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Diamond diagrams and multivariable $(\varphi, \mathcal{O}_K^\times)$-modules

Yitong Wang

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Abstract

Let p be a prime number and K a finite unramified extension of $\mathbb{Q}_p$. Let π be an admissible smooth mod p representation of $\mathrm{GL}_2(K)$ occurring in some Hecke eigenspaces of the mod p cohomology and $\overline{r}$ be its underlying global two-dimensional Galois representation. When $\overline{r}$ satisfies some Taylor–Wiles hypotheses and is sufficiently generic at p, we compute explicitly certain constants appearing in the diagram associated to π, generalizing the results of Dotto and Le. As a result, we prove that the associated étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ defined by Breuil, Herzig, Hu, Morra and Schraen is explicitly determined by the restriction of $\overline{r}$ to the decomposition group at p, generalizing the results of Breuil *et al.* and the author.

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1. Introduction

Let p be a prime number and F be a totally real number field that is unramified at places above p. Let D be a quaternion algebra with center F that is split at all places above p and at exactly one infinite place. For each compact open subgroup $U \subseteq (D \otimes_F \mathbb{A}_F^\infty)$ where $\mathbb{A}_F^\infty$ is the set of finite adèles of F, we denote by X_U the associated smooth projective algebraic Shimura curve over F.

Let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$. We fix an absolutely irreducible continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$. For w a finite place of F, we write $\overline{r}_w \overset{\mathrm{def}}{=} \overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$. We let S_D be the set of finite places where D ramifies, $S_{\overline{r}}$ be the set of finite places where $\overline{r}$ ramifies,

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and S_p the set of places above p. We fix a place $v \in S_p$ and write $K \overset{\text{def}}{=} F_v$. We assume that $p \geq 5$, that $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible, that the image of $\overline{r}(G_{F(\sqrt[5]{1})})$ in $\mathrm{PGL}_2(\mathbb{F})$ is not isomorphic to A_5, that $\overline{r}_w$ is generic in the sense of [BP12, Definition 11.7] for $w \in S_p$ and that $\overline{r}_w$ is non-scalar for $w \in S_D$. Then there is a so-called 'local factor' defined in [BD14, § 3.3] and [EGS15, § 6.5] as follows:

$$\pi \overset{\text{def}}{=} \mathrm{Hom}_{U^v}(\overline{M}^v, \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\overline{r}, \varinjlim_V H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})))[\mathfrak{m}'], \tag{1.1}$$

where the inductive limit runs over the compact open subgroups $V \subseteq (D \otimes_F \mathbb{A}_F^\infty)^\times$, and we refer to [BD14, § 3.3] and [EGS15, § 6.5] for the definitions of the compact open subgroup $U^v \subseteq (D \otimes_F \mathbb{A}_F^{\infty,v})^\times$, the (finite-dimensional) irreducible smooth representation $\overline{M}^v$ of U^v over $\mathbb{F}$, and the maximal ideal $\mathfrak{m}'$ in a certain Hecke algebra. We assume that $\overline{r}$ is modular in the sense that $\pi \neq 0$.

Then the key question is to understand the $\mathrm{GL}_2(K)$-representation π in (1.1). It is hoped that the representation π can be used to realize the mod p Langlands correspondence for $\mathrm{GL}_2(K)$. In particular, we hope that π only depends on $\overline{r}_v$ and would like to find a description of π in terms of $\overline{r}_v$. There have been many results on the representation-theoretic properties of π as above. For example, under some mild assumptions on $\overline{r}$ facts are known.

(i) We have $\pi^{K_1} \cong D_0(\overline{r}_v^\vee)$ as $K^\times \mathrm{GL}_2(\mathcal{O}_K)$-representations [Le19], where $K_1 \overset{\text{def}}{=} 1 + p\,\mathrm{M}_2(\mathcal{O}_K)$, $D_0(\overline{r}_v^\vee)$ is the (finite-dimensional) representation of $\mathrm{GL}_2(\mathcal{O}_K)$ defined in [BP12, § 13] and $K^\times$ acts on $D_0(\overline{r}_v^\vee)$ by the character $\det(\overline{r}_v^\vee)\omega^{-1}$ with ω the mod p cyclotomic character.

(ii) The Diamond diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ only depends on $\overline{r}_v$ (see [DL21]), where $I_1 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right)$. Moreover, [DL21] computed explicitly many constants appearing in the diagram when $\overline{r}_v$ is assumed to be semisimple.

However, the complete understanding of π still seems a long way off.

In this article, we generalize the computation of [DL21] and compute explicitly many constants appearing in the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ when $\overline{r}_v$ is non-semisimple, which is much more complicated than in the semisimple case. As a result, (when p is sufficiently large with respect to $f \overset{\text{def}}{=} [K : \mathbb{Q}_p]$ with some more assumptions on $\overline{r}$) we prove a local–global compatibility result for π as above which was conjectured by Breuil, Herzig, Hu, Morra and Schraen [BHH+25b, Conjecture 3.1.2].

To state the main result, we refer to [BHH+25a] for the definition of the ring A and the notion of étale $(\varphi, \mathcal{O}_K^\times)$-modules over A (see also § 7). In [BHH+25a], Breuil, Herzig, Hu, Morra and Schraen attached to π as in (1.1) an étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ over A. In [BHH+25b], they also gave a conjectural description of $D_A(\pi)$ in terms of $\overline{r}_v$ by constructing a functor $D_A^\otimes$ from the category of finite-dimensional continuous representations of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{F}$ to the category of étale $(\varphi, \mathcal{O}_K^\times)$-modules over A.

We assume, moreover, that:

(i) the framed deformation ring $R_{\overline{r}_w}$ of $\overline{r}_w$ over the Witt vectors $W(\mathbb{F})$ is formally smooth for $w \in (S_D \cup S_{\overline{r}}) \setminus S_p$;

(ii) up to twist, $\overline{r}_v$ is of the form

$$\overline{r}_v|_{I_K} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix} \quad \text{with } \max\{12, 2f+1\} \leq r_j \leq p - \max\{15, 2f+3\} \quad \forall\, j, \tag{1.2}$$

where $I_K \subseteq \mathrm{Gal}(\overline{K}/K)$ is the inertia subgroup and ω_f is the fundamental character of level f.

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Our main result is the following theorem, which verifies [BHH+25b, Conjecture 3.1.2] in our setting.

THEOREM 1.1. *Let π be as in (1.1) and keep all the assumptions on $\overline{r}$. Then we have an isomorphism of étale $(\varphi, \mathcal{O}_K^\times)$-modules*

$$D_A(\pi) \cong D_A^{\otimes}(\overline{r}_v(1)).$$

Theorem 1.1 is proved by [BHH+25b, Theorem 3.1.3] when $\overline{r}_v$ is semisimple, and proved by [Wan26, Theorem 1.1] when $\overline{r}_v$ is maximally non-split in the sense that $|W(\overline{r}_v)| = 1$, where $W(\overline{r}_v)$ is the set of Serre weights of $\overline{r}_v$ defined in [BDJ10, § 3]. The proof of Theorem 1.1 is based on the explicit computation of certain constants appearing in the Diamond diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ in the sense of [DL21], together with the results of [Wan24] on $D_A(\pi)$ and the results of [Wan26] on $D_A^{\otimes}(\overline{r}_v(1))$.

We describe certain constants in the Diamond diagram and the strategy of the proof of Theorem 1.1 in more detail.

We let $R : \pi^{I_1} \to (\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi)^{I_1}$ be the map defined as in [DL21, Definition 4.1] and we write $I \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{smallmatrix}\right)$. Given an I-character χ, we write $R\chi$ for the I-character such that $R(\pi^{I_1}[\chi]) = \pi^{I_1}[R\chi]$. In particular, by definition we have $R\chi = \chi$ if and only if χ appears in $(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi)^{I_1}$. Then we define the non-zero map $g_\chi : \pi^{I_1}[R\chi] \to \pi^{I_1}[R\chi^s]$ between one-dimensional $\mathbb{F}$-vector spaces by the formula $g_\chi(R(v)) = R(\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)v)$ for $v \in \pi^{I_1}[\chi]$, where χ^s is the conjugation of χ by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. All the constants in the Diamond diagram are then defined in terms of suitable compositions of the maps g_χ. We write $\delta(\chi) \overset{\mathrm{def}}{=} R\chi^s$.

When $\overline{r}_v$ is semisimple, for any I-character χ appearing in π^{I_1} such that $R\chi = \chi$ there exists an integer $d \geq 1$ such that $\delta^d(\chi) = \chi$. Then the composition

$$\pi^{I_1}[\chi] \xrightarrow{g_\chi} \pi^{I_1}[\delta(\chi)] \xrightarrow{g_{\delta(\chi)}} \cdots \to \pi^{I_1}[\delta^d(\chi)] = \pi^{I_1}[\chi]$$

is given by a scalar $g(\chi) \in \mathbb{F}^\times$, which is an example of the constants in the Diamond diagram. The constants $g(\chi)$ for $R\chi = \chi$ are computed explicitly by [DL21], which is enough to determine the structure of $D_A(\pi)$ and to conclude Theorem 1.1 in the semisimple case.

When $\overline{r}_v$ is non-semisimple, the situation is completely different. In fact, any I-character χ appearing in π^{I_1} converges to a distinguished I-character χ_0 in the sense that there exists $\ell(\chi) \geq 1$ such that $\delta^\ell(\chi) = \chi_0$ for all $\ell \geq \ell(\chi)$. To obtain a constant in the Diamond diagram, we consider the following two maps:

$$\pi^{I_1}[R\chi] \xrightarrow{g_\chi} \pi^{I_1}[\delta(\chi)] \xrightarrow{g_{\delta(\chi)}} \cdots \to \pi^{I_1}[\delta^{\ell(\chi)}(\chi)] = \pi^{I_1}[\chi_0],$$
$$\pi^{I_1}[R\chi] \xrightarrow{g_{R\chi}} \pi^{I_1}[\delta(R\chi)] \xrightarrow{g_{\delta(R\chi)}} \cdots \to \pi^{I_1}[\delta^{\ell(R\chi)}(R\chi)] = \pi^{I_1}[\chi_0].$$

Then the composition

$$\left(\prod_{i=\ell(R\chi)-1}^{0} g_{\delta^i(R\chi)}\right)^{-1} \circ \left(\prod_{i=\ell(\chi)-1}^{0} g_{\delta^i(\chi)}\right) : \pi^{I_1}[R\chi] \to \pi^{I_1}[R\chi]$$

is given by a scalar $g(\chi) \in \mathbb{F}^\times$, which is an example of the constants in the Diamond diagram.

The key step in the non-semisimple case is to find a suitable collection of I-characters χ such that the constants $g(\chi)$ are enough to determine the structure of $D_A(\pi)$. This is based on a detailed study of the relation between I_1-invariants of π.

Then we compute explicitly these constants $g(\chi)$ following the strategy of [DL21] and we refer to [DL21, § 1] for a more detailed introduction. The computation is much more delicate

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than in the semisimple case and takes up a substantial portion of the article. Finally, we are able to conclude Theorem 1.1 using these computations together with the results of [Wan24] on $D_A(\pi)$ and the results of [Wan26] on $D_A^{\otimes}(\overline{r}_v(1))$.

The proof of Theorem 1.1 is very computational. There may one day exist a more conceptual proof, which will hopefully avoid the genericity assumptions on $\overline{r}_v$ and the technical computations, but such proof is not known so far.

1.1 Organization of the article

In § 2, we define all the basic objects that are needed throughout this article. In § 3, we study the relation between I_1-invariants of π that are needed to form the necessary constants in the Diamond diagram. In § 5, we review the strategy of [DL21] and specialize to the non-semisimple case. In particular, the computation of the constants in the Diamond diagram can be divided into two parts: one comes from certain elements in tamely potentially Barsotti–Tate deformation rings and is the content of § 4, the other comes from the action of certain elements of the group algebra of $\mathrm{GL}_2(\mathcal{O}_K)$ on tame types and is the content of § 6. Finally, in § 7 we combine the results of the previous sections and the results of [Wan24] and [Wan26] to finish the proof of Theorem 1.1.

1.2 Notation

Let p be a prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $K \subseteq \overline{\mathbb{Q}}_p$ be the unramified extension of $\mathbb{Q}_p$ of degree $f \geq 1$ with ring of integers $\mathcal{O}_K$ and residue field $\mathbb{F}_q$ (hence $q = p^f$). We denote by $G_K \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ the absolute Galois group of K and by $I_K \subseteq G_K$ the inertia subgroup. Let $\mathbb{F}$ be a large enough finite extension of $\mathbb{F}_p$. Fix an embedding $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and let $\sigma_j \overset{\mathrm{def}}{=} \sigma_0 \circ \varphi^j$ for $j \in \mathbb{Z}$, where $\varphi : x \mapsto x^p$ is the arithmetic Frobenius on $\mathbb{F}_q$. We identify $\mathcal{J} \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{F}_q, \mathbb{F})$ with $\{0, 1, \dots, f-1\}$, which is also identified with $\mathbb{Z}/f\mathbb{Z}$ so that the addition and subtraction in $\mathcal{J}$ are modulo f. For $a \in \mathcal{O}_K$, we denote by $\overline{a} \in \mathbb{F}_q$ its reduction modulo p. For $a \in \mathbb{F}_q$, we also view it as an element of $\mathbb{F}$ via σ_0.

For F a perfect ring of characteristic p, we denote by $W(F)$ the ring of Witt vectors of F. For $x \in F$, we denote by $[x] \in W(F)$ its Techmüller lift.

Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. We view elements of $\mathcal{O}_K$ and elements of $W(\mathbb{F})$ as elements of $\mathcal{O}$ via the embedding $\mathcal{O}_K = W(\mathbb{F}_q) \overset{\sigma_0}{\hookrightarrow} W(\mathbb{F}) \hookrightarrow \mathcal{O}$.

Let $I \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{smallmatrix}\right)$ be the Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$, $I_1 \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right)$ be the pro-p Iwahori subgroup, $K_1 \overset{\mathrm{def}}{=} 1 + p\,\mathrm{M}_2(\mathcal{O}_K)$ be the first congruence subgroup, $N_0 \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)$ and $H \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{smallmatrix}\right)$.

For P a statement, we let $\delta_P \overset{\mathrm{def}}{=} 1$ if P is true and $\delta_P \overset{\mathrm{def}}{=} 0$ otherwise.

Throughout this article, we let π be as in (1.1) and $\overline{\rho} \overset{\mathrm{def}}{=} \overline{r}_v^\vee$. Twisting $\overline{\rho}$ and π using [BHH+25b, Lemma 2.9.6 and 3.1.1], we assume, moreover, that

$$\overline{\rho} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j}\,\mathrm{un}(\xi) & * \\ 0 & \mathrm{un}(\xi)^{-1} \end{pmatrix}, \tag{1.3}$$

where r_j is as in (1.2), $\xi \in \mathbb{F}^\times$, $\mathrm{un}(\xi) : G_K \to \mathbb{F}^\times$ is the unramified character sending geometric Frobenius elements to ξ, and $\omega_f : G_K \to \mathbb{F}^\times$ is such that $\omega_f(g)$ is the reduction modulo p of $g(\sqrt[q-1]{-p})/\sqrt[q-1]{-p} \in \mathcal{O}^\times$ for all $g \in G_K$ and for any choice of a $(q-1)$th root $\sqrt[q-1]{-p}$ of $-p$. In particular, p acts trivially on π.

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2. Preliminaries

We write $\underline{i}$ for an element $(i_0, \dots, i_{f-1}) \in \mathbb{Z}^f$. For $a \in \mathbb{Z}$, we write $\underline{a} \overset{\text{def}}{=} (a, \dots, a) \in \mathbb{Z}^f$. For $J \subseteq \mathcal{J}$, we define $\underline{e}^J \in \mathbb{Z}^f$ by $e_j^J \overset{\text{def}}{=} \delta_{j \in J}$. We say that $\underline{i} \le \underline{i}'$ if $i_j \le i_j'$ for all j. We define the left shift $\delta : \mathbb{Z}^f \to \mathbb{Z}^f$ by $\delta(\underline{i})_j \overset{\text{def}}{=} i_{j+1}$. Abusing notation, we still denote by $\underline{i}$ the integer $\sum_{j=0}^{f-1} i_j p^j \in \mathbb{Z}$.

A *Serre weight* is an isomorphism class of an absolutely irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{F}$. For $\lambda = (\underline{\lambda}_1, \underline{\lambda}_2) \in \mathbb{Z}^{2f}$ such that $\underline{0} \le \underline{\lambda}_1 - \underline{\lambda}_2 \le \underline{p} - \underline{1}$, we define

$$F(\lambda) \overset{\text{def}}{=} \bigotimes_{j=0}^{f-1} ((\mathrm{Sym}^{\lambda_{1,j} - \lambda_{2,j}} \mathbb{F}_q^2 \otimes_{\mathbb{F}_q} \det^{\lambda_{2,j}}) \otimes_{\mathbb{F}_q, \sigma_j} \mathbb{F}).$$

We also denote it by $(\underline{\lambda}_1 - \underline{\lambda}_2) \otimes \det^{\underline{\lambda}_2}$.

For $\lambda = (\underline{\lambda}_1, \underline{\lambda}_2) \in \mathbb{Z}^{2f}$, we define the character $\chi_\lambda : I \to \mathbb{F}^\times$ by $\left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \mapsto (\overline{a})^{\underline{\lambda}_1} (\overline{d})^{\underline{\lambda}_2}$, where $a, d \in \mathcal{O}_K^\times$ and $b, c \in \mathcal{O}_K$. In particular, if $\underline{0} \le \underline{\lambda}_1 - \underline{\lambda}_2 \le \underline{p} - \underline{1}$, then χ_λ is the I-character acting on $F(\lambda)^{I_1}$. We still denote by χ_λ its restriction to H.

We write $\underline{r} = (r_0, \dots, r_{f-1})$ with r_j as in (1.2). For $\underline{b} \in \mathbb{Z}^f$ such that $-\underline{r} \le \underline{b} \le \underline{p} - \underline{2} - \underline{r}$, we denote by $\sigma_{\underline{b}}$ the Serre weight $F(\mathfrak{t}_{(\underline{r},\underline{0})}(\underline{b}))$ (see [BHH+23, § 2.4] for $\mathfrak{t}_{(\underline{r},\underline{0})}(\underline{b}) \in \mathbb{Z}^{2f}$).

For $\overline{\rho}$ as in (1.3), we let $W(\overline{\rho})$ be the set of Serre weights of $\overline{\rho}$ defined in [BDJ10, § 3] and $J_{\overline{\rho}} \subseteq \mathcal{J}$ be the subset as in [Bre14, (17)]. Then by [Bre14, Proposition A.3] and [BHH+23, (14)], the subset $J_{\overline{\rho}} \subseteq \mathcal{J}$ is characterized by

$$W(\overline{\rho}) = \left\{ \sigma_{\underline{b}} : \begin{array}{ll} b_j = 0 & \text{if } j \notin J_{\overline{\rho}} \\ b_j \in \{0, 1\} & \text{if } j \in J_{\overline{\rho}} \end{array} \right\}.$$

In particular, $\overline{\rho}$ is semisimple if and only if $J_{\overline{\rho}} = \mathcal{J}$.

For each $J \subseteq \mathcal{J}$, as in [Wan24, § 2] we define the (distinct) Serre weight $\sigma_J \overset{\text{def}}{=} F(\lambda_J)$ with $\lambda_J \overset{\text{def}}{=} (\underline{s}^J + \underline{t}^J, \underline{t}^J)$, where

$$s_j^J \overset{\text{def}}{=} \begin{cases} r_j & \text{if } j \notin J,\ j+1 \notin J, \\ r_j + 1 & \text{if } j \in J,\ j+1 \notin J, \\ p - 2 - r_j & \text{if } j \notin J,\ j+1 \in J, \\ p - 1 - r_j & \text{if } j \in J,\ j+1 \in J,\ j \notin J_{\overline{\rho}}, \\ p - 3 - r_j & \text{if } j \in J,\ j+1 \in J,\ j \in J_{\overline{\rho}}; \end{cases} \tag{2.1}$$

$$t_j^J \overset{\text{def}}{=} \begin{cases} 0 & \text{if } j \notin J,\ j+1 \notin J, \\ -1 & \text{if } j \in J,\ j+1 \notin J, \\ r_j + 1 & \text{if } j \notin J,\ j+1 \in J \text{ or } j \in J,\ j+1 \in J,\ j \in J_{\overline{\rho}}, \\ r_j & \text{if } j \in J,\ j+1 \in J,\ j \notin J_{\overline{\rho}}. \end{cases} \tag{2.2}$$

In particular, for $J \subseteq J_{\overline{\rho}}$ we have $\sigma_J = \sigma_{\underline{e}^J}$ (see [Wan24, § 2]). We also define the Serre weight $\sigma_J^s \overset{\text{def}}{=} F(\lambda_{J^s})$ with $\lambda_{J^s} \overset{\text{def}}{=} (\underline{s}^{J^s} + \underline{t}^{J^s}, \underline{t}^{J^s})$, where $s_j^{J^s} \overset{\text{def}}{=} p - 1 - s_j^J$ and $t_j^{J^s} \overset{\text{def}}{=} r_j - t_j^J$. We write $\chi_J \overset{\text{def}}{=} \chi_{\lambda_J}$ and $\chi_{J^s} \overset{\text{def}}{=} \chi_{\lambda_{J^s}}$ so that χ_J (respectively, χ_{J^s}) is the I-character acting on $\sigma_J^{I_1}$ (respectively, $(\sigma_J^s)^{I_1}$). For each I-character χ, we denote by χ^s its conjugation by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. Then we have $\chi_J^s = \chi_{J^s}$ and $\chi_{J^s}^s = \chi_J$ for all $J \subseteq \mathcal{J}$.

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For $J \subseteq \mathcal{J}$ and $k \in \mathbb{Z}$, we write $J + k \overset{\text{def}}{=} \{j + k : j \in J\} \subseteq \mathcal{J}$. Then we define

$$\begin{aligned} &J^{\mathrm{ss}} \overset{\text{def}}{=} J \cap J_{\overline{\rho}}, && J^{\mathrm{nss}} \overset{\text{def}}{=} J \setminus J_{\overline{\rho}}, && J^{c} \overset{\text{def}}{=} \mathcal{J} \setminus J, \\ &\partial J \overset{\text{def}}{=} J \setminus (J-1), && \delta_{\mathrm{ss}}(J) \overset{\text{def}}{=} (J-1)^{\mathrm{ss}}, && J^{\delta} \overset{\text{def}}{=} J \Delta (J-1)^{\mathrm{ss}}. \end{aligned} \tag{2.3}$$

Here we recall that $J \Delta J' \overset{\text{def}}{=} (J \setminus J') \sqcup (J' \setminus J)$.

LEMMA 2.1. *For $J \subseteq \mathcal{J}$, we have:*

(i) $s_j^J = \begin{cases} s_j^{(J-1)^{\mathrm{ss}}} + \delta_{j \in J\Delta(J-1)^{\mathrm{ss}}} & \text{if } j+1 \notin J\Delta(J-1)^{\mathrm{ss}}, \\ p-2-s_j^{(J-1)^{\mathrm{ss}}} + \delta_{j \in J\Delta(J-1)^{\mathrm{ss}}} & \text{if } j+1 \in J\Delta(J-1)^{\mathrm{ss}}; \end{cases}$

(ii) $s_j^J = \begin{cases} s_j^{J^{\mathrm{ss}}} + \delta_{j \in J^{\mathrm{nss}}} & \text{if } j+1 \notin J^{\mathrm{nss}}, \\ p-2-s_j^{J^{\mathrm{ss}}} + \delta_{j \in J^{\mathrm{nss}}} & \text{if } j+1 \in J^{\mathrm{nss}}; \end{cases}$

(iii) $t_j^J = r_j - t_j^{(J-1)^{\mathrm{ss}}}$ *if* $j, j+1 \in J\Delta(J-1)^{\mathrm{ss}}$.

