sets U_{k+2}, W'_{k+1}, D'_{k+1}, W_k and D_k. Assume that U_{j+1}, W'_j, D'_j, W_{j-1} and D_{j-1} have been constructed for any $0 \leq j \leq k$. Now we build the sets U_{k+2}, W'_{k+1}, D'_{k+1}, W_k and D_k.

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We take a neighborhood $U_{k+2} \subset U_{k+1} \cap B(K, 1/(k+2))$ of K such that:

(i) there is no periodic orbit with periodic less than $\kappa_d^2 L_{k+2}$ in $\overline{U_{k+2}} \backslash K$, which is the property (1);
(ii) $f^i(\overline{U_{k+2}}) \subset U_{k+1}$, for all $-4\kappa_d^2 L_{k+2} \le i \le 4\kappa_d^2 L_{k+2}$, which is the property (2);
(iii) $\overline{W_k'} \cap f^i(\overline{U_{k+2}}) = \emptyset$ for all $-4\kappa_d^2 L_{k+2} \le i \le 4\kappa_d^2 L_{k+2}$, which implies the item $(3')$ and the last property of item $(2')$.

Consider the compact set $X_{k+1} = X \cap \overline{U_{k+1} \backslash U_{k+2}}$. Note that X_{k+1} contains no periodic orbit of period less than or equal to $\kappa_d^2 L_{k+1}$. By Lemma 2.36, there is an open set V_{k+1}' whose closure $\overline{V_{k+1}'}$ is a compact d-dimensional sub-manifold with boundary such that any point in X_{k+1} has a positive iterate contained in V_{k+1}'. Moreover, the set $\overline{V_{k+1}'}$ is disjoint from its first $\kappa_d L_{k+1}$ first iterates and $\overline{V_{k+1}'}$ is contained in a small neighborhood of $\bigcup_{i=0}^{\kappa_d^2 L_{k+1}} f^i(X_{k+1})$. By taking this neighborhood as small, we can assume that V_{k+1}' satisfies the following properties.

(i) Since $f^i(\overline{U_{k+1}}) \subset U_k$ for any $-4\kappa_d^2 L_{k+1} \le i \le 4\kappa_d^2 L_{k+1}$ and $X_{k+1} \subset \overline{U_{k+1}}$, we have that $f^i(\overline{V_{k+1}'}) \subset U_k$, for any $-2\kappa_d^2 L_{k+1} \le i \le 2\kappa_d^2 L_{k+1}$.
(ii) Since $f^i(\overline{U_{k+1}}) \cap (\overline{W_0 \cup \cdots \cup W_{k-1}}) = \emptyset$ for any $-4\kappa_d^2 L_{k+1} \le i \le 4\kappa_d^2 L_{k+1}$, we have that $f^i(\overline{V_{k+1}'}) \cap (\overline{W_0 \cup \cdots \cup W_{k-1}} \cup K) = \emptyset$ for any $-2\kappa_d^2 L_{k+1} \le i \le 2\kappa_d^2 L_{k+1}$.

Moreover, we can choose V_{k+1}' such that the diameter of all of its connected components is small enough such that all of the ith iterates of every connected component of V_{k+1}' is of diameter smaller than $\gamma_{k+1}/3$, for any $0 \le i \le \kappa_d L_{k+1}$.

Recall that $\overline{W_k'}$ and $\overline{V_{k+1}'}$ are two compact d-dimensional sub-manifolds with boundary and $\overline{V_{k+1}'}$ is disjoint from its first $\kappa_d L_{k+1}$ first iterates. By Lemma 2.34, considering $\overline{W_k'}$ and $\overline{V_{k+1}'}$ as W' and V', and considering the integer L_{k+1} as the integer T, there is an open set $S_k = W_k \cup V_{k+1}$ satisfying the following properties:

(i) $\overline{W_k}$ and $\overline{V_{k+1}}$ are two compact d-dimensional sub-manifolds with boundary;
(ii) $W_k \cup V_{k+1} \subset U \backslash K$;
(iii) W_k is a small neighborhood of W_k' and hence $\overline{W_k}$ is disjoint with its first L_k iterates; moreover, the properties $2'$, $3'$, $4'$ of W_k' implies the properties of (3), (4) and (5) of W_k; the property $(1')$ is automatically satisfied;
(iv) $V_{k+1}' \subset \bigcup_{i=0}^{\kappa_d L_{k+1}} f^{-i}(S_k)$;
(v) $\overline{W_k} \cap f^i(\overline{V_{k+1}}) = \emptyset$ for all $i = 0, \pm 1, \ldots, \pm L_{k+1}$;
(vi) $\overline{V_{k+1}}$ is contained in a small neighborhood of $V_{k+1}' \cup f(V_{k+1}') \cup \cdots \cup f^{\kappa_d L_{k+1}}(V_{k+1}')$ and disjoint from its L_{k+1} iterates; thus, we can assume that $K \cap \overline{V_{k+1}} = \emptyset$ and, for all $-\kappa_d L_{k+1} \le i \le \kappa_d L_{k+1}$, we have $f^i(\overline{V_{k+1}})) \cap \overline{(W_0 \cup \cdots \cup W_{k-1})} = \emptyset$ and $f^i(\overline{V_{k+1}})) \subset U_k$.

Moreover, by the assumption of the diameter of every connected component of W_k' and V_{k+1}', we can take W_k and V_{k+1} such that every connected component of W_k is of diameter less than γ_k and every connected component of V_{k+1} is of diameter less than $\gamma_k/2$. Then the item (7) is satisfied.

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By the fact that any point in X_{k+1} has a positive iterate contained in V'_{k+1}, and $V'_{k+1} \subset \bigcup_{i=0}^{\kappa_d L_{k+1}} f^{-i}(V_{k+1})$, one can see that any point in X_{k+1} has a positive iterate contained in S_k. By the compactness of X_{k+1}, there is a compact set $D'_{k+1} \subset S_k$ such that all such iterates are contained in $\mathrm{Int}(D'_{k+1})$. Put $W'_{k+1} = V_{k+1}$ and $D_k = (D'_k \cup D'_{k+1}) \cap W_k$. Then we have $D'_{k+1} \subset S_k = W_k \cup W'_{k+1}$. From the construction of V_{k+1}, we can see that W'_{k+1} and D'_{k+1} satisfy the properties (6), (2′)–(7′).

This finishes the construction of the sets U_{k+2}, W'_{k+1}, D'_{k+1} W_k and D_k.

