ABELIAN GROUPS, GRAPHS AND GENERALIZED KNIGHTS

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ABSTRACT. If g is a set of generators of an enumerably infinite Abelian group A, it is proved that the elements of A can be arranged in both a one-ended and an endless infinite sequence in which successive terms differ by $\pm$ an element of g, except that the one-ended arrangement is impossible if g is finite and the rank of A is 1. Let ν be a cardinal number. Consider an infinite 'chessboard' whose positions are those lattice points of ν-dimensional space which have only finitely many non-zero coordinates. A piece associated with this chessboard is a *generalized knight* if every vector obtainable from a move of the piece by permuting its components and changing the signs of a subset of them is itself a permitted move. It is ascertained which positions a given generalized knight can reach in a finite sequence of moves starting at the origin, and whether or not, if $\nu \leqslant \aleph_0$, it can trace out the whole chessboard in (i) a one-ended, (ii) an endless infinite sequence of moves visiting each position exactly once.

1. *Introduction.* Kürschák ((3), (4)) proved that an unbounded 2-dimensional chessboard can be traced out in both a one-ended and an endless infinite sequence of knight's moves so that each position is visited exactly once. Vázsonyi (6) gave a new proof of this theorem and also generalized it to n dimensions by induction on n. The following work was suggested by the problems of (i) extending Vázsonyi's result to a more general kind of piece than the knight, and (ii) including within the scope of this discussion the chessboard whose positions are those lattice points of $\aleph_0$-dimensional space which have only finitely many non-zero coordinates. Our argument for the finite-dimensional case will use the method of Vázsonyi's proof of Kürschák's theorem.

Let A be an Abelian group† and S be a subset of its elements. Then $\{S\}$ will denote the subgroup generated by the elements of S; and $\{a_1, a_2, \ldots, a_n\}$ will denote the subgroup generated by elements $a_1, a_2, \ldots, a_n$ of A. A sequence of elements of A will be called *S-admissible* if the differences between successive terms of the sequence all belong to $S \cup (-S)$. The elements of S are *independent* if there exists no relation of the form

$$\sum_{s \in S} m_s s = 0,$$

in which the m_s are integers of which all but a finite number, but not all, are zero. It is well known ((2), p. 140) that there exist maximal sets of independent elements of A and that every such set has the same cardinal number; this number is called the *rank* of A and will be denoted by $r(A)$. By saying that the elements of A can be *arranged* to form a sequence s, we shall mean that each element of A appears exactly once among the terms of s. The following theorem is proved in § 2:

THEOREM 1. *Let g be a set of generators of an enumerably infinite Abelian group A. Then the elements of A can be arranged in both a one-ended and an endless g-admissible sequence, except that, if g is finite and $r(A) = 1$, the one-ended sequential arrangement is impossible.*

† The connexion between Abelian groups and the above chessboard problem was brought to my notice by Prof. D. Rees.

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The set of those elements which appear at least once among the terms of a sequence $\xi_1, \xi_2, \ldots, \xi_n$ will be denoted by $[\xi_1, \xi_2, \ldots, \xi_n]$; and a similar notation will be used for infinite sequences. For each positive integer ν, let f_ν denote the set $[1, 2, \ldots, \nu]$; and, for each infinite cardinal number ν, select a set f_ν of ν elements amongst which all the positive integers are included. Then, for any cardinal number $\nu > 0$, we shall define a *ν-vector* to be a mapping $\mathbf{x}$ of f_ν into the set of integers, such that $\mathbf{x}(\alpha) \neq 0$ for at most finitely many $\alpha \in f_\nu$. The integer $\mathbf{x}(\alpha)$ will be called the *α-th component of* $\mathbf{x}$. *Addition* and *subtraction* of ν-vectors are defined by the rules $(\mathbf{x} \pm \mathbf{y})(\alpha) = \mathbf{x}(\alpha) \pm \mathbf{y}(\alpha)$. The Abelian group of all ν-vectors will be denoted by F_ν. The subgroup of those ν-vectors whose components have even sums will be denoted by E_ν. If ν is finite, P_ν will denote the subgroup of those ν-vectors whose components are all of the same parity. Two ν-vectors are *similar* if either is obtainable from the other by permuting its components and changing the signs of a subset of them. A subset S of F_ν is *symmetrical* if every ν-vector similar to an element of S belongs to S. A subset of F_ν is *non-trivial* if it has at least one non-zero element.