Proof. This follows from a case-by-case examination and is left as an exercise. As a sample computation, we refer to [Wan24, Lemma D.1] for the second part of (i). □

LEMMA 2.2. *We have $\sigma_{J^s} = \sigma_{(J-1)^{\mathrm{ss}}}$ if and only if $J\Delta(J-1)^{\mathrm{ss}} = \mathcal{J}$, which happens for a unique $J^* \subseteq \mathcal{J}$ if $J_{\overline{\rho}} \neq \mathcal{J}$.*

Proof. Suppose that $J\Delta(J-1)^{\mathrm{ss}} = \mathcal{J}$. Then by Lemma 2.1(i),(iii) we have $s_j^J = p-1-s_j^{(J-1)^{\mathrm{ss}}}$ and $t_j^J = r_j - t_j^{(J-1)^{\mathrm{ss}}}$ for all $j \in \mathcal{J}$, which implies $\sigma_{J^s} = \sigma_{(J-1)^{\mathrm{ss}}}$.

Conversely, suppose that $\sigma_{J^s} = \sigma_{(J-1)^{\mathrm{ss}}}$. Comparing the I_1-invariants, we get $\chi_{J^s} = \chi_{(J-1)^{\mathrm{ss}}}$. In particular, we have $\sum_{j=0}^{f-1} p^j t_j^{J^s} \equiv \sum_{j=0}^{f-1} p^j t_j^{(J-1)^{\mathrm{ss}}} \bmod (q-1)$, which implies $\underline{t}^{J^s} = \underline{t}^{(J-1)^{\mathrm{ss}}}$ since $-1 \le t_j^J \le p-3$ for all $J \subseteq \mathcal{J}$ and $j \in \mathcal{J}$ by (1.2) and (2.2). By (2.2) and using $r_j \ge 1$ for all j, we have $\{j : t_j^J \in \{r_j, r_j+1\}\} + 1 = J$ for all $J \subseteq \mathcal{J}$, hence we conclude that $J^c = (J-1)^{\mathrm{ss}}$.

If, moreover, $J_{\overline{\rho}} \neq \mathcal{J}$, then $J^* \subseteq \mathcal{J}$ is uniquely characterized by the property that $j \in J^*$ if and only if $j \notin J_{\overline{\rho}}$ or $(j \in J_{\overline{\rho}}$ and $j+1 \notin J)$. □

LEMMA 2.3. *For $J, J' \subseteq \mathcal{J}$, we have $(-1)^{\underline{t}^J + \underline{t}^{J^s} + \underline{s}^{J'}} = 1$.*

Proof. By definition we have $\underline{t}^J + \underline{t}^{J^s} = \underline{r}$. By (2.1) we also have $s_j^{J'} \not\equiv r_j \bmod 2$ if and only if $j \in \partial(J')$ or $j \in \partial((J')^c)$, which proves the result since $|\partial(J')| = |\partial((J')^c)|$. □

3. The relation between I_1-invariants

Let π be as in (1.1). We study the relation between I_1-invariants of π, which generalizes some results of [Wan24, § 5]. The main results are Propositions 3.1 and 3.4.

From now on, we identify π^{K_1} with $D_0(\overline{\rho})$ (see the introduction), which is the (finite-dimensional) representation of $\mathrm{GL}_2(\mathcal{O}_K)$ defined in [BP12, § 13]. For each $J \subseteq \mathcal{J}$, the character χ_J appears in $\pi^{I_1} = D_0(\overline{\rho})^{I_1}$ with multiplicity one by [Wan24, Lemma 4.1(ii)]. We fix a choice of $0 \neq v_J \in \pi^{I_1}$ with I-character χ_J. For each $j \in \mathcal{J}$ we define

$$Y_j \overset{\text{def}}{=} \sum_{a \in \mathbb{F}_q^\times} a^{-p^j} \begin{pmatrix} 1 & [a] \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[\![N_0]\!].$$

For $\underline{i} = (i_0, \dots, i_{f-1}) \in \mathbb{Z}^f$, we write $\underline{Y^i}$ for $\prod_{j=0}^{f-1} Y_j^{i_j}$.

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For $J, J' \subseteq \mathcal{J}$ such that $(J-1)^{\mathrm{ss}} = (J')^{\mathrm{ss}}$, we define $\mu_{J,J'} \in \mathbb{F}^\times$ as in [Wan24, (47)]. In particular, in the case $(J')^{\mathrm{nss}} \neq \mathcal{J}$ such that

$$(J')^{\mathrm{nss}} \subseteq (J-1)^{\mathrm{nss}} \Delta (J'-1)^{\mathrm{nss}}, \tag{3.1}$$

the element $\mu_{J,J'} \in \mathbb{F}^\times$ is defined by the formula (see [Wan24, Proposition 5.10])

$$\left[\prod_{j+1 \in J \Delta J'} Y_j^{s_j^{J'}} \prod_{j+1 \notin J \Delta J'} Y_j^{p-1}\right] \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} (\underline{Y}^{-\underline{e}^{(J \cap J')^{\mathrm{nss}}}} v_J) = \mu_{J,J'} v_{J'}, \tag{3.2}$$

where $\underline{Y}^{-\underline{e}^{(J \cap J')^{\mathrm{nss}}}} v_J \in D_0(\overline{\rho})$ is a suitable shift of v_J defined in [Wan24, Proposition 4.2]. As in the discussion after [Wan24, (47)], for $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$ such that $(J_1-1)^{\mathrm{ss}} = (J_2-1)^{\mathrm{ss}} = J_3^{\mathrm{ss}} = J_4^{\mathrm{ss}}$ we have $\mu_{J_1,J_3}/\mu_{J_1,J_4} = \mu_{J_2,J_3}/\mu_{J_2,J_4}$. In particular, for J, J' such that $J^{\mathrm{ss}} = (J')^{\mathrm{ss}}$, the quantity $\mu_{J'',J}/\mu_{J'',J'}$ does not depend on J'' such that $(J''-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$, and we denote it by $\mu_{*,J}/\mu_{*,J'}$.

For each $J \subseteq \mathcal{J}$, the character χ_J^s also appears in $\pi^{I_1} = D_0(\overline{\rho})^{I_1}$ with multiplicity one by [Wan24, Lemma 4.1(ii)]. We fix a choice of $0 \neq v_{J^s} \in \pi^{I_1}$ with I-character χ_J^s. Since p acts trivially on π, by rescaling the vectors v_J and v_{J^s} we assume from now on that $v_{J^s} = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) v_J$ for all $J \subseteq \mathcal{J}$. We prove some analogous results for the vectors v_{J^s}.

PROPOSITION 3.1.

(i) *For $J \subseteq \mathcal{J}$, there exists a unique element $\mu_{J^s,J^{\mathrm{ss}}} \in \mathbb{F}^\times$ such that*

$$\left[\prod_{j+1 \notin J^{\mathrm{nss}}} Y_j^{s_j^{J^{\mathrm{ss}}}} \prod_{j+1 \in J^{\mathrm{nss}}} Y_j^{p-1}\right] \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v_{J^s} = \mu_{J^s,J^{\mathrm{ss}}} v_{J^{\mathrm{ss}}}. \tag{3.3}$$

(ii) *For $J \subseteq \mathcal{J}$ such that $J^{\mathrm{nss}} \neq \mathcal{J}$, there exists a unique element $\mu_{J^s,J} \in \mathbb{F}^\times$ such that*

$$\underline{Y}^{\underline{s}^J} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v_{J^s} = \mu_{J^s,J} v_J.$$

Proof. (i) Since $v_J \in \pi^{I_1}$ has I-character χ_J, by Frobenius reciprocity there is a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant map

$$\begin{aligned} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_{J^s}^s) = \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J) &\overset{\alpha}{\twoheadrightarrow} \langle \mathrm{GL}_2(\mathcal{O}_K) v_J \rangle = \left\langle \mathrm{GL}_2(\mathcal{O}_K) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v_{J^s} \right\rangle \hookrightarrow \pi, \\ \phi &\mapsto v_J = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} v_{J^s}, \end{aligned} \tag{3.4}$$

where $\phi \in \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J)$ is supported on I such that $\phi(\mathrm{id}) = 1$. By [Wan24, Proposition 4.2], the $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation $V \overset{\mathrm{def}}{=} \langle \mathrm{GL}_2(\mathcal{O}_K) v_J \rangle \subseteq D_0(\overline{\rho})$ lies in $D_{0,J^{\mathrm{ss}}}(\overline{\rho})$ (as defined in [Wan24, Lemma 4.1(i)]), hence has socle $\sigma_{J^{\mathrm{ss}}}$. As a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J)$, V has cosocle σ_J, which equals $\sigma_{\underline{a}^J}$ with $\underline{a}^J$ as in [Wan24, (5)]. In particular, we have $V = I(\sigma_{J^{\mathrm{ss}}}, \sigma_{\underline{a}^J})$ in the notation of [Wan24, Proposition 5.1(iii)]. From the explicit formula of $\underline{a}^J$, one checks that $a_j^J = \delta_{j \in J^{\mathrm{ss}}}$ if and only if $j \notin J^{\mathrm{nss}}$. Then we deduce from [Wan24, Lemma 3.2(i)] (applied to $\lambda = \lambda_{J^s}$) that $\sigma_{J^{\mathrm{ss}}}$ corresponds to the subset $(J^{\mathrm{nss}})^c - 1$ (in the sense of [Wan24, Lemma 3.2(i)]), hence V is isomorphic to the quotient $Q(\chi_{J^s}^s, (J^{\mathrm{nss}})^c - 1)$ of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_{J^s}^s)$ in the notation of [Wan24, Lemma 3.2(iii)]. By Lemma 2.1(ii), for $j+1 \notin J^{\mathrm{nss}}$ we have

$$p - 2 - s_j^{J^s} + \delta_{j \in (J^{\mathrm{nss}})^c} = s_j^J - \delta_{j \in J^{\mathrm{nss}}} = s_j^{J^{\mathrm{ss}}}.$$

Then by [Wan24, Lemma 3.2(iii)] (applied to $\lambda = \lambda_{J^s}$), the left-hand side of (3.3) is non-zero in $\sigma_{J^{\mathrm{ss}}}$ and is the unique (up to scalar) H-eigencharacter in $\sigma_{J^{\mathrm{ss}}}$ killed by all Y_j. It follows that the left-hand side of (3.3) is a non-zero I_1-invariant of $\sigma_{J^{\mathrm{ss}}}$, hence is a scalar multiple of $v_{J^{\mathrm{ss}}}$.

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(ii) By the proof of (i) and using $J^{\mathrm{nss}} \neq \mathcal{J}$, the $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant surjection α in (3.4) is not an isomorphism, hence it maps $\sigma_J^s = \mathrm{soc}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_J))$ to zero. By [Wan24, Lemma 3.2(iii)(a)] (applied to $\lambda = \lambda_{J^s}$) we have

$$\underline{Y}^{\underline{p}-\underline{1}-\underline{s}^{J^s}} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v_{J^s} + (-1)^{f-1+\underline{s}^{J^s}+\underline{t}^{J^s}} \left(\prod_{j=0}^{f-1} (s_j^{J^s})! \right) v_{J^s} = 0.$$

Since $\underline{s}^{J^s} = \underline{p} - \underline{1} - \underline{s}^J$, this proves (ii) with

$$\mu_{J^s,J} = (-1)^{\underline{1}+\underline{s}^{J^s}+\underline{t}^{J^s}} \left(\prod_{j=0}^{f-1} (p-1-s_j^J)! \right)^{-1} = (-1)^{\underline{t}^{J^s}} \left(\prod_{j=0}^{f-1} (s_j^J)! \right), \tag{3.5}$$

where the second equality uses

$$((p-1-r)!)^{-1} \equiv (-1)^{r+1} r! \mod p \ \forall\, 0 \leq r \leq p-1, \tag{3.6}$$

and the uniqueness is clear. □

COROLLARY 3.2. *Suppose that $J_{\overline{\rho}} \neq \mathcal{J}$ and $J_{\overline{\rho}} \neq \emptyset$. Let J^* be as in Lemma 2.2. Then we have*

$$(-1)^{|J^* \cap (J^*-1)^{\mathrm{nss}}|} \mu_{(J^*-1)^{\mathrm{ss}},J^*} \mu_{J^*,(J^*-1)^{\mathrm{ss}}} = 1.$$

Proof. Note that $J_{\overline{\rho}} \neq \emptyset$ implies $(J^*)^{\mathrm{nss}} \neq \mathcal{J}$. Since $\sigma_{(J^*-1)^{\mathrm{ss}}} = \sigma_{J^*}^s$ by Lemma 2.2, we have $\underline{s}^{(J-1)^{\mathrm{ss}}} = \underline{s}^{J^s}$ and $\underline{t}^{((J-1)^{\mathrm{ss}})^s} = \underline{t}^J$. Then by (3.5) we have

$$\mu_{(J^*-1)^{\mathrm{ss}},J^*} = \mu_{(J^*)^s,J^*} = (-1)^{\underline{t}^{J^s}} \left(\prod_{j=0}^{f-1} (s_j^J)! \right),$$

$$\mu_{J^*,(J^*-1)^{\mathrm{ss}}} = \mu_{((J^*-1)^{\mathrm{ss}})^s,(J^*-1)^{\mathrm{ss}}} = (-1)^{\underline{t}^J} \left(\prod_{j=0}^{f-1} (s_j^{J^s})! \right).$$

Hence, we have

$$\mu_{(J^*-1)^{\mathrm{ss}},J^*} \mu_{J^*,(J^*-1)^{\mathrm{ss}}} = (-1)^{\underline{t}^{J^s}+\underline{t}^J} \left(\prod_{j=0}^{f-1} (s_j^J)!(s_j^{J^s})! \right) = (-1)^{\underline{t}^{J^s}+\underline{t}^J+\underline{s}^J+\underline{1}} = (-1)^f,$$

where the second equality uses (3.6) and the last equality follows from Lemma 2.3. Moreover, from the structure of J^* one can check that $|J^* \cap (J^*-1)^{\mathrm{nss}}| \equiv f \mod 2$, which completes the proof. □

LEMMA 3.3. *Let $J \subseteq \mathcal{J}$. We write $J_0 \overset{\mathrm{def}}{=} (J^{\mathrm{ss}} \sqcup (J^c - 1)^{\mathrm{nss}}) + 1$ and $c_j' \overset{\mathrm{def}}{=} 2\delta_{j \in J_0 \cap J^{\mathrm{nss}}} + p - 1 - s_j^{J_0} + \delta_{j \in J_0 \Delta J}$ for all $j \in \mathcal{J}$. Then we have for all $j \in \mathcal{J}$:*

(i) $(-1)^{\delta_{j+1 \notin J_0}} (2\delta_{j \in J_0 \cap J^{\mathrm{nss}}} + \delta_{j \in J^{\mathrm{ss}}} - \delta_{j \in J_0 \Delta J^{\mathrm{ss}}} + \delta_{j \in J_0 \Delta J}) = a_j^J$ *(defined in [Wan24, (5)]);*

(ii) $s_j^J + \delta_{j+1 \notin J_0 \Delta J} c_j' = \delta_{j+1 \in J_0 \Delta J} s_j^J + \delta_{j+1 \notin J_0 \Delta J}(p-1)$;

(iii) $\delta_{j+1 \notin J^{\mathrm{nss}}} s_j^{J^{\mathrm{ss}}} + \delta_{j+1 \in J^{\mathrm{nss}}}(p-1) + \delta_{j+1 \notin J_0 \Delta J} c_j' = \delta_{j+1 \in J_0 \cap J^{\mathrm{nss}}} p + \delta_{j+1 \in J_0 \Delta J^{\mathrm{ss}}} s_j^{J^{\mathrm{ss}}} + \delta_{j+1 \notin J_0 \Delta J^{\mathrm{ss}}}(p-1)$.

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Proof.

(i) We have

$$\begin{aligned}&(-1)^{\delta_{j+1\notin J_0}}(2\delta_{j\in J_0\cap J^{\mathrm{nss}}}+\delta_{j\in J^{\mathrm{ss}}}-\delta_{j\in J_0\Delta J^{\mathrm{ss}}}+\delta_{j\in J_0\Delta J})\\&\quad=(-1)^{\delta_{j+1\notin J_0}}(2\delta_{j\in J_0}\delta_{j\in J^{\mathrm{nss}}}+\delta_{j\in J^{\mathrm{ss}}}\\&\qquad-(\delta_{j\in J_0}+\delta_{j\in J^{\mathrm{ss}}}-2\delta_{j\in J_0}\delta_{j\in J^{\mathrm{ss}}})+(\delta_{j\in J_0}+\delta_{j\in J}-2\delta_{j\in J_0}\delta_{j\in J}))\\&\quad=(-1)^{\delta_{j\in(J^c)^{\mathrm{ss}}}+\delta_{j\in(J-1)^{\mathrm{nss}}}}\delta_{j\in J}=(-1)^{\delta_{j\in(J-1)^{\mathrm{nss}}}}\delta_{j\in J}=a_j^J,\end{aligned}$$

where the last equality follows from the explicit formula of a_j^J.

(ii) It suffices to show that $s_j^{J_0}=s_j^J+2\delta_{j\in J_0\cap J^{\mathrm{nss}}}+\delta_{j\in J_0\Delta J}$ for $j+1\notin J_0\Delta J$. This follows from a case-by-case examination similar to the proof of [Wan24, Lemma D.1].

(iii) By the definition of J_0, we have that $j+1\notin J_0\Delta J$ implies $j\notin J^{\mathrm{nss}}$. Hence, by (ii) and Lemma 2.1(ii) we have

$$\delta_{j+1\notin J_0\Delta J}c_j'=\delta_{j+1\notin J_0\Delta J}(p-1-s_j^J)=\begin{cases}s_j^{J^{\mathrm{ss}}}+1 & \text{if } j+1\in J_0\cap J^{\mathrm{nss}},\\ p-1-s_j^{J^{\mathrm{ss}}} & \text{if } j+1\notin J_0\Delta J,\ j+1\notin J_0\cap J^{\mathrm{nss}}.\end{cases}$$

Consider the decomposition $\mathcal{J}=J_1\sqcup J_2\sqcup J_3\sqcup J_4\sqcup J_5$ with $J_1\overset{\mathrm{def}}{=}(J_0\setminus J)\sqcup(J^{\mathrm{ss}}\setminus J_0)$, $J_2\overset{\mathrm{def}}{=}J_0\cap J^{\mathrm{ss}}$, $J_3\overset{\mathrm{def}}{=}J_0\cap J^{\mathrm{nss}}$, $J_4\overset{\mathrm{def}}{=}J^{\mathrm{nss}}\setminus J_0$ and $J_5\overset{\mathrm{def}}{=}(J_0)^c\cap J^c$. Then we can rewrite both sides of (iii) as

$$\begin{aligned}\mathrm{LHS}&=(\delta_{j+1\in J_1}+\delta_{j+1\in J_2}+\delta_{j+1\in J_5})s_j^{J^{\mathrm{ss}}}+(\delta_{j+1\in J_3}+\delta_{j+1\in J_4})(p-1)\\&\quad+\delta_{j+1\in J_3}(s_j^{J^{\mathrm{ss}}}+1)+(\delta_{j+1\in J_2}+\delta_{j+1\in J_5})(p-1-s_j^{J^{\mathrm{ss}}}),\\\mathrm{RHS}&=\delta_{j+1\in J_3}p+(\delta_{j+1\in J_1}+\delta_{j+1\in J_3})s_j^{J^{\mathrm{ss}}}+(\delta_{j+1\in J_2}+\delta_{j+1\in J_4}+\delta_{j+1\in J_5})(p-1),\end{aligned}$$

from which it is easy to see that the equality holds. □

PROPOSITION 3.4. *For $J\subseteq\mathcal{J}$ such that $J^{\mathrm{nss}}\neq\mathcal{J}$, we have*

$$\frac{\mu_{J^s,J}}{\mu_{J^s,J^{\mathrm{ss}}}}=\frac{\mu_{*,J}}{\mu_{*,J^{\mathrm{ss}}}}.$$

Proof. Let $J_0\overset{\mathrm{def}}{=}(J^{\mathrm{ss}}\sqcup(J^c-1)^{\mathrm{nss}})+1$. Then both the pairs (J_0,J) and (J_0,J^{ss}) satisfy (3.1). We consider the elements $B_1\overset{\mathrm{def}}{=}\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)v_{J^s}\in\pi$ and

$$B_2\overset{\mathrm{def}}{=}\left[\prod_{j+1\notin J_0\Delta J}Y_j^{2\delta_{j\in J_0\cap J^{\mathrm{nss}}}+p-1-s_j^{J_0}+\delta_{j\in J_0\Delta J}}\right]\begin{pmatrix}p&0\\0&1\end{pmatrix}(\underline{Y}^{-\underline{i}}v_{J_0})\in\pi.$$

By Proposition 3.1(ii), we have $\underline{Y}^{\underline{s}^J}B_1=\mu_{J^s,J}v_J$. By Lemma 3.3(ii) and (3.2) applied to (J_0,J), we also have

$$\underline{Y}^{\underline{s}^J}B_2=\left[\prod_{j+1\in J_0\Delta J}Y_j^{s_j^{J'}}\prod_{j+1\notin J_0\Delta J}Y_j^{p-1}\right]\begin{pmatrix}p&0\\0&1\end{pmatrix}(\underline{Y}^{-\underline{e}^{J_0\cap J^{\mathrm{nss}}}}v_J)=\mu_{J_0,J}v_J.$$

In particular, we deduce from [Wan24, Lemma 3.1(ii)] that B_1 and B_2 are H-eigenvectors with the same H-eigencharacter.

Moreover, by the proof and the notation of Proposition 3.1(i), we have $B_1\in I(\sigma_{J^{\mathrm{ss}}},\sigma_{\underline{a}^J})\cong Q(\chi^s_{J^s},(J^{\mathrm{nss}})^c-1)$. By [Wan24, Proposition 5.7] applied to $(J,J',\underline{i})=(J_0,(J_0\Delta J)-1,\underline{e}^{J_0\cap J^{\mathrm{nss}}})$ and using Lemma 3.3(i), we have $Y_{j'}B_2\in I(\sigma_{J^{\mathrm{ss}}},\sigma_{\underline{a}^J})$ for all $j'\in\mathcal{J}$. Since $I(\sigma_{J^{\mathrm{ss}}},\sigma_{\underline{a}^J})$ is

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multiplicity-free as an H-representation by [Wan24, Proposition Lemma 3.2(ii),(iii)] and using $J^{\mathrm{nss}} \neq \mathcal{J}$, combining with the first paragraph we deduce that $Y_{j'}B_1 = (\mu_{J^s,J}/\mu_{J_0,J})Y_{j'}B_2$ for all $j' \in \mathcal{J}$.

Then we have

$$\begin{aligned}
\mu_{J^s,J^{\mathrm{ss}}} v_{J^{\mathrm{ss}}} &= \Bigg[\prod_{j+1\notin J^{\mathrm{nss}}} Y_j^{s_j^{J^{\mathrm{ss}}}} \prod_{j+1\in J^{\mathrm{nss}}} Y_j^{p-1}\Bigg] B_1 \\
&= \frac{\mu_{J^s,J}}{\mu_{J_0,J}} \Bigg[\prod_{j+1\notin J^{\mathrm{nss}}} Y_j^{s_j^{J^{\mathrm{ss}}}} \prod_{j+1\in J^{\mathrm{nss}}} Y_j^{p-1}\Bigg] B_2 \\
&= \frac{\mu_{J^s,J}}{\mu_{J_0,J}} \Bigg[\prod_{j+1\notin J_0\Delta J^{\mathrm{ss}}} Y_j^{s_j^{J^{\mathrm{ss}}}} \prod_{j+1\in J_0\Delta J^{\mathrm{ss}}} Y_j^{p-1}\Bigg] \underline{Y}^{p\delta(\underline{e}^{J_0\cap J^{\mathrm{nss}}})} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} (\underline{Y}^{-\underline{e}^{J_0\cap J^{\mathrm{nss}}}} v_{J_0}) \\
&= \frac{\mu_{J^s,J}}{\mu_{J_0,J}} \Bigg[\prod_{j+1\notin J_0\Delta J^{\mathrm{ss}}} Y_j^{s_j^{J^{\mathrm{ss}}}} \prod_{j+1\in J_0\Delta J^{\mathrm{ss}}} Y_j^{p-1}\Bigg] \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v_{J_0} \\
&= \frac{\mu_{J^s,J}}{\mu_{J_0,J}} \mu_{J_0,J^{\mathrm{ss}}} v_{J^{\mathrm{ss}}},
\end{aligned}$$

where the first equality follows from Proposition 3.1(i), the third equality follows from Lemma 3.3(iii), the fourth equality follows from [Wan24, Lemma 3.1(i)], and the last equality follows from (3.2) applied to (J_0, J^{ss}). Therefore, we have $\mu_{J^s,J^{\mathrm{ss}}} = (\mu_{J^s,J}/\mu_{J_0,J})\mu_{J_0,J^{\mathrm{ss}}}$, which completes the proof. □

4. Kisin modules

Let $\overline{\rho}$ be as in (1.3). In particular, $\overline{\rho}$ is generic in the sense of [DL21, § 3.2.2]. We study the Kisin modules following [DL21] that can be used to describe the tamely potentially Barsotti–Tate deformation rings of $\overline{\rho}$. When $\overline{\rho}$ is non-semisimple, we define and compute the elements in the Kisin modules that form one part of the computation of the constants in the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$. The main result is Proposition 4.3.

