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Random fractals

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1. *Introduction*

There has been considerable interest of late in fractals, both in their occurrence in the sciences, and in their mathematical theory. A general account of fractals is given by Mandelbrot [**10**], whilst a more mathematical approach may be found in Falconer [**5**].

The main aim of this paper is to define and analyse a class of fractals generated by a random process, which perhaps provide a better model for naturally occurring fractals than sets obtained in a non-random way.

The model to be described may be regarded as a generalization of the standard 'recurrent' construction of the Cantor set or von Koch curve, but with a random element introduced at each step. We shall consider a class of fractal with an underlying net structure, but of a very general type. For example, in the one-dimensional case, we shall consider Cantor-type constructions where each component interval is replaced by a random number of sub-intervals of random length and distribution.

Of course, Brownian motion is an example of a random fractal which has been analysed very fully, but the situation here is completely different.

Given a fractal K, the Hausdorff measures and Hausdorff dimension of K are of particular interest. If $0 \leqslant s < \infty$, $\delta > 0$ and F is any subset of $\mathbb{R}^n$, write

$$\mathscr{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : F \subset \bigcup_{i=1}^{\infty} U_i, \quad 0 < |U_i| \leqslant \delta\right\}, \tag{1·1}$$

where $|\ |$ denotes the diameter of a set, and let

$$\mathscr{H}^s(F) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(F). \tag{1·2}$$

Then $\mathscr{H}^s$ is s-dimensional Hausdorff outer measure, and, moreover, is a Borel measure. There is a number $0 \leqslant \dim F \leqslant n$, known as the Hausdorff dimension of F, such that

$$\mathscr{H}^s(F) = \infty \quad \text{if} \quad s < \dim F \quad \text{and} \quad \mathscr{H}^s(F) = 0 \quad \text{if} \quad s > \dim F.$$

For further details of Hausdorff measures and dimensions, see Falconer [**5**] or Rogers [**12**].

One of our main aims is to prove limit theorems for Hausdorff dimensions of the form that, for certain random K, $\dim K$ takes a particular value, given as the solution of an expectation equation, with probability 1; see Corollary 8·6. In particular, we can do this when the probability distribution underlying each stage of the construction is independent and identical except for a scale factor. Of course, such results would be anticipated using the Kolmogorov 0–1 law.

This paper may be regarded as the generalization of the work of Hutchinson [**9**] to

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the random setting. As well as calculating dimensions, we extend the theory of self-similar sets generated by a collection of transformations to the random case, and show that there is a natural invariant probability distribution on the fractals obtained. (This involves defining a metric on Borel probabilities in a novel way.)

It turns out that it is convenient to transform the problem of estimating Hausdorff measures to an equivalent one of examining flows through random networks, in Sections 4–6. This part of the work, which may be of interest in its own right, depends on defining certain martingales to which the martingale convergence theorem may be applied to obtain limit theorems.

The area of random fractals is of considerable current interest and is developing rapidly. Since the work for this paper was completed, the author's attention has been drawn to independent work of Mauldin and Williams[13] and Graf[14] containing results that overlap those of this paper, but obtained using rather different methods. Moreover, since this paper was written, the author has devised an alternative and in some ways more direct approach that leads to more precise and more general results [15].

2. *Sequences and trees*

In this section we introduce some notation for sequences of integers and for trees.

Let $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ be a sequence of positive integers and let $|\mathbf{i}| = k$ denote the length of the sequence. If $\mathbf{i}' = (i'_1, \ldots, i'_q)$, then $\mathbf{i}, \mathbf{i}'$ is the sequence $(i_1, \ldots, i_k, i'_1, \ldots, i'_q)$ obtained by juxtaposition. Write $\mathbf{i} \prec \mathbf{j}$ to mean that the sequence $\mathbf{j}$ is an extension of $\mathbf{i}$, that is $\mathbf{j} = \mathbf{i}, \mathbf{i}'$ for some sequence $\mathbf{i}'$. We adopt a similar convention if $\mathbf{j}$ is an infinite sequence of positive integers. We assume that the null sequence $\varnothing \prec \mathbf{i}$ for any sequence $\mathbf{i}$.

A *tree* is a collection $\mathscr{I}$ of finite sequences of positive integers such that $\mathbf{i} \in \mathscr{I}$ implies that $\mathbf{i}' \in \mathscr{I}$ for any $\mathbf{i}' \prec \mathbf{i}$; the sequences of $\mathscr{I}$ may be identified with the vertices of a directed graph with $\mathbf{i}$ joined to $\mathbf{i}, i$ in the obvious way. We write $\mathscr{I}_k = \{\mathbf{i} \in \mathscr{I}: |\mathbf{i}| = k\}$ for the set of sequences of length k, and $N(\mathbf{i}) = \#\{i: \mathbf{i}, i \in \mathscr{I}\}$ for the number of outgoing edges from the vertex $\mathbf{i}$ in the graph of $\mathscr{I}$. We allow the possibility that $N(\mathbf{i}) = 0$, but always assume that $N(\mathbf{i}) < \infty$.

Let $\bar{\mathscr{I}}$ be the set of infinite sequences $\mathbf{j}$ such that $\mathbf{i} \in \mathscr{I}$ for every finite curtailment $\mathbf{i} \prec \mathbf{j}$. We may regard $\bar{\mathscr{I}}$ as a topological space in a natural way by taking as a basis of neighbourhoods $\{\mathbf{j} \in \bar{\mathscr{I}}: \mathbf{i} \prec \mathbf{j}\}_{i \in \mathscr{I}}$ The topology τ is that induced by the metric $d(\mathbf{j}, \mathbf{j}') = \inf\{2^{-k}: \mathbf{i} \prec \mathbf{j}$ and $\mathbf{i} \prec \mathbf{j}'$ for some $\mathbf{i} \in \mathscr{I}_k\}$. It is easy to show, for example, by a 'bisection' argument based on the tree $\mathscr{I}$, that $(\bar{\mathscr{I}}, \tau)$ is a compact topological space.

We say that a subset $\mathscr{S}$ of $\mathscr{I}$ *covers* $\bar{\mathscr{I}}$ if for every $\mathbf{j} \in \bar{\mathscr{I}}$ there is a finite sequence $\mathbf{i} \in \mathscr{S}$ with $\mathbf{i} \prec \mathbf{j}$. A cover is *minimal* if for every $\mathbf{j} \in \bar{\mathscr{I}}$ there is a unique $\mathbf{i} \in \mathscr{S}$ with $\mathbf{i} \prec \mathbf{j}$. Any covering collection of sequences may be reduced to a finite cover using the compactness of $(\bar{\mathscr{I}}, \tau)$. Moreover, any covering collection of sequences may be reduced to a minimal collection by taking $\{\mathbf{i} \in \mathscr{I}: \mathbf{i}' \not\prec \mathbf{i}$ for all $\mathbf{i}' \in \mathscr{S}, \mathbf{i}' \neq \mathbf{i}\}$.

3. *Networks and flows*

We outline the properties of networks and flows that we require. Fuller treatments may be found in Ford and Fulkerson[6] or Bollobás[3]. We do not yet consider a random model.

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For our purposes a *network* or *capacitated network* G comprises a tree $\mathscr{I}$ with a capacity $C_{\mathbf{i}} > 0$ assigned to each $\mathbf{i} \in \mathscr{I}$. We regard $C_{\mathbf{i}}$, as the maximum allowable flow through the edge of the directed graph $\mathscr{I}$ joining the vertices $i_1, \ldots, i_{k-1}$ and

$$i_1, \ldots, i_{k-1}, i_k = \mathbf{i}.$$

A *flow* or *positive flow* in the network G is a function $\phi: \mathscr{I} \to [0, \infty)$ such that

$$\phi(\mathbf{i}) = \sum_{i:\mathbf{i}, i \in \mathscr{I}} \phi(\mathbf{i}, i) \quad (\mathbf{i} \in \mathscr{I}) \tag{3·1}$$

$$0 \leqslant \phi(\mathbf{i}) \leqslant C_{\mathbf{i}} \quad (\mathbf{i} \in \mathscr{I}) \tag{3·2}$$

$$0 < \phi(\varnothing). \tag{3·3}$$

Intuitively ϕ represents the rate of flow of a liquid through the network. Condition (3·1) reflects the fact that the amount of liquid reacting a vertex of $\mathscr{I}$ equals that leaving it, (3·2) ensures that the flow through each edge does not exceed the edge capacity, and (3·3) is the positivity condition, that a positive amount of liquid is able to flow through the system from $\varnothing$ to infinity. We shall principally be concerned with conditions under which a positive flow through a network exists.

The main general result on the existence of flows in the 'max-flow min-cut' theorem of Ford and Fulkerson [6]. Here, this stipulates that the maximum value of $\phi(\varnothing)$, given that ϕ satisfies (3·1)–(3·3), is

$$M_0(G) \equiv \inf_{\mathscr{S}} \{ \sum_{\mathbf{i} \in \mathscr{S}} C_{\mathbf{i}} : \mathscr{S} \text{ is finite and minimal for } \mathscr{I} \}. \tag{3·4}$$

(This follows easily by applying the Ford–Fulkerson result to the finite graph obtained from the tree $\mathscr{I}$ by joining together the vertices of $\mathscr{I}_k$, and using a limiting argument to obtain a flow in the infinite tree. Minimal sequences correspond to cutsets in the obvious way.) We shall also need the quantities

$$M_k(G) = \inf_{\mathscr{S}} \{ \sum_{\mathbf{i} \in \mathscr{S}} C_{\mathbf{i}} : \mathscr{S} \text{ is finite and minimal in } \mathscr{I}, \quad |\mathbf{i}| \geqslant k \quad \text{for} \quad \mathbf{i} \in \mathscr{S} \}. \tag{3·5}$$

By the remarks at the end of section 2, $M_k(G)$ may be expressed in the alternative ways:

$$M_k(G) = \inf_{\mathscr{S}} \{ \sum_{\mathbf{i} \in \mathscr{S}} C_{\mathbf{i}} : \mathscr{S} \text{ is minimal in } \mathscr{I}, \quad |\mathbf{i}| \geqslant k \quad \text{for} \quad \mathbf{i} \in \mathscr{S} \} \tag{3·6}$$

$$= \inf_{\mathscr{S}} \{ \sum_{\mathbf{i} \in \mathscr{S}} C_{\mathbf{i}} : \mathscr{S} \text{ covers } \mathscr{I}, \quad |\mathbf{i}| \geqslant k \quad \text{for} \quad \mathbf{i} \in \mathscr{S} \}. \tag{3·7}$$

Since $M_k(G)$ increases with k, the limit

$$M(G) = \lim_{k \to \infty} M_k(G) \tag{3·8}$$

exists, $0 \leqslant M(G) \leqslant \infty$. We shall also employ sums over the vertices at each level of the tree,

$$S_k(G) = \sum_{\mathbf{i} \in \mathscr{I}_k} C_{\mathbf{i}}. \tag{3·9}$$

Clearly $S_k(G) \geqslant M_k(G)$ for each k.

The obvious criterion for the existence of a positive flow is $M_0(G) > 0$. However, since we shall be concerned with limit properties of random processes it is more convenient to have a condition involving the limiting value $M(G)$.

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LEMMA 3·1. *A positive flow (satisfying* (3·1)–(3·3)*) exists in a network G if and only if* $M(G) > 0$.

Proof. If ϕ is such a flow, then $0 < \phi(\varnothing) \leqslant M_0(G) \leqslant M_k(G)$ for all k, so $M(G) > 0$. On the other hand, if $M(G) > 0$, then $M_k(G) > 0$ for some value of k. By the Ford–Fulkerson result, there is a positive flow from $\mathscr{I}_k$ to infinity (with all the vertices of $\mathscr{I}_k$ joined), so there exists a positive flow from $\mathbf{i}$ to infinity for some $\mathbf{i} \in \mathscr{I}_k$. But there is a positive flow from $\varnothing$ to $\mathbf{i}$ (define $\phi(\mathbf{j}) = c$ if $\mathbf{j} \prec \mathbf{i}$, $\phi(\mathbf{j}) = 0$ otherwise, where $c = \min_{\mathbf{j} \prec \mathbf{i}} C_{\mathbf{j}}$), so there is a positive flow from $\varnothing$ to infinity. ∎

Given a network $G = (\mathscr{I}, C_{\mathbf{i}})$ and an integer k it is sometimes useful to consider the reduced network $G^{(k)}$ with tree $\mathscr{I}^{(k)} = \{\mathbf{i} \in \mathscr{I}: k \text{ divides } |\mathbf{i}|\}$ and capacities $\{C_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}^{(k)}\}$.