The elements of F_ν will be regarded as the positions on an unbounded ν-dimensional chessboard. A *piece* associated with this chessboard is defined by means of a set of *moves* which are elements of F_ν; a piece can move from position $\mathbf{x}$ to position $\mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x}$ is a move of the piece. We shall say that a piece can *cover* F_ν (*endlessly*, *one-endedly*) if it can describe F_ν in an (endless, one-ended) infinite sequence of moves visiting each position exactly once. A piece is a *generalized knight* if its set of moves is symmetrical. A piece is *non-trivial* if it has at least one non-zero move.

The following three theorems are proved in § 3:

THEOREM 2. *The only symmetrical subgroups of F_ν are the subgroups λF_ν, λE_ν and, if ν is finite, λP_ν, for $\lambda = 0, 1, 2, \ldots$.*

THEOREM 3. *Let d be the g.c.d. of the components of the elements of a non-trivial symmetrical subset M of F_ν. Then $\{M\}$ is the first term of the sequence dP_ν, dE_ν, dF_ν (with dP_ν omitted if ν is infinite) which contains M.*

Remark. Let M be the set of moves of a generalized knight K. Since M is symmetrical, $M = -M$. Therefore the subgroup $\{M\}$, characterized by Theorem 3, is the 'range' of K, i.e. the set of those positions which K can reach in a finite sequence of moves starting from $\mathbf{0}$.

THEOREM 4. *Let K be a non-trivial generalized knight associated with the chessboard F_ν, where ν is a positive integer or $\aleph_0$. Then K can cover F_ν endlessly if and only if*

(i) *the components of the moves of K (taken all together) are co-prime,*

(ii) *K has a move, the sum of whose components is odd, and*

(iii) *if ν is finite and greater than 1, K has a move whose components are not all of the same parity.*

K can cover F_ν one-endedly if and only if (i)–(iii) *are satisfied and, in addition, K has an infinite number of moves if $\nu = 1$.*

In Theorem 4, we observe that Condition (ii) subsumes Condition (iii) if ν is even; hence (iii) can be weakened to 'if ν is odd and greater than 1, ...'.

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2. *Proof of Theorem* 1. *Definitions.* An *xy-sequence* is a sequence with first term x and last term y. A sequence will be called *proper* if its terms are distinct.

Let S, S' be subsets of the elements of an Abelian group. Then $S - S'$ will denote the set of all elements of the form $s - s' (s \in S, s' \in S')$; and $S \ominus S'$ will denote the set of those elements of S which do not belong to S'.

LEMMA 1. *Let g be an infinite set of generators of an Abelian group A, F be a finite subset of A and x, y be elements of $A \ominus F$. Then there exists a g-admissible proper xy-sequence of elements of $A \ominus F$.*

Proof. By considering expressions for x, y as sums of integer multiples of elements of g, we can clearly construct a g-admissible xy-sequence

$$x = t_0,\ t_1,\ t_2,\ \ldots,\ t_{n-1},\ t_n = y$$

of elements of A. Since g is an infinite and $F - [t_0, t_1, \ldots, t_n]$ a finite set, we can select an element a of g which does not belong to $F - [t_0, t_1, \ldots, t_n]$. Then (since $x,\ y \notin F$ by hypothesis) it is clear that

$$x = t_0,\ t_0 + a,\ t_1 + a,\ \ldots,\ t_n + a,\ t_n = y$$

is a g-admissible sequence having no term in F. If this sequence is not already proper, select two equal terms, strike out one of them and all terms between them, and repeat this operation until the sequence becomes proper.