Up to enlarging $\mathbb{F}$, we fix an fth root $\beta \overset{\mathrm{def}}{=} \sqrt[f]{\xi} \in \mathbb{F}^\times$ of ξ (see (1.3)). We let T_K be the Lubin–Tate variable as in [Wan26, § 2]. By [Wan26, (44)], the Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-module $D_K(\overline{\rho})$ associated to $\overline{\rho}$ has the following form ($a \in \mathcal{O}_K^\times$):

$$\begin{cases}
D_K(\overline{\rho}) = \prod_{j=0}^{f-1} D_{K,\sigma_j}(\overline{\rho}) = \prod_{j=0}^{f-1} (\mathbb{F}(\!(T_K)\!)e_0^{(j)} \oplus \mathbb{F}(\!(T_K)\!)e_1^{(j)}) \\
\varphi(e_0^{(j+1)}\ e_1^{(j+1)}) = (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(\varphi^{(j)}) \\
a(e_0^{(j)}\ e_1^{(j)}) = (e_0^{(j)}\ e_1^{(j)})\,\mathrm{Mat}(a^{(j)}),
\end{cases}$$

where

$$\mathrm{Mat}(\varphi^{(j)}) = \begin{pmatrix} \beta\, T_K^{-(q-1)(r_j+1)} & \beta^{-1}d_j \\ 0 & \beta^{-1} \end{pmatrix} \tag{4.1}$$

for some $d_j \in \mathbb{F}$, and $\mathrm{Mat}(a^{(j)}) \in I_2 + \mathrm{M}_2(T_K^{q-1}\mathbb{F}[\![T_K^{q-1}]\!])$ which uniquely determines $\mathrm{Mat}(a^{(j)})$.

By [Wan26, Lemma 5.1], the Fontaine–Laffaille module $\mathrm{FL}(\overline{\rho})$ associated to $\overline{\rho}$ (see [FL82]) has the following form:

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$$\begin{cases} \mathrm{FL}(\overline{\rho}) = \prod_{j=0}^{f-1} \mathrm{FL}_{\sigma_j}(\overline{\rho}) = \prod_{j=0}^{f-1} (\mathbb{F}e_0^{(j)} \oplus \mathbb{F}e_1^{(j)}) \\ \mathrm{Fil}^k \mathrm{FL}_{\sigma_j}(\overline{\rho}) = \mathbb{F}e_0^{(j)} \text{ exactly for } 1 \le k \le r_j + 1 \\ \varphi_{r_{j+1}+1}(e_0^{(j+1)}) = \beta^{-1}(e_0^{(j)} - d_{j+1}e_1^{(j)}) \\ \varphi(e_1^{(j+1)}) = \beta\, e_1^{(j)}, \end{cases} \tag{4.2}$$

where $d_j \in \mathbb{F}$ is as in (4.1). In particular, by [Bre14, (18)] with $e^j = e_1^{(f-j)}$, $f^j = e_0^{(f-j)}$, $\alpha_j = \beta$, $\beta_j = \beta^{-1}$ and $\mu_j = d_{f+1-j}$ for all $j \in \mathcal{J}$ in [Bre14, (16)], we deduce that $d_j = 0$ if and only if $j \in J_{\overline{\rho}}$.

We fix a compatible system $(p_n)_n$ of p-power roots of $(-p)$ in $\overline{\mathbb{Q}}_p$ and define $K_\infty \overset{\text{def}}{=} \bigcup_{n\ge 0} K(p_n)$. Let $\overline{\mathcal{M}}$ be the étale φ-module over $k(\!(v)\!) \otimes_{\mathbb{F}_p} \mathbb{F}$ in the sense of [DL21, § 3.2.1] such that $\mathbb{V}^*(\overline{\mathcal{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$, where $\mathbb{V}^*$ is Fontaine's anti-equivalence of categories (see [Fon90]). Then as in the proof of [DL21, Proposition 3.3] and using (4.2), we can take

$$\begin{cases} \overline{\mathcal{M}} = \prod_{j=0}^{f-1} \overline{\mathcal{M}}^{(j)} = \prod_{j=0}^{f-1} (\mathbb{F}(\!(v)\!)e_0^{(j)} \oplus \mathbb{F}(\!(v)\!)e_1^{(j)}) \\ \varphi(e_0^{(j)}\ e_1^{(j)}) = (e_0^{(j+1)}\ e_1^{(j+1)})\,\mathrm{Mat}(\varphi^{(j)}_{\overline{\mathcal{M}}}) \end{cases}$$

with

$$\mathrm{Mat}(\varphi^{(f-1-j)}_{\overline{\mathcal{M}}}) = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} v^{r_j+1} & 0 \\ -\beta^{-2} d_j v^{r_j+1} & 1 \end{pmatrix}, \tag{4.3}$$

where $d_j \in \mathbb{F}$ is as in (4.1).

We write $\underline{W} \overset{\text{def}}{=} (S_2)^f$ and $X^*(\underline{T}) \overset{\text{def}}{=} (\mathbb{Z}^2)^f \cong \mathbb{Z}^{2f}$. For $w \in \underline{W}$ and $\lambda \in X^*(\underline{T})$, we write the corresponding jth components $w_j \in S_2$ and $\lambda_j \in \mathbb{Z}^2$ and define $w\lambda \in X^*(\underline{T})$ with $(w\lambda)_j = w_j(\lambda_j)$. We denote by $\tau(w,\lambda) : I_K \to \mathrm{GL}_2(\mathcal{O}) \subseteq \mathrm{GL}_2(E)$ the associated tame inertial type defined in [DL21, § 2.3.2], which is a two-dimensional representation of I_K over E that factors through the tame inertial quotient and extends to G_K. If, moreover, (w,μ) is a good pair (see [DL21, § 2.3.2]), which will always be the case in this article, we write $R_w(\lambda)$ for the associated tame type of K defined in [DL21, § 2.3.1], which is a smooth irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over E.

For $\chi : I \to \mathbb{F}^\times$ a character, we write $\theta^\circ(\chi) \overset{\text{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}([\chi])$, which is an $\mathcal{O}$-lattice in the principal series type $\theta(\chi) \overset{\text{def}}{=} \theta^\circ(\chi)[1/p]$, where $[\chi] : I \to \mathcal{O}^\times$ is the Techmüller lift of χ. We let $\varphi^\chi \in \theta^\circ(\chi)$ be the unique element supported on I such that $\varphi^\chi(\mathrm{id}) = 1$. If, moreover, $\chi \ne \chi^s$, we write $\sigma(\chi)$ for the unique Serre weight such that χ is the I-character acting on $\sigma(\chi)^{I_1}$. Then the cosocle of $\theta^\circ(\chi)$ is isomorphic to $\sigma(\chi)$, and we denote the image of φ^χ in $\sigma(\chi)$ by φ^χ as well.

For R an $\mathcal{O}$-algebra and τ a two-dimensional tame inertial type, we define a *Kisin module* over R of type τ and its eigenbasis as in [DL21, § 3.1], with the caveat that we only consider modules of rank two. For each Kisin module $\mathfrak{M}$ with a fixed eigenbasis, we define the matrices $A^{(j)} \in \mathrm{M}_2(R[\![v]\!])$ for $0 \le j \le f-1$ as in [DL21, § 3.4].

From now on, we write $\mu \overset{\text{def}}{=} (\underline{r}+\underline{1}, \underline{0}) \in X^*(\underline{T})$ and $\eta \overset{\text{def}}{=} (\underline{1}, \underline{0}) \in X^*(\underline{T})$. We let $w, w' \in \underline{W}$ such that (see [DL21, Proposition 3.11]):

(i) $(w_j, w'_j) \ne (\mathfrak{w}, \mathrm{id})$ for all $j \in \mathcal{J}$;

(ii) if $(w_j, w'_j) = (\mathrm{id}, \mathfrak{w})$, then $j \in J_{\overline{\rho}}$ (or, equivalently, $d_j = 0$);

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where $\mathfrak{w}$ is the unique non-trivial element in S_2. As in [DL21, § 3.5] with $s_{\overline{\rho}}=1$, we consider the tame inertial type $\tau=\tau(w,\mu-w'\eta)$. We let $\overline{\mathfrak{M}}^{\tau}$ be the Kisin module over $\mathbb{F}$ of type τ given by the matrices $\overline{A}^{(f-1-j)}=\mathrm{Mat}(\phi_{\overline{\mathcal{M}}}^{(f-1-j)})v^{-(\mu_j-w'_j\eta_j)}\dot{w}_j\in \mathrm{M}_2(\mathbb{F}[\![v]\!])$ for $j\in\mathcal{J}$, where $v^{(a,b)}\in \mathrm{M}_2(\mathbb{F}(\!(v)\!))$ denotes the diagonal matrix $\left(\begin{smallmatrix} v^a & 0\\ 0 & v^b\end{smallmatrix}\right)$ for $(a,b)\in\mathbb{Z}^2$, and $\dot{w}_j\in \mathrm{M}_2(\mathbb{F})$ denotes the corresponding permutation matrix associated to w_j. Then by [LLL+20, Proposition 3.2.1], the étale φ-module over $k(\!(v)\!)\otimes_{\mathbb{F}_p}\mathbb{F}$ associated to $\overline{\mathfrak{M}}^{\tau}$ in the sense of [LLL+20, § 3.2] is isomorphic to $\overline{\mathcal{M}}$. Concretely, from (4.3) we have (cf. [DL21, (14)])

$$\overline{A}^{(f-1-j)}=\begin{cases}\begin{pmatrix}\beta^{-1}&0\\0&\beta\end{pmatrix}\begin{pmatrix}v&0\\-\beta^{-2}d_jv&1\end{pmatrix} & \text{if } (w_j,w'_j)=(\mathrm{id},\mathrm{id}),\\ \begin{pmatrix}\beta^{-1}&0\\0&\beta\end{pmatrix}\begin{pmatrix}1&0\\0&v\end{pmatrix} & \text{if } (w_j,w'_j)=(\mathrm{id},\mathfrak{w}),\\ \begin{pmatrix}\beta^{-1}&0\\0&\beta\end{pmatrix}\begin{pmatrix}0&1\\v&-\beta^{-2}d_j\end{pmatrix} & \text{if } (w_j,w'_j)=(\mathfrak{w},\mathfrak{w}).\end{cases}$$

Let $R_0\overset{\mathrm{def}}{=}\mathcal{O}[\![X_j,Y_j,Z_j,Z'_j]\!]_{j=0}^{f-1}/(f_j)_{j=0}^{f-1}$ with $f_j=Y_j$ if $j\notin J_{\overline{\rho}}$ and $f_j=X_jY_j-p$ if $j\in J_{\overline{\rho}}$. For $\tau=\tau(w,\mu-w'\eta)$ as above, we let $\mathfrak{M}^{\tau}$ be the Kisin module over R_0 given by the matrices $A^{(f-1-j)}\in \mathrm{M}_2(R_0[\![v]\!])$ for $j\in\mathcal{J}$ such that (cf. [DL21, (15)])

$$A^{(f-1-j)}=D^{(f-1-j)}A'^{(f-1-j)} \tag{4.4}$$

with

$$D^{(f-1-j)}\overset{\mathrm{def}}{=}\begin{pmatrix}Z'_j+[\beta]^{-1}&0\\0&Z_j+[\beta]\end{pmatrix},$$

$$A'^{(f-1-j)}\overset{\mathrm{def}}{=}\begin{cases}\begin{pmatrix}v+p&0\\(X_j-[\beta^{-2}d_j])v&1\end{pmatrix} & \text{if } (w_j,w'_j)=(\mathrm{id},\mathrm{id}),\\ \begin{pmatrix}1&-Y_j\\0&v+p\end{pmatrix} & \text{if } (w_j,w'_j)=(\mathrm{id},\mathfrak{w}),\\ \begin{pmatrix}-Y_j&1\\v&X_j\end{pmatrix} & \text{if } (w_j,w'_j)=(\mathfrak{w},\mathfrak{w}) \text{ and } j\in J_{\overline{\rho}},\\ \begin{pmatrix}-p(X_j-[\beta^{-2}d_j])^{-1}&1\\v&X_j-[\beta^{-2}d_j]\end{pmatrix} & \text{if } (w_j,w'_j)=(\mathfrak{w},\mathfrak{w}) \text{ and } j\notin J_{\overline{\rho}}.\end{cases} \tag{4.5}$$

In particular, $\mathfrak{M}^{\tau}$ is a deformation of $\overline{\mathfrak{M}}^{\tau}$ to R_0.

For each $J\subseteq\mathcal{J}$, we define $s^*,w,w'\in\underline{W}$ by the formula

$$\begin{aligned}s^*_{j-1}=\mathrm{id}&\Leftrightarrow j\notin J,\\ w_j=\mathrm{id}&\Leftrightarrow j\notin J,\ j+1\notin J \text{ or } j\in J,\ j+1\in J,\\ w'_j=\mathrm{id}&\Leftrightarrow j\notin J,\ j+1\notin J \text{ or } j\in J,\ j+1\in J,\ j\notin J_{\overline{\rho}}.\end{aligned} \tag{4.6}$$

In particular, we have $w_j=s^*_js^*_{j-1}$ for all j by (4.6) and $\chi_J=\chi_{(s^*)^{-1}(\mu-w'\eta)}$ by Lemma 4.1 below, hence an isomorphism of tame types $\theta(\chi_J)\cong R_w(\mu-w'\eta)$ by [DL21, § 2.3.1] (cf. [DL21, Lemma 3.13]). Then we define $U_p(\chi_J)\overset{\mathrm{def}}{=}\prod_{j=0}^{f-1}U_p(\chi_J)_j\in R_0$ with (cf. [DL21, Proposition 3.22])

$$U_p(\chi_J)_j\overset{\mathrm{def}}{=}(p(A^{(f-1-j)}_{s^*_{j-1}(1)s^*_{j-1}(1)})^{-1}\bmod v)\in R_0, \tag{4.7}$$

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where $A^{(f-1-j)}_{s^*_{j-1}(1)s^*_{j-1}(1)}$ is the $(s^*_{j-1}(1), s^*_{j-1}(1))$th entry of the matrix $A^{(f-1-j)}$ in (4.4) for the Kisin module $\mathfrak{M}^\tau$ over R_0 of type $\tau = \tau(w, \mu - w'\eta)$.

LEMMA 4.1. *For $J \subseteq \mathcal{J}$ and $s^*, w' \in \underline{W}$ as in (4.6), we have $\chi_J = \chi_{(s^*)^{-1}(\mu - w'\eta)}$.*

Proof. Recall from § 2 that $\chi_J = \chi_{\lambda_J}$ with $\lambda_J = (\underline{s}^J + \underline{t}^J, \underline{t}^J) \in X^*(\underline{T})$. Concretely, by (2.1) and (2.2) we have

$$(\lambda_J)_j = \begin{cases} (r_j, 0) & \text{if } j \notin J,\ j+1 \notin J, \\ (r_j, -1) & \text{if } j \in J,\ j+1 \notin J, \\ (p-1, r_j+1) & \text{if } j \notin J,\ j+1 \in J, \\ (p-2, r_j+1) & \text{if } j \in J,\ j+1 \in J,\ j \in J_{\overline{\rho}}, \\ (p-1, r_j) & \text{if } j \in J,\ j+1 \in J,\ j \notin J_{\overline{\rho}}. \end{cases} \tag{4.8}$$

By (4.6) we also have

$$\begin{aligned} &((s^*)^{-1}(\mu - w'\eta))_j \\ &\quad = \begin{cases} (r_j, 0) & \text{if } s^*_j = \mathrm{id},\ w'_j = \mathrm{id}, \text{ equivalently, } j \notin J,\ j+1 \notin J, \\ (r_j+1, -1) & \text{if } s^*_j = \mathrm{id},\ w'_j = \mathfrak{w}, \text{ equivalently, } j \in J,\ j+1 \notin J, \\ (-1, r_j+1) & \text{if } s^*_j = \mathfrak{w},\ w'_j = \mathfrak{w}, \text{ equivalently, } \begin{array}{l} j \notin J,\ j+1 \notin J, \\ \text{or } j \in J,\ j+1 \in J,\ j \in J_{\overline{\rho}}, \end{array} \\ (0, r_j) & \text{if } s^*_j = \mathfrak{w},\ w'_j = \mathrm{id}, \text{ equivalently, } j \in J,\ j+1 \in J,\ j \notin J_{\overline{\rho}}. \end{cases} \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), we have

$$\lambda_J - (s^*)^{-1}(\mu - w'\eta) = (p\underline{e}^{J-1} - \underline{e}^J, \underline{0}) \quad \text{in } X^*(\underline{T}).$$

Since $x^{p\underline{e}^{J-1} - \underline{e}^J} = 1$ for all $x \in \mathbb{F}_q^\times$, we deduce the equality $\chi_J = \chi_{(s^*)^{-1}(\mu - w'\eta)}$. □

LEMMA 4.2. *Let $J \subseteq \mathcal{J}$. Then $U_p(\chi_J)$ is a product of a 1-unit of R_0, an integer power of p, the scalar $[\beta]^{|J^c| - |J|}$, and the quantity $U_p(\chi_J)' \overset{\text{def}}{=} \prod_{i=0}^{f-1} U_p(\chi_J)'_j \in R_0$ with*

$$U_p(\chi_J)'_j = \begin{cases} 1 & \text{if } j \notin J,\ j+1 \notin J \text{ or } j \in J,\ j+1 \in J, \\ Y_j & \text{if } j \in J,\ j+1 \notin J,\ j \in J_{\overline{\rho}}, \\ -X_j & \text{if } j \notin J,\ j+1 \in J,\ j \in J_{\overline{\rho}}, \\ -[\beta^{-2} d_j]^{-1} & \text{if } j \in J,\ j+1 \notin J,\ j \notin J_{\overline{\rho}}, \\ [\beta^{-2} d_j] & \text{if } j \notin J,\ j+1 \in J,\ j \notin J_{\overline{\rho}}. \end{cases} \tag{4.10}$$

In particular, we have $U_p(\chi_J) \in R_0[1/p]^\times$. Here, a 1-unit means an element of $1 + \mathfrak{m}_0$, where $\mathfrak{m}_0$ is the maximal ideal of R_0.

Proof. Let $s^*, w, w' \in \underline{W}$ be as in Lemma 4.1. By (4.5) and (4.6) we have

$$D^{(f-1-j)}_{s^*_{j-1}(1)s^*_{j-1}(1)} = \begin{cases} Z'_j + [\beta]^{-1} \in [\beta]^{-1}(1 + \mathfrak{m}_0) & \text{if } s^*_{j-1} = \mathrm{id}, \text{ equivalently, } j \notin J, \\ Z_j + [\beta] \in [\beta](1 + \mathfrak{m}_0) & \text{if } s^*_{j-1} = \mathfrak{w}, \text{ equivalently, } j \in J. \end{cases}$$

Hence, we have

$$\left(\prod_{j=0}^{f-1} D^{(f-1-j)}_{s^*_{j-1}(1)s^*_{j-1}(1)} \right)^{-1} \in [\beta]^{|J^c| - |J|}(1 + \mathfrak{m}_0). \tag{4.11}$$

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By (4.5) we also have

$$(A'^{(f-1-j)}_{s^*_{j-1}(1)s^*_{j-1}(1)} \bmod v) = \begin{cases} 1 \text{ or } p & \text{if } w_j = \mathrm{id}, \\ X_j & \text{if } (w_j, w'_j, s^*_{j-1}) = (\mathfrak{w}, \mathfrak{w}, \mathfrak{w}) \text{ and } j \in J_{\overline{\rho}}, \\ -Y_j & \text{if } (w_j, w'_j, s^*_{j-1}) = (\mathfrak{w}, \mathfrak{w}, \mathrm{id}) \text{ and } j \in J_{\overline{\rho}}, \\ X_j - [\beta^{-2} d_j] & \text{if } (w_j, w'_j, s^*_{j-1}) = (\mathfrak{w}, \mathfrak{w}, \mathfrak{w}) \text{ and } j \notin J_{\overline{\rho}}, \\ -p(X_j - [\beta^{-2} d_j])^{-1} & \text{if } (w_j, w'_j, s^*_{j-1}) = (\mathfrak{w}, \mathfrak{w}, \mathrm{id}) \text{ and } j \notin J_{\overline{\rho}}. \end{cases} \tag{4.12}$$

Since $X_j - [\beta^{-2} d_j] \in -[\beta^{-2} d_j](1 + \mathfrak{m}_0)$ for all $j \in \mathcal{J}$ and $X_j Y_j = p$ in R_0 for $j \in J_{\overline{\rho}}$, from (4.12) and (4.6) we deduce that

$$(p(A'^{(f-1-j)}_{s^*_{j-1}(1)s^*_{j-1}(1)})^{-1} \bmod v) \in p^{\mathbb{Z}_{\geq 0}} U_p(\chi_J)'_j (1 + \mathfrak{m}_0) \quad \text{in } R_0 \tag{4.13}$$

for $U_p(\chi_J)'_j$ as in (4.10). The result is then a combination of (4.11) and (4.13). □

In the rest of this section we suppose that $J_{\overline{\rho}} \neq \mathcal{J}$. Then for each $J \subseteq \mathcal{J}$, there exists $i \geq 0$ such that $\delta^i_{\mathrm{ss}}(J) = \emptyset$ (see (2.3) for δ_{ss}) and we define $\ell(J) \overset{\mathrm{def}}{=} \min\{i \geq 0 : \delta^i_{\mathrm{ss}}(J) = \emptyset\}$. We then define

$$\widetilde{U}_p(J) \overset{\mathrm{def}}{=} \frac{\prod_{i=0}^{\ell(J)-1} U_p(\chi_{\delta^i_{\mathrm{ss}}(J)})}{\prod_{i=0}^{\ell(J^{\mathrm{ss}})-1} U_p(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})} \in R_0[1/p]. \tag{4.14}$$

Since $J_{\overline{\rho}} \neq \mathcal{J}$, there is a unique decomposition of $J_{\overline{\rho}}$ into a disjoint union of intervals (in $\mathbb{Z}/f\mathbb{Z}$) not adjacent to each other $J_{\overline{\rho}} = J_1 \sqcup \cdots \sqcup J_t$. For each $1 \leq i \leq t$, we write $J_i = \{j_i, j_i + 1, \ldots, j_i + k_i\}$ with $j_i \in \mathcal{J}$ and $k_i \geq 0$. Then we define

$$A^{\mathrm{ss}}(J) \overset{\mathrm{def}}{=} \sum_{i=1}^{t} (\delta_{j_i + k_i \in \partial(J^c)}(k_i + 1)) \in \mathbb{Z}. \tag{4.15}$$

Proposition 4.3. *Suppose that $J_{\overline{\rho}} \neq \mathcal{J}$. Let $J \subseteq \mathcal{J}$. Then $\widetilde{U}_p(J)$ is a product of a 1-unit of R_0, an integer power of p, and the scalar $[U_p(J)]$ with*

$$U_p(J) = (-1)^{A(J)} \beta^{B(J)} d(J) \in \mathbb{F}^\times, \tag{4.16}$$

where

$$\begin{aligned} A(J) &\overset{\mathrm{def}}{=} A^{\mathrm{ss}}(J) + \sum_{j \notin J_{\overline{\rho}}} \delta_{j \in \partial J} \in \mathbb{Z}, \\ B(J) &\overset{\mathrm{def}}{=} \sum_{i=0}^{\ell(J)-1} (|\delta^i_{\mathrm{ss}}(J)^c| - |\delta^i_{\mathrm{ss}}(J)|) - \sum_{i=0}^{\ell(J^{\mathrm{ss}})-1} (|\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})^c| - |\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})|) \in \mathbb{Z}, \\ d(J) &\overset{\mathrm{def}}{=} \Big(\prod_{j \in J^{\mathrm{nss}}} d_j \Big)^{-1} \Big[\frac{\prod_{i=0}^{\ell(J)-1} \prod_{j \in (\delta^i_{\mathrm{ss}}(J)-1)^{\mathrm{nss}}} d_j}{\prod_{i=0}^{\ell(J^{\mathrm{ss}})-1} \prod_{j \in (\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})-1)^{\mathrm{nss}}} d_j} \Big] \in \mathbb{F}^\times. \end{aligned} \tag{4.17}$$

Proof. We write $\widetilde{U}_p(J)' \overset{\mathrm{def}}{=} \prod_{j=0}^{f-1} \widetilde{U}_p(J)'_j \in R_0[1/p]$ with

$$\widetilde{U}_p(J)'_j \overset{\mathrm{def}}{=} \frac{\prod_{i=0}^{\ell(J)-1} U_p(\chi_{\delta^i_{\mathrm{ss}}(J)})'_j}{\prod_{i=0}^{\ell(J^{\mathrm{ss}})-1} U_p(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})'_j} = \frac{\prod_{i \geq 0} U_p(\chi_{\delta^i_{\mathrm{ss}}(J)})'_j}{\prod_{i \geq 0} U_p(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})'_j},$$

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where each $U_p(\chi_{J'})'_j$ is defined in (4.10) and the equality uses $U_p(\chi_\emptyset)'_j = 1$ for all $j \in \mathcal{J}$. By Lemma 4.2, it suffices to show that $\widetilde{U}_p(J)' \in p^{\mathbb{Z}}(-1)^{A(J)}[d(J)]$ for $A(J) \in \mathbb{Z}$ and $d(J) \in \mathbb{F}^\times$ as in (4.17).