LEMMA 3·2. *Suppose that in the network G we have $N(\mathbf{i}) \leqslant c$ if $\mathbf{i} \in \mathscr{I}$, and $C_{\mathbf{i},i} \leqslant C_{\mathbf{i}}$ if $\mathbf{i}, i \in \mathscr{I}$. Then a positive flow exists in $G^{(k)}$ if and only if a positive flow exists in G.*

Proof. Clearly if a flow exists in G then its restriction to $\mathscr{I}^{(k)}$ gives a flow in $G^{(k)}$. On the other hand, if $\phi^{(k)}$ is a flow in $\mathscr{I}^{(k)}$, it is easy to see that

$$\phi(\mathbf{j}) = c^{-k} \sum_{\substack{\mathbf{i} \in \mathscr{I}_r \\ \mathbf{j} \prec \mathbf{i}}} \phi(\mathbf{i}) \quad (\mathbf{j} \in \mathscr{I})$$

constitutes a flow in $\mathscr{I}$, where r is the least multiple of k greater than or equal to $|\mathbf{j}|$. ∎

If $G = (\mathscr{I}, C_{\mathbf{i}})$ is a network, we term $G' = (\mathscr{I}', C'_{\mathbf{i}})$ a *subnetwork* of G if $\mathscr{I}'$ is a subtree of $\mathscr{I}$ (i.e. $\mathscr{I}'$ is a tree with $\mathbf{i} \in \mathscr{I}'$ implying $\mathbf{i} \in \mathscr{I}$) and $C'_{\mathbf{i}} = C_{\mathbf{i}}$ if $\mathbf{i} \in \mathscr{I}'$. Clearly, if a positive flow exists in G', then a positive flow exists in G.

4. *Random networks*

We give a random construction for capacitated networks. Several applications spring to mind, for example as a model for the growth of root systems, or the development of river systems. However, our main application will be to random net fractals, in Section 8. A general account of flows in random networks is given in Grimmett [7].

Let Ω be a set and let $G(\omega)$ be a network of the form described in the previous section for each $\omega \in \Omega$. We obtain an increasing sequence of σ-fields of subsets of Ω. Let $\mathscr{F}_1 = \sigma(N(\varnothing); C_i: 1 \leqslant i \leqslant N(\varnothing))$, and given $\mathscr{F}_k$ define

$$\mathscr{F}_{k+1} = \sigma(\mathscr{F}_k; N(\mathbf{i}): \mathbf{i} \in \mathscr{I}_k; C_{\mathbf{i},i_{k+1}}: 1 \leqslant i_{k+1} \leqslant N(\mathbf{i})).$$

Let $\mathscr{F} = \bigcup_{k=1}^{\infty} \mathscr{F}_k$, and assume that p is a probability measure on the sets in $\mathscr{F}$, making $(\Omega, \mathscr{F}, p)$ into a probability space.

We omit the proof of the next lemma which follows using standard methods.

LEMMA 4·1. *$S_k(G)$, $M_k(G)$, $M(G)$ are random variables (i.e. are $\mathscr{F}$-measurable).*

We start to impose a (scaled) common distribution at each stage of the construction. For each $\mathbf{i} = i_1, \ldots, i_k \in \mathscr{I}$ define $J_{\mathbf{i}}$ by

$$C_{\mathbf{i}} = J_{\mathbf{i}} C_{i_1,\ldots,i_{k-1}}. \tag{4·1}$$

For convenience we shall always assume the normalization

$$C_{\varnothing} = 1, \tag{4·2}$$

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so that

$$C_{\mathbf{i}} = J_{i_1} J_{i_1,i_2} \dots J_{i_1,\dots,i_k} \quad (\mathbf{i} = i_1,\dots,i_k). \tag{4·3}$$

We also assume that $0 < J_{\mathbf{i}} \leqslant 1$, giving $0 < C_{\mathbf{i},i} \leqslant C_{\mathbf{i}}$ for all $\mathbf{i}$ and i. We may regard the $J_{\mathbf{i}}$ and $N(\mathbf{i})$ as the defining random variables in the construction of the random network G, rather than the $C_{\mathbf{i}}$ and $N(\mathbf{i})$.

Assume that the process is such that for each $\mathbf{i} \in \mathscr{I}$ the random elements

$$Z(\mathbf{i}) = (N(\mathbf{i}); J_{\mathbf{i},1}, \dots, J_{\mathbf{i},N(\mathbf{i})}) \tag{4·4}$$

are independently and identically distributed. Thus the random network is generated by the random element

$$Z = (N; J_1, \dots, J_N), \tag{4·5}$$

writing $N = N(\varnothing)$ as we frequently shall. Note that we do not require the J_i to be independent of each other or to have the same distribution. We shall assume that $E(N) < \infty$ in what follows.

For a typical random (tree)-network the edges of the kth generation will be large in number and small in capacity if k is large. Thus it is natural to require identical distribution of the ratios of the capacities of consecutive edges rather than of the edge-capacities themselves.

LEMMA 4·2.

$$E(S_k(G)|\mathscr{F}_{k-1}) = S_{k-1}(G)\, E\left(\sum_{i=1}^{N} J_i\right), \tag{4·6}$$

so that in particular

$$E(S_k(G)) = \left(E\left(\sum_{i=1}^{N} J_i\right)\right)^k. \tag{4·7}$$

Proof.

$$S_k(G) = \sum_{\mathbf{i} \in \mathscr{I}_k} C_{\mathbf{i}} = \sum_{\mathbf{i} \in \mathscr{I}_{k-1}} C_{\mathbf{i}} \sum_{i=1}^{N(\mathbf{i})} J_{\mathbf{i},i}.$$

Thus

$$\begin{aligned} E(S_k(G)|\mathscr{F}_{k-1}) &= \sum_{\mathbf{i} \in \mathscr{I}_{k-1}} C_{\mathbf{i}} E\left(\sum_{i=1}^{N(\mathbf{i})} J_{\mathbf{i},i}\right) \\ &= S_{k-1}(G)\, E\left(\sum_{i=1}^{N} J_i\right). \end{aligned}$$

Iteration of (4·6) yields (4·7). ∎

THEOREM 4·3. *Suppose that in the random network model described above*

$$E\left(\sum_{i=1}^{N} J_i\right) < 1.$$

Then flow through the network is impossible a.s.

Proof. Let $\gamma = E(\sum_{i=1}^{N} J_i) < 1$. By (4·6) $\{\gamma^{-k} S_k(G); \mathscr{F}_k\}$ is a martingale, with $E(\gamma^{-k} S_k(G)) = 1$. By the martingale convergence theorem (see, for example, Hall and Heyde [8]), we have $\gamma^{-k} S_k(G) \to S$ a.s., where S is a non-negative random variable on $(\Omega, \mathscr{F}, p)$ with $E(S) \leqslant 1$. In particular $0 \leqslant S < \infty$ and so $M_k(G) \leqslant S_k(G) \to 0$ with probability 1. By Lemma 3·1 flow through the network is impossible a.s. ∎

Our main aim in the next two sections is to prove a converse to Theorem 4·3, that if $E(\sum_{i=1}^{N} J_i) > 1$ a positive flow exists a.s.

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It is useful to have an alternative characterization of $E(\sum_{i=1}^{N} J_i)$.

LEMMA 4·4.

$$\sup_{0\leqslant\rho\leqslant 1} \rho^k E(\#\mathbf{i}\in\mathscr{I}_k\colon C_{\mathbf{i}}\geqslant\rho^k) \leqslant \left(E\left(\sum_1^N J_i\right)\right)^k \leqslant k \sup_{0\leqslant\rho\leqslant 1} \rho^{k-1} E(\#\mathbf{i}\in\mathscr{I}_k\colon C_{\mathbf{i}}\geqslant\rho^k)$$

for $k = 1, 2, \ldots$.

Proof. Since $(\#\mathbf{i}\in\mathscr{I}_k\colon C_{\mathbf{i}}\geqslant\rho^k) \leqslant \sum_{\mathbf{i}\in\mathscr{I}_k} C_{\mathbf{i}}\rho^{-k}$ we get

$$E(\#\mathbf{i}\in\mathscr{I}_k\colon C_{\mathbf{i}}\geqslant\rho^k) \leqslant \rho^{-k} E\Big(\sum_{\mathbf{i}\in\mathscr{I}_k} C_{\mathbf{i}}\Big) \leqslant \rho^{-k}\left(E\left(\sum_1^N J_i\right)\right)^k$$

by (4·7). On the other hand,

$$C_{\mathbf{i}} = k\int_0^1 \rho^{k-1}\chi_{(C_{\mathbf{i}}\geqslant\rho^k)}\,d\rho,$$

so using Fubini's theorem,

$$\begin{aligned} E\Big(\sum_{\mathbf{i}\in\mathscr{I}_k} C_{\mathbf{i}}\Big) &= k\int_0^1 \rho^{k-1} E\Big(\sum_{\mathbf{i}\in\mathscr{I}_k}\chi_{(C_{\mathbf{i}}\geqslant\rho^k)}\Big)\,d\rho \\ &= k\int_0^1 \rho^{k-1} E(\#\mathbf{i}\in\mathscr{I}_k\colon C_{\mathbf{i}}\geqslant\rho^k)\,d\rho, \end{aligned}$$

which, with (4·7), gives the right-hand estimate. ∎

5. *Random networks generated by a branching process*

In this section we examine a model for random networks based on a Galton–Watson branching process. This model underlies the much more general case discussed in the next section. Full details of branching processes may be found in Athreya and Ney [1]; we state the properties that we require as we proceed.

It is clear that the tree underlying the random network described in the previous section may be regarded as a random tree representing a branching process. Thus a typical individual $\mathbf{i}\in\mathscr{I}$ has $N(\mathbf{i})$ offspring where the distribution of $N(\mathbf{i})$, specified in $Z(\mathbf{i})$, is identical and independent for each $\mathbf{i}\in\mathscr{I}$. The number of offspring of the individuals of $\mathscr{I}_0, \mathscr{I}_1, \ldots, \mathscr{I}_{k-1}$ is determined by the σ-fields $\mathscr{F}_k$, and the whole branching process by $\mathscr{F}$, as in the standard setting. We often identify a branching process with its random tree. We obtain some general properties of branching processes which we then apply to random networks.

We first consider a branching process without extinction, so that $N(\mathbf{i})$ is always positive. We are particularly concerned with quantities such as

$$\prod_{r=0}^{k} N(i_1, \ldots, i_r)^{-1} = (N(\varnothing)\,N(i_1)\,N(i_1, i_2)\ldots N(i_1, \ldots, i_r))^{-1}.$$

Notice that for any $\mathbf{i}\in\mathscr{I}_k$

$$1 = \sum_{i\colon \mathbf{i}, i\in\mathscr{I}_{k+1}} N(\mathbf{i})^{-1},$$

so if $\mathscr{S}$ is any finite minimal set of sequences in $\mathscr{I}$

$$1 = \sum_{\mathbf{i}\in\mathscr{S}} \prod_{r=0}^{|\mathbf{i}|-1} N(i_1, \ldots, i_r)^{-1}. \tag{5·1}$$

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For real t, and $k = 1, 2, \ldots$, define

$$Y_k^t = \sum_{\mathbf{i} \in \mathscr{I}_k} \prod_{r=0}^{k-1} N(i_1, \ldots, i_r)^t \tag{5·2}$$

$$= \sum_{\mathbf{i} \in \mathscr{I}_{k-1}} \left(\prod_{r=0}^{k-1} N(i_1, \ldots, i_r)^t \right) N(\mathbf{i}). \tag{5·3}$$

Thus Y_k^t is an r.v. on $\mathscr{F}_k$.