LEMMA 2. *Let g be an infinite set of generators of an enumerably infinite Abelian group A. Then the elements of A can be arranged in both a one-ended and an endless g-admissible sequence.*

Proof. Let $x_1, x_2, x_3, \ldots$ be an enumeration of the elements of A. Then, by Lemma 1, we can construct a g-admissible proper $x_1 x_2$-sequence s_1 of elements of A. Let S_1 be the set of terms of s_1, and $f(3)$ be the least value of i for which $x_i \notin S_1$. Then, by Lemma 1, we can construct a g-admissible proper $x_2 x_{f(3)}$-sequence s_2 of elements of $A \ominus (S_1 \ominus [x_2])$. In general, for $n \geqslant 3$, suppose that we have defined finite sequences $s_1, s_2, \ldots, s_{n-1}$ and that $x_{f(n)}$ is the last term of s_{n-1}. Define $f(n+1)$ to be the least value of i for which $x_i \notin S_1 \cup S_2 \cup \ldots \cup S_{n-1}$, where S_j denotes the set of terms of s_j. Then, by Lemma 1, we can construct a g-admissible proper $x_{f(n)} x_{f(n+1)}$-sequence s_n of elements of

$$A \ominus ((S_1 \cup S_2 \cup \ldots \cup S_{n-1}) \ominus [x_{f(n)}]).$$

By combining $s_1, s_2, s_3, \ldots$ in an obvious manner, we obtain an arrangement of the elements of A in a one-ended g-admissible sequence.

To obtain the endless sequential arrangement, we use an adaptation of the above procedure, in which s_1 is defined to be an $x_1 x_2$-sequence, s_2 to be an $x_{f(3)} x_1$-sequence, s_3 to be an $x_2 x_{f(4)}$-sequence and s_n to be an $x_{f(n+1)} x_{f(n-1)}$-sequence for $n = 4, 6, 8, \ldots$ and an $x_{f(n-1)} x_{f(n+1)}$-sequence for $n = 5, 7, 9, \ldots$, i.e. we attach finite sequences on the right and left alternately.

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LEMMA 3. *Let g be a finite set of generators of an Abelian group A of rank r. Then we can find an enumeration $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s$ of the elements of g and positive integers $m_1, m_2, \ldots, m_s$ such that each element of A is uniquely expressible in the form*

$$\sum_{i=1}^{r} p_i a_i + \sum_{j=1}^{s} q_j b_j,$$

where the p_i, q_j are integers and

$$0 \leqslant q_j < m_j \quad (j = 1, 2, \ldots, s).$$

Proof. Let $a_1, a_2, \ldots, a_\rho$ be a maximal collection of independent elements of g. Then $a_1, a_2, \ldots, a_\rho$ is clearly also a maximal collection of independent elements of A; hence $\rho = r$. Let $b_1, b_2, \ldots, b_s$ be any enumeration of the remaining elements of g. Write $A_0 \equiv \{a_1, a_2, \ldots, a_r\}$ and

$$A_\theta \equiv \{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_\theta\} \quad (\theta = 1, 2, \ldots, s).$$

Define m_θ to be the least positive integer such that $m_\theta b_\theta \in A_{\theta-1}$ $(\theta = 1, 2, \ldots, s)$. Such an m_θ exists, since $a_1, a_2, \ldots, a_r$ is a maximal collection of independent elements of g and $\{a_1, a_2, \ldots, a_r\} \subset A_{\theta-1}$. Let $\mathscr{S}_0$ denote the statement that each element of A_0 is uniquely expressible in the form $\Sigma p_i a_i$, and $\mathscr{S}_\theta$ denote, for $\theta = 1, 2, \ldots, s$, the statement that each element of A_θ is uniquely expressible in the form

$$\Sigma p_i a_i + \sum_{j=1}^{\theta} q_j b_j,$$

where $0 \leqslant q_j < m_j$ $(j = 1, 2, \ldots, \theta)$. Then $\mathscr{S}_0$ is true since the a_i are independent; and $\mathscr{S}_{\theta-1}$ implies $\mathscr{S}_\theta$ $(\theta = 1, 2, \ldots, s)$ since the distinct cosets of $A_{\theta-1}$ in A_θ are clearly $A_{\theta-1} + qb_\theta$ $(q = 0, 1, \ldots, m_\theta - 1)$. Therefore $\mathscr{S}_s$ is true, i.e. the lemma is proved.