We fix $j \in \mathcal{J}$ and compute $\widetilde{U}_p(J)'_j$. By definition, for $i \geq 0$ we have

$$\begin{aligned} j \in \delta_{\mathrm{ss}}^i(J) &\Leftrightarrow (j+i \in J, \text{ and } j+i' \in J_{\overline{\rho}} \text{ for } 0 \leq i' \leq i-1), \\ j \in \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}}) &\Leftrightarrow (j+i \in J, \text{ and } j+i' \in J_{\overline{\rho}} \text{ for } 0 \leq i' \leq i). \end{aligned} \tag{4.18}$$

We let $0 \leq k \leq f-1$ be the unique integer such that $j+i \in J_{\overline{\rho}}$ for $1 \leq i \leq k$, and $j+k+1 \notin J_{\overline{\rho}}$. We also write it as $k(j)$ to emphasize its dependence on j. We separate the following two cases.

Case 1. Assume that $j \in J_{\overline{\rho}}$. Then we write

$$\begin{aligned} S_1(J)_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta_{\mathrm{ss}}^i(J)})'_j = Y_j\} \\ &= \#\{i \geq 0 : j \in \delta_{\mathrm{ss}}^i(J),\ j+1 \notin \delta_{\mathrm{ss}}^i(J)\} = \left(\sum_{i'=0}^{k} \delta_{j+i' \in \partial J}\right) + \delta_{j+k+1 \in J}, \\ S_2(J)_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta_{\mathrm{ss}}^i(J)})'_j = -X_j\} \\ &= \#\{i \geq 0 : j \notin \delta_{\mathrm{ss}}^i(J),\ j+1 \in \delta_{\mathrm{ss}}^i(J)\} = \sum_{i'=0}^{k} \delta_{j+i' \in \partial(J^c)}, \\ S_1(J^{\mathrm{ss}})_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})})'_j = Y_j\} \\ &= \#\{i \geq 0 : j \in \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}}),\ j+1 \notin \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})\} = \left(\sum_{i'=0}^{k-1} \delta_{j+i' \in \partial J}\right) + \delta_{j+k \in J}, \\ S_2(J^{\mathrm{ss}})_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})})'_j = -X_j\} \\ &= \#\{i \geq 0 : j \notin \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}}),\ j+1 \in \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})\} = \sum_{i'=0}^{k-1} \delta_{j+i' \in \partial(J^c)}, \end{aligned}$$

where in each formula the first equality follows from (4.10) and the second equality follows from (4.18). In particular, we have

$$\begin{aligned} &(S_1(J)_j - S_1(J^{\mathrm{ss}})_j) - (S_2(J)_j - S_2(J^{\mathrm{ss}})_j) = (\delta_{j+k \in \partial J} + \delta_{j+k+1 \in J} - \delta_{j+k \in J}) - \delta_{j+k \in \partial(J^c)} \\ &\quad = (\delta_{j+k \in J}(1 - \delta_{j+k+1 \in J}) + \delta_{j+k+1 \in J} - \delta_{j+k \in J}) - (1 - \delta_{j+k \in J})\delta_{j+k+1 \in J} = 0. \end{aligned}$$

Then using $X_j Y_j = p$ in R_0 since $j \in J_{\overline{\rho}}$, we deduce that

$$\widetilde{U}_p(J)'_j = Y_j^{S_1(J)_j - S_1(J^{\mathrm{ss}})_j}(-X_j)^{S_2(J)_j - S_2(J^{\mathrm{ss}})_j} \in (-1)^{\delta_{j+k \in \partial(J^c)}} p^{\mathbb{Z}}. \tag{4.19}$$

Case 2. Assume that $j \notin J_{\overline{\rho}}$. Then we write

$$\begin{aligned} N_1(J)_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta_{\mathrm{ss}}^i(J)})'_j = -[\beta^{-2} d_j]^{-1}\} \\ &= \#\{i \geq 0 : j \in \delta_{\mathrm{ss}}^i(J),\ j+1 \notin \delta_{\mathrm{ss}}^i(J)\} = \delta_{j \in \partial J}, \\ N_2(J)_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta_{\mathrm{ss}}^i(J)})'_j = [\beta^{-2} d_j]\} \\ &= \#\{i \geq 0 : j \notin \delta_{\mathrm{ss}}^i(J),\ j+1 \in \delta_{\mathrm{ss}}^i(J)\} = \delta_{j \in \partial(J^c)} + \sum_{i'=2}^{k+1} \delta_{j+i' \in J}, \end{aligned}$$

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$$\begin{aligned}
N_1(J^{\mathrm{ss}})_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})'_j = -[\beta^{-2} d_j]^{-1}\} \\
&= \#\{i \geq 0 : j \in \delta^i_{\mathrm{ss}}(J^{\mathrm{ss}}),\ j+1 \notin \delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})\} = 0, \\
N_2(J^{\mathrm{ss}})_j &\overset{\mathrm{def}}{=} \#\{i \geq 0 : U_p(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})'_j = [\beta^{-2} d_j]\} \\
&= \#\{i \geq 0 : j \notin \delta^i_{\mathrm{ss}}(J^{\mathrm{ss}}),\ j+1 \in \delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})\} = \sum_{i'=1}^{k} \delta_{j+i' \in J},
\end{aligned}$$

where in each formula the first equality follows from (4.10) and the second equality follows from (4.18). In particular, we have

$$\begin{aligned}
N(J)_j &\overset{\mathrm{def}}{=} (N_2(J)_j - N_2(J^{\mathrm{ss}})_j) - (N_1(J)_j - N_1(J^{\mathrm{ss}})_j) \\
&= (\delta_{j \in \partial(J^c)} + \delta_{j+k+1 \in J} - \delta_{j+1 \in J}) - \delta_{j \in \partial J} \\
&= (1 - \delta_{j \in J})\delta_{j+1 \in J} + \delta_{j+k+1 \in J} - \delta_{j+1 \in J} + \delta_{j \in J}(1 - \delta_{j+1 \in J}) = \delta_{j+k+1 \in J} - \delta_{j \in J}.
\end{aligned}$$

Hence, we have

$$\widetilde{U}_p(J)'_j = (-1)^{N_1(J)_j + N_1(J^{\mathrm{ss}})_j} [\beta^{-2} d_j]^{N(J)_j} = (-1)^{\delta_{j \in \partial J}} [\beta^{-2} d_j]^{N(J)_j}. \tag{4.20}$$

By the definition of $d(J)$, we have $d(J) = \prod_{j \notin J_{\overline{\rho}}} d_j^{M(J)_j}$ with (for each $j \notin J_{\overline{\rho}}$)

$$\begin{aligned}
M(J)_j &= -\delta_{j \in J} + \left(\sum_{i=0}^{\ell(J)-1} \delta_{j+1 \in \delta^i_{\mathrm{ss}}(J)} \right) - \left(\sum_{i=0}^{\ell(J^{\mathrm{ss}})-1} \delta_{j+1 \in \delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})} \right) \\
&= -\delta_{j \in J} + \left(\sum_{i=0}^{k(j)} \delta_{j+i+1 \in J} \right) - \left(\sum_{i=0}^{k(j)-1} \delta_{j+i+1 \in J} \right) \\
&= \delta_{j+k(j)+1 \in J} - \delta_{j \in J} = N(J)_j.
\end{aligned} \tag{4.21}$$

By the definition of $k(j)$, we have that $j + k(j) + 1$ is the first place after j that is not in $J_{\overline{\rho}}$, hence we have

$$\sum_{j \notin J_{\overline{\rho}}} N(J)_j = \sum_{j \notin J_{\overline{\rho}}} (\delta_{j+k(j)+1 \in J} - \delta_{j \in J}) = 0. \tag{4.22}$$

Combining (4.19)–(4.21) and (4.22), we deduce that

$$\begin{aligned}
\widetilde{U}_p(J)' &= \prod_{j=0}^{f-1} \widetilde{U}_p(J)'_j \\
&\in p^{\mathbb{Z}} (-1)^{\sum_{j \in J_{\overline{\rho}}} \delta_{j+k(j) \in \partial(J^c)} + \sum_{j \notin J_{\overline{\rho}}} \delta_{j \in \partial J}} [\beta]^{-2 \sum_{j \notin J_{\overline{\rho}}} N(J)_j} \left(\prod_{j \notin J_{\overline{\rho}}} [d_j]^{N(J)_j} \right) \\
&= p^{\mathbb{Z}} (-1)^{A(J)} [d(J)]
\end{aligned}$$

for $A(J) \in \mathbb{Z}$ and $d(J) \in \mathbb{F}^\times$ as in (4.17), which completes the proof. □

5. Constants in the Diamond diagrams

We review the strategy of [DL21] to compute the constants in the diagram $(\pi^{I_1} \hookrightarrow \pi^{K_1})$. When $\overline{\rho}$ is non-semisimple, we specialize the general formula to some particular constants that are needed to prove the main result, Theorem 1.1 (see Example 5.3).

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For $0 \le i \le q-1$, we define (with the convention that $0^0 \overset{\text{def}}{=} 1$)

$$\begin{aligned} S_i &\overset{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q^\times} [\lambda]^i \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{O}[\mathrm{GL}_2(\mathbb{F}_q)], \\ S_i^+ &\overset{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q^\times} [\lambda]^i \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \in \mathcal{O}[\mathrm{GL}_2(\mathbb{F}_q)]. \end{aligned} \tag{5.1}$$

Recall that we identify π^{K_1} with $D_0(\overline{\rho})$. As in [DL21, Definition 4.1], we let $R : D_0(\overline{\rho})^{I_1} \to (\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\overline{\rho}))^{I_1}$ be the unique map defined as follows: if $\chi : I \to \mathbb{F}^\times$ is an I-character such that $D_0(\overline{\rho})^{I_1}[\chi] \neq 0$, then $R|_{D_0(\overline{\rho})^{I_1}[\chi]}$ is given by $S_{i(\chi)}$ for some unique $0 \le i(\chi) \le q-1$, except when χ appears in $(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\overline{\rho}))^{I_1}$, in which case $R|_{D_0(\overline{\rho})^{I_1}[\chi]}$ is the identity. Given χ, we write $R\chi$ for the I-character such that $R(D_0(\overline{\rho})^{I_1}[\chi]) = D_0(\overline{\rho})^{I_1}[R\chi]$. Then we define $g_\chi : D_0(\overline{\rho})^{I_1}[R\chi] \to D_0(\overline{\rho})^{I_1}[R\chi^s]$ by the formula $g_\chi(R(v)) = R(\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) v)$ for $v \in D_0(\overline{\rho})^{I_1}[\chi]$. In particular, we have $g_{\chi^s} = g_\chi^{-1}$ for all χ.

Example 5.1. Suppose that $J_{\overline{\rho}} \neq \mathcal{J}$. We let $J \subseteq \mathcal{J}$ such that $J \neq J^*$ (see Lemma 2.2 for J^*, which implies $J^{\mathrm{nss}} \neq \mathcal{J}$). By [Wan24, Lemma 4.1(iii)] we have

$$R\chi_J = \chi_{J^{\mathrm{ss}}} xs, \quad R\chi_J^s = \chi_{(J-1)^{\mathrm{ss}}}. \tag{5.2}$$

In particular, we have $R\chi_J = \chi_J$ if and only if $J = J^{\mathrm{ss}}$ and we have $R\chi_J^s \neq \chi_J^s$ by Lemma 2.2.

Recall that $0 \neq v_J \in D_0(\overline{\rho})^{I_1}[\chi_J]$ and $0 \neq v_{J^s} = (\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) v_J \in D_0(\overline{\rho})^{I_1}[\chi_{J^s}]$ as defined in § 3. By (3.2) with $J' = (J-1)^{\mathrm{ss}}$ and [BHH+25a, Lemma 3.2.2.5(i)], we have

$$R(v_{J^s}) = S_{i(\chi_J^s)} v_{J^s} = (-1)^{f-1} P_1(\chi_J) \mu_{J,(J-1)^{\mathrm{ss}}} v_{(J-1)^{\mathrm{ss}}} \tag{5.3}$$

with

$$\begin{aligned} i(\chi_J^s) &= \sum_{j+1 \in J \Delta (J-1)^{\mathrm{ss}}} (p - 1 - s_j^{(J-1)^{\mathrm{ss}}}) p^j, \\ P_1(\chi_J) &\overset{\text{def}}{=} \prod_{j+1 \in J \Delta (J-1)^{\mathrm{ss}}} (p - 1 - s_j^{(J-1)^{\mathrm{ss}}})! \in \mathbb{F}^\times. \end{aligned} \tag{5.4}$$

If $J \subseteq J_{\overline{\rho}}$, then we have $R(v_J) = v_J = v_{J^{\mathrm{ss}}}$ by definition. If $J \nsubseteq J_{\overline{\rho}}$, then by Proposition 3.1(i) and [BHH+25a, Lemma 3.2.2.5(i)] we have

$$R(v_J) = S_{i(\chi_J)} v_J = (-1)^{f-1} P_2(J) \mu_{J^s, J^{\mathrm{ss}}} v_{J^{\mathrm{ss}}} \tag{5.5}$$

with

$$\begin{aligned} i(\chi_J) &= \sum_{j+1 \notin J^{\mathrm{nss}}} (p - 1 - s_j^{J^{\mathrm{ss}}}) p^j, \\ P_2(J) &\overset{\text{def}}{=} \prod_{j+1 \notin J^{\mathrm{nss}}} (p - 1 - s_j^{J^{\mathrm{ss}}})! \in \mathbb{F}^\times. \end{aligned} \tag{5.6}$$

By the definition of g_{χ_J} and combining (5.3) and (5.5), we get

$$g_{\chi_J}(v_{J^{\mathrm{ss}}}) = \begin{cases} (-1)^{f-1} P_1(\chi_J) \mu_{J,(J-1)^{\mathrm{ss}}} v_{(J-1)^{\mathrm{ss}}} & \text{if } J \subseteq J_{\overline{\rho}}, \\ \dfrac{(-1)^{f-1} P_1(\chi_J) \mu_{J,(J-1)^{\mathrm{ss}}}}{(-1)^{f-1} P_2(J) \mu_{J^s, J^{\mathrm{ss}}}} v_{(J-1)^{\mathrm{ss}}} & \text{if } J \nsubseteq J_{\overline{\rho}}. \end{cases} \tag{5.7}$$

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For $J \not\subseteq J_{\overline{\rho}}$ we define (see above (4.14) for $\ell(J)$)

$$\gamma(J) \overset{\text{def}}{=} (-1)^{|J\cap(J-1)^{\text{nss}}|}\frac{\mu_{*,J}}{\mu_{*,J^{\text{ss}}}}\left[\frac{\prod_{i=0}^{\ell(J)-1}(-1)^{f-1}\mu_{\delta_{\text{ss}}^i(J),\delta_{\text{ss}}^{i+1}(J)}}{\prod_{i=0}^{\ell(J^{\text{ss}})-1}(-1)^{f-1}\mu_{\delta_{\text{ss}}^i(J^{\text{ss}}),\delta_{\text{ss}}^{i+1}(J^{\text{ss}})}}\right]. \tag{5.8}$$

Consider the following two maps:

$$D_0(\overline{\rho})^{I_1}[R\chi_J] \xrightarrow{g_{\chi_J}} D_0(\overline{\rho})^{I_1}[R\chi_{\delta_{\text{ss}}(J)}] \xrightarrow{g_{\chi_{\delta_{\text{ss}}(J)}}} \cdots \to D_0(\overline{\rho})^{I_1}[R\chi_\emptyset],$$
$$D_0(\overline{\rho})^{I_1}[R\chi_J] = D_0(\overline{\rho})^{I_1}[R\chi_{J^{\text{ss}}}] \xrightarrow{g_{\chi_{J^{\text{ss}}}}} D_0(\overline{\rho})^{I_1}[R\chi_{\delta_{\text{ss}}(J^{\text{ss}})}] \xrightarrow{g_{\chi_{\delta_{\text{ss}}(J^{\text{ss}})}}} \cdots \to D_0(\overline{\rho})^{I_1}[R\chi_\emptyset].$$

Suppose that the composition

$$\left(\prod_{i=\ell(J^{\text{ss}})-1}^{0} g_{\chi_{\delta_{\text{ss}}^i(J^{\text{ss}})}}\right)^{-1} \circ \left(\prod_{i=\ell(J)-1}^{0} g_{\chi_{\delta_{\text{ss}}^i(J)}}\right) : D_0(\overline{\rho})^{I_1}[R\chi_J] \to D_0(\overline{\rho})^{I_1}[R\chi_J] \tag{5.9}$$

is given by the scalar $g(J) \in \mathbb{F}^\times$. Applying the first formula of (5.7) to $\delta_{\text{ss}}^i(J^{\text{ss}})$ for $0 \le i \le \ell(J^{\text{ss}}) - 1$ and $\delta_{\text{ss}}^i(J)$ for $1 \le i \le \ell(J) - 1$, and applying the second formula of (5.7) to J, we get

$$\begin{aligned}\gamma(J) &= (-1)^{|J\cap(J-1)^{\text{nss}}|}\frac{\mu_{*,J}}{\mu_{*,J^{\text{ss}}}}(-1)^{f-1}\mu_{J^s,J^{\text{ss}}}(P_2(J)/P_1(J))g(J)\\ &= (-1)^{f-1+|J\cap(J-1)^{\text{nss}}|}\mu_{J^s,J}(P_2(J)/P_1(J))g(J),\end{aligned} \tag{5.10}$$

where $P_1(J) \overset{\text{def}}{=} (\prod_{i=0}^{\ell(J)-1} P_1(\chi_{\delta_{\text{ss}}^i(J)}))/(\prod_{i=0}^{\ell(J^{\text{ss}})-1} P_1(\chi_{\delta_{\text{ss}}^i(J^{\text{ss}})}))$ and the last equality follows from Proposition 3.4.

Lemma 5.2. *We have $\mu_{\emptyset,\emptyset} = (-1)^{f-1}\xi$ (see (1.3) for ξ).*

Proof. By the proof of [DL21, Lemma 4.17], the map $g_{\chi_\emptyset}$ is given by the reduction modulo $\mathfrak{m}_0$ of $U_p(\chi_\emptyset) \in R_0$ (see (4.7)), which equals $\beta^f = \xi$ by the explicit computation given in proof of Lemma 4.2. Then we conclude using (5.7) with $J = \emptyset$. □

We let M_∞ be the $\mathcal{O}[\mathrm{GL}_2(K)]$-module as in [DL21, § 6.2]. Then M_∞ has a minimal arithmetic action of R_∞ in the sense of [DL21, § 4.2], where R_∞ is a suitable power series ring over $R_{\overline{\rho}}$, the universal framed $\mathcal{O}$-deformation ring for $\overline{\rho}$. In particular, we have $M_\infty^\vee[\mathfrak{m}_\infty] \cong \pi$ for π as in (1.1), where $(-)^\vee$ is the Pontrjagin dual and $\mathfrak{m}_\infty$ is the maximal ideal of R_∞. We let $M_\infty(-)$ be the corresponding patching functor.

Let θ be a non-scalar tame type. For each Serre weight $\sigma \in \mathrm{JH}(\overline{\theta})$ where $\overline{\theta}$ is the reduction modulo ϖ of any $\mathcal{O}$-lattice in θ, we fix a lattice θ^σ in θ with irreducible cosocle σ. Such a lattice is unique up to homothety and we rescale it when necessary. For $\chi : I \to \mathbb{F}^\times$ a character such that $\chi \ne \chi^s$ and $\sigma(\chi) \in \mathrm{JH}(\overline{\theta})$, we write θ^χ for $\theta^{\sigma(\chi)}$. We let $\mathrm{pr}_{\chi^s} : \theta(\chi^s)^{R\chi^s} \to \sigma(R\chi^s)$ and $\mathrm{pr}_\chi : \theta(\chi^s)^{R\chi} \to \sigma(R\chi)$ be the normalized surjections as in [DL21, (23), (24)].

We fix a tame inertial type $\tau_0 \overset{\text{def}}{=} \tau(w, \mu - w\eta)$ with associated tame type $\theta_0 \overset{\text{def}}{=} R_w(\mu - w\eta)$ for some fixed $w \in \underline{W}$ satisfying $w_j = \mathfrak{w}$ for $j \in J_{\overline{\rho}}$ and $w_0 w_1 \cdots w_{f-1} = \mathfrak{w}$ if $J_{\overline{\rho}} \ne \mathcal{J}$. Then we have $W(\overline{\rho}) \subseteq \mathrm{JH}(\overline{\theta}_0)$ by [DL21, Proposition 3.11] and θ_0 is a cuspidal type if $J_{\overline{\rho}} \ne \mathcal{J}$. Moreover, the ring R_0 defined above (4.4) is a power series ring over $R_{\overline{\rho}}^{\tau_0}$ (see [DL21, § 3.5.1]), where $R_{\overline{\rho}}^{\tau_0}$ is the quotient of $R_{\overline{\rho}}$ parametrizing potentially crystalline lifts of $\overline{\rho}$ with Hodge–Tate weights $(1, 0)$ in each embedding and inertial type τ_0. In particular, all the arguments of [DL21, § 4] still hold, replacing the so-called central type $\theta = R_{\underline{\mathfrak{w}}}(\mu - \underline{\mathfrak{w}}\eta)$ with the type θ_0.

For any $\mathcal{O}$-lattice θ_0° in θ_0, the patched module $M_\infty(\theta_0^\circ)$ is supported on $R_\infty(\tau_0) \overset{\text{def}}{=} R_\infty \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}^{\tau_0}$. We let $Q(\chi^s)^{R\chi^s}$ (respectively, $Q(\chi^s)^{R\chi}$) be the quotient of $\theta(\chi^s)^{R\chi^s}/\varpi$ (respectively, $\theta(\chi^s)^{R\chi}/\varpi$) as in [DL21, Proposition 4.18]. Then the surjection pr_{χ^s} (respectively, pr_χ) factors

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through $Q(\chi^s)^{R\chi^s}$ (respectively, $Q(\chi^s)^{R\chi}$). As in the discussion after [DL21, (27)], we fix a surjection $\alpha : \theta_0^{R\chi^s} \to Q(\chi^s)^{R\chi^s}$ which induces a surjection $\alpha : \theta_0^{R\chi} \to Q(\chi^s)^{R\chi}$. Then as in [DL21, (29)] there is a commutative diagram

$$\begin{array}{ccccc} M_\infty(\theta_0^{R\chi}) & \xhookrightarrow{\iota} & M_\infty(\theta_0^{R\chi^s}) & \xrightarrow{\mathrm{pr}_{\chi^s}\circ\alpha} & M_\infty(\sigma(R\chi^s))/\mathfrak{m}_\infty \\ & \searrow^{\widetilde{U}_p(\chi)} & \cong\downarrow\widetilde{h}_\chi & & \cong\downarrow\overline{h}_\chi \\ & & M_\infty(\theta_0^{R\chi}) & \xrightarrow{\mathrm{pr}_\chi\circ\alpha} & M_\infty(\sigma(R\chi))/\mathfrak{m}_\infty \end{array}$$

where we refer to [DL21, § 4.4] for the maps ι, $\widetilde{h}_\chi$, $\overline{h}_\chi$ and the element $\widetilde{U}_p(\chi) \in R_\infty(\tau_0)$. Moreover, by [DL21, Lemma 5.5] and the definition of $\widetilde{U}_p'(\chi)$ in [DL21, Definition 3.22], we deduce that $\widetilde{U}_p(\chi)\widetilde{U}_p(\chi^s)$ is a product of an integer power of p and a 1-unit of $R_\infty(\tau_0)$ (i.e. an element of $1+\mathfrak{m}_\infty(\tau_0)$, where $\mathfrak{m}_\infty(\tau_0)$ is the maximal ideal of $R_\infty(\tau_0)$). In particular, $\widetilde{U}_p(\chi) \in R_\infty(\tau_0)[1/p]^\times$.