LEMMA 5·1. *Let $\mathscr{I}$ be a branching process with $N > 0$. Then for any real t*

$$E(Y_k^t | \mathscr{F}_{k-1}) = Y_{k-1}^t E(N^{t+1}) \quad (k = 1, 2, \ldots). \tag{5·4}$$

Proof. From (5·3)

$$Y_k^t = \sum_{\mathbf{i} \in \mathscr{I}_{k-1}} \left(\prod_{r=0}^{k-2} N(i_1, \ldots, i_r)^t \right) N(\mathbf{i})^{t+1}$$

$$= \sum_{\mathbf{i} \in \mathscr{I}_{k-2}} \left(\prod_{r=0}^{k-2} N(i_1, \ldots, i_r)^t \right) \sum_{i : \mathbf{i}, i \in \mathscr{I}_{k-1}} N(\mathbf{i}, i)^{t+1}.$$

Thus

$$E(Y_k^t | \mathscr{F}_{k-1}) = \sum_{\mathbf{i} \in \mathscr{I}_{k-2}} \left(\prod_{r=0}^{k-2} N(i_1, \ldots, i_r)^t \right) E\left(\sum_{i : \mathbf{i}, i \in \mathscr{I}_{k-1}} N(\mathbf{i}, i)^{t+1} \,\middle|\, \mathscr{F}_{k-1} \right)$$

$$= \sum_{\mathbf{i} \in \mathscr{I}_{k-2}} \left(\prod_{r=0}^{k-2} N(i_1, \ldots, i_r)^t \right) N(\mathbf{i})\, E(N^{t+1}),$$

since for each $\mathbf{i} \in \mathscr{I}_{k-2}$ there are $N(\mathbf{i})$ integers i with $\mathbf{i}, i \in \mathscr{I}_{k-1}$, with $N(\mathbf{i})$ a random variable on $\mathscr{F}_{k-1}$, and $N(\mathbf{i}, i)$ with the distribution of N independently for each such i. Thus (5·4) follows on comparison with (5·3). ∎

COROLLARY 5·2. *For each real t*

$$E(N^{t+1})^{-k}\, Y_k^t \to W$$

a.s., where W is a non-negative r.v. with $E(W) \leqslant 1$. If $\eta < E(N^{t+1})^{-1}$ then there exists $0 < \lambda < 1$ such that a.s.

$$0 \leqslant \eta^k Y_k^t \leqslant c\lambda^k \quad (k = 0, 1, 2, \ldots) \tag{5·5}$$

for some $c < \infty$.

Proof. By Lemma 5·1 $\{E(N^{t+1})^{-k}\, Y_k^t; \mathscr{F}_k\}$ is a non-negative martingale. By the martingale convergence theorem it converges a.s. to a r.v. W with $E(W) \leqslant 1$. In particular, $E(N^{t+1})^{-k}\, Y_k^t$ is a.s. bounded in k, so (5·5) follows. ∎

LEMMA 5·3. *Let $\mathscr{I}$ be a branching process with $N > 0$. Suppose $a > 0$ satisfies*

$$-\log a < E(\log N) \tag{5·6}$$

Let $\mathscr{U}_k$ be the (random) set of sequences

$$\mathscr{U}_k = \left\{ \mathbf{i} \in \mathscr{I}_k : \prod_{r=0}^{k-1} N(i_1, \ldots, i_r)^{-1} \geqslant a^k \right\}. \tag{5·7}$$

Then there exists $0 < \lambda < 1$ such that with probability 1

$$\sum_{\mathbf{i} \in \mathscr{U}_k} \prod_{r=0}^{k-1} N(i_1, \ldots, i_r)^{-1} < c\lambda^k \quad (k = 1, 2, \ldots) \tag{5·8}$$

for some $c < \infty$.

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Proof. For a moment, assume that N is bounded above. Then if $t > 0$

$$\begin{aligned} a^{-t}E(N^{-t}) &= a^{-t}E(\exp(-t\log N)) \\ &= \exp(-t(\log a + E(\log N)))\, E(\exp t(E(\log N) - \log N)) \\ &\leqslant \exp(-t(\log a + E(\log N)))\, E(1 + t(E(\log N) - \log N) + c_1 t^2) \\ &= \exp(-t(\log a + E(\log N)))\,(1 + c_1 t^2) \end{aligned}$$

for some c_1, since N is bounded and non-zero.

By (5·6)

$$a^{-t}E(N^{-t}) < 1 \tag{5·9}$$

if t is chosen sufficiently small. This also holds if N is unbounded by considering $N' = \min\{m, N\}$, where m is chosen large enough to get $-\log a < E(\log N')$. By (5·7)

$$\begin{aligned} \sum_{\mathbf{i}\in\mathscr{U}_k} \prod_{r=0}^{k-1} N(i_1, \ldots, i_r)^{-1} &\leqslant \sum_{\mathbf{i}\in\mathscr{U}_k} a^{-tk} \prod_{r=0}^{k-1} N(i_1, \ldots, i_r)^{-1-t} \\ &\leqslant a^{-tk}\, Y_k^{-1-t}. \end{aligned}$$

The result follows using Corollary 5·2 with (5·9). ∎

We now return to random networks.

LEMMA 5·4. *Let G be a random network constructed as described in the previous section with $N > 0$. Suppose that $C_{\mathbf{i}} \geqslant a^{|\mathbf{i}|}$ if $\mathbf{i}\in\mathscr{I}$, for some $a > 0$. If $-\log a \leqslant E(\log N)$, then a positive flow through the network exists a.s.*

Proof. We apply our results on branching processes to the random trees $\mathscr{I}$ underlying the random network.

For each k write $\mathscr{T}_k = \bigcup_{q=k}^{\infty} \mathscr{U}_q$ with $\mathscr{U}_q$ as in (5·7). If $\mathscr{S}$ is any finite minimal collection of sequences such that $|\mathbf{i}| \geqslant k$ if $\mathbf{i}\in\mathscr{S}$,

$$\begin{aligned} \sum_{\mathbf{i}\in\mathscr{S}} C_{\mathbf{i}} &\geqslant \sum_{i\in\mathscr{S}} a^{|\mathbf{i}|} \\ &\geqslant \sum_{\mathbf{i}\in\mathscr{S}\setminus\mathscr{T}_k} a^{|\mathbf{i}|} \\ &\geqslant \sum_{\mathbf{i}\in\mathscr{S}\setminus\mathscr{T}_k} \prod_{r=0}^{|\mathbf{i}|-1} N(i_1, \ldots, i_r)^{-1} \\ &\geqslant \sum_{\mathbf{i}\in\mathscr{S}} \prod_{r=0}^{|\mathbf{i}|-1} N(i_1, \ldots, i_r)^{-1} - \sum_{\mathbf{i}\in\mathscr{T}_k} \prod_{r=0}^{|\mathbf{i}|-1} N(i_1, \ldots, i_r)^{-1} \\ &\geqslant 1 - \sum_{q=k}^{\infty} \sum_{\mathbf{i}\in\mathscr{U}_q} \prod_{r=0}^{q-1} N(i_1, \ldots, i_r)^{-1} \\ &\geqslant 1 - c\sum_{q=k}^{\infty} \lambda^q, \end{aligned}$$

for some constant $0 < \lambda < 1$, where $c < \infty$ a.s. is independent of the choice of $\mathscr{S}$, using (5·1) and Lemma 5·3. Thus

$$M_k(G) \geqslant 1 - c_1\lambda^k.$$

Letting $k\to\infty$, it follows that $M(G) \geqslant 1$ a.s. By Lemma 3·1 a flow through the network exists a.s. ∎

LEMMA 5·5. *Let G be a random network constructed as in the previous section, with $N > 0$. Suppose that $C_{\mathbf{i}} \geqslant a^{|\mathbf{i}|}$ if $\mathbf{i}\in\mathscr{I}$, where $a > 0$. Then if $aE(N) > 1$, a positive flow exists a.s.*

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Proof. By replacing N by $N' = \min\{m, N\}$ where m is large enough to make $aE(N') > 1$, we may take N to be bounded. For $k \geqslant 1$, consider the branching process with tree $\mathscr{I}^{(k)} = \{\mathbf{i} \in \mathscr{I} : k \text{ divides } |\mathbf{i}|\}$. By standard theory, this process is realized by letting $N^{(k)}(\mathbf{i}) = \#\{\mathbf{j}, \mathbf{j} \in \mathscr{I}^{(k)}, |\mathbf{j}| = k\}$ be i.i.d. for each $\mathbf{i} \in \mathscr{I}^{(k)}$ with probability generating function the k-fold composition $f \circ \ldots \circ f$, where f is the probability generating function of N. Moreover, $N^{(k)}/E(N)^k \to W$ a.s. as $k \to \infty$, where W is a r.v. on $\mathscr{F}$ with $0 < W < \infty$. Thus, given $\epsilon > 0$,

$$\left| \frac{1}{k} E(\log N^{(k)}) - \log E(N) \right| < \epsilon \tag{5.10}$$

if k is sufficiently large, by the bounded convergence theorem.

By Lemma 3·2 flow through the network G will be possible if a positive flow exists through the network with tree $\mathscr{I}^{(k)}$ generated by $N^{(k)}$ and with capacities $\{C_{\mathbf{i}} : \mathbf{i} \in \mathscr{I}^{(k)}\}$. But

$$\begin{aligned} E(\log N^{(k)}) &\geqslant k(\log E(N) - \epsilon) \\ &> k(-\log a) \\ &= -\log a^k, \end{aligned}$$

if ϵ is small enough, since $\log E(N) > -\log a$. Applying Lemma 5·4 to this network, positive flow is possible a.s. ∎

Next we allow the possibility of the underlying branching process becoming extinct, so that N may be any non-negative integer. Suppose the branching process with progeny distribution N has probability generating function

$$f(u) = E(u^N) = \sum_{m=0}^{\infty} p(N = m) u^m.$$

The extinction probability q of the process, that is the probability that $\mathscr{I}_k$ is empty for all sufficiently large k, equals the smallest non-negative root of the equation $f(u) = u$. If $E(N) \leqslant 1$ then $q = 1$ (except in the trivial case when $N = 1$ a.s.) but if $E(N) > 1$ then $0 \leqslant q < 1$. Moreover, $q = 0$ if and only if $p(N = 0) = 0$. From now on q will always denote the extinction probability of the branching process generated by N.

We extend the previous result to cover the possibility of extinction, which results in the network terminating.

THEOREM 5·6. *Let G be a random network with $N \geqslant 0$. Suppose that $C_{\mathbf{i}} \geqslant a^{|\mathbf{i}|}$ if $\mathbf{i} \in \mathscr{I}$, where $a > 0$ and $aE(N) > 1$. Then with probability q the network terminates and with probability $1 - q$ a positive flow through G exists.*

Proof. The network terminates if $\mathscr{I}$ becomes extinct, and this happens with probability q.

Otherwise, if $0 \leqslant q < 1$, define a tree

$$\mathscr{I}' = \{\mathbf{i} \in \mathscr{I} : \text{there exists } \mathbf{j} \in \tilde{\mathscr{I}} \text{ with } \mathbf{i} \prec \mathbf{j}\}.$$

Thus $\mathscr{I}'$ consists of those individuals of the branching process whose descendents never die out. By a standard result (see Athreya and Ney [1], §I·12), the random tree $\mathscr{I}'$ has exactly the same distribution (down to any generation) as that of the branching process with progeny distribution N' having probability p' given by

$$\begin{aligned} &p'(N' = 0) = 0, \\ &p'(N' = m) = (1-q)^{-1} \sum_{r=m}^{\infty} p(N = r) \binom{r}{m} (1-q)^m q^{r-m} \quad (m \geqslant 1). \end{aligned}$$

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In particular $E'(N') = E(N)$. Let G' be the random network obtained using the tree $\mathscr{I}'$ and the capacities $\{C_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}'\}$. Then $aE(N') > 1$, so a flow through G' exists a.s. by Lemma 5·5. |

COROLLARY 5·7. *Let G be a random network with $N \geqslant 0$. Suppose that $C_{\mathbf{i}} = a^{|\mathbf{i}|}$ if $\mathbf{i} \in \mathscr{I}$, for some $a > 0$. Then the network terminates with probability q. Otherwise, if $aE(N) < 1$ flow is impossible a.s. and if $aE(N) > 1$ a positive flow exists a.s.*

Proof. This combines Theorems 4·3 and 5·6, noting that $E(\sum_1^N J_i) = aE(N)$. |

6. *Flows in general random networks*

This section extends the results of the last section to the general model for random networks with commonly distributed capacity ratios introduced in Section 4. Thus we now let the $Z(\mathbf{i})$ of (4·4) be any i.i.d. elements.