Definitions. A *graph* Γ consists (for the purposes of this paper) of a set $V(\Gamma)$ of elements called *vertices* of Γ and a symmetric relation $R(\Gamma) \subset V(\Gamma) \times V(\Gamma)$; two vertices related under $R(\Gamma)$ are said to be *adjacent*. A *spanning subgraph* of Γ is a graph Δ such that $V(\Delta) = V(\Gamma)$ and $R(\Delta) \subset R(\Gamma)$. An *isomorphism* between two graphs is a one-to-one correspondence between their vertices which preserves adjacency and non-adjacency.

If $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are graphs, their *product* $\Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$ is the graph Γ defined† by

(i) $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2) \times \ldots \times V(\Gamma_n)$;

(ii) $((\xi_1, \xi_2, \ldots, \xi_n), (\eta_1, \eta_2, \ldots, \eta_n)) \in R(\Gamma)$ if and only if, for some i,

$$(\xi_i, \eta_i) \in R(\Gamma_i),$$

$$\xi_j = \eta_j \quad (j \neq i).$$

We shall blur distinctions between $\Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$ and graphs obtained by bracketing this product in various ways, since all these graphs are naturally isomorphic.

The symbols G, H, $K(m)$ (where m is a positive integer) will denote the graphs defined as follows:

(i) $V(G)$ is the set of all integers;

(ii) $V(H)$ is the set of all non-negative integers;

† This definition is taken from (5).

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(iii) $V(K(m)) = [0, 1, 2, \ldots, m-1]$;

(iv) in each case, two vertices p, q are adjacent if and only if $|p-q| = 1$.

The product of r copies of G will be denoted by G^r. A graph is a *way* if it is isomorphic to G, H or some $K(m)$; in the first two cases, it is an *endless* or *one-ended* way, respectively. A *spanning way* of a graph Γ is a spanning subgraph of Γ which is a way.

In Lemmas 4–6, we assume that $r > 0$, each $m_i > 0$ and $s \geqslant 0$.

LEMMA 4 (Vázsonyi). *G^r has an endless and, if $r > 1$, a one-ended spanning way.*†

Proof. G is an endless spanning way of itself. For $r > 1$, make the inductive hypothesis that G^{r-1} has an endless spanning way W. Then $G^r = G^{r-1} \times G$, which has the spanning subgraph $W \times G$, and one can easily verify diagrammatically that the product of two endless ways has both a one-ended ((6), Fig. 3) and an endless ((1), Fig. 17; (3), Fig. 3; (6), Fig. 1) spanning way.

LEMMA 5. *$K(m_1) \times K(m_2) \times \ldots \times K(m_r)$ has a spanning way.*

This lemma admits a similar inductive proof.

LEMMA 6. *$G^r \times K(m_1) \times K(m_2) \times \ldots \times K(m_s)$ has an endless and, if $r > 1$, a one-ended spanning way.*

Proof. By Lemma 4, G^r has an endless spanning way W_0 and, if $r > 1$, a one-ended spanning way W_1. This proves Lemma 6 if $s = 0$. If $s > 0$, $K(m_1) \times K(m_2) \times \ldots \times K(m_s)$ has, by Lemma 5, a (finite) spanning way W. One can easily verify diagrammatically that $W_0 \times W$ and (if $r > 1$) $W_1 \times W$ have an endless and a one-ended spanning way respectively. This clearly proves the lemma.