For our purposes, we consider a cycle of characters but in a different order from that in [DL21, § 4.5]. Namely, we consider the I-characters $\psi_0, \psi_1, \dots, \psi_n$ and $\psi_0', \psi_1', \dots, \psi_m'$ appearing in $D_0(\overline{\rho})^{I_1}$ such that $R\psi_0 = R\psi_0'$, $R\psi_n^s = R\psi_m'^s$, $R\psi_i^s = R\psi_{i+1}$ for $0 \le i \le n-1$ and $R\psi_i'^s = R\psi_{i+1}'$ for $0 \le i \le m-1$. Here we allow $m = -1$, in which case we are reduced to the situation considered in [DL21, § 4.5]. We fix a surjection $\alpha_0 : \theta_0^{R\psi_0^s} \to Q(\psi_0^s)^{R\psi_0^s}$. Similarly to [DL21, (32)], we define the surjection $\alpha_0' : \theta_0^{R\psi_0'^s} \to Q(\psi_0'^s)^{R\psi_0'^s}$ by the following commutative diagram (if $m \ge 0$).

$$\begin{array}{ccccc} \theta_0^{R\psi_0} & \xrightarrow{\alpha_0} & Q(\psi_0^s)^{R\psi_0} & \xrightarrow{\mathrm{pr}_{\psi_0}} & \sigma(R\psi_0) \\ \| & & & & \| \\ \theta_0^{R\psi_0'} & \xrightarrow{\alpha_0'} & Q(\psi_0'^s)^{R\psi_0'} & \xrightarrow{\mathrm{pr}_{\psi_0'}} & \sigma(R\psi_0') \end{array}$$

Then as in [DL21, (32)] we define the surjections $\alpha_i : \theta_0^{R\psi_i^s} \to Q(\psi_i^s)^{R\psi_i^s}$ for $1 \le i \le n$ inductively by the commutative diagram

$$\begin{array}{ccccc} \theta_0^{R\psi_{i+1}} & \xrightarrow{\alpha_{i+1}} & Q(\psi_{i+1}^s)^{R\psi_{i+1}} & \xrightarrow{\mathrm{pr}_{\psi_{i+1}}} & \sigma(R\psi_{i+1}) \\ \| & & & & \| \\ \theta_0^{R\psi_i^s} & \xrightarrow{\alpha_i} & Q(\psi_i^s)^{R\psi_i^s} & \xrightarrow{\mathrm{pr}_{\psi_i^s}} & \sigma(R\psi_i^s) \end{array}$$

and we define the surjections $\alpha_i' : \theta_0^{R\psi_i'^s} \to Q(\psi_i'^s)^{R\psi_i'^s}$ for $1 \le i \le m$ inductively in a similar way.

Analogously to the picture in [DL21, § 4.5], we give a picture for $n = 1$ and $m = 0$, where all the patched modules are cyclic R_∞-modules, as follows.

$$\begin{array}{ccccccc} M_\infty(\theta_0^{R\psi_0}) & \hookrightarrow & M_\infty(\theta_0^{R\psi_0^s}) & \hookrightarrow & M_\infty(\theta_0^{R\psi_1^s}) & \xrightarrow{\mathrm{pr}_{\psi_1^s}\circ\alpha_1} & M_\infty(\sigma(R\psi_1^s))/\mathfrak{m}_\infty \\ & \searrow^{\widetilde{U}_p(\psi_1)} & & \searrow^{\widetilde{U}_p(\psi_1)} & \cong\downarrow\widetilde{h}_{\psi_1} & & \cong\downarrow\overline{h}_{\psi_1} \\ & & M_\infty(\theta_0^{R\psi_0}) & \hookrightarrow & M_\infty(\theta_0^{R\psi_0^s}) & \substack{\xrightarrow{\mathrm{pr}_{\psi_1}\circ\alpha_1} \\ \xrightarrow{\mathrm{pr}_{\psi_0^s}\circ\alpha_0'}} & M_\infty(\sigma(R\psi_0^s))/\mathfrak{m}_\infty \\ & & & \searrow^{\widetilde{U}_p(\psi_0)} & \cong\downarrow\widetilde{h}_{\psi_0} & & \cong\downarrow\overline{h}_{\psi_0} \\ & & & & M_\infty(\theta_0^{R\psi_0}) & \substack{\xrightarrow{\mathrm{pr}_{\psi_0}\circ\alpha_0} \\ \xrightarrow{\mathrm{pr}_{\psi_0'}\circ\alpha_0'}} & M_\infty(\sigma(R\psi_0))/\mathfrak{m}_\infty \\ & & & \nearrow_{\widetilde{U}_p(\psi_0')} & \cong\uparrow\widetilde{h}_{\psi_0'} & & \cong\uparrow\overline{h}_{\psi_0'} \\ & & M_\infty(\theta_0^{R\psi_0'}) & \hookrightarrow & M_\infty(\theta_0^{R\psi_0'^s}) & \xrightarrow{\mathrm{pr}_{\psi_0'^s}\circ\alpha_0'} & M_\infty(\sigma(R\psi_0'^s))/\mathfrak{m}_\infty \end{array}$$

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Suppose that $\mathrm{pr}_{\psi_n^s}\circ\alpha_n = c(\psi,\psi')\,\mathrm{pr}_{\psi_m'^s}\circ\alpha_m'$ for $c(\psi,\psi')\in\mathbb{F}^\times$. Suppose that the composition

$$\overline{h}_{\psi_n}^{-1}\circ\cdots\circ\overline{h}_{\psi_0}^{-1}\circ\overline{h}_{\psi_0'}\circ\cdots\circ\overline{h}_{\psi_m'}: M_\infty(\sigma(R\psi_n^s))/\mathfrak{m}_\infty = M_\infty(\sigma(R\psi_m'^s))/\mathfrak{m}_\infty \to M_\infty(\sigma(R\psi_n^s))/\mathfrak{m}_\infty \tag{5.11}$$

is given by multiplication by $h(\psi,\psi')\in\mathbb{F}^\times$. Analogously to [DL21, (34)], there exists $\nu\in\mathbb{Z}$ such that the element

$$p^{-\nu}\left(\prod_{i=0}^{n}\widetilde{U}_p(\psi_i)\right)^{-1}\left(\prod_{i=0}^{m}\widetilde{U}_p(\psi_i')\right)\in R_\infty(\tau_0)[1/p]^\times \tag{5.12}$$

lies in $R_\infty(\tau_0)$ and reduces to $h(\psi,\psi')c(\psi,\psi')^{-1}$ modulo $\mathfrak{m}_\infty(\tau_0)$.

Example 5.3. Suppose that $J_{\overline{\rho}}\neq\mathcal{J}$. We let $J\subseteq\mathcal{J}$ such that $J\not\subseteq J_{\overline{\rho}}$ and $J\neq J^*$ (see Lemma 2.2 for J^*). Then we take $n=\ell(J)-1$ (see above (4.14) for $\ell(J)$), $m=\ell(J^{\mathrm{ss}})-1$, $\psi_i=\chi_{\delta_{\mathrm{ss}}^{n-i}(J)}^s$ for $0\le i\le n$ and $\psi_i'=\chi_{\delta_{\mathrm{ss}}^{m-i}(J^{\mathrm{ss}})}^s$ for $0\le i\le m$ (using (5.2)), and write $c(J)$ for $c(\psi,\psi')\in\mathbb{F}^\times$. Let $g(J),U_p(J)\in\mathbb{F}^\times$ be as in (5.9) and (4.16). Then by (5.11) and [DL21, Proposition 4.16] we have $g(J)=h(\psi,\psi')$. Moreover, using [DL21, Lemma 5.5] together with the definition of $\widetilde{U}_p(J)$ in (4.14), Proposition 4.3 and (5.12), we deduce that $U_p(J)=h(\psi,\psi')c(J)^{-1}$, hence $g(J)=U_p(J)c(J)$. Combining with (5.10), we conclude that (see (5.8) for $\gamma(J)$)

$$\gamma(J)=U_p(J)c'(J) \tag{5.13}$$

with $c'(J)\overset{\mathrm{def}}{=}(-1)^{f-1+|J\cap(J-1)^{\mathrm{nss}}|}\mu_{J^s,J}(P_2(J)/P_1(J))c(J)$.

6. Computation of constants

Throughout this section, we suppose that $J_{\overline{\rho}}\neq\mathcal{J}$, or equivalently that $\overline{\rho}$ is non-semisimple. We compute $c'(J)$ defined in (5.13) for $J\not\subseteq J_{\overline{\rho}}$ and $J\neq J^*$ (see Lemma 2.2 for J^*). The main results are Propositions 6.7 and 6.12. Together with the results of § 4, we finish the computation of all the constants in the diagram $(\pi^{I_1}\hookrightarrow\pi^{K_1})$ that we need.

6.1 Relation between S-operators

For $a\in\mathbb{Z}$, we define $a_j\in\{0,1,\dots,p-1\}$ for $0\le j\le f-1$ by writing $a=\sum_{j=0}^{f-1}a_jp^j+Q(q-1)$ for some $Q\in\mathbb{Z}$ and we define $a_q\overset{\mathrm{def}}{=}\sum_{j=0}^{f-1}a_jp^j\in\{0,1,\dots,q-2\}$. If $(q-1)\nmid a$, we write S_a (respectively, S_a^+) for the operators S_{a_q} (respectively, $S_{a_q}^+$) defined in (5.1). For $0\neq b\in K$, we let $u\in\mathbb{Z}$ be such that $b\in p^u\mathcal{O}_K^\times$, then we define the *leading term* of b to be the element of $\mathbb{F}_q^\times$ that is the reduction modulo p of $p^{-u}b\in\mathcal{O}_K^\times$.

For $a,b\in\mathbb{Z}$, we define

$$\begin{aligned}u(a,b)&\overset{\mathrm{def}}{=}(p-1)^{-1}\sum_{j=0}^{f-1}(p-1-(a_j+b_j-(a+b)_j))\in\mathbb{Z},\\ J(a,b)&\overset{\mathrm{def}}{=}(-1)^{f-1+u(a,b)}\prod_{j=0}^{f-1}(a_j!b_j!((a+b)_j!)^{-1})\in\mathbb{F}_q^\times.\end{aligned} \tag{6.1}$$

More generally, for $a_1,\dots,a_n\in\mathbb{Z}$, we define

$$\begin{aligned}u(a_1,\dots,a_n)&\overset{\mathrm{def}}{=}(p-1)^{-1}\sum_{j=0}^{f-1}\left(\sum_{i=1}^{n}(p-1-(a_i)_j)-\left(p-1-\left(\sum_{i=1}^{n}a_i\right)_j\right)\right)\in\mathbb{Z},\\ J(a_1,\dots,a_n)&\overset{\mathrm{def}}{=}(-1)^{(n-1)(f-1)+u(a_1,\dots,a_n)}\prod_{j=0}^{f-1}\left(\left(\prod_{i=1}^{n}(a_i)_j!\right)\left(\left(\sum_{i=1}^{n}a_i\right)_j!\right)^{-1}\right)\in\mathbb{F}_q^\times.\end{aligned} \tag{6.2}$$

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LEMMA 6.1.

(i) *Let $0 < a, b < q-1$ such that $a+b \neq q-1$. Then we have*

$$S_a^+ S_b^+ = \widetilde{J}(a,b) S_{a+b}^+, \quad S_a S_b^+ = \widetilde{J}(a,b) S_{a+b},$$

where $0 \neq \widetilde{J}(a,b) \in \mathcal{O}_K$ has leading term $J(a,b)$.

(ii) *Let $0 < a_1, \dots, a_n < q-1$ such that $(q-1)$ does not divide $\sum_{i=1}^k a_i$ for all $1 \leq k \leq n-1$. Then we have*

$$S_{a_1}^+ \dots S_{a_n}^+ = \widetilde{J}(a_1, \dots, a_n) S_{a_1+\dots+a_n}^+ \qquad \text{if } (q-1) \nmid \sum_{i=1}^n a_i,$$

$$S_{a_1}^+ \dots S_{a_n}^+ = J'(a_1, \dots, a_n) S_0^+ + \widetilde{J}(a_1, \dots, a_n) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{if } (q-1) \mid \sum_{i=1}^n a_i,$$

where $0 \neq \widetilde{J}(a_1, \dots, a_n) \in \mathcal{O}_K$ has leading term $J(a_1, \dots, a_n)$, and $J'(a_1, \dots, a_n) \in \mathcal{O}_K$.

Proof. Part (i) is [DL21, Lemmas 2.3 and 2.4]. The first formula of (ii) follows from (i) by induction, and the second formula of (ii) is [DL21, Proposition 2.5]. □

LEMMA 6.2. *Let v be an H-eigenvector in an $\mathcal{O}[\mathrm{GL}_2(\mathbb{F}_q)]$-module with H-eigencharacter χ. Then $S_i v$ has H-eigencharacter $\chi^s \alpha^{-i}$ and $S_i^+ v$ has H-eigencharacter $\chi \alpha^i$, where α is the H-character $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mapsto ad^{-1}$.*

Proof. This is [DL21, Lemma 2.7]. □

LEMMA 6.3. *Let $J \subseteq \mathcal{J}$ and v be an I-eigenvector in an $\mathcal{O}[\mathrm{GL}_2(\mathbb{F}_q)]$-module with I-character $[\chi_J]$. Let $a, b \in \mathbb{Z}$ such that $(q-1)$ does not divide $a, a-b, a-b-\underline{s}^J$ (see (2.1) for $\underline{s}^J$, which also denotes the integer $\sum_{j=0}^{f-1} s_j^J p^j \in \mathbb{Z}$). Then we have (see (2.2) for $\underline{t}^J$)*

$$S_a S_b v = (-1)^{\underline{t}^J} \widetilde{J}(a, -b-\underline{s}^J) S_{a-b-\underline{s}^J} v,$$

where $0 \neq \widetilde{J}(a, -b-\underline{s}^J) \in \mathcal{O}_K$ has leading term $J(a, -b-\underline{s}^J)$.

Proof. This is [DL21, Proposition 2.8] with $\chi = \chi_J$. □

LEMMA 6.4. *Suppose that $J \neq J^*$. Then we have (see (5.4) for $i(\chi_J^s)$ and see (2.3) for J^δ)*

$$J(i(\chi_J^s), -\underline{s}^J) = (-1)^{1+\sum_{j+1 \in J^\delta} s_j^{(J-1)^{\mathrm{ss}}}} \left[\frac{\prod_{j+1 \in J^\delta, j \notin J^\delta} (-1)(s_j^{(J-1)^{\mathrm{ss}}} + 1)}{\prod_{j+1 \notin J^\delta, j \in J^\delta} (s_j^{(J-1)^{\mathrm{ss}}} + 1)} \right].$$

Proof. We write $a \overset{\mathrm{def}}{=} i(\chi_J^s) \in \mathbb{Z}$ and $b \overset{\mathrm{def}}{=} -\underline{s}^J \in \mathbb{Z}$ so that $a_j = \delta_{j+1 \in J^\delta}(p-1-s_j^{(J-1)^{\mathrm{ss}}})$ and $b_j = p-1-s_j^J$ for each $j \in \mathcal{J}$. By Lemma 2.1(i) we have

$$a_j + b_j = \begin{cases} (p-1-s_j^{(J-1)^{\mathrm{ss}}}) + (p-1-s_j^J) = p - \delta_{j \in J^\delta} & \text{if } j+1 \in J^\delta, \\ p-1-s_j^J = p-1-s_j^{(J-1)^{\mathrm{ss}}} - \delta_{j \in J^\delta} & \text{if } j+1 \notin J^\delta, \end{cases} \tag{6.3}$$

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hence we have

$$\begin{aligned} a+b &\equiv \sum_{j=0}^{f-1}(a_j+b_j)p^j \\ &= \sum_{j+1\in J^\delta}(p-\delta_{j\in J^\delta})p^j + \sum_{j+1\notin J^\delta}(p-1-s_j^{(J-1)^{\mathrm{ss}}}-\delta_{j\in J^\delta})p^j \\ &= \Big(\sum_{j+1\notin J^\delta}(p-1-s_j^{(J-1)^{\mathrm{ss}}})p^j\Big) + \Big(\sum_{j+1\in J^\delta}p^{j+1}-\sum_{j\in J^\delta}p^j\Big) \\ &\equiv \sum_{j+1\notin J^\delta}(p-1-s_j^{(J-1)^{\mathrm{ss}}})p^j \mod (q-1), \end{aligned} \tag{6.4}$$

which implies $(a+b)_j=0$ if $j+1\in J^\delta$ and $(a+b)_j=p-1-s_j^{(J-1)^{\mathrm{ss}}}$ if $j+1\notin J^\delta$.

Then by (6.1) using (6.3) and (6.4), we obtain both

$$\begin{aligned} u(a,b) &= (p-1)^{-1}\sum_{j=0}^{f-1}(p-1-(a_j+b_j-(a+b)_j)) \\ &= (p-1)^{-1}\Big(\sum_{j+1\notin J^\delta}(p-1+\delta_{j\in J^\delta})+\sum_{j+1\in J^\delta}(-1+\delta_{j\in J^\delta})\Big) \\ &= (p-1)^{-1}((p-1)\#\{j: j+1\notin J^\delta\}+\#\{j: j\in J^\delta\}-\#\{j: j+1\in J^\delta\}) \\ &= f-|J^\delta| \end{aligned}$$

and

$$\begin{aligned} J(i(\chi_J^s),-\underline{s}^J) &= (-1)^{f-1+u(a,b)}\prod_{j=0}^{f-1}(a_j!b_j!((a+b)_j!)^{-1}) \\ &= (-1)^{f-1+u(a,b)}\left[\frac{\prod_{j+1\in J^\delta,j\in J^\delta}(a_j!(p-1-a_j)!)\prod_{j+1\in J^\delta,j\notin J^\delta}(a_j!(p-a_j)!)}{\prod_{j+1\notin J^\delta,j\in J^\delta}(a+b)_j}\right] \\ &= (-1)^{f-1+f-|J^\delta|+\sum_{j+1\in J^\delta}(s_j^{(J-1)^{\mathrm{ss}}}+1)}\left[\frac{\prod_{j+1\in J^\delta,j\notin J^\delta}(s_j^{(J-1)^{\mathrm{ss}}}+1)}{\prod_{j+1\notin J^\delta,j\in J^\delta}(p-1-s_j^{(J-1)^{\mathrm{ss}}})}\right] \\ &= (-1)^{1+\sum_{j+1\in J^\delta}s_j^{(J-1)^{\mathrm{ss}}}}\left[\frac{\prod_{j+1\in J^\delta,j\notin J^\delta}(-1)(s_j^{(J-1)^{\mathrm{ss}}}+1)}{\prod_{j+1\notin J^\delta,j\in J^\delta}(s_j^{(J-1)^{\mathrm{ss}}}+1)}\right], \end{aligned}$$

where the third equality follows from (3.6) and the last equality uses $\#\{j: j+1\in J^\delta, j\notin J^\delta\}=\#\{j: j+1\notin J^\delta, j\in J^\delta\}$. □

Lemma 6.5. *Let $J\not\subseteq J_{\overline{\rho}}$. Then for each $j\in\mathcal{J}$ we have (see (5.6) for $i(\chi_J)$)*

$$(i(\chi_J)+\underline{s}^J)_j=\delta_{j+1\in J^{\mathrm{nss}}}(p-1-s_j^{J^{\mathrm{ss}}}).$$

Proof. This follows from the computation

$$\begin{aligned} i(\chi_J)+\underline{s}^J &\equiv \sum_{j=0}^{f-1}(i(\chi_J)_j+s_j^J)p^j \\ &= \sum_{j+1\notin J^{\mathrm{nss}}}(p-1-s_j^{J^{\mathrm{ss}}}+s_j^J)p^j+\sum_{j+1\in J^{\mathrm{nss}}}s_j^Jp^j \end{aligned}$$

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$$\begin{aligned}
&= \sum_{j+1\notin J^{\mathrm{nss}}} (p-\delta_{j\notin J^{\mathrm{nss}}})p^j + \sum_{j+1\in J^{\mathrm{nss}}} (p-1-s_j^{J^{\mathrm{ss}}}-\delta_{j\notin J^{\mathrm{nss}}})p^j \\
&= \Big(\sum_{j+1\in J^{\mathrm{nss}}} (p-1-s_j^{J^{\mathrm{ss}}})p^j\Big) + \Big(\sum_{j+1\notin J^{\mathrm{nss}}} p^{j+1} - \sum_{j\notin J^{\mathrm{nss}}} p^j\Big) \\
&\equiv \sum_{j+1\in J^{\mathrm{nss}}} (p-1-s_j^{J^{\mathrm{ss}}})p^j \mod (q-1),
\end{aligned}$$

where the second equality follows from Lemma 2.1(ii). □

6.2 The case $(J-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$

LEMMA 6.6. *Let $\emptyset \neq J \subseteq \mathcal{J}$ such that $J \neq J^*$ and $(J-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$ (which implies $J \not\subseteq J_{\overline{\rho}}$ and $J^{\mathrm{nss}} \neq \mathcal{J}$). Then we have (see § 4 for $\theta^\circ(\chi_J)$ and φ^{χ_J})*

$$S_{i(\chi_J^s)} S_0 \varphi^{\chi_J} = \widetilde{c}(\chi_J) S_{i(\chi_J)} \varphi^{\chi_J} \quad \text{in } \theta^\circ(\chi_J),$$

where $0 \neq \widetilde{c}(\chi_J) \in \mathcal{O}_K$ has leading term $c(\chi_J) \overset{\mathrm{def}}{=} (-1)^{\underline{t}^J} J(i(\chi_J^s), -\underline{s}^J)$.

Proof. Since $(J-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$, we deduce from (5.6) and (6.4) that $i(\chi_J^s) - \underline{s}^J \equiv i(\chi_J) \mod (q-1)$. Moreover, our assumption implies $J^{\mathrm{nss}} \notin \{\emptyset, \mathcal{J}\}$, hence $(q-1) \nmid i(\chi_J^s)$ and $(q-1) \nmid i(\chi_J)$. Then we conclude using Lemma 6.3. □

Recall that $c'(J) = (-1)^{f-1+|J\cap(J-1)^{\mathrm{nss}}|} \mu_{J^s,J}(P_2(J)/P_1(J))c(J)$ for $J \not\subseteq J_{\overline{\rho}}$ and $J \neq J^*$ with $\mu_{J^s,J}$ defined in Proposition 3.1(ii), $P_i(J)$ defined in Example 5.1 for $i \in \{1,2\}$ and $c(J)$ defined in Example 5.3.