LEMMA 6·1. *Let G be a random network. If*

$$\rho E(\#i: J_i \geqslant \rho) > 1 \tag{6·1}$$

for some $0 < \rho \leqslant 1$, then a flow through G exists with positive probability.

Proof. For any realization of the process, define a tree

$$\mathscr{I}' = \{\mathbf{i} = i_1, \ldots, i_k \in \mathscr{I}: J_{i_1, \ldots, i_r} \geqslant \rho \text{ for all } 1 \leqslant r \leqslant k\}.$$

Let G' be the network with tree $\mathscr{I}'$ and capacities $\{C_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}'\}$; then $C_{\mathbf{i}} \geqslant \rho^{|\mathbf{i}|}$ if $\mathbf{i} \in \mathscr{I}'$. We may regard $\mathscr{I}'$ as a branching process with progeny distribution given by

$$N' = \{\#i: J_i \geqslant \rho\}.$$

Certainly $E(N') > 1$ by (6·1), so $\mathscr{I}'$ has extinction probability $q < 1$; but also $\rho E(N') > 1$, so there is a probability of $1 - q > 0$ of a flow through G' by Theorem 5·6. As G' is a subnetwork of G, the result follows. |

LEMMA 6·2. *Let G be a random network. If*

$$\rho^k E(\#\mathbf{i} \in \mathscr{I}_k: C_{\mathbf{i}} \geqslant \rho^k) > 1 \tag{6·2}$$

for some $0 < \rho \leqslant 1$ and some positive integer k, then there is positive probability of a positive flow through G.

Proof. By replacing G by the network with tree

$$\mathscr{I}' = \{\mathbf{i} = i_1, \ldots, i_k \in \mathscr{I}: 1 \leqslant i_r \leqslant m \quad \text{for} \quad 1 \leqslant r \leqslant k\}$$

and capacities $\{C_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}'\}$ where m is sufficiently large, we may assume N to be bounded. Consider the network $G^{(k)}$ with tree $\mathscr{I}^{(k)} = \{\mathbf{i} \in \mathscr{I}: k \text{ divides } |\mathbf{i}|\}$ and capacities $\{C_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}^{(k)}\}$. $G^{(k)}$ may be regarded as the random network generated by taking $Z = (\#\mathbf{i} \in \mathscr{I}_k; C_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}_k)$ in (4·5). Applying Lemma 6·1 to $G^{(k)}$, there is positive probability of a flow in $G^{(k)}$ so by Lemma 3·2 there is positive probability of a flow in G. |

THEOREM 6·3. *Let G be a random network. The network terminates with probability q. Otherwise, if $E(\sum_{i=1}^N J_i) < 1$ positive flow is a.s. impossible, and if $E(\sum_{i=1}^N J_i) > 1$ a positive flow through G exists a.s.*

Proof. The network terminates if $\mathscr{I}$ becomes extinct, which happens with probability q.

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Otherwise, if $E(\sum_{i=1}^{N} J_i) < 1$, flow is almost surely impossible, by Theorem 4·3. On the other hand, if $E(\sum_{i=1}^{N} J_i) > 1$, we may, using Lemma 4·4, find $0 < \rho \leqslant 1$ and a positive integer k such that $\rho^k E(\#\mathbf{i} \in \mathscr{I}_k : C_{\mathbf{i}} \geqslant \rho^k) > 1$. By Lemma 6·2 there is a probability $s > 0$ of flow in G. By virtue of the self-similar construction there is a probability of s of a flow from $\mathbf{i}$ to infinity, independently for each $\mathbf{i} \in \mathscr{I}_k$. Hence, letting $N^{(k)} = (\#\mathbf{i} : \mathbf{i} \in \mathscr{I}_k)$,

$$\begin{aligned} p\,(\text{a positive flow through } K \text{ exists}) &\geqslant \sum_{m=0}^{\infty} p(N^{(k)} = m)\,(1-(1-s)^m) \\ &\geqslant (1-(1-s)^r) \sum_{m=r}^{\infty} p(N^{(k)} = m) \\ &\geqslant (1-(1-s)^r) \end{aligned}$$

for any r, since $N^{(k)} \to \infty$ a.s. given that $\mathscr{I}$ does not become extinct. Thus a positive flow exists with probability $1-q$. ∎

7. *Net fractals*

This section describes the construction and basic properties of a class of fractals (not as yet random) obtained by generalizing the classical construction of the 'middle third' Cantor set. Such sets, which occur frequently in theory and in practice, will be termed *net fractals*; it is high time that this class of set had a name, and this has been chosen because of the closely related net measures (see Falconer[5], chapter 5 or Rogers[12], §2·7).

Let $\mathscr{I}$ be a tree and let $\bar{\mathscr{I}}$ be the associated set of infinite sequences (see Section 2). Let $\{I_{\mathbf{i}} : \mathbf{i} \in \mathscr{I}\}$ be a collection of compact subsets of $\mathbb{R}^n$, partially ordered by inclusion and indexed by $\mathscr{I}$, so that $I_{\mathbf{i}'} \subset I_{\mathbf{i}}$ whenever $\mathbf{i} \prec \mathbf{i}'$. In particular there is a set $I_{\varnothing}$ with $I_{\mathbf{i}} \subset I_{\varnothing}$ for all $\mathbf{i} \in \mathscr{I}$. (We do not at this juncture demand that $I_{\mathbf{i}}$ and $I_{\mathbf{i}'}$ be disjoint if $\mathbf{i}$ and $\mathbf{i}'$ are incomparable under $\prec$.)

Write $|\ |$ for the diameter of subsets of $\mathbb{R}^n$. We always assume that $|I_{\mathbf{i}}| > 0$ if $\mathbf{i} \in \mathscr{I}$. Usually we have

$$|I_{i_1,\dots,i_r}| \to 0 \quad \text{as} \quad r \to \infty \quad \text{if} \quad \mathbf{i} = i_1, i_2, \dots, \in \bar{\mathscr{I}}. \tag{7·1}$$

If this is the case, define functions $f_r : \bar{\mathscr{I}} \to \mathbb{R}$ by $f_r(i_1, i_2, \dots) = |I_{i_1,\dots,i_r}|$, then $\{f_r\}$ is a sequence of continuous functions decreasing to zero on the compact metric space $(\bar{\mathscr{I}}, \tau)$. By Dini's theorem convergence is uniform, so given $\delta > 0$ there exists $k(\delta)$ such that

$$|I_{\mathbf{i}}| \leqslant \delta \quad \text{whenever} \quad |\mathbf{i}| \geqslant k(\delta). \tag{7·2}$$

Moreover, $\bigcap_{r=1}^{\infty} I_{i_1,\dots,i_r}$ is a single point of $\mathbb{R}^n$ for each $\mathbf{i} = i_1, i_2, \dots$, and the mapping $\phi : \bar{\mathscr{I}} \to \mathbb{R}^n$ defined by $\phi(\mathbf{i}) = \bigcap_{r=1}^{\infty} I_{i_1,\dots,i_r}$ is continuous on $(\bar{\mathscr{I}}, \tau)$. We term the set $K = \phi(\bar{\mathscr{I}})$ a *net fractal* constructed on the tree $\mathscr{I}$ with the sets $\{I_{\mathbf{i}} : \mathbf{i} \in \mathscr{I}\}$; in general such a set will be a fractal by any reasonable definition. Note that ϕ need not be injective.

LEMMA 7·1. $K = \bigcap_{k=0}^{\infty} \bigcup_{\mathbf{i} \in \mathscr{I}_k} I_{\mathbf{i}}$ *and is compact.*

Proof. Clearly $K \subset \bigcup_{\mathbf{i} \in \mathscr{I}_k} I_{\mathbf{i}}$ for all k. On the other hand, suppose $x \in \bigcup_{\mathbf{i} \in \mathscr{I}_k} I_{\mathbf{i}}$ for each k, say $x \in I_{\mathbf{i}(k)}$ where $\mathbf{i}(k) \in \mathscr{I}_k$. Since $\mathscr{I}_k$ is finite for all k, a standard diagonal argument leads to an infinite sequence $\mathbf{i} = (i_1, i_2, \dots)$ such that for each r we have

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$(i_1, \ldots, i_r) \prec \mathbf{i}(k)$ for infinitely many values of k. Thus $\mathbf{i} \in \tilde{\mathscr{I}}$ and $x = \bigcap_{r=1}^{\infty} I_{i_1,\ldots,i_r} \in K$. Finally, K is compact as the intersection of a decreasing sequence of compact sets. Alternatively, K is the image of a compact set under the continuous function ϕ. |

Observe that a given net fractal K may be constructed on many different trees and with many different collections of sets $\{I_\mathbf{i}\}$. Whilst it is possible to define K purely in terms of $\mathscr{I}$ and a mapping ϕ, such a definition gives little information about the metric structure of the set. In any case net fractals are almost invariably obtained from some such nest of sets.

We say that $\mathscr{I}$ and $\{I_\mathbf{i}: \mathbf{i} \in \mathscr{I}\}$ provide a *proper* construction for the net fractal K if $K \subset \bigcup_{\mathbf{i} \in \mathscr{S}} I_\mathbf{i}$ implies that $\mathscr{S}$ is a complete collection of sequences. Equivalently, this is the case if for any minimal set of sequences $\mathscr{S}$, each set of the collection $\{I_\mathbf{i}: \mathbf{i} \in \mathscr{S}\}$ contains a point of K that lies in no other set of the collection, or again equivalently, if this holds just for $\mathscr{S} = \mathscr{I}_k$, $k = 1, 2, \ldots$. We shall say that K is a *proper net fractal*, without reference to the underlying sets, when it is clear what construction is being used.

We now introduce the 'net measures' which are defined in terms of the sets $\{I_\mathbf{i}: \mathbf{i} \in \mathscr{I}\}$ and are therefore easier to work with than Hausdorff measures when studying particular net fractals. However, given certain conditions, the net measures and Hausdorff measures are closely related.

Given $0 \leqslant s < \infty$ we define an outer measure on subsets of K in an analogous way to Hausdorff measure but using covers from $\{I_\mathbf{i}: \mathbf{i} \in \mathscr{I}\}$. If k is a positive integer and $F \subset K$, write

$$\mathscr{M}_k^s(F) = \inf_{\mathscr{S}} \Big\{ \sum_{\mathbf{i} \in \mathscr{S}} |I_\mathbf{i}|^s : F \subset \bigcup_{\mathbf{i} \in \mathscr{S}} I_\mathbf{i} \quad \text{and} \quad |\mathbf{i}| \geqslant k \quad \text{if} \quad \mathbf{i} \in \mathscr{S} \Big\} \tag{7·3}$$

and

$$\mathscr{M}^s(F) = \lim_{k \to \infty} \mathscr{M}_k^s(F). \tag{7·4}$$

Then $\mathscr{M}^s$ is an outer measure of Hausdorff type on subsets of K. Standard methods (see Rogers[12], chapter 1 or Falconer[5], §1·1) show that the $\mathscr{M}^s$-measurable sets include the Borel sets. By (7·2) any covering used in the definition of $\mathscr{M}_{k(\delta)}^s$ is an admissible covering in the definition of $\mathscr{H}_\delta^s$, so

$$\mathscr{H}_\delta^s(F) \leqslant \mathscr{M}_{k(\delta)}^s(F),$$

leading to

$$\mathscr{H}^s(F) \leqslant \mathscr{M}^s(F) \tag{7·5}$$

for $F \subset K$. (Note that an alternative to (7·3) is to define $\mathscr{M}_\delta^s$ as an infimum over covers with $|I_\mathbf{i}| \leqslant \delta$ and to let $\delta \to 0$ to get the same outer measure $\mathscr{M}^s$.)