Note that $\bigcap_{k\geq 0} U_k = K$ and $z \notin \overline{U_1}$ are obviously satisfied by the choice of U_k. This finishes the proof of Lemma 7.1. □

7.2. *Construction of perturbation domains.* We take $L_k = 10dN_k$ for all $k \geq 0$, and take a small number $\delta > 0$ such that for any two different hyperbolic periodic points $q_1, q_2 \in \mathrm{Per}_{N_0}(f) \cap X$, we have $W^{\sigma_1}_\delta(q_1) \cap W^{\sigma_2}_\delta(q_2) = \emptyset$, where $\sigma_i \in \{u, s\}$. Take a decreasing sequence of positive constants $(\gamma_k)_{k\geq 0}$ such that for any subset of M whose diameter is smaller than γ_k, then its ith iterate is of diameter smaller than γ for any $0 \leq i \leq L_k$. By Lemma 7.1, we get the sequences $(U_k)_{k\geq 0}$, $(W_k)_{k\geq 0}$ and $(D_k)_{k\geq 0}$. We still use the notation $X_k = X \cap \overline{(U_k \backslash U_{k+1})}$ for all $k \geq 0$.

Now we build the perturbation domains for the family (X_k). The techniques are mainly from [**8**, §§4.1 and 4.2]. First, we build the perturbation domains that covers the points which are not on the local stable manifolds of periodic orbits with period less than or equal to N_0. The proof is essentially due to [**8**, Corollaire 4.1]. They deal with a family of perturbation domains with the same order, thus the union forms a perturbation domain. Here we have a sequence of perturbation domains with different orders, however, the construction of each perturbation domain can be separated.

LEMMA 7.3. *There is a perturbation domain B_k of order N_k for $(f, \mathcal{U}_k)$ for each $k \geq 0$, such that the sequence $(B_k)_{k\geq 0}$ satisfies the following properties.*

(1) *The supports of the perturbations domains B_k are pairwise disjoint, contained in U, and also contained in $U_{k-1} \backslash \overline{U_{k+2}}$.*

(2) *Any point of $X_0 \backslash (\bigcup_{p\in \mathrm{Per}_{N_0}(f)} W^s_\delta(p))$ has a positive iterate in the interior of one tile of the perturbation domain B_0 and any point of X_k has a positive iterate in the interior of one tile of the perturbation domain $B_{k-1} \cup B_k$ for $k \geq 1$.*

As a consequence, for any $k \geq 0$, there is a finite family of tiles $\mathcal{C}_k$ associated to B_k and a family of compact sets $\mathcal{D}_k$ contained in the interior of tiles of $\mathcal{C}_k$ such that:

(i) *each tile of $\mathcal{C}_k$ contains exactly one element of $\mathcal{D}_k$, for all $k \geq 0$ and each element of $\mathcal{D}_k$ is contained in a tile of $\mathcal{C}_k$;*

(ii) *any point of $X_0 \backslash (\bigcup_{p\in \mathrm{Per}_{N_0}(f)} W^s_\delta(p))$ has a positive iterate in the interior of one element of $\mathcal{D}_0$ and any point of X_k has a positive iterate in the interior of one element of $\mathcal{D}_{k-1} \cup \mathcal{D}_k$ for $k \geq 1$.*

Moreover, the diameter of any connected component of B_k is smaller than γ.

Proof. Consider the sequence of open sets $(W_k)_{k\geq 0}$ and the sequence of compact sets $(D_k)_{k\geq 0}$ obtained by Lemma 7.1. Moreover, by Lemma 7.1(7), the diameters of

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components of each W_k can be chosen small enough such that all their first L_k iterates are contained in a perturbation domain of order L_k by Theorem 2.32.

Assume that W is a component of W_k, and put $D = D_k \cap W$. By assumption, W is contained in a chart of perturbation $\varphi : W \to \mathbb{R}^d$. We can tile W with tiles of proper size such that any cube that intersects $\varphi(D)$ is contained in $\varphi(W)$. We do the same thing for all other components of W_k that has non-empty intersection with D_k and we get a finite family $\mathcal{P}_0$ of perturbation domains, each of them being an open set, pairwise disjoint, contained in W_k, and the union of their closure contains D_k in its interior. Denote by Φ_0 the family of perturbation charts in the construction of $\mathcal{P}_0$.

Repeat the construction for $f^{2iN_k}(W_k)$ and $f^{2iN_k}(D_k)$, $i \in \{1, \ldots, 5d-1\}$, and we get the families $\mathcal{P}_i$ of perturbation domains contained in $f^{2iN_k}(W_k)$, pairwise disjoint and the union of their closure contains $f^{2iN_k}(D_k)$ in its interior. Denote by Φ_i the family of perturbation charts corresponding to $\mathcal{P}_i$. Consider the family $f^{-2iN_k}(\mathcal{P}_i)$ contained in W_k. The union of the closure of all cubes of $f^{-2iN_k}(\mathcal{P}_i)$ contains D_k in its interior. By a C^1 small perturbation of Φ_i, we can suppose that a point in D_k can only be contained on the boundary of at most d different cubes of all cubes contained in† $\bigcup_{i=0}^{5d-1} f^{-2iN_k}(\mathcal{P}_i)$. Since there are at least $5d$ families of cubes, we get that any point of D_k is contained in the interior of at least $4d$ families of such cubes.

We replace every cube in $\mathbb{R}^d$ by another one with the same center and homothetic with rate $\rho < 1$ close to 1. Then we get the families $\mathcal{P}_{i,\rho}$ of perturbation domains whose closures are pairwise disjoint. If we choose ρ close enough to 1, then any point of D_k is still contained in the interior of a cube of at least $4d$ families of $(f^{-2dN_k}(\mathcal{P}_{i,\rho}))_{0\le k\le 5d-1}$. By the compactness of D_k, for each i, there is a finite family Γ_i of tiles of the domains $f^{-2iN_k}(\mathcal{P}_{i,\rho})$, such that the union Σ_i of the tiles of Γ_i satisfies that any point of D_k is contained in the interior of at least $4d$ compact $(f^{-2iN_k}(\Sigma_i))_{0\le k\le 5d-1}$.

By another C^1 small perturbation of Φ_i, we can suppose that any point of D_k is contained on the boundary of the tiles of at most d families of $(f^{-2iN_k}(\Gamma_i))_{0\le k\le 5d-1}$. Any point is contained in at least $4d$ families of tiles, hence any point is contained in the interior of at least one of these tiles. Define B_k and $\mathcal{C}_k$ to be the union of the families $\mathcal{P}_{i,\rho}$ and the union of the families Γ_i, respectively.

