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**ARTICLE**

# On a new matrix Schwarzian derivative

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**Abstract**

We give a new definition of matrix Schwarzian derivative, which is simpler than the Lagrange Schwarzian derivative and also provides an alternative to other definitions which appear in the literature. Some basic properties are discussed, in particular, analogs of Möbius invariance and the result of a change of independent variable, these being the two properties of the scalar Schwarzian derivative often considered to account for its universality. We then use our new definition of matrix Schwarzian derivative to construct new Schwarzian matrix ordinary and partial differential equation hierarchies: a Schwarzian matrix second Painlevé hierarchy and a Schwarzian matrix Korteweg–de Vries hierarchy, respectively. In addition, we define a new matrix second Painlevé hierarchy.

## 1. Introduction

As is well-known, the Schwarzian derivative $\{f;x\}$ of a complex-valued function $f(x)$ with respect to the complex independent variable $x$ is defined as

$$\{f;x\} = \left(\frac{f_{xx}}{f_x}\right)_x - \frac{1}{2}\left(\frac{f_{xx}}{f_x}\right)^2, \tag{1.1}$$

where we use subscripts to denote derivatives, or equivalently,

$$\{f;x\} = \frac{f_{xxx}}{f_x} - \frac{3}{2}\left(\frac{f_{xx}}{f_x}\right)^2. \tag{1.2}$$

It is important in a wide variety of contexts, both classically, from the study of linear second-order ordinary differential equations (ODEs) to conformal mappings, and, more recently, in Teichmüller theory and integrable systems. For an introduction to the Schwarzian derivative, the reader is referred to [11], as well as to the interesting article [21]. According to these references, the Schwarzian derivative appeared in an 1869 paper by Schwarz [23], but also in an 1834 paper by Kummer (reprinted in 1887 [14]) and even earlier in a paper by Lagrange dating from 1781 [15]. Schwarz's interest [11] was in finding a differential operator invariant under the group of fractional linear transformations, or Möbius transformations. According to [20], this is one of the two properties of the Schwarzian derivative which account for its universality, the second being the way it transforms under a change of independent

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variable, that is, if $f(x) = g(\tau)$, $\tau = \tau(x)$, then the Schwarzian derivative transforms as

$$\{f;x\} = \{g;\tau\}(\tau_x)^2 + \{\tau;x\}. \tag{1.3}$$

The aim of [20] was to define the Schwarzian derivative of a square matrix function $F(x)$. Referred to as the Lagrange Schwarzian derivative, and denoted by $LS(F(x))$, this definition involved the square root of the matrix $F_x$. Various properties of this Lagrange Schwarzian derivative were discussed in [20], in particular, an analog of Möbius invariance and the result of a change of independent variable. The discussion of the relationship between the Lagrange Schwarzian derivative and Newton systems involved the factorization of the Schrödinger operator under the Miura map; this was pursued further in [17], where the connection with matrix versions of the Korteweg–de Vries (KdV), modified KdV (mKdV), and Schwarzian KdV hierarchies was made. An alternative definition of matrix Schwarzian derivative was given in [4] and also used to define a Schwarzian matrix KdV hierarchy; moreover, an extended chain of integrable matrix hierarchies, starting at (potential) KdV, was given.

The present article arises from the recent derivation in [22] of a matrix extension of the Schwarzian second Painlevé equation, which naturally led to the identification of a matrix analog of the Schwarzian derivative. It is this new matrix Schwarzian derivative that we present in the current article, the layout of which is as follows. In Section 2, we define this new matrix Schwarzian derivative and discuss some of its basic properties, including new matrix analogs of the two "universality" properties, that is, Möbius invariance and the result of a change of independent variable. In Section 3, we use this new matrix Schwarzian derivative to define new Schwarzian matrix ODE and partial differential equation (PDE) hierarchies: a new Schwarzian matrix second Painlevé hierarchy and a new Schwarzian matrix KdV hierarchy, respectively. In addition, we define a new matrix second Painlevé hierarchy. This last hierarchy has a greatly enhanced parameter dependence when compared to the matrix second Painlevé hierarchy given in [9]. In Section 3, we also present as examples the first members of these three hierarchies. In the case of the first hierarchy, the resulting Schwarzian matrix second Painlevé equation is, in fact, more general than that given in [22]. Similarly, our Schwarzian matrix KdV equation, which occurs as the first member of the second hierarchy, also generalizes previously-known results, as does our new enhanced matrix second Painlevé equation obtained from the third hierarchy. In Section 4, we give a brief summary of our results and discuss their implications for future work.