PROPOSITION 6.7. *Let $\emptyset \neq J \subseteq \mathcal{J}$ such that $J \neq J^*$ and $(J-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$. Then we have (see (5.13) for $c'(J)$ and see (4.17) for $A(J)$)*

$$c'(J) = (-1)^{A(J)}.$$

Proof. By (3.5), using (3.6) and Lemma 2.1(ii), we have

$$\mu_{J^s,J} = (-1)^{\underline{t}^{J^s} + \sum_{j+1\in J^{\mathrm{nss}}} (s_j^{J^{\mathrm{ss}}} + \delta_{j\in J^{\mathrm{nss}}})} \left[\frac{\prod_{j+1\notin J^{\mathrm{nss}}} (s_j^{J^{\mathrm{ss}}} + \delta_{j\in J^{\mathrm{nss}}})!}{\prod_{j+1\in J^{\mathrm{nss}}} (s_j^{J^{\mathrm{ss}}} + \delta_{j\notin J^{\mathrm{nss}}})!}\right]. \tag{6.5}$$

Since $(J-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$, we have $\delta_{\mathrm{ss}}^{i+1}(J) = \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})$ for all $i \geq 0$, hence we have

$$\begin{aligned}
P_2(J)/P_1(J) &= P_2(J)/P_1(\chi_J) \\
&= \frac{\prod_{j+1\notin J^{\mathrm{nss}}} (p-1-s_j^{J^{\mathrm{ss}}})!}{\prod_{j+1\in J^{\mathrm{nss}}} (p-1-s_j^{J^{\mathrm{ss}}})!} = (-1)^{\underline{s}^{J^{\mathrm{ss}}} + \underline{1}} \left[\frac{\prod_{j+1\in J^{\mathrm{nss}}} (s_j^{J^{\mathrm{ss}}})!}{\prod_{j+1\notin J^{\mathrm{nss}}} (s_j^{J^{\mathrm{ss}}})!}\right].
\end{aligned} \tag{6.6}$$

Recall from Example 5.3 that $c(J) = c(\psi, \psi')$ with $\psi_i = \chi^s_{\delta_{\mathrm{ss}}^{n-i}(J)}$ for $0 \leq i \leq n \overset{\mathrm{def}}{=} \ell(J) - 1$ and $\psi'_i = \chi^s_{\delta_{\mathrm{ss}}^{m-i}(J^{\mathrm{ss}})}$ for $0 \leq i \leq m \overset{\mathrm{def}}{=} \ell(J^{\mathrm{ss}}) - 1$. Since $(J-1)^{\mathrm{ss}} = J^{\mathrm{ss}}$, we have $n = m+1$ and $\psi_{i+1} = \psi'_i$ (hence $\overline{h}_{\psi_{i+1}} = \overline{h}_{\psi'_i}$) for $0 \leq i \leq m$, hence $c(J)$ is also equal to $c(\psi, \psi')$ with $n = 0$, $m = -1$ and $\psi_0 = \chi_J^s$ by definition. Then we deduce from Lemma 6.4 to Lemma 6.6 that

$$c(J) = c(\chi_J) = (-1)^{1+\underline{t}^J + \sum_{j+1\in J^{\mathrm{nss}}} s_j^{J^{\mathrm{ss}}}} \left[\frac{\prod_{j+1\in J^{\mathrm{nss}}, j\notin J^{\mathrm{nss}}} (-1)(s_j^{J^{\mathrm{ss}}} + 1)}{\prod_{j+1\notin J^{\mathrm{nss}}, j\in J^{\mathrm{nss}}} (s_j^{J^{\mathrm{ss}}} + 1)}\right]. \tag{6.7}$$

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Moreover, we have

$$\begin{aligned}&|J\cap(J-1)^{\mathrm{nss}}|+\#\{j:j+1\in J^{\mathrm{nss}},j\in J^{\mathrm{nss}}\}+\#\{j:j+1\in J^{\mathrm{nss}},j\notin J^{\mathrm{nss}}\}\\&\quad=|J^{\mathrm{nss}}\cap(J-1)|+\#\{j:j+1\in J^{\mathrm{nss}},j\in J^{\mathrm{nss}}\}+\#\{j:j+1\notin J^{\mathrm{nss}},j\in J^{\mathrm{nss}}\}\\&\quad=|J^{\mathrm{nss}}\cap(J-1)|+|J^{\mathrm{nss}}|\equiv\#\{j\in J^{\mathrm{nss}},j+1\notin J\}=A(J)\mod 2,\end{aligned}\tag{6.8}$$

where the last equality follows from the definition of $A(J)$ using $(J-1)^{\mathrm{ss}}=J^{\mathrm{ss}}$.

By the definition of $c'(J)$ and combining (6.5), (6.6) and (6.7), we deduce that $c'(J)=(-1)^d$ with

$$\begin{aligned}d&=(f-1+|J\cap(J-1)^{\mathrm{nss}}|)+\Big(\underline{t}^{J^s}+\sum_{j+1\in J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}}+\delta_{j\in J^{\mathrm{nss}}})\Big)\\&\quad+(\underline{s}^{J^{\mathrm{ss}}}+\underline{1})+\Big(1+\underline{t}^{J}+\sum_{j+1\in J^{\mathrm{nss}}}s_j^{J^{\mathrm{ss}}}+\#\{j:j+1\in J^{\mathrm{nss}},j\notin J^{\mathrm{nss}}\}\Big)\\&\equiv\underline{t}^{J}+\underline{t}^{J^s}+\underline{s}^{J^{\mathrm{ss}}}+A(J)\equiv A(J)\mod 2,\end{aligned}$$

where the first congruence follows from (6.8) and the second congruence follows from Lemma 2.3. This completes the proof. □

Remark 6.8. When $(J-1)^{\mathrm{ss}}=J^{\mathrm{ss}}$, we are in the situation of [Hu16], and the constant $g(J)=U_p(J)c(J)$ (hence the constant $\gamma(J)=U_p(J)c'(J)$) can be computed by [Hu16, Theorem 4.7]. Here we remark that the term $(-1)^{e(\tau)(r_0,\dots,r_{f-1})}$ is missing in the formula of [Hu16, Theorems 4.5 and 4.7].

6.3 The case $(J-1)^{\mathrm{ss}}\neq J^{\mathrm{ss}}$

Lemma 6.9. *Let $\emptyset\neq J\subseteq J_{\overline{P}}$ (which implies $J\neq J^*$ and $(J-1)^{\mathrm{ss}}\neq J^{\mathrm{ss}}$). Then we have*

$$S_{i(\chi_J^s)}S_0\varphi^{\chi_J}=\widetilde{c}(\chi_J)S^+_{i^+(\chi_J)}\varphi^{\chi_J}\quad\text{in }\theta^\circ(\chi_J),$$

where $i^+(\chi_J)\overset{\mathrm{def}}{=}q-1-i(\chi_J^s)$ and $0\neq\widetilde{c}(\chi_J)\in\mathcal{O}_K$ has leading term $c(\chi_J)\overset{\mathrm{def}}{=}J(i(\chi_J^s),-\underline{s}^J)$.

Proof. The proof is the same as [DL21, Lemmas 5.11 and 5.28]. □

Lemma 6.10. *Let $J\subseteq\mathcal{J}$ such that $J\not\subseteq J_{\overline{P}}$, $J\neq J^*$ and $(J-1)^{\mathrm{ss}}\neq J^{\mathrm{ss}}$. Then we have*

$$S_{i(\chi_J^s)}S_0\varphi^{\chi_J}=\widetilde{c}(\chi_J)S^+_{i^+(\chi_J)}S_{i(\chi_J)}\varphi^{\chi_J}\quad\text{in }\theta(\chi_J),\tag{6.9}$$

where $i^+(\chi_J)\overset{\mathrm{def}}{=}i(\chi_J)-i(\chi_J^s)+\underline{s}^J$, and $0\neq\widetilde{c}(\chi_J)\in K$ has leading term

$$c(\chi_J)\overset{\mathrm{def}}{=}(-1)^{\underline{t}^J}\frac{J(i(\chi_J^s),-\underline{s}^J)}{J(i^+(\chi_J),-i(\chi_J)-\underline{s}^J)}.$$

Proof. Recall from Example 5.1 that $R\chi_J=\chi_{J^{\mathrm{ss}}}$, $R\chi_J^s=\chi_{(J-1)^{\mathrm{ss}}}$ and $R\chi_J^s\neq\chi_J^s$. Then the assumption $J\not\subseteq J_{\overline{P}}$ implies $J\neq(J-1)^{\mathrm{ss}}$, hence $R\chi_J^s\neq\chi_J$, and the assumption $(J-1)^{\mathrm{ss}}\neq J^{\mathrm{ss}}$ implies $R\chi_J\neq R\chi_J^s$. As in the proof of [DL21, Lemma 5.11] (using Lemma 6.2 and [DL21, Lemma 2.11]), equality (6.9) holds for some $0\neq\widetilde{c}(\chi_J)\in K$.

Next we compute $\widetilde{c}(\chi_J)\in K$. The assumption $J\not\subseteq J_{\overline{P}}$ implies $J^\delta\neq\emptyset$, hence $(q-1)\nmid i(\chi_J^s)$. The assumption $J\neq J^*$ implies $J^\delta\neq\mathcal{J}$, hence $(q-1)\nmid(i(\chi_J^s)-\underline{s}^J)$ by (6.4). Then by Lemma 6.3 we have

$$S_{i(\chi_J^s)}S_0\varphi^{\chi_J}=\widetilde{J}_1S_{i(\chi_J^s)-\underline{s}^J}\varphi^{\chi_J},\tag{6.10}$$

where $0\neq\widetilde{J}_1\in\mathcal{O}_K$ has leading term $J_1\overset{\mathrm{def}}{=}(-1)^{\underline{t}^J}J(i(\chi_J^s),-\underline{s}^J)$.

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We choose $0 < z < q-1$ such that none of the numbers

$$z - i(\chi_J^s) + \underline{s}^J,\ z - i(\chi_J^s),\ z + i^+(\chi_J),\ z + i^+(\chi_J) - i(\chi_J)$$

are multiples of $(q-1)$, which is possible since $q \geq 7$. On the one hand, by Lemma 6.3 we have

$$S_z S_{i(\chi_J^s)-\underline{s}^J} \varphi^{\chi_J} = \widetilde{J}_2 S_{z-i(\chi_J^s)} \varphi^{\chi_J}, \tag{6.11}$$

where $0 \neq \widetilde{J}_2 \in \mathcal{O}_K$ has leading term $J_2 \stackrel{\text{def}}{=} (-1)^{\underline{t}^J} J(z, -i(\chi_J^s))$. On the other hand, by Lemmas 6.1(i) and 6.3 we have

$$S_z S^+_{i^+(\chi_J)} S_{i(\chi_J)} \varphi^{\chi_J} = \widetilde{J}_3 S_{z+i^+(\chi_J)} S_{i(\chi_J)} \varphi^{\chi_J} = \widetilde{J}_3 \widetilde{J}_4 S_{z-i(\chi_J^s)} \varphi^{\chi_J}, \tag{6.12}$$

where $0 \neq \widetilde{J}_3 \in \mathcal{O}_K$ has leading term $J_3 \stackrel{\text{def}}{=} J(z, i^+(\chi_J))$ and $0 \neq \widetilde{J}_4 \in \mathcal{O}_K$ has leading term $J_4 \stackrel{\text{def}}{=} (-1)^{\underline{t}^J} J(z + i^+(\chi_J), -i(\chi_J) - \underline{s}^J)$. Since $S_{z-i(\chi_J^s)} \varphi^{\chi_J} \neq 0$ as explained in the proof of [DL21, Lemma 5.28], we deduce. combining (6.9)–(6.11) and (6.12), that $0 \neq \widetilde{c}(\chi_J) = (\widetilde{J}_1 \widetilde{J}_2)/(\widetilde{J}_3 \widetilde{J}_4) \in K$ has leading term

$$c(\chi_J) = \frac{J_1 J_2}{J_3 J_4} = \frac{(-1)^{\underline{t}^J} J(i(\chi_J^s), -\underline{s}^J) J(z, -i(\chi_J^s))}{J(z, i^+(\chi_J)) J(z + i^+(\chi_J), -i(\chi_J) - \underline{s}^J)} = (-1)^{\underline{t}^J} \frac{J(i(\chi_J^s), -\underline{s}^J)}{J(i^+(\chi_J), -i(\chi_J) - \underline{s}^J)},$$

where the last equality follows from the formula (applied with $a = z$, $b = i^+(\chi_J)$, $c = -i(\chi_J) - \underline{s}^J$)

$$J(a,b) J(a+b,c) = J(a, b+c) J(b,c) \quad \text{for } a, b, c \in \mathbb{Z},$$

which can be deduced from the explicit formula (6.1). □

For $J \subseteq \mathcal{J}$ such that $J \neq J^*$ and $(J-1)^{\text{ss}} \neq J^{\text{ss}}$, we write

$$i^+(J) \stackrel{\text{def}}{=} \sum_{i=0}^{\ell(J)-1} i^+(\chi_{\delta_{\text{ss}}^i(J)}) \in \mathbb{Z},$$

$$\beta(J) \stackrel{\text{def}}{=} J(i^+(\chi_J), i^+(\chi_{\delta_{\text{ss}}(J)}), \dots, i^+(\chi_{\delta_{\text{ss}}^{\ell(J)-1}(J)})) \in \mathbb{F}^\times,$$

where each $i^+(\chi_{J'})$ is defined in either Lemma 6.9 or Lemma 6.10.

PROPOSITION 6.11. *Let $J \subseteq \mathcal{J}$ such that $J \not\subseteq J_{\overline{\rho}}$, $J \neq J^*$ and $(J-1)^{\text{ss}} \neq J^{\text{ss}}$. Then we have*

$$c(J) = \frac{\beta(J)}{\beta(J^{\text{ss}})} \frac{\prod_{i=0}^{\ell(J)-1} c(\chi_{\delta_{\text{ss}}^i(J)})}{\prod_{i=0}^{\ell(J^{\text{ss}})-1} c(\chi_{\delta_{\text{ss}}^i(J^{\text{ss}})})},$$

where $c(J)$ is defined in Example 5.3 and each $c(\chi_{J'})$ is defined in either Lemma 6.9 or Lemma 6.10.

Proof. Recall from Example 5.3 that $c(J) = c(\psi, \psi')$ with $\psi_i = \chi^s_{\delta_{\text{ss}}^{n-i}(J)}$ for $0 \leq i \leq n \stackrel{\text{def}}{=} \ell(J) - 1$ and $\psi'_i = \chi^s_{\delta_{\text{ss}}^{m-i}(J^{\text{ss}})}$ for $0 \leq i \leq m \stackrel{\text{def}}{=} \ell(J^{\text{ss}}) - 1$. Recall from § 5 the maps pr_χ, α_i and α'_i. If $x \in \theta_0^{R\psi_n^s}$ is an H-eigenvector such that $\text{pr}_{\psi_n^s} \alpha_n(x) = \varphi^{R\psi_n^s} \in \sigma(R\psi_n^s)$, then the proof of [DL21, Proposition 5.14] (using Lemmas 6.9, 6.10 and the cuspidality of θ_0) and the argument that follows show that

$$\text{pr}_{\psi_0} \alpha_0 \left(\prod_{i=n}^{0} [c(\chi_{\delta_{\text{ss}}^i(J)})] S^+_{i^+(\chi_{\delta_{\text{ss}}^i(J)})}(x) \right) = \varphi^{R\psi_0} = \varphi^{\chi_\emptyset} \in \sigma(\chi_\emptyset). \tag{6.13}$$

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Similarly, if $x' \in \theta_0^{R\psi'^s_m}$ is an H-eigenvector such that $\mathrm{pr}_{\psi'^s_m}\alpha'_m(x') = \varphi^{R\psi'^s_m} \in \sigma(R\psi'^s_m)$, then we have

$$\mathrm{pr}_{\psi'_0}\alpha'_0\Big(\prod_{i=m}^{0}[c(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})]S^+_{i^+(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})}(x')\Big) = \varphi^{R\psi'_0} = \varphi^{\chi_\emptyset} \in \sigma(\chi_\emptyset). \tag{6.14}$$

We compare H-eigencharacters of (6.13) and (6.14) using Lemma 6.2. Since both x and x' have H-eigencharacters $R\psi^s_n = R\psi'^s_m = \chi_{J^{\mathrm{ss}}}$ by (5.2), we deduce that $i^+(J) \equiv i^+(J^{\mathrm{ss}}) \bmod (q-1)$. Since $R\chi_{\delta^i_{\mathrm{ss}}(J)} \neq \chi_\emptyset$ for $0 \leq i \leq n-1$ and $R\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})} \neq \chi_\emptyset$ for $0 \leq i \leq m-1$ by the choice of n and m, we deduce that $(q-1) \nmid i^+(\delta^i_{\mathrm{ss}}(J))$ for $1 \leq i \leq n$ and $(q-1) \nmid i^+(\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}}))$ for $1 \leq i \leq m$. Moreover, since θ_0 is a cuspidal type, we have $S^+_0(x) = S^+_0(x') = 0$ by [DL21, Lemma 2.9]. Hence, by Lemma 6.1(ii) we have

$$\begin{aligned}\Big(\prod_{i=n}^{0} S^+_{i^+(\chi_{\delta^i_{\mathrm{ss}}(J)})}\Big)(x) &= \widetilde{\beta}(J)S^+_{i^+(J)}(x),\\ \Big(\prod_{i=m}^{0} S^+_{i^+(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})}\Big)(x) &= \widetilde{\beta}(J^{\mathrm{ss}})S^+_{i^+(J^{\mathrm{ss}})}(x),\end{aligned} \tag{6.15}$$

where $0 \neq \widetilde{\beta}(J) \in \mathcal{O}_K$ (respectively, $0 \neq \widetilde{\beta}(J^{\mathrm{ss}}) \in \mathcal{O}_K$) has leading term $\beta(J)$ (respectively, $\beta(J^{\mathrm{ss}})$). Combining (6.13)–(6.15) and the argument following the proof of [DL21, Proposition 5.14], we deduce that

$$c(J) = \frac{\beta(J)}{\beta(J^{\mathrm{ss}})}\frac{\prod_{i=0}^{n} c(\chi_{\delta^i_{\mathrm{ss}}(J)})}{\prod_{i=0}^{m} c(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})},$$

which completes the proof. □

Proposition 6.12. *Let $J \subseteq \mathcal{J}$ such that $J \nsubseteq J_{\overline{p}}$, $J \neq J^*$ and $(J-1)^{\mathrm{ss}} \neq J^{\mathrm{ss}}$. Then we have*

$$c'(J) = (-1)^{A(J)}.$$

Proof. By the definition $c'(J)$ and Proposition 6.11 we have

$$c'(J) = (-1)^{f-1+|J\cap(J-1)^{\mathrm{nss}}|}\mu_{J^s,J}\frac{P_2(J)}{P_1(J)}\frac{\beta(J)}{\beta(J^{\mathrm{ss}})}\frac{\prod_{i=0}^{\ell(J)-1} c(\chi_{\delta^i_{\mathrm{ss}}(J)})}{\prod_{i=0}^{\ell(J^{\mathrm{ss}})-1} c(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})}. \tag{6.16}$$

Recall from the proof of Proposition 6.11 that $i^+(J) \equiv i^+(J^{\mathrm{ss}}) \bmod (q-1)$, hence by (6.2) we have

$$\frac{\beta(J)}{\beta(J^{\mathrm{ss}})} = \frac{(-1)^{u(J)}}{(-1)^{u(J^{\mathrm{ss}})}}\frac{((-1)^{f-1}\prod_{j=0}^{f-1}(i^+(\chi_J)_j)!)\prod_{i=1}^{\ell(J)-1}((-1)^{f-1}\prod_{j=0}^{f-1}(i^+(\chi_{\delta^i_{\mathrm{ss}}(J)})_j)!)}{\prod_{i=0}^{\ell(J^{\mathrm{ss}})-1}((-1)^{f-1}\prod_{j=0}^{f-1}(i^+(\chi_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})_j)!)}, \tag{6.17}$$

where $u(J) \overset{\mathrm{def}}{=} u(i^+(\chi_J), i^+(\chi_{\delta_{\mathrm{ss}}(J)}), \ldots, i^+(\chi_{\delta^{\ell(J)-1}_{\mathrm{ss}}(J)})) \in \mathbb{Z}$, and similarly for $u(J^{\mathrm{ss}})$. Moveover, by Lemmas 6.9, 6.10 and (6.1) we have

$$\begin{aligned}\prod_{j=0}^{f-1}(i^+(\chi_{J'})_j)! &= \prod_{j=0}^{f-1}(p-1-i(\chi^s_{J'})_j)! \quad \text{for } J' \subseteq J_{\overline{p}},\\ \frac{\prod_{j=0}^{f-1}(i^+(\chi_J)_j)!}{J(i^+(\chi_J), -i(\chi_J)-\underline{s}^J)} &= (-1)^{f-1+u'(J)}\frac{\prod_{j=0}^{f-1}(p-1-i(\chi^s_J)_j)!}{\prod_{j=0}^{f-1}((-i(\chi_J)-\underline{s}^J)_j)!},\end{aligned} \tag{6.18}$$

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where $u'(J) \overset{\text{def}}{=} u(i^+(\chi_J), -i(\chi_J) - \underline{s}^J)$. Combining (6.16)–(6.18), the definition of $c(\chi_J)$ in Lemma 6.10 and the definition of each $c(\chi'_J)$ for $J' \subseteq J_{\overline{\rho}}$ in Lemma 6.9, we deduce that

$$c'(J) = (-1)^{U(J)} \alpha'(J) \alpha(J),$$

where

$$\begin{aligned} U(J) &\overset{\text{def}}{=} u(J) - u(J^{\text{ss}}) - u'(J) \in \mathbb{Z}, \\ \alpha'(J) &\overset{\text{def}}{=} (-1)^{|J \cap (J-1)^{\text{nss}}| + \underline{t}^J} \mu_{J^s, J} P_2(J) \Big/ \Big(\prod_{j=0}^{f-1} ((-i(\chi_J) - \underline{s}^J)_j)! \Big) \in \mathbb{F}^\times, \\ \alpha(J) &\overset{\text{def}}{=} \Big(\prod_{i=0}^{\ell(J)-1} \alpha(\chi_{\delta_{\text{ss}}^i(J)}) \Big) \Big/ \Big(\prod_{i=0}^{\ell(J^{\text{ss}})-1} \alpha(\chi_{\delta_{\text{ss}}^i(J^{\text{ss}})}) \Big) \in \mathbb{F}^\times, \end{aligned} \tag{6.19}$$

with (for each $J' \neq J^*$)

$$\alpha(\chi_{J'}) \overset{\text{def}}{=} J(i(\chi^s_{J'}), -\underline{s}^{J'}) \Big((-1)^{f-1} \prod_{j=0}^{f-1} (p - 1 - i(\chi^s_{J'}))! \Big) \Big/ P_1(\chi_{J'}) \in \mathbb{F}^\times.$$

Then the proposition follows from an explicit computation of the constants $U(J)$, $\alpha'(J)$ and $\alpha(J)$, which is given in Lemma 6.13. □

6.4 Explicit computations

Here we prove Lemma 6.13, which will finish the proof of Proposition 6.12. To state the result, for $J \subseteq \mathcal{J}$ and $j \in \mathcal{J}$ we define (in $\mathbb{F}^\times$)

$$\alpha'(J)_j \overset{\text{def}}{=} \begin{cases} 1 & \text{if } j \notin J^{\text{nss}},\ j+1 \notin J^{\text{nss}}, \\ r_j + 1 & \text{if } j \in J^{\text{nss}},\ j+1 \notin J, \\ p - 1 - r_j & \text{if } j \in J^{\text{nss}},\ j+1 \in J^{\text{ss}}, \\ -(r_j!(r_j+1)!)^{-1} & \text{if } j \notin J,\ j+1 \in J^{\text{nss}}, \\ -((r_j+1)!(r_j+2)!)^{-1} & \text{if } j \in J^{\text{ss}},\ j+1 \in J^{\text{nss}}, \\ (r_j!)^{-2} & \text{if } j \in J^{\text{nss}},\ j+1 \in J^{\text{nss}}. \end{cases} \tag{6.20}$$

LEMMA 6.13. *Let $J \subseteq \mathcal{J}$ such that $J \nsubseteq J_{\overline{\rho}}$, $J \neq J^*$ and $(J-1)^{\text{ss}} \neq J^{\text{ss}}$. Let $U(J) \in \mathbb{Z}$, $\alpha'(J)$, $\alpha(J) \in \mathbb{F}^\times$ be as in (6.19). Then we have:*

(i) $U(J) = A^{\text{ss}}(J)$ *(see (4.15) for $A^{\text{ss}}(J)$);*

(ii) $\alpha'(J) = (-1)^{|(\partial J)^{\text{nss}}|} \prod_{j=0}^{f-1} \alpha'(J)_j$;

(iii) $\alpha(J) \prod_{j=0}^{f-1} \alpha'(J)_j = 1$.