Then the compact set D_k is covered by the interior of the tiles of the family $f^{-2iN_k}(\Gamma_i)$. We can take all of the components of the intersection of $f^{2iN_k}(D_k)$ and the elements of the family Γ_i, and this is the family $\mathcal{D}_k$.

Finally, by the assumption that $L_k = 10dN_k$ and the choice of W_k in Lemma 7.1, the supports of perturbation domains $(B_k)_{k\ge 0}$ are pairwise disjoint and are contained in U. Also, the support of the perturbation domain B_k is also contained in $U_{k-1}\setminus\overline{U_{k+2}}$. By the choice of the sequence $(\gamma_k)_{k\ge 0}$, one can see that the diameter of any connected component of B_k is smaller than γ. This finishes the proof of Lemma 7.3. □

We also have to construct perturbation domains that cover the stable and unstable manifolds of periodic orbits contained in $X_0 \cap \mathrm{Per}_{N_0}(f)$. By the assumption of

† In [**8**], the sets of $\bigcup_{i=0}^{5d-1} f^{-2iN_k}(\mathcal{P}_i)$ on general position. We do not introduce this definition in our paper. The reader can refer to [**8**, §3.3] for more details.

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hyperbolicity of periodic orbits, $X_0 \cap \mathrm{Per}_{N_0}(f)$ is a finite set. By [**8**, Proposition 4.2], we can construct in the following way.

LEMMA 7.4. (Bonatti and Crovisier [**8**, Proposition 4.2]) *For any periodic orbit $Q \subset X \cap \mathrm{Per}_{N_0}(f)$, any neighborhood V of Q, there are a neighborhood W of Q, two perturbation domains B_s and B_u of order N_0 for $(f, \mathcal{U}_0)$, two finite families of tiles $\mathcal{C}_s$ and $\mathcal{C}_u$ associated to B_s and B_u, respectively, two finite families of compact sets $\mathcal{D}_s$ and $\mathcal{D}_u$, and an integer $n_0(Q)$, such that:*

(1) *V contains $\overline{W}$ and $\bigcup_{0\le i\le N_0-1} f^i(B_s \cup B_u)$;*
(2) *$f^i(B_s) \cap f^j(B_u) = \emptyset$ for all $0 \le i,\ j \le N_0 - 1$;*
(3) *each element of $\mathcal{D}_s$ is contained in the interior of an element of $\mathcal{C}_s$, and each element of $\mathcal{D}_u$ is contained in the interior of an element of $\mathcal{C}_u$; moreover, each tile of $\mathcal{C}_s$ and $\mathcal{C}_u$ contains exactly an element of $\mathcal{D}_s \cup \mathcal{D}_u$;*
(4) *for any two pairs $D_s \in \mathcal{D}_s$ and $D_u \in \mathcal{D}_u$, there is $n \in \{0, \dots, n_0(Q)\}$, such that $f^n(D_s) \cap D_u \ne \emptyset$;*
(5) *for any point $z \in W \setminus W^s_{\mathrm{loc}}(Q)$, there is $n > 0$ and $D \in \mathcal{D}_u$, such that $f^n(z) \in \mathrm{Int}(D)$ and $f^i(z) \in V$ for all $0 \le i \le n$; moreover, if $f(z) \notin W$, then $n \le n_0(Q)$;*
(6) *for any point $z \in W \setminus W^u_{\mathrm{loc}}(Q)$, there is $n > 0$ and $D \in \mathcal{D}_s$, such that $f^{-n}(z) \in \mathrm{Int}(D)$ and $f^{-i}(z) \in V$ for all $0 \le i \le n$; moreover, if $f^{-1}(z) \notin W$, then $n \le n_0(Q)$.*

Moreover, the diameter of any connected component of B_s and B_u is smaller than γ.

7.3. *Choice of a pseudo-orbit.* By Lemma 7.3, we have the sequences of perturbation domains $(B_k)_{k\ge0}$, tiles $(\mathcal{C}_k)_{k\ge0}$ and families of compact sets $(\mathcal{D}_k)_{k\ge0}$. Note that there are only finitely many periodic orbits contained in $\mathrm{Per}_{N_0}(f) \cap X$, and they are all outside $\overline{U_1}$. Hence, for each periodic orbit Q contained in X with period no more than N_0, we can take an open neighborhood $V(Q) \subset U$ that are pairwise disjoint, disjoint from $\overline{U_1}$ and disjoint from $f^i(B_k)$ for any $0 \le i \le N_k - 1$ and any $k \ge 0$. By Lemma 7.4, we have for each Q the open set $W(Q)$, the perturbation domains $B_s(Q)$ and $B_u(Q)$, the families of tiles $\mathcal{C}_s(Q)$ and $\mathcal{C}_u(Q)$, the families of compact sets $\mathcal{D}_s(Q)$ and $\mathcal{D}_u(Q)$ and the number $n_0(Q)$. By the choice of $V(Q)$, we have that $f^i(B_\sigma(Q)) \cap f^j(B_k) = \emptyset$ for any $\sigma = s, u$, any $0 \le i \le N_k - 1$ and any $k \ge 0$.

We take the union of $(B_s(Q), \mathcal{C}_s(Q), \mathcal{D}_s(Q))$, $(B_u(Q), \mathcal{C}_u(Q), \mathcal{D}_u(Q))$ and $(B_0, \mathcal{C}_0, \mathcal{D}_0)$, and to simplify the notation, we still denote the union by $(B_0, \mathcal{C}_0, \mathcal{D}_0)$. By Remark 2.33, we know the newly created $(B_0, \mathcal{C}_0, \mathcal{D}_0)$ is still a perturbation domain of order N_0 for $(f, \mathcal{U}_0)$. Denote D'_k the union of the compact sets of the family $\mathcal{D}_k$ for each $k \ge 0$. By a similar argument as in [**8**, §4.3], we assume that z is not in any of the perturbation domains that we have chosen.

Recall that the support of the perturbation domain B_k is $\mathrm{supp}(B_k) = \bigcup_{0\le n\le N_k-1} f^n(B_k)$. From the above constructions, the supports of the perturbation domains $(B_k)_{k\ge0}$ are pairwise disjoint and are contained in U. Moreover, we have that $\mathrm{supp}(B_k) \subset U_{k-1} \setminus \overline{U_{k+2}}$ for any $k \ge 0$.