## 2. A matrix Schwarzian derivative: Definition and properties

Let us begin with our new definition of matrix Schwarzian derivative.

**Definition 2.1** (Matrix Schwarzian derivative). Given a square matrix function $\rho$ of $x$ such that $\rho_x$ is nonsingular, then, for any two nonsingular square matrix functions $z$ and $y$ of $x$ such that $zy = \rho_x$ and $z_x y = z y_x$, we define its Schwarzian derivative $S(\rho)$ as

$$S(\rho) = \left(z^{-1}\rho_{xx}y^{-1}\right)_x - \frac{1}{2}\left(z^{-1}\rho_{xx}y^{-1}\right)^2, \tag{2.1}$$

or equivalently,

$$S(\rho) = z^{-1}\left(\rho_{xxx} - \frac{3}{2}\rho_{xx}\rho_x^{-1}\rho_{xx}\right)y^{-1}. \tag{2.2}$$

**Remark 2.2.** We may also write

$$S(\rho) = \left(z^{-1}\rho_{xx}\rho_x^{-1}z\right)_x - \frac{1}{2}\left(z^{-1}\rho_{xx}\rho_x^{-1}z\right)^2, \qquad \text{where} \qquad z_x = \frac{1}{2}\rho_{xx}\rho_x^{-1}z, \tag{2.3}$$

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or

$$S(\rho) = \left(y\rho_x^{-1}\rho_{xx}y^{-1}\right)_x - \frac{1}{2}\left(y\rho_x^{-1}\rho_{xx}y^{-1}\right)^2, \qquad \text{where} \qquad y_x = \frac{1}{2}y\rho_x^{-1}\rho_{xx}, \tag{2.4}$$

formulations obtained by, respectively, substituting for $y = z^{-1}\rho_x$ and $z = \rho_x y^{-1}$ in (2.1) and $z_x y = z y_x$.

**Remark 2.3.** The formulation (2.1) is simpler than the Lagrange Schwarzian derivative introduced in [20] with fewer assumptions being made on $\rho$. In [20], $\rho$ is assumed to be symmetric and such that $\rho_x$ is positive definite, although this last condition can be relaxed [17]. For our definition of matrix Schwarzian derivative, it is sufficient to assume that $\rho$ is such that the linear equation $z_x = \frac{1}{2}\rho_{xx}\rho_x^{-1}z$ admits a fundamental matrix solution $z$ in some domain (see, e.g., [11]). We may then take $y = z^{-1}\rho_x$. We assume $\rho$ to be analytic with $\rho_x$ nonsingular in some domain $D$, so that $\frac{1}{2}\rho_{xx}\rho_x^{-1}$ is also analytic in $D$. A solution $z$, locally analytic and nonsingular in $D$, can then be constructed. This solution may be multi-valued but, if necessary, we may restrict our attention to some subdomain of $D$ in order to obtain a single-valued solution. It is worth noting that, even if $z$ is multi-valued, the quantity $z^{-1}z_x$ might be single-valued, and so also the matrix Schwarzian derivative, since this can be rewritten in the form $S(\rho) = 2(z^{-1}z_x)_x - 2(z^{-1}z_x)^2$ (and also, indeed, $S(\rho) = 2(y_x y^{-1})_x - 2(y_x y^{-1})^2$). We note that Definition 2.1 also provides an alternative definition of matrix Schwarzian derivative to that presented in [4].

We now consider some properties of the above-defined matrix Schwarzian derivative.

**Proposition 2.4.** *The matrix Schwarzian derivative given in Definition 2.1 is defined up to conjugation by a constant matrix.*

*Proof.* Given a square matrix function $\rho = \rho(x)$ such that $\rho_x$ is nonsingular, we consider defining its Schwarzian derivative as $S(\rho) = z^{-1}R(\rho)y^{-1}$, and alternatively as $T(\rho) = s^{-1}R(\rho)t^{-1}$, where

