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On Morita's p-adic gamma function

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Y. Morita proved that, for each prime number p, one can define a p-adic continuous function $\Gamma_p(x)$ from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, interpolating the sequence

$$n \to (-1)^n \prod_m m,$$

where m runs through the integers m prime to p with $1 \leqslant m < n$. Our aim is to show how this result is related to Dwork's result on the radius of convergence of

$$\exp(X + (X^p/p)).$$

1. *Introduction and notations*

Let p be a prime, $\mathbb{N}, \mathbb{R}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ as usual (2). The absolute value (resp. the valuation) on $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ is denoted by $|.|$ (resp. $v(.)$), and normalized by $|p| = p^{-1}$ (resp. $v(p) = 1$). The sequence $n \to n!$ cannot be the restriction to $\mathbb{N}$ of a p-adic continuous function from $\mathbb{Z}_p$ to $\mathbb{C}_p$. However, Morita (10) proved that the sequence

$$n \to (-1)^n \prod_m m = \Gamma_p(n),$$

m prime to p, $1 \leqslant m < n$, can be interpolated by a continuous p-adic function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ denoted by $\Gamma_p(x)$. We shall prove, by means of the formal Laplace transform ((5) or (3)), that this result can be deduced from Dwork's estimate of the p-adic absolute value of the coefficients $b_n/n!$ of the series

$$\sum_{n \geqslant 0} \frac{b_n}{n!} X^n = \exp\left(X + \frac{X^p}{p}\right).$$

We actually prove that the result of Dwork is equivalent to the existence of a locally analytic function $x \to \Gamma_p(x)$ on $\mathbb{Z}_p$, with local analyticity radius $\rho_p = p^{-(1/p)-(1/(p-1))}$, bounded by $1/\rho_p$ on

$$W_{\rho_p}(\mathbb{Z}_p) = \{x \in \mathbb{C}_p;\ |x - z| < \rho_p \text{ for all } z \in \mathbb{Z}_p\},$$

interpolating the sequence $n \to (-1)^{pn}((pn)!/p^n(n!)) = \Gamma_p(pn)$. Let us set

$$B(a, \rho^+) = \{X \in \mathbb{C}_p;\ |X - a| < \rho\}, \quad \text{for} \quad a \in \mathbb{C}_p \quad \text{and} \quad \rho \in \mathbb{R}_+ = \{x \in \mathbb{R};\ x > 0\}.$$

Recall (3) that the formal Laplace transform $\mathscr{L}(\tilde{f}(X))$ of

$$\tilde{f}(X) = \sum_{n \geqslant 0} \frac{a_n}{n!} X^n \in \mathbb{C}_p[[X]]$$

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is

$$f(X) = \mathscr{L}(\tilde{f}(X)) = \sum_{n\geq 0} a_n X^n \in C_p[[X]]. \tag{1}$$

The following properties of $\mathscr{L}$ are obvious from the definition: $\mathscr{L}$ is a $\mathbb{C}_p$-linear map from $\mathbb{C}_p[[X]]$ onto $\mathbb{C}_p[[X]]$, $\mathscr{L}$ is continuous for the (p, X)-adic topology on $\mathbb{C}_p[[X]]$. Let

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}, \quad \binom{x}{0} = 1.$$

2. *Morita's p-adic Γ-function*

Definition 1. (Morita (10)). Let $\Gamma_p(n)$ be defined, for $n \in \mathbb{N}$, by

$$\Gamma_p(0) = 1, \tag{2}$$

$$\Gamma_p(n) = (-1)^n \prod_{0<m<n} m, \quad m \text{ prime to } p. \tag{3}$$

LEMMA 1. *For* $1 \leqslant i \leqslant p$, *we have*

$$\Gamma_p(np+i) = (-1)^{pn+i} \frac{(pn+i-1)!}{p^n(n!)}. \tag{4}$$

Proof. Obvious from definition 1.