LEMMA 7.5. *There is an infinitely long pseudo-orbit $Y = (y_0, y_1, \dots)$ for f contained in X that has jumps only in tiles of $(\mathcal{C}_k)_{k\ge0}$ with $y_0 = z$ and $d(y_n, K) \to 0$ as $n \to \infty$. Moreover, for each $k \ge 0$, there is a minimal number l_k, such that $y_i \in U_k$ for all $i \ge l_k$.*

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Proof. By the former constructions, any point $x \in X_0$ has a positive iterate contained in the union of the interior of the compact set D_0' and the open sets $W(Q)$ for all periodic orbits $Q \subset X \cap \mathrm{Per}_{N_0}(f)$. Any point $x \in X_k$ has a positive iterate contained in the union of the interior of compact sets D_k' for $k \geq 1$. By the compactness of the sets X_k, there are integers T_k, compact sets $\tilde{D}_k \subset D_k'$ and compact sets $\tilde{W}(Q) \subset W(Q)$ such that:

(i) all points $x \in X_0$ will enter the union of $\tilde{D}_0$ and $\tilde{W}(Q)$ for all $Q \subset X \cap \mathrm{Per}_{N_0}(f)$ in time bounded by T_0;

(ii) all points $x \in X_k$ will enter in $\tilde{D}_k$ for $k \geq 1$ in time bounded by T_k.

We can assume that T_0 is larger than $n_0(Q)$, for any $Q \subset X \cap \mathrm{Per}_{N_0}(f)$ (recall that $n_0(Q)$ is obtained from Lemma 7.4).

Setting of the constants. For $k \geq 0$, set η_k to be smaller than half of the minimum of the distances between a point in $(\bigcup_{Q \subset X \cap \mathrm{Per}_{N_0}(f)} \tilde{W}(Q)) \cup (\bigcup_{0 \leq i \leq k} \tilde{D}_i)$ and a point in the complement of $(\bigcup_{Q \subset X \cap \mathrm{Per}_{N_0}(f)} W(Q)) \cup (\bigcup_{0 \leq i \leq k} D_i')$. Moreover, we also assume that η_k is smaller than half of the minimum of the distances between a point in $f(\overline{M \backslash U_k})$ to a point in U_{k+1}, and smaller than the minimum of the distances between a point in a compact set $D \in \mathcal{D}_k$ and a point on the boundary of the tile $C \in \mathcal{C}_k$ that contains D. Then for any $k \geq 0$, there is a number $0 < \varepsilon_k < \eta_k$ such that, for any ε_k-pseudo-orbit $(x_0, \ldots, x_{T_k})$, we have $d(x_i, f^i(x_0)) < \frac{1}{2}\eta_k$ and $d(x_i, f^{i-T_k}(x_{T_k})) < \frac{1}{2}\eta_k$ for all $0 \leq i \leq T_k$. For each ε_k, there is a number $\delta_k \in (0, \frac{1}{3}\varepsilon_k)$ such that, for any two points $x, y \in M$, if $d(x, y) < \delta_k$, then $d(f(x), f(y)) < \frac{1}{3}\varepsilon_k$. Without loss of generality, we can assume that the sequences $(\eta_k)_{k \geq 0}$, $(\varepsilon_k)_{k \geq 0}$ and $(\delta_k)_{k \geq 0}$ are strictly decreasing sequences.

The sets $\tilde{X}_k$ and the pseudo-orbits Z_k. Now we take a finite δ_k-dense set $\tilde{X}_k$ of X_k for any $k \geq 0$ such that $z \in \tilde{X}_0$. For any $k \geq 0$, take a δ_k-pseudo-orbit $(y_1^k, \ldots, y_{m_k}^k)$ in $X \backslash K$ such that $y_1^k = z$ and $d(y_{m_k}^k, K) < \delta_k$. Then we project this pseudo-orbit to the set $\bigcup_{i \geq 0} \tilde{X}_i$: if $y_j^k \in X_i \backslash X_{i+1}$, then there is $z_j^k \in \tilde{X}_i$ such that $d(y_j^k, z_j^k) < \delta_i$. Then the pseudo-orbit $Z_k = (z_1^k, \ldots, z_{m_k}^k)$ is a pseudo-orbit contained in $\bigcup_{i \geq 0} \tilde{X}_i$ that connects z to K, where $z_1^k = z$.

Recall that $(U_k)_{k \geq 0}$ is a sequence of decreasing neighborhoods of K and $X_k = X \cap \overline{(U_k \backslash U_{k+1})}$. Hence, if $y_j^k, y_{j+1}^k \in U_i$, then we have $d(f(z_j^k), z_{j+1}^k) \leq d(f(z_j^k), f(y_j^k)) + d(f(y_j^k), y_{j+1}^k) + d(y_{j+1}^k, z_{j+1}^k) < \frac{1}{3}\varepsilon_i + \delta_k + \delta_i < \frac{2}{3}\varepsilon_i + \frac{1}{3}\varepsilon_k$. Thus, $d(f(z_j^k), z_{j+1}^k) < \varepsilon_i$ when $k \geq i$. For any $k \geq 0$, by cutting some part of Z_k, we can assume that $z_j^k \neq z_l^k$ for any $j \neq l$. Then for any $k \geq 0$, there is a minimal integer $l(m, k)$, such that $z_i^k \in U_m$ for all $i > l(m, k)$.

The infinitely long pseudo-orbit Z. Since $\tilde{X}_k$ is a finite set for any $k \geq 0$, one can extract a subsequence (Z_k^1) of (Z_k) such that all pseudo-orbits in this subsequence have the same piece before staying in U_1, that is to say, $(z_1^k, \ldots, z_{l(1,k)}^k)$ are equal to each other for any $Z_k \in \{Z_k^1\}$. Similarly, there is a subsequence (Z_k^2) of (Z_k^1) such that all pseudo-orbits in this subsequence have the same piece before staying in U_2. We can continue

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this process, and finally, by taking the limit, we can get an infinitely long pseudo-orbit $Z = (z_1, z_2, \ldots)$ such that $z_1 = z$, $d(z_n, K) \to 0$ as $n \to \infty$. Moreover, if $z_j, z_{j+1} \in X_i$, then $d(f(z_j), z_{j+1}) < \varepsilon_i$, since Z is a limit set of (Z_k).

By the analysis of [**8**, Lemma 4.6], the pseudo-orbit $Z = (z_1, z_2, \ldots)$ has the property stated in the following claim. We omit the proof here since it follows exactly the proof of Lemma 4.6 in [**8**].