$$R(\rho) = \rho_{xxx} - \frac{3}{2}\rho_{xx}\rho_x^{-1}\rho_{xx}, \tag{2.5}$$

and where $z$, $y$, $s$, and $t$ are nonsingular square matrix functions of $x$ such that $\rho_x = zy = st$, $z_x y = z y_x$, and $s_x t = s t_x$. Since $\rho_{xx} = (zy)_x = 2z_x y$ and $\rho_{xx} = (st)_x = 2s_x t$, we must have $z_x y = s_x t$, and so $z_x(z^{-1}st) = s_x t$, which then gives $z_x z^{-1} = s_x s^{-1}$. Thus, $(s^{-1}z)_x = -s^{-1}s_x s^{-1}z + s^{-1}z_x = -s^{-1}z_x z^{-1}z + s^{-1}z_x = 0$ and so $s^{-1}z = P$, some nonsingular constant matrix $P$. Then $T(\rho) = s^{-1}R(\rho)t^{-1} = s^{-1}zS(\rho)yt^{-1} = s^{-1}zS(\rho)(s^{-1}z)^{-1} = PS(\rho)P^{-1}$, that is, any alternatively-defined Schwarzian derivative $T(\rho)$ must be equal to $S(\rho)$ conjugated by some constant matrix $P$. □

**Remark 2.5.** The above result corresponds to the fact that, given any solution $s$ of $s_x = \frac{1}{2}\rho_{xx}\rho_x^{-1}s$, then $z = sP$ gives a solution of $z_x = \frac{1}{2}\rho_{xx}\rho_x^{-1}z$. Since $s$ and $z$ are assumed nonsingular, then so is $P$. Then $zy = st$ gives $t = Py$, and so $T(\rho) = s^{-1}R(\rho)t^{-1} = PS(\rho)P^{-1}$.

**Remark 2.6.** Proposition 2.4 is the analog of the result in [20] that the (more restrictive) Lagrange Schwarzian derivative is defined up to conjugation by orthogonal matrices.

**Lemma 2.7.** *Let $\rho = -r^{-1}$. Then $S(\rho) = S(r)$.*

*Proof.* Let $S(\rho)$ be as in Definition 2.1, that is, $S(\rho) = z^{-1}R(\rho)y^{-1}$, where $R(\rho)$ is as given by (2.5). Substituting $\rho = -r^{-1}$ in (2.5), we find $R(\rho) = r^{-1}R(r)r^{-1}$ and so $S(\rho) = (rz)^{-1}R(r)(yr)^{-1}$. We now define $s = rz$ and $t = yr$, so that we have $S(\rho) = s^{-1}R(r)t^{-1}$. It remains to show that $s$ and $t$ satisfy $r_x = st$ and $s_x t = s t_x$. First of all, we find $0 = \rho_x - zy = r^{-1}r_x r^{-1} - (r^{-1}s)(tr^{-1}) = r^{-1}(r_x - st)r^{-1}$, and so $r_x = st$. Secondly, we have $0 = z_x y - z y_x = (r^{-1}s)_x tr^{-1} - r^{-1}s(tr^{-1})_x = r^{-1}(s_x t - s t_x)r^{-1} - r^{-1}(r_x r^{-1}st - str^{-1}r_x)r^{-1} = r^{-1}(s_x t - s t_x)r^{-1}$, where we have used $r_x = st$, and so $s_x t = s t_x$. □

**Lemma 2.8.** *Let $\rho = ArB$, where $A$ and $B$ are nonsingular constant matrices. Then $S(\rho) = S(r)$.*

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*Proof.* Again we take $S(\rho) = z^{-1}R(\rho)y^{-1}$, where $R(\rho)$ is given by (2.5). Substituting $\rho = ArB$ in (2.5), we find $R(\rho) = AR(r)B$ and so $S(\rho) = (A^{-1}z)^{-1}R(r)(yB^{-1})^{-1}$. Defining $s = A^{-1}z$ and $t = yB^{-1}$, we obtain $S(\rho) = s^{-1}R(r)t^{-1}$. We now need to show that $s$ and $t$ satisfy $r_x = st$ and $s_x t = st_x$. First of all, we obtain $0 = \rho_x - zy = A(r_x - st)B$, and so $r_x = st$. Secondly, $0 = z_x y - zy_x = A(s_x t - st_x)B$, and thus $s_x t = st_x$. □

**Lemma 2.9.** *Let $\rho = r + C$, where $C$ is a constant matrix. Then $S(\rho) = S(r)$.*

*Proof.* With $S(\rho) = z^{-1}R(\rho)y^{-1}$ and $R(\rho)$ given by (2.5), we see that $R(\rho) = R(r)$ and so $S(\rho) = z^{-1}R(r)y^{-1}$, where, since $\rho = r + C$, $r_x = zy$ (and $z_x y = zy_x$). [That is, we take $S(r) = s^{-1}R(r)t^{-1}$ with $s = z$ and $y = t$.] □