Proof of Theorem 1. If g is infinite, the required sequential arrangements exist by Lemma 2. Let $r(A) = r$. If g is finite, then $r \geqslant 1$ (since a finitely generated periodic Abelian group is finite) and there exist, by Lemma 3, an enumeration $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s$ of the elements of g and positive integers $m_1, m_2, \ldots, m_s$ such that

$$(p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s) \to \sum_{i=1}^{r} p_i a_i + \sum_{j=1}^{s} q_j b_j$$

is a one-to-one mapping of $V(G^r \times K(m_1) \times K(m_2) \times \ldots \times K(m_s))$ onto A. Therefore, by Lemma 6, the elements of A can be arranged in an endless and, if $r > 1$, a one-ended g-admissible sequence.

To prove the exception, suppose that g is finite and $r(A) = 1$. Since g is finite, A is, by the fundamental decomposition theorem for finitely generated Abelian groups ((2), p. 148), the direct sum of a finite number of cyclic subgroups, of which clearly (cf. (2), p. 150) exactly $r(A) = 1$ are infinite. Therefore A is the direct sum of a cyclic infinite subgroup ($\{a\}$, say) and a finite subgroup T. For any non-negative integer n, we will write

$$T_n = \bigcup_{p=-n}^{n} (T + pa), \quad {}_+T_n = \bigcup_{p=n+1}^{\infty} (T + pa), \quad {}_-T_n = \bigcup_{p=n+1}^{\infty} (T - pa).$$

Select an integer N such that $g \cup (-g) \subset T_{2N+1}$. Then, if $x_1, x_2, x_3, \ldots$ (or $\ldots, x_3, x_2, x_1$) were a g-admissible sequential arrangement of the elements of A, $x_i - x_{i+1}$ would belong

† This lemma and its proof are taken from (6).

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to T_{2N+1} for every i. Therefore, there would be no pair x_i, x_{i+1} of which one belonged to ${}_+T_N$ and the other belonged to ${}_-T_N$. Therefore, for any m such that $T_N \subset [x_1, x_2, \ldots, x_m]$, the set $[x_{m+1}, x_{m+2}, \ldots]$ would be contained in either ${}_+T_N$ or ${}_-T_N$. This clearly contradicts the assumption that the sequence (x_i) includes every element of A.

3. *Proof of Theorems* 2–4. Throughout this section, ν will denote a non-zero cardinal number. The ν-vector whose αth component is 1 and whose remaining components are 0 will be denoted by $\mathbf{e}_\alpha$.

LEMMA 7. *Let H be a symmetrical subgroup of F_ν, and let $H \cap \{\mathbf{e}_1\} = \{\sigma\mathbf{e}_1\}$. Then, if p is a component of an element of H, σ divides $2p$.*

Proof. Let p be a component of an element $\mathbf{x}$ of H. Then we can write

$$\mathbf{x} = p\mathbf{e}_\alpha + p_2\mathbf{e}_\beta + p_3\mathbf{e}_\gamma + \ldots + p_k\mathbf{e}_\omega,$$

where $\alpha, \beta, \gamma, \ldots, \omega$ are distinct elements of f_ν. The vectors

$$p\mathbf{e}_1 \pm (p_2\mathbf{e}_2 + p_3\mathbf{e}_3 + \ldots + p_k\mathbf{e}_k),$$

being similar to $\mathbf{x}$, must also belong to H; hence, by addition, $2p\mathbf{e}_1 \in H$. Therefore σ divides $2p$.

Definition. A subgroup of F_ν will be called *divisorless* if it is not an integer multiple of any other subgroup.