For convenience of exposition, we introduce a dimension of K in an analogous way to Hausdorff dimension but based on the net measures $\mathscr{M}^s$ and so intrinsic to the construction of K. Thus

$$\dim_M K = \inf\{s: \mathscr{M}^s(K) = 0\} = \sup\{s: \mathscr{M}^s(K) = \infty\}. \tag{7·6}$$

It is clear from (7·5) that

$$\dim K \leqslant \dim_M K. \tag{7·7}$$

Much of this section is devoted to finding conditions which allow inequalities (7·5) and (7·7) to be reversed.

First, assume that

$$I_\mathbf{i} = \overline{\operatorname{int} I_\mathbf{i}} \quad (\mathbf{i} \in \mathscr{I}) \tag{7·8}$$

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(the bar denoting closure in $\mathbb{R}^n$), and also that the open sets

$$\{\operatorname{int} I_{\mathbf{i}} : \mathbf{i} \in \mathscr{I}\} \text{ form a net,} \tag{7·9}$$

that is $\operatorname{int} I_{\mathbf{i}} \supset \operatorname{int} I_{\mathbf{i}'}$ if $i \prec i'$, but $\operatorname{int} I_{\mathbf{i}} \cap \operatorname{int} I_{\mathbf{i}'} = \varnothing$ if neither $\mathbf{i} \prec \mathbf{i}'$ nor $\mathbf{i}' \prec \mathbf{i}$. These conditions permit $I_{\mathbf{i}}$ and $I_{\mathbf{i}'}$ to overlap 'a little but not too much' when $\mathbf{i}$ and $\mathbf{i}'$ are incomparable under $\prec$; this is the case in many important examples.

Next we require a condition to ensure that the sets $I_{\mathbf{i}}$ do not become small too rapidly. Assume that there is a constant $a > 0$ such that

$$a\,|I_{\mathbf{i}}| \leqslant |I_{\mathbf{i},i}| \leqslant |I_{\mathbf{i}}| \quad \text{if} \quad \mathbf{i}, i \in \mathscr{I}. \tag{7·10}$$

Finally assume that there are $\lambda > 0$ and $\nu \geqslant 1$, independent of $\mathbf{i}$, such that

$$\operatorname{inradius}(I_{\mathbf{i}}) \geqslant \lambda\,|I_{\mathbf{i}}|^{\nu} \quad (\mathbf{i} \in \mathscr{I}). \tag{7·11}$$

Very often we can take $\nu = 1$, so that

$$\operatorname{inradius}(I_i) \geqslant \lambda\,|I_{\mathbf{i}}| \quad (\mathbf{i} \in \mathscr{I}), \tag{7·12}$$

but it is useful to allow the possibility of $\nu > 1$ enabling estimates of dimension to be made in 'non-linear' cases. These conditions control the rate at which the sets $I_{\mathbf{i}}$ can become elongated as $|\mathbf{i}| \to \infty$, and ensure that the $\{I_{\mathbf{i}}\}$ provide reasonably efficient coverings for calculating $\mathscr{H}^s$.

If K is a net fractal constructed from sets $\{I_{\mathbf{i}} : \mathbf{i} \in \mathscr{I}\}$ satisfying (7·1) and (7·8)–(7·11), we shall say that conditions (CN_ν) hold.

We show that under such conditions $\mathscr{H}^s$ may be estimated in terms of $\mathscr{M}^s$. The proof is a modified form of that in Hutchinson [9]; see also Moran [11]. We require a simple geometric lemma.

LEMMA 7·2. *Suppose $c_1 > 0$ and $\nu \geqslant 1$, and let $U \subset \mathbb{R}^n$ be any set with $|U| = \rho > 0$. Let $\{O_\alpha\}$ be a family of disjoint open sets, and suppose that each O_α contains a ball of radius $c_1\rho^\nu$ and is contained in a ball of radius ρ. Then at most $3^n c_1^{-n}\rho^{(1-\nu)n}$ of the $\{\overline{O_\alpha}\}$ intersect U.*

Proof. Choose $x \in U$ and let B be the closed ball of centre x and radius 3ρ. Any set $\overline{O_\alpha}$ that meets U is contained in B. Thus if the closures of q disjoint O_α intersect U we get, summing the volumes of the corresponding interior balls of radius $c_1\rho^\nu$, that

$$q(c_1\rho^\nu)^n \leqslant (3\rho)^n$$

leading to the result. ∎

THEOREM 7·3. *Let $\{I_{\mathbf{i}} : \mathbf{i} \in \mathscr{I}\}$ satisfy conditions (CN_ν), $\nu \geqslant 1$. Then there is a constant $c > 0$ such that*

$$c\mathscr{M}^{s+(\nu-1)n}(F) \leqslant \mathscr{H}^s(F) \leqslant \mathscr{M}^s(F)$$

for any $F \subset K$.

Proof. We already have the right-hand inequality (7·5).

Suppose that U is a set with $|U| = \rho \leqslant |I_\varnothing|$. Curtail each infinite sequence

$$\mathbf{i} = (i_1, i_2, \ldots) \in \tilde{\mathscr{I}}$$

at the least value of k for which

$$a\rho \leqslant |I_{i_1,\ldots,i_k}| \leqslant \rho; \tag{7·13}$$

such a k exists by (7·1) and (7·10). Let $\mathscr{S}$ denote the set of finite sequences obtained in this way. It is easy to see that $\mathscr{S}$ is minimal. It follows from the net property (7·9)

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that the collection of open sets $\{\operatorname{int} I_{\mathbf{i}}: \mathbf{i} \in \mathscr{S}\}$ is disjoint. Each such $\operatorname{int} I_{\mathbf{i}}$ contains a ball of radius $\lambda |I_{\mathbf{i}}|^{\nu} \geqslant \lambda a^{\nu} \rho^{\nu}$ by (7·11) and (7·13), and is contained in a ball of radius $|I_{\mathbf{i}}| \leqslant \rho$ by (7·13). By Lemma 7·2, at most $q\rho^{(1-\nu)n}$ of the sets $\{\overline{\operatorname{int} I_{\mathbf{i}}}: \mathbf{i} \in \mathscr{S}\} = \{I_{\mathbf{i}}: \mathbf{i} \in \mathscr{S}\}$ meet U, where $q = 3^n(\lambda a^{\nu})^{-n}$. Since $K \subset \bigcup_{\mathbf{i} \in \mathscr{S}} I_{\mathbf{i}}$, we may, for any U with $|U| \leqslant |I_{\varnothing}|$, select a collection $\mathscr{C}$ of at most $q|U|^{(1-\nu)n}$ sets from $\{I_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}\}$ with diameters at most $|U|$ and which cover $U \cap K$. Thus

$$\sum_{\mathbf{i} \in \mathscr{C}} |I_{\mathbf{i}}|^{s+(\nu-1)n} \leqslant q|U|^{(1-\nu)n} |U|^{s+(\nu-1)n} = q|U|^{s},$$

so that

$$\mathscr{M}_k^{s+(\nu-1)n}(F) \leqslant q \mathscr{H}_{\delta}^{s}(F)$$

for any $k \leqslant (\log \delta + \log |I_{\varnothing}|)/\log a$, iterating (7·10). Letting $\delta \to 0$ gives the result. ∎

COROLLARY 7·4. *Let K be a net fractal constructed from a collection of sets satisfying (CN_{ν}), $\nu \geqslant 1$. Then*

$$\dim_M K - (\nu-1)n \leqslant \dim K \leqslant \dim_M K.$$

In particular, if (CN_1) holds,

$$\dim K = \dim_M K.$$

Finally in this section we connect the ideas of net fractals and flows in networks.

THEOREM 7·5. *Let K be a proper net fractal constructed using the tree $\mathscr{I}$ and the sets $\{I_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}\}$. Fix $0 < s < \infty$ and let G be the network with tree $\mathscr{I}$ and capacities $C_{\mathbf{i}} = |I_{\mathbf{i}}|^s$. With the notation of Sections 3 and 7,*

$$\mathscr{M}_k^s(K) = M_k(G), \tag{7·14}$$

$$\mathscr{M}^s(K) = M(G), \tag{7·15}$$

$$\sum_{\mathbf{i} \in \mathscr{I}_k} |I_{\mathbf{i}}|^s = S_k(G). \tag{7·16}$$

Thus $\mathscr{M}^s(K) > 0$ if and only if a positive flow exists through G.

Proof. Since the construction is proper (7·3) gives

$$\begin{aligned}\mathscr{M}_k^s(K) &= \inf\Big\{\sum_{\mathbf{i} \in \mathscr{S}} |I_{\mathbf{i}}|^s : \mathscr{S} \text{ is complete, } |\mathbf{i}| \geqslant k\Big\} \\ &= \inf\Big\{\sum_{\mathbf{i} \in \mathscr{S}} C_{\mathbf{i}} : \mathscr{S} \text{ is complete, } |\mathbf{i}| \geqslant k\Big\} \\ &= M_k(G).\end{aligned}$$

Letting $k \to \infty$ gives (7·15), and (7·16) is immediate. Lemma 3·1 states that a positive flow exists through G if and only if $\mathscr{M}^s(K) = M(G) > 0$. ∎

8. *Random net fractals*

We randomize the construction of the last section. Our main aim is to find an expression for the almost sure Hausdorff dimension of fractals constructed in a 'statistically self-similar' manner, see Theorem 8·5.

Fix $I_{\varnothing}$. Let Ω be a set and suppose $\mathscr{I}(\omega)$ and $\{I_{\mathbf{i}}(\omega): \mathbf{i} \in \mathscr{I}(\omega)\}$ provide a proper construction for a net fractal $K(\omega)$ for each $\omega \in \Omega$. We define an increasing sequence

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of σ-fields of subsets of Ω. Let $\mathscr{F}_1 = \sigma(N(\varnothing), |I_{i_1}|: 1 \leqslant i_1 \leqslant N(\varnothing))$, and given $\mathscr{F}_k$ define

$$\mathscr{F}_{k+1} = \sigma(\mathscr{F}_k; N(\mathbf{i}): \mathbf{i} \in \mathscr{I}_k; |I_{\mathbf{i},i_{k+1}}|: 1 \leqslant i_{k+1} \leqslant N(\mathbf{i})).$$

Let

$$\mathscr{F} = \bigcup_{k=1}^{\infty} \mathscr{F}_k,$$

and assume that p is a probability measure making $(\Omega, \mathscr{F}, p)$ into a probability space.

For fixed s we associate a random network G with the random net fractal K, taking the tree $\mathscr{I}$ and capacities $C_{\mathbf{i}} = |I_{\mathbf{i}}|^s$. Then the σ-fields $\mathscr{F}_k$, $\mathscr{F}$ are precisely those introduced in Section 4 in terms of networks; replacing $|I_{\mathbf{i}}|$ by $|I_{\mathbf{i}}|^s$ does not alter the σ-fields generated.

LEMMA 8·1. *With K a proper random net fractal as above, $\mathscr{M}^s_k(K)$, $\mathscr{M}^s(K)$ and $\dim_M K$ are random variables (i.e. are $\mathscr{F}$-measurable).*

Proof. For given s, the measurability of $\mathscr{M}^s_k(K)$ and $\mathscr{M}^s(K)$ follows from Lemma 4·1 using (7·14) and (7·15). But

$$\{\omega: \dim_M K(\omega) > s\} = \bigcup_{j=1}^{\infty} \{\omega: \mathscr{M}^{s+1/j}(K(\omega)) > 0\},$$

so $\dim_M K$ is also an r.v. ∎

Notice that, as far as dimension calculations are concerned, there is no need to impose measurability conditions on the relative positions of the I_i. Provided we have a proper net fractal it has no bearing on the dimension.

Now we impose a degree of (statistical) self-similarity on the random fractal K. If K is constructed from the tree $\mathscr{I}$ and sets $\{I_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}\}$, define

$$|I_{\mathbf{i}}| = L_{\mathbf{i}} |I_{i_1,\ldots,i_{k-1}}| \quad (\mathbf{i} = i_1, \ldots, i_k \in \mathscr{I}). \tag{8·1}$$

From now on we assume the normalization

$$|I_{\varnothing}| = 1, \tag{8·2}$$

so that

$$|I_{\mathbf{i}}| = L_{i_1} L_{i_1,i_2} \ldots L_{i_1,\ldots,i_k} \quad (\mathbf{i} = i_1, \ldots, i_k). \tag{8·3}$$

Clearly $0 < L_{\mathbf{i}} \leqslant 1$ for $\mathbf{i} \in \mathscr{I}$. We may take $N(\mathbf{i})$ and $\{L_{\mathbf{i}}: \mathbf{i} \in \mathscr{I}\}$ to be the r.v.s underlying the σ-fields $\mathscr{F}_k$, $\mathscr{F}$ rather than $N(\mathbf{i})$ and $\{|I_{\mathbf{i}}|\}$.