LEMMA 2. *The generating function of the sequence* $n \to (-1)^n \Gamma_p(n)$,

$$F(X) = \sum_{n\geq 0} (-1)^{n+1} \Gamma_p(n+1) X^n, \tag{5}$$

is the formal Laplace transform of

$$\tilde{F}(X) = \sum_{i=1}^{p} X^{i-1} \exp(X^p/p). \tag{6}$$

Proof.

$$\sum_{i=1}^{p} X^{i-1} \exp(X^p/p) = \sum_{n\geq 0} \sum_{i=1}^{p} \frac{X^{pn+i-1}}{p^n(n!)}. \tag{7}$$

So

$$\left(\sum_{i=1}^{p} X^{i-1} \exp(X^p/p)\right) = \sum_{i=1}^{p} \sum_{n\geq 0} \frac{(pn+i-1)!}{p^n(n!)} X^{pn+i-1}, \tag{8}$$

the conclusion follows from lemma 1. ∎

Let

$$\sum_{n\geq 0} \frac{b_n}{n!} X^n \tag{9}$$

be the formal Taylor series at the origin of $\exp(X + X^p/p)$. It is clear that

$$\sum_{n\geq 0} (b_n/n!) X^n \in \mathbb{Q}[[X]].$$

LEMMA 3 (*Dwork* (6)). *The coefficients* b_n *of the Taylor series of* $\exp(X + (X^p/p))$ *satisfy the following p-adic estimate*

$$v(b_n/n!) \geqslant -n \frac{2p-1}{p^2(p-1)}. \tag{10}$$

For the proof see (9) or (6).

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LEMMA 4. *If* $\mathscr{L}(\tilde{f}(X)) = f(X)$, *then*

$$\mathscr{L}((\exp(\alpha X)\tilde{f}(X)) = \frac{1}{1-\alpha X} f\left(\frac{X}{1-\alpha X}\right), \quad \alpha \in \mathbb{C}_p. \tag{11}$$

Proof. Let $f(X) = \sum_{n\geqslant 0} a_n (X^n/n!)$; then

$$(\exp \alpha X)\tilde{f}(X) = \sum_{n\geqslant 0} \left(\sum_{k=0}^{n} \alpha^k a_{n-k} \binom{n}{k} \right) \frac{X^n}{n!}.$$

This implies

$$\mathscr{L}((\exp(\alpha X)\tilde{f}(X)) = \sum_{n\geqslant 0} \left(\sum_{k=0}^{n} a_{n-k} \alpha^k \binom{n}{k} \right) X^n = \sum_{n\geqslant 0} a_n \frac{X^n}{(1-\alpha X)^{n+1}}. \quad \blacksquare$$

THEOREM 1. *The following identity holds*

$$F(X) = \sum_{n\geqslant 0} (-1)^{n+1} \Gamma_p(n+1) X^n = \sum_{i=1}^{p} \sum_{n\geqslant 0} b_n^{(i)} \frac{X^n}{(1+X)^{n+1}} \tag{12}$$

with

$$b_n^{(i)} = b_{n-i+1} \frac{n!}{(n+1-i)!}.$$

Proof. $F(X) = \mathscr{L}(\exp(-X)(\Sigma_{i=1}^{p} X^{i-1} \exp(X + (X^p/p))))$. Applying Lemmas 4 and 2, we obtain

$$F(X) = \sum_{i=1}^{p} \sum_{n\geqslant 0} b_n \frac{X^{n+i-1}}{(1+X)^{n+i}} \frac{(n+i-1)!}{n!}. \quad \blacksquare$$

COROLLARY 1. *For* $n \in \mathbb{N}$, *we have*

$$\Gamma_p(n+1) = -\sum_{i=1}^{p} \sum_{k} (-1)^k b_k^{(i)} \binom{n}{k}. \tag{13}$$

Proof

$$F(X) = \sum_{i=1}^{p} \sum_{n\geqslant 0} b_n^{(i)} \frac{X^n}{(1+X)^{n+1}} = \sum_{i=1}^{p} \sum_{n\geqslant 0} b_n^{(i)} \sum_{k\geqslant 0} X^{n+k} \binom{-n-k}{k}$$

$$= \sum_{i=1}^{p} \sum_{n\geqslant 0} X^n \sum_{k=0}^{n} (-1)^{n-k} b_k^{(i)} \binom{n}{k}. \quad \blacksquare$$