CLAIM 7.6. *There is a strictly increasing sequence $t_0 = 1, t_1, \ldots$ such that for $j > 0$, z_{t_j} is contained in a compact set E_j of $\bigcup_{k\geq 0} \mathcal{D}_k$. Moreover, for any $j \geq 0$:*

(i) *if $E_j \in \mathcal{D}_0$, then either $t_j - t_{j-1} < T_1$ or there is $Q \subset X \cap \mathrm{Per}_{N_0}(f)$ such that $E_{j-1} \in \mathcal{D}_s(Q)$ and $E_j \in \mathcal{D}_u(Q)$;*

(ii) *if $E_j \in \mathcal{D}_k$ for some $k \geq 1$, then $t_j - t_{j-1} < T_k$.*

Construction of the pseudo-orbit Y from Z. Now we replace some part of Z to get an infinitely long pseudo-orbit that connects $\tilde{U}$ to K, accumulates to K in the future and has jumps only in the tiles of the perturbation domains. Using Claim 7.6, we construct Y as follows.

(i) If $E_j \in \mathcal{D}_0$ and $t_j - t_{j-1} < T_1$ or if $E_j \in \mathcal{D}_k$ where $k \geq 1$, we replace the piece of pseudo-orbit $(z_{t_{j-1}+1}, \ldots, z_{t_j})$ by the piece of true orbit $(f(z_{t_{j-1}}), f^2(z_{t_{j-1}}), \ldots, f^{t_j - t_{j-1}}(z_{t_{j-1}}))$.

(ii) If $E_j \in \mathcal{D}_0$ and $t_j - t_{j-1} \geq T_1$, we have that there is $Q \subset X \cap \mathrm{Per}_{N_0}(f)$ such that $E_{j-1} \in \mathcal{D}_s(Q)$ and $E_j \in \mathcal{D}_u(Q)$. By Lemma 7.4, there is a piece of true orbit $(x, f(x), \ldots, f^t(x))$ such that $x \in E_{j-1}$, $f^t(x) \in E_j$ and $t \leq n_0(Q) < T_0$. Then we replace the piece of pseudo-orbit $(z_{t_{j-1}+1}, \ldots, z_{t_j})$ by the piece of true orbit $(f(x), f^2(x), \ldots, f^t(x))$.

Then we get a new pseudo-orbit $Y = (y_0, y_1, \ldots)$.

The property of the pseudo-orbit Y. From the construction of the pseudo-orbit Y, we can see that two nearby points y_i, y_{i+1} satisfy the following properties.

(i) If $y_i \notin \mathcal{D}_k$ for any $k \geq 0$, then $f(y_i) = y_{i+1}$.

(ii) If there exists $k \geq 0$ such that $y_i \in \mathcal{D}_k$, then y_i and $f^{-1}(y_{i+1})$ are in the same tile of $\mathcal{C}_k$.

This implies that Y has jumps only in tiles of $(\mathcal{C}_k)_{k\geq 0}$ with $y_0 = z$ and $d(y_n, K) \to 0$ as $n \to \infty$. Moreover, there is a minimal number l_k such that $y_i \in U_k$ for all $i \geq l_k$ and all $k \geq 0$.

This finishes the proof of Lemma 7.5. □

Remark 7.7. In Lemma 7.5, we only need to guarantee that the pseudo-orbit Y obtained has jumps only in the tiles of $(\mathcal{C}_k)_{k\geq 0}$. We do not have to consider the scale of jumps at each step.

7.4. *The connecting processes.* We take the infinitely long pseudo-orbit $Y = (y_0, y_1, \ldots)$ with $y_0 = z$ contained in X from Lemma 7.5. Recall that Y has jumps only in tiles of $(\mathcal{C}_k)_{k\geq 0}$ and $d(y_n, K) \to 0$ as $n \to \infty$. Moreover, for each $k \geq 0$, there is a minimal integer l_k, such that $y_i \in U_k$ for all $i \geq l_k$. We have the following lemma.

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LEMMA 7.8. *For each $k \geq 0$, there are a diffeomorphism f_k, an infinitely long pseudo-orbit $Y_k = (y_0^k, y_1^k, \ldots)$ of f_k with $y_0^k = z$, and two sequences of positive integers $(m_k)_{k\geq 0}$ and $(n_k)_{k\geq 0}$ such that, for any $k \geq 0$, the following properties are satisfied.*

(1) *There is $\phi_k \in \mathcal{V}_k$ such that $\phi_k|_{M\setminus \mathrm{supp}(B_k)} = \mathrm{Id}|_{M\setminus \mathrm{supp}(B_k)}$, and $f_k = f_{k-1} \circ \phi_k$, where we put $f = f_{-1}$.*
(2) *The integer m_k is the smallest positive integer, such that $f_{k+1}^{m_k}(z) \in U_k$. Moreover, we have $m_k < m_{k+1}$.*
(3) *The piece of pseudo-orbit $(y_0^{k+1}, y_1^{k+1}, \ldots, y_{m_k-1}^{k+1})$ of f_{k+1} coincides with $(z, f_{k+1}(z), \ldots, f_{k+1}^{m_k-1}(z))$.*
(4) *We have $n_k \leq l_{k+2}$ and $y_{n_k+i}^k = y_{l_{k+2}+i}$, for all $i \geq 0$.*
(5) *The pseudo-orbit Y_k of f_k has jumps only in the tiles $\{\mathcal{C}_{k+1}, \mathcal{C}_{k+2}, \ldots\}$.*

Proof. We build the sequences by induction. We construct f_{k+1}, Y_{k+1}, n_{k+1} and m_k after f_k, Y_k, n_k and m_{k-1} has been built.

The constructions of f_0, Y_0, n_0: the case $n = 0$. Recall that the pseudo-orbit Y of f has jumps only in tiles of $(\mathcal{C}_k)_{k\geq 0}$ and l_k is the minimal integer such that $y_i \in U_k$ for all $i \geq l_k$.

We consider the finite pseudo-orbit $(y_0, y_1, \ldots, y_{l_2})$, which also has jumps only in titles of $(\mathcal{C}_k)_{k\geq 0}$ since it is a piece of Y. Moreover, we have that $y_0 = z$ and $y_0, y_{l_k} \notin \mathrm{supp}(B_0)$ by the former constructions. By Lemma 2.33, there are a diffeomorphism $f_0 \in \mathcal{U}_0$, a positive integer n_0 and a new pseudo-orbit $Y_0^0 = (\hat{y}_0^0, \hat{y}_1^0, \ldots, \hat{y}_{n_0}^0)$ of f_0, satisfying the following three properties:

(i) $\hat{y}_0^0 = y_0 = z$ and $\hat{y}_{n_0}^0 = y_{l_2}$;
(ii) the diffeomorphism f_0 coincides with f outside $\mathrm{supp}(B_0)$, hence there is $\phi_0 \in \mathcal{V}_0$ such that $\phi_0|_{M\setminus \mathrm{supp}(B_0)} = \mathrm{Id}|_{M\setminus \mathrm{supp}(B_0)}$, and $f_0 = f \circ \phi_0$, which is item (1);
(iii) the pseudo-orbit Y_0^0 has only jumps in the tiles $\{\mathcal{C}_1, \mathcal{C}_2, \ldots\}$;
(iv) $n_0 \leq l_2$.