Assume for the remainder of this section that the random elements

$$W(\mathbf{i}) = (N(\mathbf{i}); L_{\mathbf{i},1}, \ldots, L_{\mathbf{i},N(\mathbf{i})}) \tag{8·4}$$

are i.i.d. for each $\mathbf{i} \in \mathscr{I}$. Thus the distribution of $\mathscr{I}$ and $|I_{\mathbf{i}}|$ are determined by

$$W(\varnothing) = (N; L_1, \ldots, L_N).$$

We do not require that the L_i are independent of each other or have the same distribution. As before, q will be the extinction probability of the branching process generated by N.

LEMMA 8·2. *$E(\sum_{i=1}^N L_i^s)$ is continuous from above and non-increasing in s. Where it is finite it is continuous on open intervals, and, provided that $p(\sup_{1 \leqslant i \leqslant N} L_i < 1) > 0$, is strictly decreasing.*

Proof. This follows from the monotone convergence theorem and that $0 < L_{\mathbf{i}} \leqslant 1$. ∎

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Usually there is a unique solution to the equation $E(\sum_{i=1}^N L_i^s) = 1$, which gives the almost sure value of $\dim_M K$.

LEMMA 8·3.

$$E(\sum_{\mathbf{i}\in\mathscr{I}_k} |I_{\mathbf{i}}|^s \mathscr{F}_{k-1}) = (\sum_{\mathbf{i}\in\mathscr{I}_{k-1}} |I_{\mathbf{i}}|^s)\left(E\left(\sum_{i=1}^N L_i^s\right)\right).$$

Proof. This is a translation of Lemma 4·2 to the present setting, using (7·16). ∎

The upper estimates for the net measures are straight forward.

THEOREM 8·4. *Let K be a proper random net fractal. Suppose that*

$$E\left(\sum_{i=1}^N L_i^s\right) \leqslant 1. \tag{8·5}$$

Them $\mathscr{M}^s(K) < \infty$ a.s. Moreover, $\mathscr{H}^s(K) < \infty$ a.s. provided that (7·2) holds a.s.; this will certainly be the case if $E(\sum_{i=1}^N L_i^t) < 1$ for some $t > 0$.

Proof. By Lemma 8·3 and (8·5), $\{E(\sum_{\mathbf{i}\in\mathscr{I}_k} |I_{\mathbf{i}}|^s); \mathscr{F}_{k-1}\}$ is a positive supermartingale, so by the convergence theorem, $\mathscr{M}_k^s(K) \leqslant \sum_{\mathbf{i}\in\mathscr{I}_k} |I_{\mathbf{i}}|^s$ is bounded a.s. and $\mathscr{M}^s(K) < \infty$ a.s. Similarly, if $E(\sum_{i=1}^N L_i^t) < 1$ for some t, then $\sum_{\mathbf{i}\in\mathscr{I}_k} |I_{\mathbf{i}}|^s \to 0$ a.s. giving (7·2) and thus (7·5). ∎

THEOREM 8·5. *Let K be a proper random net fractal. Then $K = \varnothing$ with probability q, and $\dim_M K = d$ with probability $1-q$, where*

$$d = \min\left\{s: E\left(\sum_{i=1}^N L_i^s\right) \leqslant 1\right\}.$$

Proof. Clearly, if the branching process becomes extinct then $K = \varnothing$. Otherwise $E(N) > 1$ and $0 < d \leqslant \infty$. If G is the random network with tree $\mathscr{I}$ and capacities $C_{\mathbf{i}} = |I_{\mathbf{i}}|^s$ we have $J_{\mathbf{i}} = L_{\mathbf{i}}^s$ in the setting of Section 4. Thus G is the random network obtained taking $Z(\mathbf{i}) = (N(\mathbf{i}); L_{\mathbf{i},1}^s, \ldots, L_{\mathbf{i},N(\mathbf{i})}^s)$ in (4·4). If $0 < s < d$ then $E(\sum_{i=1}^N J_i) > 1$, so given that the process does not become extinct a positive flow exists through G a.s. by Theorem 6·3. By Theorem 7·5 $\mathscr{M}^s(K) > 0$, so $\dim_M K \geqslant d$ a.s. The opposite inequality comes from Theorem 8·4. ∎

We couch the result in terms of Hausdorff dimensions.

COROLLARY 8·6. *Let K be a proper random net fractal in $\mathbb{R}^n$ with $0 < L_{\mathbf{i}} < 1$, where inradius $(I_{\mathbf{i}}) \geqslant \lambda |I_{\mathbf{i}}|^\nu$ for constants $\lambda > 0$ and $\nu \geqslant 1$, and where (7·8), (7·9) hold, for all $\mathbf{i}$ for all realizations of the process. Then $K = \varnothing$ with probability q, and*

$$d - (\nu - 1)n \leqslant \dim K \leqslant d$$

with probability $1-q$, where $d = \min\{s: E(\sum_{i=1}^N L_i^s) \leqslant 1\}$. In particular, if $\nu = 1$ then $\dim K = d$.

Proof. First consider the special case with $0 < a \leqslant L_i \leqslant b < 1$ for constants a, b. Then the result follows by combining Theorem 8·5 with Corollary 7·4, noting that in this case (CN_ν) holds for any realization of K.

For the general case $0 < L_{\mathbf{i}} < 1$, suppose $0 < s < d$. We may find $\epsilon > 0$ such that

$$E\left(\sum_{i=1}^N L_i^s: \epsilon \leqslant L_i \leqslant 1-\epsilon\right) > 1.$$

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By taking ϵ small enough we can arrange for the probability of extinction of the branching process generated by $N' = \{\#i: \epsilon \leqslant L_i \leqslant 1-\epsilon\}$ to be as close as we like to q. Let K' be the random net fractal obtained taking

$$\mathscr{I}' = \{\mathbf{i} = i_1, \ldots, i_k \in \mathscr{I}: \epsilon < L_{i_1,\ldots,i_r} \leqslant 1-\epsilon \quad \text{for} \quad 1 \leqslant r \leqslant k\}$$

and $I'_{\mathbf{i}} = I_{\mathbf{i}}$ if $\mathbf{i} \in \mathscr{I}'$. Then $s-(\nu-1)n \leqslant \dim K' \leqslant \dim K$, applying the special case to $K' \subset K$.

On the other hand, by summing volumes we see that $\lambda^n \sum_{\mathbf{i}\in\mathscr{I}_k} L_{\mathbf{i}}^{\nu n} \leqslant c$, so by Lemma 8·3 $(E(\sum_{i=1}^N L_i^{\nu n}))^k \leqslant c\lambda^{-n}$ for all k, giving

$$E\left(\sum_{i=1}^N L_i^{2\nu n}\right) < E\left(\sum_{i=1}^N L_i^{\nu n}\right) \leqslant 1.$$

It follows from Theorem 8·4 that $\dim K \leqslant d$. ∎

9. *A metric on probability measures*

We introduce a metric on the Borel probability measures on a metric space that gives a rather strong topology. We require a special case for our study of random fractals, though the metric may be of interest in its own right.

Let (Y, d) be a compact metric space and let $\mathscr{B}$ denote the Borel subsets of Y. Let $\mathscr{P}$ denote the class of Borel probability measures on (Y, d). If $A \subset Y$, let

$$A_\delta = \{y: \inf_{w\in A} d(y, w) \leqslant \delta\}$$

be the δ-parallel body of A. Then A_δ is closed if $\delta > 0$. Define $\Delta: \mathscr{P}\times\mathscr{P} \to [0,\infty)$ by

$$\Delta(\mu_1, \mu_2) = \sup_{\varnothing \neq A\in\mathscr{B}} \inf\{\delta: \mu_1(A_\delta) \geqslant \mu_2(A) \quad \text{and} \quad \mu_2(A_\delta) \geqslant \mu_1(A)\}. \tag{9·1}$$

(To avoid confusion later we use μ's for probability measures in this context.) We show that Δ is a complete metric on $\mathscr{P}$.

In some ways Δ is reminiscent of the Hausdorff metric on compact subsets of a metric space, see Falconer [5], p. 37.

THEOREM 9·1. *$(\mathscr{P}, \Delta)$ is a complete metric space.*

Proof. Clearly $\Delta(\mu_1, \mu_2) = \Delta(\mu_2, \mu_1)$. Moreover $\Delta(\mu_1, \mu_2) \geqslant 0$, and if $\Delta(\mu_1, \mu_2) = 0$ then $\mu_1(A) = \mu_2(A)$ if A is closed, using continuity of probability measures from above, and hence if A is Borel.

For the triangle inequality, suppose that $\Delta(\mu_1, \mu_2) < \delta_1$ and $(\mu_2, \mu_3) < \delta_2$. If A is a Borel set, $A_{\delta_1+\delta_2} \supset (A_{\delta_2})_{\delta_1}$, and

$$\mu_1(A_{\delta_1+\delta_2}) \geqslant \mu_1((A_{\delta_2})_{\delta_1}) \geqslant \mu_2(A_{\delta_2}) \geqslant \mu_3(A).$$

Similarly $\mu_3(A_{\delta_1+\delta_2}) \geqslant \mu_1(A)$, so $\Delta(\mu_1, \mu_3) \leqslant \delta_1 + \delta_2$.

Now let $\mu_r \in \mathscr{P}$ be a Cauchy sequence, say $\Delta(\mu_s, \mu_t) \leqslant \delta$ if $s, t \geqslant r(\delta)$. For any $A \subset Y$ define

$$\mu(A) = \lim_{\delta\to 0} \varliminf_{s\to\infty} \mu_s(A_\delta) = \lim_{\delta\to 0} \varlimsup_{s\to\infty} \mu_s(A_\delta). \tag{9·2}$$

The two expressions are equal since

$$\mu_s(A_\delta) \leqslant \mu_r((A_\delta)_\delta) \leqslant \mu_r(A_{2\delta}) \quad \text{if} \quad r, s \geqslant r(\delta). \tag{9·3}$$

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We must check that μ is a probability measure. First, if $\{A(j)\}_{j=1}^{\infty}$ is any sequence of subsets of Y, we have

$$\mu_s\left(\left(\bigcup_{j=1}^{\infty} A(j)\right)_\delta\right) \leqslant \mu_s\left(\bigcup_{j=1}^{\infty} (A(j))_{2\delta}\right) \leqslant \sum_{j=1}^{\infty} \mu_s(A(j)_{2\delta})$$

for any s. Taking $\overline{\lim}_{\delta\to 0} \lim_{s\to\infty}$ we get that $\mu(\bigcup_{j=1}^{\infty} A(j)) \leqslant \sum_{j=1}^{\infty} \mu(A(j))$, so that μ is an outer measure on subsets of Y. Now suppose that $A, B \subset Y$ are positively separated, that is $d(A,B) = \inf\{d(y,w): y\in A, w\in B\} = \delta_0 > 0$. If $\delta < \frac{1}{2}\delta_0$ then

$$A_\delta \cap B_\delta = \varnothing, \quad \text{so} \quad \mu_s((A\cup B)_\delta) = \mu_s(A_\delta \cup B_\delta) = \mu_s(A_\delta) + \mu_s(B_\delta)$$

for all s. Taking $\lim_{\delta\to 0} \underline{\lim}_{s\to 0}$ it follows that $\mu(A\cup B) = \mu(A) + \mu(B)$. Thus μ is a *metric* outer measure and so is a Borel measure (see Falconer [5], p. 5–6). Clearly $\mu(Y) = \lim_{\delta\to 0} \underline{\lim}_{s\to 0} \mu_s(Y) = 1$, so μ is a probability measure.