THEOREM 2. *The sequence* $n \to \Gamma_p(n)$ *is the restriction to* $\mathbb{N}$ *of a unique locally analytic function on* $\mathbb{Z}_p$, $\Gamma_p(x)$, *with local analyticity radius*

$$\rho_p = p^{-(1/p)-(1/(p-1))}. \tag{14}$$

Proof. Recall, (1), that a function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ is locally analytic on $\mathbb{Z}_p$, if, for each point $\alpha \in \mathbb{Z}_p$, there exist $\rho_\alpha \in \mathbb{R}_+$ such that, on $B(\alpha, \rho_\alpha^+) \cap \mathbb{Z}_p$, f is the restriction of a function $f_\alpha(x) = \Sigma_{n\geqslant 0} a_n(\alpha)(x-\alpha)^n$, analytic on $B(\alpha, \rho_\alpha^+)$. The local analyticity radius of f is

$$\rho = \inf_{\alpha \in \mathbb{Z}_p} \rho_\alpha > 0 \quad \text{(because } \mathbb{Z}_p \text{ is compact).} \tag{15}$$

Take the series

$$\Gamma_p(x+1) = -\sum_{i=1}^{p} \sum_{n\geqslant 0} (-1)^n b_n^{(i)} \binom{x}{n}. \tag{16}$$

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By lemma 3 we get

$$\left(\sum_{i=1}^{p} b_n^{(i)}\right) \geqslant -(n-p+1)\frac{2p-1}{p^2(p-1)} + v((n-p+1)!) \tag{17}$$
$$\geqslant (n-p+1)((1/p)-(1/p^2)) - \left[\frac{\log n}{\log p}\right] - 1,$$

where $[a]$ is the integral part of $a \in \mathbb{R}_+$, that is $[a] \in \mathbb{N}$ and $a-1 < [a] \leqslant a$. Then (17) implies

$$\lim_{n \to +\infty} \left| \sum_{i=1}^{p} b_n^{(i)} \right| = 0. \tag{18}$$

By Mahler's theorem (1) this implies that (16) defines the unique continuous function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ (actually to $\mathbb{Q}_p$), interpolating the sequence $n \to \Gamma_p(n)$.

Actually (17) gives us a stronger result. Let $1/p < \rho = p^{-\alpha} < 1$, $\alpha \in \mathbb{R}_+$, $1 > \alpha > 0$. Let $W_\rho(\mathbb{Z}_p) = \{x \in \mathbb{C}_p;\ |x-z| < \rho \text{ for all } z \in \mathbb{Z}_p\}$. We have

$$\inf_{x \in W_\rho(\mathbb{Z}_p)} v\left(\binom{x}{n}\right) > \left(\left[\frac{n}{p}\right](1-\alpha) + \sum_{i \geqslant 2}\left[\frac{n}{p^i}\right]\right), \tag{19}$$

and

$$\inf_{x \in W_\rho(\mathbb{Z}_p)} v\left(\binom{x}{n}\right) > -\left[\frac{n}{p}\right]\left(1-\alpha+\frac{1}{p-1}\right). \tag{20}$$

From (20) and (17), we obtain

$$\lim_{n \to +\infty} \sup_{x \in W_\rho(\mathbb{Z}_p)} \left| \sum_{i=1}^{p} b_n^{(i)} \binom{x}{n} \right| = 0$$

if, and only if,

$$\alpha > \frac{1}{p-1} + \frac{1}{p}. \tag{21}$$

However, we can get more:

THEOREM 3. *The following two propositions are equivalent.*

(i) *The sequence $n \to \Gamma_p(pn)$ is the restriction of a unique locally analytic function on $\mathbb{Z}_p$, $\Gamma_p(px)$, with local analyticity radius $r_p = p^{1-(1/p)-(1/(p-1))}$, and bounded by $p^{((1/p)+1/(p-1))}$ on $W_{r_p}(\mathbb{Z}_p)$.*

(ii) $\exp(X + (X^p/p)) = \Sigma_{n \geqslant 0} b_n X^n/n!$, *with* $v(b_n/n!) \geqslant -n\dfrac{2p-1}{p^2(p-1)}$.