In the rest of this subsection, we prove Lemma 6.13. We start with some more notation that is needed in the proof. For $J \subseteq \mathcal{J}$ and $j \in \mathcal{J}$, we define

$$\begin{aligned} I(J)_j^1 &\overset{\text{def}}{=} \{i \geq 0 : j+1 \in \delta_{\text{ss}}^i(J),\ j+1 \notin \delta_{\text{ss}}^{i+1}(J)\}, \\ I(J)_j^2 &\overset{\text{def}}{=} \{i \geq 0 : j+1 \notin \delta_{\text{ss}}^i(J),\ j+1 \in \delta_{\text{ss}}^{i+1}(J)\}, \\ I(J)_j^3 &\overset{\text{def}}{=} \{i \geq 0 : j+1 \in \delta_{\text{ss}}^i(J),\ j+1 \in \delta_{\text{ss}}^{i+1}(J)\}, \\ I(J)_j^4 &\overset{\text{def}}{=} \{i \geq 0 : j+1 \notin \delta_{\text{ss}}^i(J),\ j+1 \notin \delta_{\text{ss}}^{i+1}(J)\}. \end{aligned}$$

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Since $j \in \delta_{\mathrm{ss}}^{i+1}(J)$ if and only if $(j+1 \in \delta_{\mathrm{ss}}^{i}(J)$ and $j \in J_{\overline{\rho}})$, by (2.1) we have

$$s_j^{\delta_{\mathrm{ss}}^{i+1}(J)} = \begin{cases} r_j + \delta_{j \in J_{\overline{\rho}}} & \text{if } i \in I(J)_j^1, \\ p-2-r_j & \text{if } i \in I(J)_j^2, \\ p-2-r_j-\delta_{j \in J_{\overline{\rho}}} & \text{if } i \in I(J)_j^3, \\ r_j & \text{if } i \in I(J)_j^4. \end{cases} \tag{6.21}$$

Then we define

$$I(J)_j^{t+4} \overset{\text{def}}{=} I(J)_j^t \cap \{i \geq 0 : j \notin (\delta_{\mathrm{ss}}^i(J))^\delta\} \quad \text{for } t \in \{1,2\},$$
$$I(J)_j^{t+4} \overset{\text{def}}{=} I(J)_j^t \cap \{i \geq 0 : j \in (\delta_{\mathrm{ss}}^i(J))^\delta\} \quad \text{for } t \in \{3,4\}.$$

We also define

$$i(J)_j^t \overset{\text{def}}{=} |I(J)_j^t| - |I(J^{\mathrm{ss}})_j^t| \quad \text{for } t \in \{1,2\},$$
$$i(J)_j^t \overset{\text{def}}{=} (|I(J)_j^{t+2}| - |I(J^{\mathrm{ss}})_j^{t+2}|) - (|I(J)_j^{t+4}| - |I(J^{\mathrm{ss}})_j^{t+4}|) \quad \text{for } t \in \{3,4\}.$$

Finally, for $t \in \{1,2,5,6,7,8\}$ we define

$$I^\circ(J)_j^t \overset{\text{def}}{=} I(J)_j^t \cap \{0\}.$$

If, moreover, $j+1 \in J_{\overline{\rho}}$, we let $k \geq 0$ be such that $j+i+1 \in J_{\overline{\rho}}$ for $0 \leq i \leq k$ and $j+k+2 \notin J_{\overline{\rho}}$. Then we define (for $t \in \{1,2,5,6,7,8\}$)

$$I^{\geq}(J)_j^t \overset{\text{def}}{=} \{i \geq 0 : i+k \in I(J)_j^t\}.$$

Proof of Lemma 6.13(i). Recall that $U(J) = u(J) - u(J^{\mathrm{ss}}) - u'(J)$ with $u(J)$ and $u(J^{\mathrm{ss}})$ defined in (6.17) and $u'(J)$ defined in (6.18). By (6.1) we have

$$\begin{aligned} u'(J) &= (p-1)^{-1} \sum_{j=0}^{f-1} (p-1-(i^+(\chi_J)_j + (-i(\chi_J) - \underline{s}^J)_j - (-i(\chi_J^s))_j)) \\ &= (p-1)^{-1} \sum_{j=0}^{f-1} ((p-1-i^+(\chi_J)_j) - (p-1-(i(\chi_J) + \underline{s}^J)_j) + (p-1-i(\chi_J^s)_j)). \end{aligned} \tag{6.22}$$

By Lemma 6.9 we have $p-1-i^+(\chi_{J'})_j = i(\chi_{J'}^s)_j$ for $\emptyset \neq J' \subseteq J_{\overline{\rho}}$ and $j \in \mathcal{J}$. Hence, by (6.2) we have

$$\begin{aligned} u(J) &= (p-1)^{-1} \sum_{j=0}^{f-1} \left((p-1-i^+(\chi_J)_j) + \left(\sum_{i=1}^{\ell(J)-1} i(\chi^s_{\delta_{\mathrm{ss}}^i(J)})_j \right) - (p-1-i^+(J)_j) \right), \\ u(J^{\mathrm{ss}}) &= (p-1)^{-1} \sum_{j=0}^{f-1} \left(\left(\sum_{i=0}^{\ell(J^{\mathrm{ss}})-1} i(\chi^s_{\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})})_j \right) - (p-1-i^+(J^{\mathrm{ss}})_j) \right). \end{aligned} \tag{6.23}$$

Moreover, from the proof of Proposition 6.11 we have $i^+(J) \equiv i^+(J^{\mathrm{ss}}) \bmod (q-1)$. Combining (6.22) and (6.23), we deduce that

$$U(J) = u(J) - u(J^{\mathrm{ss}}) - u'(J) = (p-1)^{-1} \sum_{j=0}^{f-1} U(J)_j,$$

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where (for each $j \in \mathcal{J}$)

$$U(J)_j \overset{\text{def}}{=} \sum_{i=0}^{\ell(J)-1} i(\chi^s_{\delta^i_{\mathrm{ss}}(J)})_j - \sum_{i=0}^{\ell(J^{\mathrm{ss}})-1} i(\chi^s_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})_j - (i(\chi_J) + \underline{s}^J)_j \in \mathbb{Z}. \tag{6.24}$$

By (5.4), we have

$$\begin{aligned}\sum_{i=0}^{\ell(J)-1} i(\chi^s_{\delta^i_{\mathrm{ss}}(J)})_j &= \sum_{i\geq 0} \delta_{j\in(\delta^i_{\mathrm{ss}}(J))^\delta}(p-1-s_j^{\delta^{i+1}_{\mathrm{ss}}(J)}) = \sum_{i\in I(J)^1_j \sqcup I(J)^2_j} (p-1-s_j^{\delta^{i+1}_{\mathrm{ss}}(J)}) \\ &= |I(J)^1_j|(p-1-r_j-\delta_{j\in J_{\overline{p}}}) + |I(J)^2_j|(r_j+1),\end{aligned} \tag{6.25}$$

where the last equality follows from (6.21). Similarly, we have

$$\sum_{i=0}^{\ell(J^{\mathrm{ss}})-1} i(\chi^s_{\delta^i_{\mathrm{ss}}(J^{\mathrm{ss}})})_j = |I(J^{\mathrm{ss}})^1_j|(p-1-r_j-\delta_{j\in J_{\overline{p}}}) + |I(J^{\mathrm{ss}})^2_j|(r_j+1). \tag{6.26}$$

Moreover, by Lemma 6.5 and (2.1) we have

$$(i(\chi_J) + \underline{s}^J)_j = \delta_{j+1\in J^{\mathrm{nss}}}(p-1-s_j^{J^{\mathrm{ss}}}) = \delta_{j+1\in J^{\mathrm{nss}}}(p-1-r_j-\delta_{j\in J^{\mathrm{ss}}}). \tag{6.27}$$

Combining (6.24)–(6.26) and (6.27), we deduce that

$$U(J)_j = i(J)^1_j(p-1-r_j-\delta_{j\in J_{\overline{p}}}) + i(J)^2_j(r_j+1) - \delta_{j+1\in J^{\mathrm{nss}}}(p-1-r_j-\delta_{j\in J^{\mathrm{ss}}}). \tag{6.28}$$

To compute each $U(J)_j$ explicitly, we separate the following cases.

Case 1. $j+1 \in J_{\overline{p}}$. Let $k \geq 0$ such that $j+i+1 \in J_{\overline{p}}$ for $0 \leq i \leq k$ and $j+k+2 \notin J_{\overline{p}}$. By (4.18) we have

$$\begin{aligned} j+1 \in \delta^i_{\mathrm{ss}}(J) &\Leftrightarrow (0 \leq i \leq k+1 \text{ and } j+i+1 \in J), \\ j+1 \in \delta^i_{\mathrm{ss}}(J^{\mathrm{ss}}) &\Leftrightarrow (0 \leq i \leq k \text{ and } j+i+1 \in J). \end{aligned} \tag{6.29}$$

In particular, for $t \in \{1,2\}$ we have $i \in I(J)^t_j \Leftrightarrow i \in I(J^{\mathrm{ss}})^t_j$ for $0 \leq i \leq k-1$, hence

$$|I(J)^t_j| - |I(J^{\mathrm{ss}})^t_j| = |I^{\geq}(J)^t_j| - |I^{\geq}(J^{\mathrm{ss}})^t_j|. \tag{6.30}$$

We denote $\mathrm{ch}^1_J \overset{\text{def}}{=} (\delta_{j+k+1\in J}, \delta_{j+k+2\in J}) \in \{0,1\}^2$. Then from the definition with a case-by-case examination we obtain the following table, where the second and third columns use (6.29), the fourth column uses (6.30), and the last column uses (6.28).

ch^1_J	$I^{\geq}(J)^{1,2}_j$	$I^{\geq}(J^{\mathrm{ss}})^{1,2}_j$	$i(J)^{1,2}_j$	$U(J)_j$
$(1,1)$	$\{1\},\emptyset$	$\{0\},\emptyset$	$0,0$	0
$(1,0)$	$\{0\},\emptyset$	$\{0\},\emptyset$	$0,0$	0
$(0,1)$	$\{1\},\{0\}$	$\emptyset,\emptyset$	$1,1$	$p-\delta_{j\in J_{\overline{p}}}$
$(0,0)$	$\emptyset,\emptyset$	$\emptyset,\emptyset$	$0,0$	0

We provide a detailed argument for one case. The other cases, as well as the tables in the proof of Lemma 6.13(iii), are similar and straightforward. Suppose that $j+k+1 \in J$ and $j+k+2 \in J$. For $i \geq 0$, we deduce from (6.29) that $j+1 \in \delta^{k+i}_{\mathrm{ss}}(J)$ if and only if $i \in \{0,1\}$. It follows that $k+i \in I(J)^1_j$ if and only if $i=1$, and that $k+i \notin I(J)^2_j$ for all $i \geq 0$. Hence, $I^{\geq}(J)^1_j = \{1\}$ and $I^{\geq}(J)^2_j = \emptyset$. Similarly, we deduce from (6.29) that $j+1 \in \delta^{k+i}_{\mathrm{ss}}(J^{\mathrm{ss}})$ if and only if $i=0$. It follows that $k+i \in I(J^{\mathrm{ss}})^1_j$ if and only if $i=1$, and that $k+i \notin I(J^{\mathrm{ss}})^2_j$ for all $i \geq 0$. Hence, $I^{\geq}(J^{\mathrm{ss}})^1_j = \{0\}$ and $I^{\geq}(J^{\mathrm{ss}})^2_j = \emptyset$. Then by (6.30) we have $i(J)^1_j = 1-1=0$ and $i(J)^2_j = 0-0=0$. Finally, by (6.28) we have $U(J)_j = 0+0-0=0$.

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Case 2. $j+1\notin J_{\overline{\rho}}$. In this case, we have $j+1\notin\delta_{\mathrm{ss}}^i(J)$ for $i\geq 1$ and $j+1\notin\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})$ for $i\geq 0$. Hence, $I(J)_j^1=\{0\}$ if $j+1\in J$, $I(J)_j^1=\emptyset$ if $j+1\notin J$, and $I(J)_j^2=I(J^{\mathrm{ss}})_j^1=I(J^{\mathrm{ss}})_j^2=\emptyset$. By (6.28) we have

$$\begin{aligned}U(J)_j&=\delta_{j+1\in J}(p-1-r_j-\delta_{j\in J_{\overline{\rho}}})-\delta_{j+1\in J^{\mathrm{nss}}}(p-1-r_j-\delta_{j\in J^{\mathrm{ss}}})\\&=-\delta_{j+1\in J}(\delta_{j\in J_{\overline{\rho}}}-\delta_{j\in J^{\mathrm{ss}}})=-\delta_{j+1\in J,\,j\in J_{\overline{\rho}}\setminus J}.\end{aligned}$$

As in § 4, we decompose $J_{\overline{\rho}}$ into a disjoint union of intervals (in $\mathbb{Z}/f\mathbb{Z}$) not adjacent to each other, $J_{\overline{\rho}}=J_1\sqcup\cdots\sqcup J_t$, and for each $1\leq i\leq t$ we write $J_i=\{j_i,j_i+1,\dots,j_i+k_i\}$ with $j_i\in\mathcal{J}$ and $k_i\geq 0$. Combining Case 1 and Case 2, we get (for each $i\in\mathbb{Z}/t\mathbb{Z}$)

$$U(J)_j=\begin{cases}p\delta_{j_i+k_i\in\partial(J^c)} & \text{if } j=j_i-1,\\ (p-1)\delta_{j_i+k_i\in\partial(J^c)} & \text{if } j=j_i,j_i+1,\dots,j_i+k_i-1,\\ -\delta_{j_i+k_i\in\partial(J^c)} & \text{if } j=j_i+k_i,\\ 0 & \text{if } j=j_i+k_i+1,j_i+k_i+2,\dots,j_{i+1}-2.\end{cases}$$

Hence, we conclude that

$$\begin{aligned}U(J)&=(p-1)^{-1}\sum_{j=0}^{f-1}U(J)_j=(p-1)^{-1}\sum_{i=1}^{t}\Big(\sum_{j=j_i-1}^{j_{i+1}-2}U(J)_j\Big)\\&=(p-1)^{-1}\sum_{i=1}^{t}(\delta_{j_i+k_i\in\partial(J^c)}(p+k_i(p-1)-1))=\sum_{i=1}^{t}(\delta_{j_i+k_i\in\partial(J^c)}(k_i+1)),\end{aligned}$$

which equals $A^{\mathrm{ss}}(J)$ by (4.15). □

Proof of Lemma 6.13(ii). As in (6.5) we have

$$\mu_{J^s,J}=(-1)^{\underline{t}^{J^s}+\sum_{j+1\in J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}}+\delta_{j\in J^{\mathrm{nss}}})}\left[\frac{\prod_{j+1\notin J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}}+\delta_{j\in J^{\mathrm{nss}}})!}{\prod_{j+1\in J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}}+\delta_{j\notin J^{\mathrm{nss}}})!}\right].\tag{6.31}$$

By (5.6) and using (3.6), we have

$$P_2(J)=(-1)^{\sum_{j+1\notin J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}}+1)}\Big(\prod_{j+1\notin J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}})!\Big)^{-1}.\tag{6.32}$$

By Lemma 6.5 and using $(p-1)!\equiv -1 \bmod p$, we have

$$\prod_{j=0}^{f-1}((-i(\chi_J)-\underline{s}^J)_j)!=(-1)^{f-|J^{\mathrm{nss}}|}\prod_{j+1\in J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}})!\tag{6.33}$$

Combining (6.31), (6.32) and (6.33), we deduce that

$$\begin{aligned}\alpha'(J)&=(-1)^{|J\cap(J-1)^{\mathrm{nss}}|+\underline{t}^J}\mu_{J^s,J}P_2(J)\Big/\Big(\prod_{j=0}^{f-1}((-i(\chi_J)-\underline{s}^J)_j)!\Big)\\&=(-1)^d\left[\frac{\prod_{j\notin J^{\mathrm{nss}},j+1\notin J^{\mathrm{nss}}}(1)\prod_{j\in J^{\mathrm{nss}},j+1\notin J^{\mathrm{nss}}}(s_j^{J^{\mathrm{ss}}}+1)}{\prod_{j\notin J^{\mathrm{nss}},j+1\in J^{\mathrm{nss}}}(-(s_j^{J^{\mathrm{ss}}})!(s_j^{J^{\mathrm{ss}}}+1)!)\prod_{j\in J^{\mathrm{nss}},j+1\in J^{\mathrm{nss}}}((s_j^{J^{\mathrm{ss}}})!)^2}\right],\end{aligned}\tag{6.34}$$

where (using Lemma 2.3)

$$\begin{aligned}d&=|J\cap(J-1)^{\mathrm{nss}}|+\underline{t}^J+\underline{t}^{J^s}+\underline{s}^{J^{\mathrm{ss}}}+\underline{1}+f-|J^{\mathrm{nss}}|\\&\equiv|J^{\mathrm{nss}}|-|J\cap(J-1)^{\mathrm{nss}}|=|(\partial J)^{\mathrm{nss}}|\mod 2.\end{aligned}$$

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Moreover, a case-by-case examination using (2.1) shows that each term on the right-hand side of (6.34) agrees with the corresponding $\alpha'(J)_j$ defined in (6.20). For example, in the case $j \in J^{\mathrm{nss}}$ and $j+1 \notin J^{\mathrm{nss}}$, we deduce from (2.1) that $s_j^{J^{\mathrm{ss}}} + 1 = r_j + 1$ if $j+1 \notin J$ and $s_j^{J^{\mathrm{ss}}} + 1 = p - 1 - r_j$ if $j+1 \in J^{\mathrm{ss}}$. Hence, we conclude that $\alpha'(J) = (-1)^{|(\partial J)^{\mathrm{nss}}|} \prod_{j=0}^{f-1} \alpha'(J)_j$. □

Proof of Lemma 6.13(iii). For $J' \neq J^*$, by (5.4) and using (3.6), we have

$$\begin{aligned} \prod_{j=0}^{f-1} (p-1-i(\chi_{J'}^s)_j)! &= (-1)^{f-|J'^\delta|} \Big(\prod_{j+1 \in J'^\delta} (s_j^{(J'-1)^{\mathrm{ss}}})! \Big), \\ P_1(\chi_{J'}) &= (-1)^{\sum_{j+1 \in J'^\delta} (s_j^{(J'-1)^{\mathrm{ss}}} + 1)} \Big(\prod_{j+1 \in J'^\delta} (s_j^{(J'-1)^{\mathrm{ss}}})! \Big)^{-1}. \end{aligned} \tag{6.35}$$

Combining (6.35) with Lemma 6.4, we deduce that

$$\begin{aligned} \alpha(\chi_{J'}) &= J(i(\chi_{J'}^s), -\underline{s}^J)((-1)^{f-1} \prod_{j=0}^{f-1} (p-1-i(\chi_{J'}^s))!)/P_1(\chi_{J'}) \\ &= \left[\frac{\prod_{j+1 \in J'^\delta, j \notin J'^\delta} (-1)(s_j^{(J'-1)^{\mathrm{ss}}} + 1)}{\prod_{j+1 \notin J'^\delta, j \in J'^\delta} (s_j^{(J'-1)^{\mathrm{ss}}} + 1)} \right] \Big(\prod_{j+1 \in J'^\delta} (s_j^{(J'-1)^{\mathrm{ss}}})! \Big)^2. \end{aligned}$$

Then we have (using (6.21))

$$\begin{aligned} \prod_{i=0}^{\ell(J)-1} \alpha(\chi_{\delta_{\mathrm{ss}}^i(J)}) &= \left[\frac{\prod_{i \in I(J)_j^5 \sqcup I(J)_j^6} (-1)(s_j^{\delta_{\mathrm{ss}}^{i+1}(J)} + 1)}{\prod_{i \in I(J)_j^7 \sqcup I(J)_j^8} (s_j^{\delta_{\mathrm{ss}}^{i+1}(J)} + 1)} \right] \Big(\prod_{i \in I(J)_j^1 \sqcup I(J)_j^2} (s_j^{\delta_{\mathrm{ss}}^{i+1}(J)})! \Big)^2 \\ &= \left[\frac{(p-1-r_j-\delta_{j \in J_{\bar{p}}})^{|I(J)_j^5|} (r_j+1)^{|I(J)_j^6|}}{(p-1-r_j-\delta_{j \in J_{\bar{p}}})^{|I(J)_j^7|} (r_j+1)^{|I(J)_j^8|}} \right] ((r_j + \delta_{j \in J_{\bar{p}}})!)^{2|I(J)_j^1|} ((p-2-r_j)!)^{2|I(J)_j^2|}. \end{aligned}$$

A similar formula holds with each J replaced with J^{ss}. Hence, we have

$$\alpha(J) = \Big(\prod_{i=0}^{\ell(J)-1} \alpha(\chi_{\delta_{\mathrm{ss}}^i(J)}) \Big) \Big/ \Big(\prod_{i=0}^{\ell(J^{\mathrm{ss}})-1} \alpha(\chi_{\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})}) \Big) = \prod_{j=0}^{f-1} \alpha(J)_j, \tag{6.36}$$

where (for each $j \in \mathcal{J}$)

$$\alpha(J)_j \overset{\mathrm{def}}{=} ((r_j + \delta_{j \in J_{\bar{p}}})!)^{2i(J)_j^1} ((p-2-r_j)!)^{2i(J)_j^2} (p-1-r_j-\delta_{j \in J_{\bar{p}}})^{i(J)_j^3} (r_j+1)^{i(J)_j^4} \in \mathbb{F}^\times. \tag{6.37}$$

To compute each $\alpha(J)_j$ explicitly, we separate the following cases.

Case 1. $j+1 \in J_{\bar{p}}$, $j \in J_{\bar{p}}$. Let $k \geq 0$ such that $j+i+1 \in J_{\bar{p}}$ for $0 \leq i \leq k$ and $j+k+2 \notin J_{\bar{p}}$. By (4.18) we have

$$\begin{aligned} j+1 \in \delta_{\mathrm{ss}}^i(J) &\Leftrightarrow (0 \leq i \leq k+1 \text{ and } j+i+1 \in J), \\ j \in \delta_{\mathrm{ss}}^i(J) &\Leftrightarrow (0 \leq i \leq k+2 \text{ and } j+i \in J), \\ j+1 \in \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}}) &\Leftrightarrow (0 \leq i \leq k \text{ and } j+i+1 \in J), \\ j \in \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}}) &\Leftrightarrow (0 \leq i \leq k+1 \text{ and } j+i \in J). \end{aligned} \tag{6.38}$$

In particular, for $t \in \{1,2,5,6,7,8\}$ we have $i \in I(J)_j^t \Leftrightarrow i \in I(J^{\mathrm{ss}})_j^t$ for $0 \leq i \leq k-1$, hence

$$|I(J)_j^t| - |I(J^{\mathrm{ss}})_j^t| = |I^{\geq}(J)_j^t| - |I^{\geq}(J^{\mathrm{ss}})_j^t|. \tag{6.39}$$

We denote $\mathrm{ch}_J^2 \overset{\mathrm{def}}{=} (\delta_{j+k \in J}, \delta_{j+k+1 \in J}, \delta_{j+k+2 \in J}) \in \{0,1\}^3$. Then from the definition with a case-by-case examination we obtain the following table, where the second and third columns use

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(6.38), the fourth column uses (6.39), and the last column follows from (6.37) using the equality $((r_j+1)!(p-2-r_j)!)^2=1$ in $\mathbb{F}$.

ch_J^2	$I^{\geq}(J)_j^{1,2,5,6,7,8}$	$I^{\geq}(J^{\mathrm{ss}})_j^{1,2,5,6,7,8}$	$i(J)_j^{1,2,3,4}$	$\alpha(J)_j$
$(1,1,1)$	$\{1\},\emptyset,\{1\},\emptyset,\emptyset,\{2\}$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\{1\}$	$0,0,0,0$	1
$(0,1,1)$	$\{1\},\emptyset,\{1\},\emptyset,\{0\},\{2\}$	$\{0\},\emptyset,\emptyset,\emptyset,\emptyset,\{1\}$	$0,0,0,0$	1
$(1,1,0)$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\{1\}$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\{1\}$	$0,0,0,0$	1
$(0,1,0)$	$\{0\},\emptyset,\emptyset,\emptyset,\emptyset,\{1\}$	$\{0\},\emptyset,\emptyset,\emptyset,\emptyset,\{1\}$	$0,0,0,0$	1
$(1,0,1)$	$\{1\},\{0\},\emptyset,\emptyset,\emptyset,\{2\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$1,1,0,0$	1
$(0,0,1)$	$\{1\},\{0\},\emptyset,\{0\},\emptyset,\{2\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$1,1,0,0$	1
$(1,0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$0,0,0,0$	1
$(0,0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1

In particular, since $\alpha'(J)_j=1$ by (6.20), we deduce that $\alpha(J)_j\alpha'(J)_j=1$.