Now we consider the infinitely long pseudo-orbit $Y_0 = (y_0^0, y_1^0, \ldots)$ of f_0 which is a composition of Y_0^0 and $(y_{l_2}, y_{l_2+1} \cdots)$. That is to say $y_i^0 = \hat{y}_i^0$ when $0 \leq i \leq n_0$ and $y_i^0 = y_{l_2+i-n_0}$ when $i > n_0$. Then the diffeomorphism f_0, the pseudo-orbit Y_0 and the integer n_0 satisfy the following properties:

(i) Y_0 has jumps only in the tiles $\{\mathcal{C}_1, \mathcal{C}_2, \ldots\}$, which is item (5);
(ii) $n_0 \leq l_2$ and $y_{n_0+i}^0 = y_{l_2+i}$ for any $i \geq 0$, which is item (4).

Note that we do not have to check items (2) and (3) for the case $k = 0$.

The constructions f_{k+1}, Y_{k+1}, n_{k+1} and m_k: the case $n = k + 1$. We assume that f_k, Y_k, m_{k-1} and n_k have been built. Then we have that the infinitely long pseudo-orbit $Y_k = (y_0^k, y_1^k, \ldots)$ of f_k with $y_0^k = z$, has only jumps in the tiles $\{\mathcal{C}_{k+1}, \mathcal{C}_{k+2}, \ldots\}$. Moreover, the piece of the pseudo-orbit $(y_0^k, y_1^k, \ldots, y_{m_{k-1}-1}^k)$ coincides with $(z, f_k(z), \ldots, f_k^{m_{k-1}-1}(z))$ and the piece of the pseudo-orbit $(y_{n_k}^k, y_{n_k+1}^k, \ldots)$ coincides with $(y_{l_{k+2}}, y_{l_{k+2}+1}, \ldots)$.

Similarly to the construction in the case $k = 0$, we consider the finite pseudo-orbit $(y_0^k, y_1^k, \ldots, y_{n_k}^k, \ldots, y_{n_k+l_{k+3}-l_{k+2}}^k)$ of f_k which also has only jumps in the tiles $\{\mathcal{C}_{k+1}, \mathcal{C}_{k+2}, \ldots\}$ since it is a piece of Y_k. Note that $y_{n_k+l_{k+3}-l_{k+2}}^k = y_{l_{k+3}}$ and

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$\mathrm{supp}(B_{k+1}) \cap \overline{U_{k+3}} = \emptyset$. By Lemma 2.33, there are a diffeomorphism f_{k+1}, a positive integer n_{k+1} and a new pseudo-orbit $Y^0_{k+1} = (\hat{y}^{k+1}_0, \hat{y}^{k+1}_1, \ldots, \hat{y}^{k+1}_{n_{k+1}})$ of f_{k+1}, satisfying the following three properties.

(i) $\hat{y}^{k+1}_0 = z$ and $\hat{y}^{k+1}_{n_{k+1}} = y_{l_{k+3}}$;
(ii) the diffeomorphism f_{k+1} coincides with f_k outside $\mathrm{supp}(B_{k+1})$, hence there is $\phi_{k+1} \in \mathcal{V}_{k+1}$, such that $\phi_{k+1}|_{M\setminus \mathrm{supp}(B_{k+1})} = \mathrm{Id}|_{M\setminus \mathrm{supp}(B_{k+1})}$ and $f_{k+1} = f_k \circ \phi_{k+1}$, which is item (1);
(iii) the pseudo-orbit Y^0_{k+1} has only jumps in the tiles $\{\mathcal{C}_{k+2}, \mathcal{C}_2, \ldots\}$;
(iv) $n_{k+1} \le l_{k+3}$.

Similarly to the case when $k = 0$, we consider the infinitely long pseudo-orbit $Y_{k+1} = (y^{k+1}_0, y^{k+1}_1, \ldots)$ of f_{k+1} which is a composition of Y^0_{k+1} and $(y_{l_{k+3}}, y_{l_{k+3}+1} \cdots)$. That is to say $y^{k+1}_i = \hat{y}^{k+1}_i$ when $0 \le i \le n_{k+1}$ and $y^{k+1}_i = y_{l_{k+3}+i-n_{k+1}}$ when $i > n_{k+1}$. Then the diffeomorphism f_{k+1}, the pseudo-orbit Y_{k+1} and the integer n_{k+1} satisfy the following properties:

(i) Y_{k+1} has jumps only in the tiles $\{\mathcal{C}_{k+2}, \mathcal{C}_{k+3}, \ldots\}$, which is item (5);
(ii) $n_{k+1} \le l_{k+3}$ and $y^0_{n_{k+1}+i} = y_{l_{k+3}+i}$ for any $i \ge 0$, which is item (4).

Then we take the smallest integer m_k, such that $f^{m_k+1}_{k+1}(z) \in U_{k+1}$. To be precise, we take m_k in the following way:

(i) we take $m_0 = 1$;
(ii) when $k \ge 1$, we take m_k such that $f^{m_k+1}_{k+1}(z) \in U_{k+1}$ and, for all $0 \le i < m_k$, we have $f^i_{k+1}(z) \notin U_k$.

Since $\mathrm{supp}(B_{k+1}) \subset U_k \setminus \overline{U_{k+3}}$, the diffeomorphism f_{k+1} coincides with f_k on the piece of orbit $(z, f_k(z), \ldots, f^{m_{k-1}-1}_k(z))$. By Lemma 7.1(2), we have that $m_{k-1} < m_k$, which is the property (2). The property (3) is satisfied by the choice of m_k. This finishes the proof of Lemma 7.8. □

End of the proof of Proposition 3. Now we consider the sequences $(f_k)_{k\ge 0}$, $(Y_k)_{k\ge 0}$, $(m_k)_{k\ge 0}$ and $(n_k)_{k\ge 0}$ from Lemma 7.8. Recall that $\mathcal{U}_k = f \circ \mathcal{V}_k$ where $\mathcal{V}_k$ satisfies the property (F) in Definition 2.24. Then the sequence of diffeomorphism $f_k = f \circ \phi_0 \circ \cdots \circ \phi_k$ converges to a diffeomorphism $g \in \mathcal{U}$. Since the supports of all perturbation domains of $(B_k)_{k\le 0}$ are contained in U, we have that $g = f|_{M\setminus U}$. Moreover, since the diameter of any connected component of the perturbation domains is smaller than γ, we have that the C^0 distance of g and f is smaller than γ.