Proof.

$$\Gamma_p(pn) = (-1)^{pn}\frac{(p(n-1)+p-1)!}{p^{n-1}(n-1)!} = (-1)^{pn}\frac{(pn)!}{p^n n!}. \tag{22}$$

From (22) it is clear that

$$\sum_{n \geqslant 0} \frac{(pn)!}{p^n n!} X^{pn} = \mathscr{L}(\exp(X^p/p)) = \mathscr{L}(\exp(-X)\exp(X + (X^p/p))). \tag{23}$$

We obtain, as in corollary 1,

$$\Gamma_p(pn) = \sum_k (-1)^k b_k \binom{pn}{k}. \tag{24}$$

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Consider the series

$$\Gamma_p(px) = \sum_{n \geqslant 0} (-1)^n b_n \binom{px}{n}. \tag{25}$$

We obtain, as in (19),

$$\sup_{x \in W_{r,p}(\mathbb{Z}_p)} \left| b_n \binom{px}{n} \right| \leqslant |b_n/n!|\, p^{-[n/p]((1/p)+(1/(p-1)))}. \tag{26}$$

Assuming (ii), we obtain

$$\begin{aligned} \sup_{x \in W_{r,p}(\mathbb{Z}_p)} \left| b_n \binom{px}{n} \right| &\leqslant p^{((1/p)+1/(p-1))((n/p)-[n/p])} \\ &< p^{((1/p)+1/(p-1))} \end{aligned} \tag{27}$$

and, for all $0 \leqslant r < r_p$,

$$\lim_{n \to +\infty} \sup_{x \in W_r(\mathbb{Z}_p)} \left| b_n \binom{px}{n} \right| = 0,$$

so (i) is true.

Assuming (i), we obtain

$$\begin{aligned} |b_n/n!| &< p^{([n/p]+1)((1/p)+1/(p-1))} \\ &\leqslant p^{(n(2p-1)/p^2(p-1))}, \end{aligned}$$

because $b_n/n! \in \mathbb{Q}$, so (ii) is true.

This theorem shows the close connections that exist between $\exp(X + (X^p/p))$ and $\Gamma_p(x)$. In fact, what formulas (24) and (25) say is that the coefficient of the Mahler expansion (1) of the function $x \to \Gamma_p(px)$ are given by the coefficients b_n of the Taylor expansion of $\mathscr{L}(\exp(X + (X^p/p)))$. This suggests the possibility of a direct proof of the Gross–Koblitz formula ((4), (7), (8)) which connects Gauss sums and the p-adic Γ-function of Morita.

REFERENCES

(1) Amice, Y. Interpolation p-adique. *Bull. Soc. Math. France*, **92** (1964), pp. 117–180 (Thèse Sc. Math., Paris 1964).

(2) Amice, Y. *Les nombres p-adiques* (Paris, Presses Universitaires de France, 1975: Collection Sup., 'Le Mathématicien', 14).

(3) Barsky, D. Transformation de Cauchy p-adique et algèbre d'Iwasawa. *Math. Annalen* **232** (1978), 255–266.

(4) Boyarsky, B. p-adic gamma function and Dwork cohomology. *Trans. Amer. Math. Soc.* **257** (1980), 359–369.

(5) Diamond, J. On the value of p-adic L functions at positive integers (to appear).

(6) Dwork, B. On the rationality of the zeta function of an algebraic variety. *Amer. J. of Math.* **82** (1960), 631–648.

(7) Gross, B. and Koblitz, N. Gauss sums and the p-adic Γ function. *Annals of Math.* **109** (1979), 569–581.

(8) Katz, N. Graduate course, at Princeton University, spring 1978.

(9) Monsky, P. *p-adic analysis and zeta function.* Tokyo, Kinokuniya Bookstore, 1970 (Lectures in Math.; Kyoto University, 4).

(10) Morita, Y. A p-adic analogue of the Γ-function. *J. Fac. Sc. University Tokyo*, section 1, **22** (1975), 255–266.

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