If $r, s \geqslant r(\delta)$ then $\mu_r(A) \leqslant \mu_s(A_\delta)$, so

$$\mu_r(A) \leqslant \underline{\lim}_{s\to\infty} \mu_s(A_\delta) \leqslant \lim_{\eta\to 0} \underline{\lim}_{s\to\infty} \mu_s((A_\delta)_\eta) = \mu(A_\delta). \tag{9·4}$$

On the other hand,

$$\mu(A) \leqslant \underline{\lim}_{s\to\infty} \mu_s(A_\delta) \leqslant \mu_r(A_{2\delta})$$

if $r \geqslant r(\delta)$, using (9·2). Thus $\Delta(\mu_r, \mu) \leqslant 2\delta$ if $r \geqslant r(\delta)$, so $\mu_r \to \mu$ in $(\mathscr{P}, \Delta)$. ∎

LEMMA 9·2. *Δ-convergence in $\mathscr{P}$ implies weak convergence.*

Proof. Suppose that $\Delta(\mu_r, \mu) \to 0$. Let $A \subset Y$ be a Borel set. Then given $\delta > 0$, $\mu_r(A) \leqslant \mu(A_\delta)$ for r sufficiently large, so $\overline{\lim}_{r\to\infty} \mu_r(A) \leqslant \inf_{\delta\to 0} \mu(A_\delta) = \mu(\bar{A})$. Applying this to the complement of A and using the fact that μ_r, μ are probability measures, $\mu(\operatorname{int} A) \leqslant \underline{\lim}_{r\to\infty} \mu_r(A)$. Thus

$$\lim_{r\to\infty} \mu_r(A) = \mu(A) \tag{9·5}$$

if A is a Borel set with $\mu(\operatorname{int} A) = \mu(\bar{A})$.

Suppose that $f\colon Y \to \mathbb{R}$ is continuous. Given $\epsilon > 0$, we may find $a_1 < a_2 < \ldots < a_k$ such that

$$f(y) - \epsilon < \sum_{j=1}^{k-1} a_j \chi_{A_j}(y) \equiv g(y) \leqslant f(y) \quad (y\in Y), \tag{9·6}$$

where $A_j = f^{-1}(a_j, a_{j+1}]$ is a Borel set for $1 \leqslant j \leqslant k-1$. Then

$$\overline{A_j} \setminus \operatorname{int} A_j \subset f^{-1}\{a_j, a_{j+1}\},$$

so by choosing the a_j outside a countable subset of $\mathbb{R}$ we may assume that

$$\mu(\operatorname{int} A_j) = \mu(\overline{A_j}).$$

By (9·5) $\int g\,d\mu_r \to \int g\,d\mu$ as $r\to\infty$, so by (9·6) $|\int f\,d\mu_r - \int f\,d\mu| < 2\epsilon$ if r is sufficiently large. ∎

It is clear that the metric case $(\mathscr{P}, \Delta)$ is in general far from being compact. For example, if $y_1 \neq y_2$ are points of Y, we may define a sequence of probability measures by $\mu_r(y_1) = 1/r$, $\mu_r(y_2) = 1 - 1/r$. Then $\Delta(\mu_r, \mu_s) = d(y_1, y_2)$ for all $r \neq s$, so $\{\mu_r\}$ has no convergent subsequence.

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10. *Fractals generated by a random set of transformations*

Hutchinson [9] showed that self-similar fractals may be obtained as the unique invariant sets under certain transformations. This section describes the natural generalization to the random setting.

Let D be a compact subset of $\mathbb{R}^n$. Let Ω be a set and for each $\omega \in \Omega$ let $\mathscr{I}(\omega)$ be a tree with $T_{\mathbf{i}}(\omega)\colon D \to D$ a contractive homeomorphism for each $\mathbf{i} \in \mathscr{I}(\omega)$. Thus $L(T_{\mathbf{i}}) < 1$, where L is the Lipschitz constant of a transformation,

$$L(T) = \sup\{|T(w_1) - T(w_2)|/|w_1 - w_2| : w_1 \neq w_2 \in D\}.$$

Let $(\Omega, \mathscr{F}, p)$ be a probability space and suppose that the elements

$$S(\mathbf{i}) = (N(\mathbf{i}); T_{\mathbf{i},1}, \ldots, T_{\mathbf{i},N(\mathbf{i})})$$

are i.i.d. in the sense that the measurability and probability of $\{\omega : S(\mathbf{i}) \in A\}$ is independent of $\mathbf{i}$.

Assume that there is a compact set $I_\varnothing \subset D$ with

$$\overline{\operatorname{int} I_\varnothing} = I_\varnothing \tag{10·1}$$

such that the following open set condition always holds:

$$\bigcup_{i=1}^{N} T_i(\operatorname{int} I_\varnothing) \subset \operatorname{int} I_\varnothing, \tag{10·2}$$

with the union disjoint. (The mappings $T_{\mathbf{i}}$ are lifted to mappings on subsets of D in the usual manner.)

This situation leads to a random net fractal of the form described in Section 8 in a natural way. For $\mathbf{i} \in \mathscr{I}$, let

$$I_{i_1,\ldots i_k} = T_{i_1} \circ T_{i_1,i_2} \circ \ldots \circ T_{i_1,\ldots,i_k}(I_\varnothing). \tag{10·3}$$

It is clear that (7·8) and (7·9) always hold, using (10·1), (10·2), the identical distribution of the $S(\mathbf{i})$ and the fact that the $T_{\mathbf{i}}$ are homeomorphisms. Moreover the construction is always proper.

Thus the set $K = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{i} \in \mathscr{I}_k} I_{\mathbf{i}}$ is a random net fractal on Ω, and its dimension may be calculated or estimated as in Section 8, assuming appropriate measurability conditions. In the usual case, where the $T_{\mathbf{i}}$ are, say, diffeomorphisms, conditions (7·10) and (7·11) hold. The fact that $(N(\mathbf{i}); L(T_{\mathbf{i},1}), \ldots, L(T_{\mathbf{i},N(\mathbf{i})}))$ are i.i.d. and the inequalities $L_{\mathbf{i}} \leqslant L(T_{\mathbf{i}})$ enable us to apply the ideas of that section. We state the dimension result in the most common and convenient case, where the $T_{\mathbf{i}}$ are always *similarities*. (A mapping T is a similarity if

$$|T(w_1) - T(w_2)| = c|w_1 - w_2| \quad (w_1, w_2 \in \mathbb{R}^n)$$

for some constant c.)

THEOREM 10·1. *Suppose that the $T_{\mathbf{i}}$ are similarities with the $L_{\mathbf{i}} = L(T_{\mathbf{i}})$ satisfying the measurability conditions of Section 8, and the underlying branching process has extinction probability q. Then $K = \varnothing$ with probability q, and*

$$\dim K = \min\left\{s : E\left(\sum_{i=1}^{N} L(T_i)^s\right) \leqslant 1\right\}$$

with probability $1 - q$.

Proof. This is immediate from Corollary 8·6, since $L_i = L(T_i)$ for similarities. ∎

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The remainder of this section is concerned with identifying the probability distribution of the random fractals K. Assume that the $T_{\mathbf{i}}$ satisfy the stronger contraction condition

$$L(T_{\mathbf{i}}) \leqslant c \tag{10·4}$$

for a constant $c < 1$ for all $\mathbf{i}$ and throughout Ω. To avoid notation becoming excessively cumbersome, we assume from now on that $N(\mathbf{i}) = N$ is constant for all $\mathbf{i}$ and throughout Ω so that the underlying tree $\mathscr{I}$ is fixed. Thus for each $\mathbf{i} \in \mathscr{I}$ the elements

$$S(\mathbf{i}) = (T_{\mathbf{i},1}, \ldots, T_{\mathbf{i},N}) \tag{10·5}$$

are i.i.d. on $(\Omega, \mathscr{F}, p)$. (The nature of the technical modifications needed to carry the work through to the case of random $N(\mathbf{i})$ should be clear.)

Let Y be the collection of non-empty compact subsets of D, endowed with the Hausdorff metric d. Thus if $E, F \in Y$

$$d(E, F) = \inf\{\delta : E \subset F_\delta \quad \text{and} \quad F \subset E_\delta\},$$

where E_δ, F_δ denote δ-parallel bodies. The Blaschke selection theorem (see, for example, Falconer [5], theorem 3·16) states that (Y, d) is compact.

Write V for the set of N-tuples of mappings from D to D. Define $U\colon Y^N \times V \to Y$ by

$$U(F_1, \ldots, F_N, S(\varnothing)) = \bigcup_{i=1}^{N} T_i(F_i) \tag{10·6}$$

with the T_i of (10·5) lifted to set mappings in the usual way. Suppose that U is measurable in the sense that

$$\{(F_1, \ldots, F_N, \omega) : U(F_1, \ldots, F_N, S(\varnothing)) \in A\} \in \mathscr{B}^N \times \mathscr{F} \tag{10·7}$$

if $A \in \mathscr{B}$, where $\mathscr{B}$ is the class of Borel subsets of (Y, d). It is not hard to see that (10·7) holds if the mapping $\mathbb{R}^n \times \ldots \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ given by

$$(w_1, \ldots, w_N, \omega) \to (T_1(w_1), \ldots, T_N(w_N))$$

is Borel measurable in the usual sense. We may 'iterate' U to get a mapping

$$U^{(2)}\colon Y^{N^2} \times V^{1+N} \to Y$$

given by

$$U(U(F_{1,1}, \ldots, F_{1,N}, S(1)), \ldots, U(F_{N,1}, \ldots, F_{N,N}, S(N)), S(\varnothing)) = \bigcup_{\mathbf{i} \in \mathscr{I}_2} T_{i_1} \circ T_{i_1,i_2}(F_{i_1,i_2}), \tag{10·8}$$

and similarly we obtain $U^{(k)}\colon Y^{N^k} \times V^{1+N+\ldots+N^{k-1}} \to Y$ given by

$$\bigcup_{\mathbf{i} \in \mathscr{I}_k} T_{i_1} \circ \ldots \circ T_{i_1,\ldots,i_k}(F_{i_1,\ldots,i_k}).$$

A routine check using (10·7) and the identical distribution of the $S(\mathbf{i})$ sees that the $U^{(k)}$ are measurable with respect to the σ-field $\mathscr{B}^{N^k} \times \mathscr{F}$.

We use the mapping U to induce a transformation ψ on the Borel probability measures $\mathscr{P}$ on (Y, d). If $\mu \in \mathscr{P}$ write μ^N for the N-fold product $\mu \times \ldots \times \mu$, and define

$$\psi(\mu)A = (\mu^N \times p)\{(F_1, \ldots, F_N, S(\varnothing)) : U(F_1, \ldots, F_N, S(\varnothing)) \in A\} \tag{10·9}$$

$$= \int \mu^N\{(F_1, \ldots, F_N) : U(F_1, \ldots, F_N, S(\varnothing)) \in A\}\, dp \tag{10·10}$$

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for $A \in \mathscr{B}$. By (10·7) $\psi(\mu)$ is a Borel probability measure. Thus

$$\int f(G)\, d\psi(\mu)(G) = \int f(U(F_1, \ldots, F_N, S(\varnothing))\, d\mu(F_1) \ldots d\mu(F_N)\, dp \tag{10·11}$$

for any Borel measurable function $f\colon Y \to \mathbb{R}$. We may use (10·11) and the fact that the $S(\mathbf{i})$ are i.i.d. to find the iterates $\psi^{(k)}$ of ψ:

$$\begin{aligned}\int f(G)\, d\psi^{(2)}(\mu)(G) &= \int f(U(F_1, \ldots, F_N, S(\varnothing)))\, d\psi(\mu)(F_1) \ldots d\psi(\mu)(F_N)\, dp \\ &= \int f(U(U(F_{1,1}, \ldots, F_{1,N}, S(1)), \ldots, U(F_{N,1}, \ldots, F_{N,N}, S(N)), S(\varnothing)) \\ &\qquad d\mu(F_{1,1}) \ldots d\mu(F_{N,N})\, dp.\end{aligned}$$

Using (10·8) we see that

$$\psi^{(2)}(\mu)A = (\mu^{N^2} \times p)\{((F_{\mathbf{i}}\colon \mathbf{i} \in \mathscr{I}_2), \omega)\colon \bigcup_{i_1, i_2} T_{i_1} \circ T_{i_1, i_2}(F_{i_1, i_2}) \in A\}. \tag{10·12}$$

Similarly we may show that

$$\psi^{(k)}(\mu)A = (\mu^{N^k} \times p)\{((F_{\mathbf{i}}\colon \mathbf{i} \in \mathscr{I}_k), \omega)\colon \bigcup_{\mathbf{i} \in \mathscr{I}_k} T_{i_1} \circ \ldots \circ T_{i_1, \ldots, i_k}(F_{i_1, \ldots, i_k}) \in A\}. \tag{10·13}$$

THEOREM 10·2. *Given* (10·4), ψ *is a contraction mapping on* $(\mathscr{P}, \Delta)$.