Since $\mathrm{supp}(B_{k+1}) \subset U_k \setminus \overline{U_{k+3}}$, by Lemma 7.8(2) and (3), the piece of orbit $(z, f_k(z), \ldots, f^{m_{k-1}-1}_k(z))$ is also a piece of orbit of f_n, when $n \ge k+1$. This implies that the limit of the sequence of pseudo-orbits Y_k is the positive orbit of z under g since the sequence $(m_k)_{k\ge 0}$ is strictly increasing. By Lemma 7.8(4), we can see that $\mathrm{Orb}^+(z, g)$ has only finitely many points outside U_k for any $k \ge 0$ (bounded by n_k), hence $\omega(z, g) \subset K$. This finishes the proof of Proposition 3. □

8. *Proofs of the applications*

In this section, we give the proofs of the applications of the main theorem.

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8.1. *Structural stability and hyperbolicity.* To prove Corollary 1.3, we use some of the results in [**46**, **51**–**53**]. We take two steps: first we prove that the statement is true for a residual subset of $\mathrm{Diff}^1(M)$ and then we prove it for all diffeomorphisms in $\mathrm{Diff}^1(M)$.

Recall that an invariant compact set Λ is *shadowable*, if for any $\varepsilon > 0$, there is $\delta > 0$ such that any ε-pseudo-orbit $(x_0, x_1, \ldots, x_n)$ in Λ can be δ-shadowed by a segment of orbit $(x, f(x), \ldots, f^n(x))$ in M, i.e. there exists $x \in M$ such that $d(f^i(x), x_i) < \delta$ for any $0 \le i \le n$.

Assume that $H(p)$ is the homoclinic class of a hyperbolic periodic point p of a diffeomorphism $f \in \mathrm{Diff}^1(M)$. We state two properties as follows.

(P1) There are $m \in \mathbb{N}$, $C > 0$ and $0 < \lambda < 1$ such that $H(p)$ admits an (m, λ)-dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = \mathrm{Ind}(p)$. And for any periodic point q homoclinically related to p, where $\tau(q)$ denotes the period of q, the following are satisfied:

$$\prod_{0 \le i < \tau(q)/m} \|Df^m|_{E(f^{im}(q))}\| < C\lambda^{\tau(q)},$$

$$\prod_{0 \le i < \tau(q)/m} \|Df^{-m}|_{F(f^{-im}(q))}\| < C\lambda^{\tau(q)}.$$

(P2) We have that $H(p)$ is shadowable and every periodic pseudo-orbit can be shadowed by a periodic orbit.

Now we state the following two lemmas, whose proofs will be omitted.

LEMMA 8.1. (Wen [**51**, Theorem 1.2]) *Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$. If a homoclinic class $H(p)$ is structurally stable, then property (P1) is satisfied for $H(p)$.*

LEMMA 8.2. (Wen [**51**, Proposition 7]) *Assume that f is a diffeomorphism in $\mathrm{Diff}^1(M)$ and p is a hyperbolic periodic point. If the two properties (P1) and (P2) are satisfied for $H(p)$, then $H(p)$ is hyperbolic.*

Proof of Corollary 1.3. We take a residual subset $\mathcal{B} \subset \mathrm{Diff}^1(M)$ such that any $f \in \mathcal{B}$ satisfies the conclusion of Theorem A and Lemma 2.38. We claim that, for any $f \in \mathcal{B}$, if a homoclinic class $H(p)$ of f is structurally stable, then it is hyperbolic. In fact, by Lemma 8.1, property (P1) is satisfied for $H(p)$. Consider the splitting $T_{H(p)}M = E \oplus F$ stated in property (P1). By the two inequalities in property (P1), the Lyapunov exponents of all periodic points homoclinically related to p are uniformly bounded away from zero. Then by Theorem A and Remark 1.2, we can see that the bundle E is contracting and the bundle F is expanding. Hence $H(p)$ is hyperbolic.

Now we assume that f is an arbitrary diffeomorphism in $\mathrm{Diff}^1(M)$. If a homoclinic class $H(p)$ of f is structurally stable, then there is a C^1 neighborhood $\mathcal{U}$ of f, such that $H(p_h)$ is conjugated to $H(p)$ through a homomorphism ϕ_h for any $h \in \mathcal{U}$. For a diffeomorphism $g \in \mathcal{B} \cap \mathcal{U}$, we know that $H(p_g)$ is structurally stable. Then $H(p_g)$ is hyperbolic by the argument above, hence $H(p_g)$ satisfies property (P2). It is easy to see that property (P2) is invariant under conjugacy, thus is satisfied by $H(p)$ since $H(p)$ is conjugated to $H(p_g)$. Property (P1) is satisfied by $H(p)$ by Lemma 8.1. Then by Lemma 8.2, we have that $H(p)$ is hyperbolic. This finishes the proof of Corollary 1.3.

This finishes the proof of Corollary 1.3. □

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8.2. *Partial hyperbolicity.* Now we give the proofs of Corollaries 1.4 and 1.5.

Proof of Corollary 1.4. We assume that $T_{H(p)}M = E \oplus F$ where $\dim(E)$ is smaller than the smallest index of periodic orbits contained in $H(p)$. In particular, we have that $\dim(E) < \mathrm{Ind}(p)$. Assume by contradiction that E is not contracting. By Theorem A, there are periodic orbits $\mathrm{Orb}(q_n) \subset H(p)$ such that $\mathrm{Ind}(q_n) = \dim(E)$, which contradicts the assumption that $\dim(E) < \mathrm{Ind}(q_n)$. This proves Corollary 1.4. □

Proof of Corollary 1.5. We assume that f satisfies the properties in Lemma 2.38 and Theorem A. For a homoclinic class $H(p)$ of f, we denote by $j \geq 1$ and $l \leq d-1$ the smallest and the largest index of periodic point contained in $H(p)$. By Lemma 2.38(4) (see also [**1**, Theorem 1]), for any $j \leq i \leq l$, there are periodic orbits of index i contained in $H(p)$. Moreover, for any $j \leq i \leq l$, the hyperbolic periodic points with index i are dense in $H(p)$. By [**47**, Theorem A], $H(p)$ admits a dominated splitting with index i. By Remark 2.7, we have that $H(p)$ admits a dominated splitting $T_{H(p)}M = E^{cs} \oplus E^c_1 \oplus \cdots \oplus E^c_n \oplus E^{cu}$, where $\dim(E^{cs}) = j$, $n = l - j$, and $\dim(E^c_i) = 1$ for all $1 \leq i \leq n$. Since $H(p)$ contains hyperbolic periodic points with index i for all $j \leq i \leq l$, we can see easily that the central bundle E^c_i is neither contracting nor expanding, for any $1 \leq i \leq n$.