Case 2. $j+1\in J_{\overline{\rho}},\ j\notin J_{\overline{\rho}}$. Let $k\geq 0$ such that $j+i+1\in J_{\overline{\rho}}$ for $0\leq i\leq k$ and $j+k+2\notin J_{\overline{\rho}}$. By (4.18) we have

$$\begin{aligned} j+1\in\delta_{\mathrm{ss}}^i(J)&\Leftrightarrow(0\leq i\leq k+1 \text{ and } j+i+1\in J),\\ j\in\delta_{\mathrm{ss}}^i(J)&\Leftrightarrow(i=0 \text{ and } j\in J),\\ j+1\in\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})&\Leftrightarrow(0\leq i\leq k \text{ and } j+i+1\in J),\\ j\in\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})&\Leftrightarrow(\text{impossible}). \end{aligned} \tag{6.40}$$

We denote $\mathrm{ch}_J^3\overset{\mathrm{def}}{=}(\delta_{j\in J},\delta_{j+1\in J},\delta_{j+2\in J})\in\{0,1\}^3$.

If $k=0$, then from the definition with a case-by-case examination we obtain the following table, where the second and third columns use (6.40), and the last column follows from (6.37) using the equality $(r_j!(p-2-r_j)!)^2=(r_j+1)^{-2}$ in $\mathbb{F}$.

ch_J^3	$I(J)_j^{1,2,5,6,7,8}$	$I(J^{\mathrm{ss}})_j^{1,2,5,6,7,8}$	$i(J)_j^{1,2,3,4}$	$\alpha(J)_j$
$(1,1,1)$	$\{1\},\emptyset,\{1\},\emptyset,\{0\},\emptyset$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$0,0,-1,0$	$(p-1-r_j)^{-1}$
$(0,1,1)$	$\{1\},\emptyset,\{1\},\emptyset,\emptyset,\emptyset$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1
$(1,1,0)$	$\{0\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$0,0,-1,0$	$(p-1-r_j)^{-1}$
$(0,1,0)$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1
$(1,0,1)$	$\{1\},\{0\},\{1\},\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$1,1,1,0$	$(p-1-r_j)^{-1}$
$(0,0,1)$	$\{1\},\{0\},\{1\},\{0\},\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$1,1,1,1$	-1
$(1,0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,-1$	$(r_j+1)^{-1}$
$(0,0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1

If $k\geq 1$, then for $t\in\{1,2,5,6,7,8\}$ we have $i\in I(J)_j^t\Leftrightarrow i\in I(J^{\mathrm{ss}})_j^t$ for $1\leq i\leq k-1$, hence

$$|I(J)_j^t|-|I(J^{\mathrm{ss}})_j^t|=(|I^{\circ}(J)_j^t|-|I^{\circ}(J^{\mathrm{ss}})_j^t|)+(|I^{\geq}(J)_j^t|-|I^{\geq}(J^{\mathrm{ss}})_j^t|). \tag{6.41}$$

By (6.40) and a case-by-case examination, we have

$$\begin{aligned} I^{\geq}(J)_j^{1,2,5,6,7,8}&=\begin{cases}\{1\},\{0\},\{1\},\{0\},\emptyset,\emptyset & \text{if } j+k+1\notin J,\ j+k+2\in J\\ \emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset & \text{otherwise,}\end{cases}\\ I^{\geq}(J^{\mathrm{ss}})_j^{1,2,5,6,7,8}&=\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset. \end{aligned} \tag{6.42}$$

If $(\delta_{j+k+1\in J},\delta_{j+k+2\in J})\neq(0,1)$, then from the definition with a case-by-case examination we obtain the following table, where the second and third columns use (6.40), the fourth column uses (6.41) and (6.42), and the last column follows from (6.37).

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ch_J^3	$I^\circ(J)_j^{1,2,5,6,7,8}$	$I^\circ(J^{\mathrm{ss}})_j^{1,2,5,6,7,8}$	$i(J)_j^{1,2,3,4}$	$\alpha(J)_j$
$(1,1,1)$	$\emptyset,\emptyset,\emptyset,\emptyset,\{0\},\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,-1,0$	$(p-1-r_j)^{-1}$
$(0,1,1)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1
$(1,1,0)$	$\{0\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$0,0,-1,0$	$(p-1-r_j)^{-1}$
$(0,1,0)$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1
$(1,0,1)$	$\emptyset,\{0\},\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\{0\},\emptyset,\{0\},\emptyset,\emptyset$	$0,0,0,-1$	$(r_j+1)^{-1}$
$(0,0,1)$	$\emptyset,\{0\},\emptyset,\{0\},\emptyset,\emptyset$	$\emptyset,\{0\},\emptyset,\{0\},\emptyset,\emptyset$	$0,0,0,0$	1
$(1,0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,-1$	$(r_j+1)^{-1}$
$(0,0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1

If $(\delta_{j+k+1\in J}, \delta_{j+k+2\in J}) = (0,1)$, then we have a similar table with the first three columns unchanged. By (6.41) and (6.42), each number of the fourth column should be added by 1. Then by (6.37) and using the equality $(r_j!(p-2-r_j)!)^2(r_j+1)(p-1-r_j) = -1$ in $\mathbb{F}$, each $\alpha(J)_j$ of the last column should be multiplied by (-1).

Combining the cases $k=0$ and $k\geq 1$ and using the definition of $\alpha'(J)_j$ in (6.20), we conclude that $\alpha(J)_j\alpha'(J)_j = -1$ if $j+k+1\notin J$ and $j+k+2\in J$, and $\alpha(J)_j\alpha'(J)_j = 1$ otherwise.

Case 3. $j+1\notin J_{\overline{\rho}},\ j\notin J_{\overline{\rho}}$. In this case, by (4.18) we have $j, j+1\notin \delta_{\mathrm{ss}}^i(J)$ for all $i\geq 1$ and $j, j+1\notin \delta_{\mathrm{ss}}^i(J^{\mathrm{ss}})$ for all $i\geq 0$. We denote $\mathrm{ch}_J^4 \overset{\mathrm{def}}{=} (\delta_{j\in J}, \delta_{j+1\in J}) \in \{0,1\}^2$. Then from the definition with a case-by-case examination we obtain the following table.

ch_J^4	$I(J)_j^{1,2,5,6,7,8}$	$I(J^{\mathrm{ss}})_j^{1,2,5,6,7,8}$	$i(J)_j^{1,2,3,4}$	$\alpha(J)_j$
$(1,1)$	$\{0\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$1,0,0,0$	$(r_j!)^2$
$(1,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,-1$	$(r_j+1)^{-1}$
$(0,1)$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$1,0,1,0$	$(r_j!)^2(p-1-r_j)$
$(0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1

Together with (6.20), we conclude that $\alpha(J)_j\alpha'(J)_j = 1$.

Case 4. $j+1\notin J_{\overline{\rho}},\ j\in J_{\overline{\rho}}$. In this case, by (4.18) we have

$$\begin{aligned} j+1\in\delta_{\mathrm{ss}}^i(J) &\Leftrightarrow (i=0 \text{ and } j+1\in J),\\ j\in\delta_{\mathrm{ss}}^i(J) &\Leftrightarrow (i\in\{0,1\} \text{ and } j+i\in J),\\ j+1\in\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}}) &\Leftrightarrow (\text{impossible}),\\ j\in\delta_{\mathrm{ss}}^i(J^{\mathrm{ss}}) &\Leftrightarrow (i=0 \text{ and } j\in J). \end{aligned} \tag{6.43}$$

Then from the definition with a case-by-case examination we obtain the following table, where the second and third columns use (6.43) and the last column follows from (6.37).

ch_J^4	$I(J)_j^{1,2,5,6,7,8}$	$I(J^{\mathrm{ss}})_j^{1,2,5,6,7,8}$	$i(J)_j^{1,2,3,4}$	$\alpha(J)_j$
$(1,1)$	$\{0\},\emptyset,\{0\},\emptyset,\emptyset,\{1\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$1,0,1,0$	$((r_j+1)!)^2(p-2-r_j)$
$(1,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{0\}$	$0,0,0,0$	1
$(0,1)$	$\{0\},\emptyset,\emptyset,\emptyset,\emptyset,\{1\}$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$1,0,0,-1$	$((r_j+1)!)^2(r_j+1)^{-1}$
$(0,0)$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset$	$0,0,0,0$	1

Together with (6.20), we conclude that $\alpha(J)_j\alpha'(J)_j = -1$ if $j\notin J,\ j+1\in J$, and $\alpha(J)_j\alpha'(J)_j = 1$ otherwise.

To give a summary of Cases 1–4, we write $J_{\overline{\rho}} = J_1\sqcup\cdots\sqcup J_t$ as in § 4 where each $J_i = \{j_i, j_i+1,\dots,j_i+k_i\}$ with $j_i\in\mathcal{J}$ and $k_i\geq 0$. Then we have (for each $i\in\mathbb{Z}/t\mathbb{Z}$)

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$$\alpha(J)_j\alpha'(J)_j=\begin{cases}(-1)^{\delta_{j_i+k_i\in\partial(J^c)}} & \text{if } j=j_i-1,\\ 1 & \text{if } j=j_i,j_i+1,\dots,j_i+k_i-1,\\ (-1)^{\delta_{j_i+k_i\in\partial(J^c)}} & \text{if } j=j_i+k_i,\\ 1 & \text{if } j=j_i+k_i+1,j_i+k_i+2,\dots,j_{i+1}-2.\end{cases}$$

Therefore, we conclude from (6.36) that

$$\alpha(J)\prod_{j=0}^{f-1}\alpha'(J)_j=\prod_{j=0}^{f-1}(\alpha(J)_j\alpha'(J)_j)=(-1)^{2\#\{j:j+1\notin J_{\overline{\rho}},\ j+1\in J,\ j\in J_{\overline{\rho}},\ j\notin J\}}=1,$$

which completes the proof. □

7. The main result

We combine the results of the previous sections and the results of [Wan24] and [Wan26] to finish the proof of Theorem 1.1.

We recall the definition of the ring A. We let $\mathfrak{m}_{N_0}$ be the maximal ideal of $\mathbb{F}[\![N_0]\!]$. Then we have $\mathbb{F}[\![N_0]\!]=\mathbb{F}[\![Y_0,\dots,Y_{f-1}]\!]$ and $\mathfrak{m}_{N_0}=(Y_0,\dots,Y_{f-1})$. Consider the multiplicative subset $S\overset{\text{def}}{=}\{(Y_0\cdots Y_{f-1})^n:n\geq 0\}$ of $\mathbb{F}[\![N_0]\!]$. Then $A\overset{\text{def}}{=}\widehat{\mathbb{F}[\![N_0]\!]_S}$ is the completion of the localization $\mathbb{F}[\![N_0]\!]_S$ with respect to the $\mathfrak{m}_{N_0}$-adic filtration

$$F_n(\mathbb{F}[\![N_0]\!]_S)=\bigcup_{k\geq 0}\frac{1}{(Y_0\cdots Y_{f-1})^k}\mathfrak{m}_{N_0}^{kf-n},$$

where $\mathfrak{m}_{N_0}^m\overset{\text{def}}{=}\mathbb{F}[\![N_0]\!]$ if $m\leq 0$. We denote by F_nA $(n\in\mathbb{Z})$ the induced filtration on A and endow A with the associated topology [LO96, § 1.3]. There is an $\mathbb{F}$-linear action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![N_0]\!]$ given by multiplication on $N_0\cong\mathcal{O}_K$, and an $\mathbb{F}$-linear Frobenius φ on $\mathbb{F}[\![N_0]\!]$ given by multiplication by p on $N_0\cong\mathcal{O}_K$. They extend canonically by continuity to commuting continuous $\mathbb{F}$-linear actions of φ and $\mathcal{O}_K^\times$ on A. Then an étale $(\varphi,\mathcal{O}_K^\times)$-module over A is by definition a finite free A-module D endowed with a semilinear Frobenius φ and a commuting continuous semilinear action of $\mathcal{O}_K^\times$ such that the image of φ generates D over A.

Let $\overline{\rho}$ be as in (1.3). We refer to [BHH+25b] for the definition of the étale $(\varphi,\mathcal{O}_K^\times)$-module $D_A^\otimes(\overline{\rho})$ over A. By [Wan26, (46)], $D_A^\otimes(\overline{\rho})$ has rank 2^f and is equipped with an A-basis such that:

(i) the corresponding matrix $\mathrm{Mat}(\varphi)\in\mathrm{GL}_{2^f}(A)$ (with its rows and columns indexed by the subsets of $\mathcal{J}$) for the φ-action is given by

$$\mathrm{Mat}(\varphi)_{J',J+1}=\begin{cases}\nu_{J+1,J'}\prod_{j\notin J}(Y_j/\varphi(Y_j))^{r_j+1} & \text{if } J'\subseteq J,\\ 0 & \text{if } J'\nsubseteq J,\end{cases}\tag{7.1}$$

with (see (4.1) for β and d_j)

$$\nu_{J,J'}\overset{\text{def}}{=}\beta^{|J^c|-|J|}\prod_{j\in(J-1)\setminus J'}d_j \text{ for } J'\subseteq J-1;$$

(ii) the corresponding matrices $\mathrm{Mat}(a)$ for the $\mathcal{O}_K^\times$-action satisfy $\mathrm{Mat}(a)\in\mathrm{I}_{2^f}+\mathrm{M}_{2^f}(F_{1-p}A)$ for all $a\in\mathcal{O}_K^\times$.

In particular, since $d_j=0$ if and only if $j\in J_{\overline{\rho}}$, we deduce that $\nu_{J,J'}\neq 0$ if and only if $(J-1)^{\mathrm{ss}}\subseteq J'\subseteq J-1$. We then extend the definition of $\nu_{J,J'}$ to all $J,J'\subseteq\mathcal{J}$ such that $(J-1)^{\mathrm{ss}}=(J')^{\mathrm{ss}}$ by

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the formula

$$\nu_{J,J'} \overset{\text{def}}{=} \beta^{|J^c|-|J|}\Big(\prod_{j\in(J-1)^{\text{nss}}} d_j\Big)\Big/\Big(\prod_{j\in(J')^{\text{nss}}} d_j\Big). \tag{7.2}$$

Then for $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$ such that $(J_1-1)^{\text{ss}} = (J_2-1)^{\text{ss}} = J_3^{\text{ss}} = J_4^{\text{ss}}$ we have $\nu_{J_1,J_3}/\nu_{J_1,J_4} = \nu_{J_2,J_3}/\nu_{J_2,J_4}$. We define $\nu_{*,J}/\nu_{*,J'}$ for $J^{\text{ss}} = (J')^{\text{ss}}$ in a similar way as $\mu_{*,J}/\mu_{*,J'}$.

If, moreover, $J_{\overline{\rho}} \neq \mathcal{J}$, then for $J \subseteq \mathcal{J}$ we define

$$\nu(J) \overset{\text{def}}{=} \frac{\nu_{*,J}}{\nu_{*,J^{\text{ss}}}}\left[\frac{\prod_{i=0}^{\ell(J)-1} \nu_{\delta_{\text{ss}}^i(J),\delta_{\text{ss}}^{i+1}(J)}}{\prod_{i=0}^{\ell(J^{\text{ss}})-1} \nu_{\delta_{\text{ss}}^i(J^{\text{ss}}),\delta_{\text{ss}}^{i+1}(J^{\text{ss}})}}\right]. \tag{7.3}$$

By definition, we have $\nu(J) = \beta^{B(J)} d(J)$ for $B(J) \in \mathbb{Z}$ and $d(J) \in \mathbb{F}^\times$ as in (4.17).

Let π be as in (1.1). We refer to [BHH+25a] for the definition of the étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ over A. By [Wan24, Proposition C.3(i),(iii) and Corollary C.4], the twisted dual étale $(\varphi, \mathcal{O}_K^\times)$-module $\mathrm{Hom}_A(D_A(\pi), A)(1)$ has rank 2^f and is equipped with an A-basis such that:

(i) the corresponding matrix $\mathrm{Mat}(\varphi)' \in \mathrm{GL}_{2^f}(A)$ for the φ-action is given by

$$\mathrm{Mat}(\varphi)'_{J',J+1} = \begin{cases} \gamma_{J+1,J'} \prod_{j\notin J}(Y_j/\varphi(Y_j))^{r_j+1} & \text{if } J^{\text{ss}} \subseteq J' \subseteq J, \\ 0 & \text{otherwise,} \end{cases} \tag{7.4}$$

where for $J, J' \subseteq \mathcal{J}$ such that $(J-1)^{\text{ss}} = (J')^{\text{ss}}$ we define

$$\gamma_{J,J'} \overset{\text{def}}{=} (-1)^{f-1+\delta_{(J')^{\text{nss}}=\mathcal{J}}+|(J'\cap(J'-1))^{\text{nss}}|}\mu_{J,J'}; \tag{7.5}$$

(ii) the corresponding matrices $\mathrm{Mat}(a)'$ for the $\mathcal{O}_K^\times$-action satisfy $\mathrm{Mat}(a)'_{J,J} \in 1 + F_{1-p}A$ for all $a \in \mathcal{O}_K^\times$ and $J \subseteq \mathcal{J}$, which uniquely determines $\mathrm{Mat}(a)'$.

Note that when $J_{\overline{\rho}} \neq \mathcal{J}$, $J \not\subseteq J_{\overline{\rho}}$ and $J \neq J^*$ (see Lemma 2.2 for J^*) we have (see (5.8) for $\gamma(J)$)

$$\gamma(J) = \frac{\gamma_{*,J}}{\gamma_{*,J^{\text{ss}}}}\left[\frac{\prod_{i=0}^{\ell(J)-1} \gamma_{\delta_{\text{ss}}^i(J),\delta_{\text{ss}}^{i+1}(J)}}{\prod_{i=0}^{\ell(J^{\text{ss}})-1} \gamma_{\delta_{\text{ss}}^i(J^{\text{ss}}),\delta_{\text{ss}}^{i+1}(J^{\text{ss}})}}\right], \tag{7.6}$$

where $\gamma_{*,J}/\gamma_{*,J^{\text{ss}}}$ is defined in a similar way to $\mu_{*,J}/\mu_{*,J^{\text{ss}}}$.

LEMMA 7.1. *Suppose that $J_{\overline{\rho}} \neq \mathcal{J}$. Let $B \in \mathrm{M}_{2^f}(\mathbb{F})$ with its rows and columns indexed by the subsets of $\mathcal{J}$ such that:*

(i) *$B_{J,J'} \in \mathbb{F}^\times$ if and only if $(J-1)^{\text{ss}} = (J')^{\text{ss}}$;*

(ii) *$B_{J_1,J_3}/B_{J_1,J_4} = B_{J_2,J_3}/B_{J_2,J_4}$ for all $J_1, J_2, J_3, J_4 \subseteq \mathcal{J}$ such that $(J_1-1)^{\text{ss}} = (J_2-1)^{\text{ss}} = J_3^{\text{ss}} = J_4^{\text{ss}}$.*

We define $B_{*,J}/B_{*,J^{\text{ss}}}$ in a similar way to $\mu_{*,J}/\mu_{*,J^{\text{ss}}}$. Then up to conjugation by diagonal matrices, B is uniquely determined by the quantities

$$\begin{cases} B_{\emptyset,\emptyset}, \\ B(J^*) \overset{\text{def}}{=} B_{(J^*-1)^{\text{ss}},J^*} B_{J^*,(J^*-1)^{\text{ss}}}, \\ B(J) \overset{\text{def}}{=} \dfrac{B_{*,J}}{B_{*,J^{\text{ss}}}}\left[\dfrac{\prod_{i=0}^{\ell(J)-1} B_{\delta_{\text{ss}}^i(J),\delta_{\text{ss}}^{i+1}(J)}}{\prod_{i=0}^{\ell(J^{\text{ss}})-1} B_{\delta_{\text{ss}}^i(J^{\text{ss}}),\delta_{\text{ss}}^{i+1}(J^{\text{ss}})}}\right] & \text{for } J \not\subseteq J_{\overline{\rho}} \text{ and } J \neq J^*. \end{cases}$$

Proof. First, it is easy to check that conjugation by a diagonal matrix does not change these quantities.

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Next, given such a matrix B, after conjugation we may assume that $B_{J,\delta_{\mathrm{ss}}(J)}=1$ for all $J\neq\emptyset$. Indeed, if we let $Q\in \mathrm{GL}_{2^f}(\mathbb{F})$ be the diagonal matrix with $Q_{J,J}=\prod_{i=0}^{\ell(J)-1} B_{\delta_{\mathrm{ss}}^i(J),\delta_{\mathrm{ss}}^{i+1}(J)}$, then $Q^{-1}BQ$ satisfies this property. In particular, this determines the entries $B_{J,J'}$ with $J'\subseteq J_{\overline{\rho}}$.

Then for J, J' such that $(J-1)^{\mathrm{ss}}=(J')^{\mathrm{ss}}$ and $J'\not\subseteq J_{\overline{\rho}}$, the entry $B_{J,J'}$ is determined by

$$B_{J,J'}=\begin{cases} B(J')B_{J,(J')^{\mathrm{ss}}} & \text{if } J'\neq J^*,\\ B(J^*)B_{J,(J-1)^{\mathrm{ss}}}/B_{(J^*-1)^{\mathrm{ss}},(J-1)^{\mathrm{ss}}} & \text{if } J'=J^*.\end{cases}$$

This completes the proof. □

Suppose that the matrices $(\gamma_{J,J'}),(\nu_{J,J'})\in \mathrm{M}_{2^f}(\mathbb{F})$ are conjugated by the diagonal matrix Q. Then the matrices $(\gamma_{J,J'}\delta_{(J-1)^{\mathrm{ss}}\subseteq J'\subseteq J-1})$ and $(\nu_{J,J'}\delta_{(J-1)^{\mathrm{ss}}\subseteq J'\subseteq J-1})$ are also conjugated by Q.

Proof of the main result. The case $J_{\overline{\rho}}=\mathcal{J}$ is proved by [BHH+25b, Theorem 3.1.3]. The case $J_{\overline{\rho}}=\emptyset$ is proved by [Wan26, Theorem 1.1]. In the rest of the proof we assume that $J_{\overline{\rho}}\notin\{\emptyset,\mathcal{J}\}$.

As in the proof of [Wan26, Theorem 1.1], it suffices to show that $\mathrm{Hom}_A(D_A(\pi),A)(1)\cong D_A^{\otimes}(\overline{\rho})$. By [Wan24, Proposition C.3(iii) and Corollary C.4], it suffices to compare the matrices $\mathrm{Mat}(\varphi)$ (see (7.1)) and $\mathrm{Mat}(\varphi)'$ (see (7.4)). Then by (7.6), Lemma 7.1 and the sentence that follows, it suffices to show that:

(i) $\gamma_{\emptyset,\emptyset}=\nu_{\emptyset,\emptyset}$;

(ii) $\gamma_{(J^*-1)^{\mathrm{ss}},J^*}\gamma_{J^*,(J^*-1)^{\mathrm{ss}}}=\nu_{(J^*-1)^{\mathrm{ss}},J^*}\nu_{J^*,(J^*-1)^{\mathrm{ss}}}$;

(iii) $\gamma(J)=\nu(J)$ for $J\not\subseteq J_{\overline{\rho}}$ and $J\neq J^*$ (see (5.8) for $\gamma(J)$ and (7.3) for $\nu(J)$).

Indeed, by Lemma 5.2 and (7.5) we have $\gamma_{\emptyset,\emptyset}=\xi$ (see (1.3) for ξ), which equals $\nu_{\emptyset,\emptyset}$ by (7.2). By Corollary 3.2 and (7.5) we have $\gamma_{(J^*-1)^{\mathrm{ss}},J^*}\gamma_{J^*,(J^*-1)^{\mathrm{ss}}}=1$, which equals $\nu_{(J^*-1)^{\mathrm{ss}},J^*}\nu_{J^*,(J^*-1)^{\mathrm{ss}}}$ by (7.2). Finally, for $J\subseteq\mathcal{J}$ such that $J\not\subseteq J_{\overline{\rho}}$ and $J\neq J^*$, by (5.13) and Propositions 4.3, 6.7 and 6.12 we have $\gamma(J)=\beta^{B(J)}d(J)$ (see (4.17) for $B(J)$ and $d(J)$), which equals $\nu(J)$ by definition. This completes the proof. □

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Conflicts of interest

None.

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