Proof. Let $F_i, G_i \in Y$ $(1 \leqslant i \leqslant N)$ with $\bigcup_{i=1}^N T_i(G_i) \in A$ for some $A \in \mathscr{B}$ and some $S(\varnothing) = (T_1, \ldots, T_N)$. Then

$$d(\bigcup T_i(F_i), A) \leqslant d(\bigcup T_i(F_i), \bigcup T_i(G_i)) \leqslant \max_{1 \leqslant i \leqslant N} d(T_i(F_i), T_i(G_i)) \leqslant c \max_{1 \leqslant i \leqslant N} d(F_i, G_i). \tag{10·14}$$

Given $\mu_1, \mu_2 \in \mathscr{P}$ with $\Delta(\mu_1, \mu_2) < \delta$, we have

$$\mu_2^N\{(F_1, \ldots, F_N)\colon \bigcup T_i(F_i) \in A\}$$

$$\leqslant \mu_1^N\{(F_1, \ldots, F_N)\colon d(F_i, G_i) \leqslant \delta \text{ for some } G_i \text{ with } \bigcup T_i(G_i) \in A\}$$

(this follows on extending the defining property of Δ to the product measure)

$$\leqslant \mu_1^N\{(F_1, \ldots, F_N)\colon \bigcup T_i(F_i) \in A_{c\delta}\}$$

using (10·14). In integrating with respect to p and using the form (10·10), we get

$$\psi(\mu_2)A \leqslant \psi(\mu_1)A_{c\delta}.$$

Together with the opposite inequality, this gives $\Delta(\psi(\mu_1), \psi(\mu_2)) \leqslant c\delta$, so ψ is a contraction mapping of ratio at most c. ∎

COROLLARY 10·3. *There exists a unique* $\mu_0 \in \mathscr{P}$ *satisfying*

$$\psi(\mu_0)(A) = \mu_0(A)$$

for all $A \in \mathscr{B}$. *Moreover the iterates* $\{\psi^{(k)}(\mu)\}$ *converge to* μ_0 *in* Δ *for any* $\mu \in \mathscr{P}$.

Proof. ψ is a contraction mapping on $(\mathscr{P}, \Delta)$, a complete metric space by Theorem 9·1. The result follows by the contraction mapping theorem. ∎

We know from the measurability properties of the $U^{(k)}$ that for each k the mapping $\Omega \to Y$ given by

$$\omega \mapsto \bigcup_{\mathbf{i} \in \mathscr{I}_k} T_{i_1} \circ \ldots \circ T_{i_1, \ldots, i_k}(I_\varnothing) = \bigcup_{\mathbf{i} \in \mathscr{I}_k} I_{\mathbf{i}} \tag{10·15}$$

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is measurable in the sense that the inverse images of Borel subsets of Y lie in $\mathscr{F}$. At each point of Ω the images (10·15) decrease with k and converge in (Y, d) to a compact set K, by virtue of the open set condition. It follows that the mapping $\Omega \to Y$ given by

$$\omega \mapsto K = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{i} \in \mathscr{I}_k} I_{\mathbf{i}}$$

is Borel measurable.

THEOREM 10·4. *The invariant measure μ_0 is given by*

$$\mu_0(A) = p(K \in A) \quad (A \in \mathscr{B}). \tag{10·16}$$

Proof. First take A to be open in (Y, d). Define μ by

$$\mu(A) = 0 \quad \text{if} \quad I_{\varnothing} \notin A$$
$$= 1 \quad \text{if} \quad I_{\varnothing} \in A.$$

By (10·13)

$$\psi^{(k)}(\mu)(A) = p\Big(\bigcup_{\mathbf{i} \in \mathscr{I}_k} I_{\mathbf{i}} \in A\Big).$$

As $k \to \infty$ the L.H.S. converges to $\mu_0(A)$ by Corollary 10·3, and the R.H.S. converges to $p(K \in A)$. Thus (10·16) holds for open sets and so for Borel sets A.

Combining Corollary 10·3 and Theorem 10·4, we see that we can obtain the probability distribution of the fractals K by taking any initial probability distribution and iterating it under ψ. This fact is useful when producing computer realizations of random fractals. We may take any initial set F (such as a line segment) with probability 1, and repeatedly choose $S(\mathbf{i}) = (T_{\mathbf{i},1}, \ldots, T_{\mathbf{i},N})$ according to the random distribution. The sets

$$\bigcup_{\mathbf{i} \in \mathscr{I}_k} T_{i_1} \circ \ldots \circ T_{i_1, \ldots, i_k}(F)$$

converge in (Y, d) to a realization of the random fractal K. In particular, if the $S(\mathbf{i})$ are specified using the 'initiator–generator' form (see Mandelbrot [**10**]), this provides an effective way of drawing random fractals, especially random curves.

11. *Examples and generalizations*

We describe some particular random fractals to which the theory may be applied. Further examples will readily spring to mind.

First we point out that our theory specializes to the non-random situation studied by Moran [**11**] and Hutchinson [**9**].

Example 11·1. Suppose that the elements $W(\mathbf{i})$ in (8·4) are constant, so independent of $\mathbf{i}$. The net fractal construction leads to a constant fractal K such that

$$d - (\nu - 1)n \leqslant \dim K \leqslant d,$$

where d satisfies $\sum_1^N L_i^d = 1$, by Corollary 8·6.

The next two examples are randomizations of the familiar 'middle-third' Cantor set construction.

Example 11·2. Divide the unit interval into three equal parts and retain each part independently with probability p. Repeat this with the parts that remain, and so on. The extinction probability of the branching process is 1 if $p \leqslant \frac{1}{3}$ and is the solution of

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$(1-p+pu)^3 = u$ lying between 0 and 1 if $p > \frac{1}{3}$; this is the probability that $K = \varnothing$. Otherwise $E(\sum_1^N L_i^s) = 3^{-s}E(N) = 3^{1-s}p$, so that $\dim K = 1 + \log p/\log 3$ a.s.

Example 11·3. Remove from the unit interval a central portion so that the remaining two parts have lengths $0 < L_1 = L_2 < \frac{1}{2}$ distributed according to the probability density function f. From each of these parts remove a proportion distributed in the same way, etc. This time extinction cannot occur. A calculation shows that

$$E(L_1^s + L_2^s) = 2\int_0^{\frac{1}{2}} u^s f(u)\,du,$$

so equating this to 1 gives the almost sure dimension of the resulting fractal K. In the case where the $L_1 = L_2$ is uniformly distributed over $(0, \frac{1}{2})$, $\dim K$ is the solution of $s+1 = 2^{1-s}$, i.e. $\dim K = 0{\cdot}457$.

We now give two examples in the plane, first a random von Koch curve and then a random set of high connectivity.

Example 11·4. Let L be a random variable taking values in $(\frac{1}{3}, \frac{1}{2})$ with probability density function f. Let F be a line segment in $\mathbb{R}^2$, let F_1 and F_4 be (random) subintervals of lengths L obtained by removing a central portion F_0 from F, and let F_2 and F_3 be the other two sides of the equilateral triangle based on F_0, always on the same side of F. Let T_i $(1 \leqslant i \leqslant 4)$ be the random similarities that map F onto F_i; to specify the T_i completely, assume that they preserve orientation, with T_1, T_4 having no rotational component and with T_2, T_3 rotating through $\frac{1}{3}\pi$, $-\frac{1}{3}\pi$ respectively. The T_i satisfy the open set condition (10·2) with $I_\varnothing$ the isosceles triangle based on F with base angles of $\frac{1}{6}\pi$. (We need $\frac{1}{3} < L$ for such a condition to hold.) Then

$$E\left(\sum_1^4 L(T_i)^s\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} f(u)\,(2u^s + 2(1-2u)^s)\,du,$$

enabling us to find the a.s. dimension of the net fractal K. If L is uniformly distributed on $(\frac{1}{3}, \frac{1}{2})$ then $\dim K = 1{\cdot}144$ a.s. Let P_1 denote the (random) polygonal curve $\bigcup_1^4 F_i$. Replace each side of P_1 by an independently selected copy of P_1, scaled so that the ends of the copy coincide with the endpoints of the side it replaces, to get a polygonal curve P_2. Replace each side of P_2 in the same way to get P_3, and so on. Each P_i is a polygonal curve with endpoints the same as those of F, and by the theory of Section 10 the P_i converge to a realization of the random fractal K in the Hausdorff metric. Thus K is a.s. an unrectifiable Jordan curve. This approach is useful for drawing random fractal curves using a computer.

Example 11·5. Let $2 \leqslant M$ and $0 \leqslant N \leqslant M^2$ be integers with some random distribution. Divide the unit square into M^2 equal squares and select N of these. Repeat this independently for each of the selected squares and continue, to get a random net fractal K. The probability of the process becoming extinct is determined by the distribution of N. Otherwise, the a.s. dimension of K may be found by solving $E(NM^{-s}) = 1$. By selecting squares in an appropriate way we can arrange for the sequence of homology groups of the sets $\bigcup_{i \in \mathscr{I}_k} I_i$ to be strictly increasing, so that the limiting set K has infinite connectivity.

The ideas in this paper could be generalized in many directions. Most of the work could be set in a general metric space rather than in $\mathbb{R}^n$, or indeed on a differential manifold where Hausdorff dimension is well-defined in terms of coordinate charts.

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The random constructions could be extended to the recurrent sets studied by Dekking [4] and Bedford [2].

The self-similarity conditions on the fractals or networks could be relaxed using more sophisticated probability limit theorems. For example, we could require the $Z(\mathbf{i})$ or $W(\mathbf{i})$ merely to be asymptotically i.i.d. as $k \to \infty$, or just impose limiting conditions on the means and variances involved. It would then be possible to apply the theory of mixinggales – asymptotic martingales – see Hall and Heyde [8].

REFERENCES

[1] K. B. ATHREYA and P. E. NEY. *Branching Processes* (Springer-Verlag, 1972).
[2] T. BEDFORD. Ph.D. thesis (University of Warwick, 1984).
[3] B. BOLLOBÁS. *Graph Theory, an Introductory Course* (Springer-Verlag, 1979).
[4] F. M. DEKKING. Recurrent sets. *Advances in Mathematics* **44** (1982), 78–104.
[5] K. J. FALCONER. *The Geometry of Fractal Sets* (Cambridge University Press, 1985).
[6] L. R. FORD and D. R. FULKERSON. *Flows in Networks* (Princeton University Press, 1962).
[7] G. R. GRIMMETT. Random flows: network flows and electrical flows through random media. *Surveys in Combinatorics 1985* (ed. I. Anderson) (Cambridge University Press, 1985), 59–95.
[8] P. HALL and C. C. HEYDE. *Martingale Limit Theory and its Application* (Academic Press, 1980).
[9] J. E. HUTCHINSON. Fractals and self-similarity. *Indiana University Math. J.* **30** (1981), 713–47.
[10] B. B. MANDELBROT. *The Fractal Geometry of Nature* (W. H. Freeman, 1982).
[11] P. A. P. MORAN. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.* **42** (1946), 15–23.
[12] C. A. ROGERS. *Hausdorff Measures* (Cambridge University Press, 1970).
[13] R. D. MAULDIN and S. C. WILLIAMS. Random constructions: asymptotic geometric and topological properties. *Trans. Amer. Math. Soc.* **295** (1986), 325–346.
[14] S. GRAF. Statistically self-similar fractals (to appear).
[15] K. J. FALCONER. Cut-set sums and tree processes (to appear).

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