LEMMA 8. *Every divisorless symmetrical subgroup of F_ν contains $2F_\nu$.*

Proof. Let H be a divisorless symmetrical subgroup of F_ν, and let $H \cap \{\mathbf{e}_1\} = \{\sigma\mathbf{e}_1\}$. By the definition of 'divisorless', $H \neq [\mathbf{0}]$. Moreover, by Lemma 7, σ divides twice every component of every element of H. Therefore, since H is divisorless, $\sigma = \pm 1$ or ± 2. Therefore $2\mathbf{e}_1 \in H$, and hence, since H is symmetrical, $2\mathbf{e}_\alpha \in H$ for every $\alpha \in f_\nu$. Therefore $2F_\nu \subset H$.

Definition. For each non-negative integer $r \leqslant \nu$, O_r will denote the set of those ν-vectors which have exactly r odd components.

LEMMA 9. *Every divisorless symmetrical subgroup of F_ν is the union of a subset of the O_r's.*

Proof. Let H be a divisorless symmetrical subgroup of F_ν, and let $\mathbf{x} \in H$. Then, by the symmetry of H and Lemma 8, H includes every ν-vector of the form $\mathbf{x}' + 2\mathbf{y}$, where $\mathbf{x}'$ is similar to $\mathbf{x}$, i.e. every ν-vector having the same number of odd components as $\mathbf{x}$.

LEMMA 10. *If a subgroup H of F_ν is the union of a subset of the O_r's, then $H = F_\nu$ or E_ν or $2F_\nu$ or (if ν is finite) P_ν.*

Proof. Since H includes $\mathbf{0}$, it must contain O_0. Therefore either (i) $H = O_0 = 2F_\nu$, or (ii) ν is finite and $H = O_0 \cup O_\nu = P_\nu$, or (iii) there is a positive integer $\rho < \nu$ such that $H \supset O_\rho$. The third alternative implies that $\nu \geqslant 2$ and that the ν-vectors

$$\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_\rho, \quad \mathbf{e}_2 + \mathbf{e}_3 + \ldots + \mathbf{e}_{\rho+1}$$

belong to H, whence, by subtraction, $\mathbf{e}_1 - \mathbf{e}_{\rho+1} \in H$. Therefore $H \supset O_2$ and hence $H \supset \{O_2\}$, which is easily seen to be E_ν. Therefore, since the index of E_ν in F_ν is 2, $H = E_\nu$ or F_ν.

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Proof of Theorem 2. The subgroups specified in the theorem are clearly symmetrical; and, since a symmetrical subgroup is clearly an integer multiple of a divisorless symmetrical subgroup, it must, by Lemmas 9 and 10, be one of the subgroups specified.

Proof of Theorem 3. If $\nu = 1$, then $\{M\} = dF_1 = dP_1$. We will therefore henceforward assume that $\nu \geqslant 2$. Since M is symmetrical, so is $\{M\}$. Therefore, by Theorem 2, $\{M\} = \lambda F_\nu$ or λE_ν or (if ν is finite) λP_ν for some non-negative integer λ. But it is clear that d is the g.c.d. of the components of the elements of $\{M\}$; therefore (since $\nu \geqslant 2$) $\lambda = d$. If ν is finite and $M \subset dP_\nu$, then $\{M\} \subset dP_\nu$. Therefore $\{M\} = dP_\nu$, since neither dE_ν nor dF_ν is properly contained in dP_ν if $\nu \geqslant 2$. If either ν is finite and $M \not\subset dP_\nu$ or ν is infinite, we are left with the possibilities that $\{M\} = dE_\nu$ or dF_ν. Therefore $\{M\} = dE_\nu$ if $M \subset dE_\nu$, and $\{M\} = dF_\nu$ if not.

Proof of Theorem 4. Let M be the set of moves of K. Then, by Theorem 3, (i)–(iii) are necessary and sufficient conditions for $\{M\}$ to be F_ν, an obviously necessary condition for K to be able to cover F_ν. If $\{M\} = F_\nu$, then, by Theorem 1, a piece whose set of moves is $M \cup (-M)$ can cover F_ν both one-endedly and endlessly, except that it cannot do so one-endedly if M is finite and $\nu = 1$. But $M \cup (-M) = M$, since M is symmetrical. The theorem is therefore proved.

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