Now we consider whether the two bundles E^{cs} and E^{cu} are hyperbolic. Here E^{cs} is hyperbolic means it is contracting and E^{cu} is hyperbolic means it is expanding.

Case 1: both E^{cs} and E^{cu} are hyperbolic In this case, the splitting $T_{H(p)}M = E^{cs} \oplus E^c_1 \oplus \cdots \oplus E^c_n \oplus E^{cu}$ is a partially hyperbolic splitting. We put $E^s = E^{cs}$ and $E^u = E^{cu}$. Then the smallest and the largest index of periodic orbits contained in $H(p)$ are $\dim(E^s)$ and $d - \dim(E^u)$, respectively.

Case 2: at least one bundle of E^{cs} and E^{cu} is not hyperbolic Without loss of generality, we assume that the bundle E^{cs} is not contracting. By Theorem A, there are periodic orbits $\mathrm{Orb}(q_n)$ with arbitrarily long period and index j, whose largest Lyapunov exponent along E^{cs} converges to zero. Moreover, such periodic orbits form a dense set in $H(p)$. Then by the Franks' lemma and a genericity argument as in §4.7, the homoclinic class $H(p)$ can be accumulated by periodic orbits with index $j-1$, see [**23**, Lemma 2.1]. Therefore, by [**47**, Theorem A] and Remark 2.7, E^{cs} has a dominated splitting $E^{cs} = E^s \oplus E^c_0$ with $\dim(E^c_0) = 1$. By Corollary 1.4, we have that E^s is contracting. Symmetrically, if the bundle E^{cu} is not expanding, then it admits a finer dominated splitting $E^{cu} = E^c_{n+1} \oplus E^u$ such that E^u is expanding and $\dim(E^c_{n+1}) = 1$.

By the analysis above, we can obtain that $H(p)$ admits a partially hyperbolic splitting $T_{H(p)}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u$ with every center bundle of dimension one. Moreover, we have that $E^{cs} = E^s$ or $E^{cs} = E^s \oplus E^c_1$ and symmetrically $E^{cu} = E^u$ or $E^{cu} = E^c_k \oplus E^u$. Hence, the smallest index of periodic orbits contained in $H(p)$ is $\dim(E^s)$ or $\dim(E^s) + 1$ and the largest is $d - \dim(E^u)$ or $d - \dim(E^u) - 1$.

This ends the proof of Corollary 1.5. □

8.3. *Lyapunov stable homoclinic classes.* Now we give the proofs of Corollaries 1.6 and 1.7.

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Proof of Corollary 1.6. From Corollary 1.4, we only have to prove the case where $\dim(E)$ equals the largest index of periodic orbits contained in $H(p)$. The idea of the proof follows from [**41**] and [**4**, §3]. We just give an explanation here and for more details, the reader should refer to [**41**] and [**4**, §3].

Assume that $f \in \mathcal{R}$ satisfies Theorem A where $\mathcal{R}$ is the residual set in Lemma 2.38. There is a neighborhood $\mathcal{U}$ of f such that Lemma 2.38(3), (4) and (6) are satisfied for $(f, H(p), \mathcal{U})$. We can assume that p has the largest index among the periodic points contained in $H(p)$, hence $\dim(E) = \mathrm{Ind}(p)$. Assume that the bundle F is not expanding, then by the conclusion of Theorem A, we can get a sequence of periodic orbits $\mathrm{Orb}(q_n)$ homoclinically related to $\mathrm{Orb}(p)$ with arbitrarily long period such that the smallest Lyapunov exponent of $\mathrm{Orb}(q_n)$ along the bundle F can be arbitrarily close to zero. By [**28**, Lemma 2.3], we can assume that all of the eigenvalues of $\|Df\|$ along $\mathrm{Orb}(q_n)$ are real. Then by [**30**, Theorem 1] (see also [**4**, Theorem 2.5]) and a proper construction of a path of diffeomorphism (see [**4**]), there is a diffeomorphism $g \in \mathcal{U}$ and a periodic point q of g with index larger than $\dim(E)$ such that $W^s(q) \cap W^u(p_g) \neq \emptyset$. By C^1 small perturbation, we can assume that $W^s(q)$ intersects $W^u(p_g)$ transversely. This property is persistent under C^1 perturbation, since $\mathrm{Ind}(q) > \mathrm{Ind}(p_g)$. Hence, there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of g, such that for any $h \in \mathcal{V}$, we have $W^s(q_h) \pitchfork W^u(p_h) \neq \emptyset$. Take a diffeomorphism $h \in \mathcal{V} \cap \mathcal{R}$, we have that $H(p_h)$ is Lyapunov stable by Lemma 2.38(6). Then $W^u(p_h)$ is contained in $H(p_h)$. By the fact $W^s(q_h) \pitchfork W^u(p_h) \neq \emptyset$, we have that $q_h \in H(p_h)$. This contradicts Lemma 2.38(4) by the choice of $\mathcal{U}$, since $\mathrm{Ind}(q_h) > \mathrm{Ind}(p_h) = \mathrm{Ind}(p)$. □

Proof of Corollary 1.7. We assume that the second item does not happen. By Lemma 2.38, all periodic orbits contained in $H(p)$ have the same index. By [**41**], $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus F$ such that $\dim(E) = \mathrm{Ind}(p)$. By Corollary 1.6, we have that the bundle F is expanding. With the same argument as f^{-1} and the bundle E, we get that E is contracting for f. Hence, the splitting $T_{H(p)}M = E \oplus F$ is hyperbolic. Then $H(p)$ is a hyperbolic chain recurrence class by Lemma 2.38(2). Hence, by a standard argument using the shadowing lemma, $H(p)$ is an isolated chain recurrence class. By [**2**, Theorem 5], since M is connected, the homoclinic class $H(p)$ is in fact the whole manifold, hence f is Anosov. □

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