**Proposition 2.10.** *The matrix Schwarzian derivative given by Definition 2.1 is invariant under the Möbius-style transformations*

$$M_1: \quad \rho = (Ar + B)(Cr + D)^{-1}, \tag{2.6}$$

$$M_2: \quad \rho = -(rC - A)^{-1}(rD - B), \tag{2.7}$$

*where $A$, $B$, $C$, and $D$ are constant matrices such that $C$ and $(AC^{-1}D - B)$ are nonsingular.*

*Proof.* The transformation $M_1$ can be written

$$\rho = AC^{-1} + (AC^{-1}D - B)\Big( - (r + C^{-1}D)^{-1}\Big)C^{-1}, \tag{2.8}$$

that is, in terms of the transformations discussed in Lemmas 2.7–2.9, all of which leave this matrix Schwarzian derivative invariant. It then follows that $M_1$ also leaves this matrix Schwarzian derivative invariant. Invariance under $M_2$ can be shown by observing that it is just the inverse transformation of $M_1$, or by writing $M_2$ as

$$\rho = -C^{-1}D + C^{-1}\Big( - (r - AC^{-1})^{-1}\Big)(AC^{-1}D - B), \tag{2.9}$$

that is, once again in terms of the transformations discussed in Lemmas 2.7–2.9. □

**Remark 2.11.** From (2.8) and (2.9), we see that the condition that the matrix $(AC^{-1}D - B)$ be nonsingular means that the transformations $M_1$ and $M_2$ are non-constant, since $(AC^{-1}D - B)$ cannot be zero. (The matrix $(B - AC^{-1}D)$ has an interpretation in the theory of quasideterminants, see, e.g., the discussion in [6].)

**Remark 2.12.** Proposition 2.10, while very similar to Theorem 4.2 in [4], is in fact different from it. In Proposition 2.10, we show the invariance of the matrix Schwarzian derivative itself; in [4], the authors' interest is in the invariance of their non-Abelian KdV singularity manifold equation, and their Schwarzian derivative is not itself invariant. (The same remark holds with respect to Lemmas 2.7–2.9 when compared to Propositions 4.4 and 4.5 in [4].) These results are consistent since Definition 2.1 does not include the definition used in [4], as the condition $z_x y = zy_x$ excludes the choice $z = \rho_x$ and $y = I$ (except when $\rho_{xx} = 0$ and so $S(\rho) = 0$). For the Lagrange Schwarzian derivative, the result corresponding to Proposition 2.10 is Proposition 1, part i), in [20].

**Proposition 2.13.** *Under the change of independent variable $\rho(x) = r(\tau)$, $\tau = \tau(x)$, the matrix Schwarzian derivative transforms as*

$$S(\rho) = \widetilde{S}(r)(\tau_x)^2 + \{\tau; x\}I, \tag{2.10}$$

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*where $\widetilde{S}(r)$ is the matrix Schwarzian derivative of $r$ with respect to $\tau$, that is,*

$$\widetilde{S}(r) = s^{-1}\widetilde{R}(r)t^{-1} = s^{-1}\left(r_{\tau\tau\tau} - \frac{3}{2}\rho_{\tau\tau}\rho_{\tau}^{-1}\rho_{\tau\tau}\right)t^{-1}, \tag{2.11}$$

*for nonsingular square matrices $s(\tau)$ and $t(\tau)$ such that $r_\tau = st$ and $s_\tau t = st_\tau$, and where*

$$\{\tau; x\} = \frac{\tau_{xxx}}{\tau_x} - \frac{3}{2}\left(\frac{\tau_{xx}}{\tau_x}\right)^2 \tag{2.12}$$

*is the scalar Schwarzian derivative of $\tau$ with respect to $x$.*

*Proof.* Under the change of independent variable $\rho(x) = r(\tau)$, $\tau = \tau(x)$, we find that $R(\rho)$ defined by (2.5) transforms as

$$R(\rho) = \widetilde{R}(r)(\tau_x)^3 + r_\tau\tau_x\{\tau; x\}. \tag{2.13}$$

Thus, defining $s = (\tau_x)^{-\frac{1}{2}}z$ and $t = (\tau_x)^{-\frac{1}{2}}y$, we see that, under this change of independent variable, we may write the matrix Schwarzian derivative $S(\rho)$ as

$$S(\rho) = z^{-1}R(\rho)y^{-1} = s^{-1}\widetilde{R}(r)t^{-1}(\tau_x)^2 + s^{-1}r_\tau t^{-1}\{\tau; x\}. \tag{2.14}$$

We now show that $s$ and $t$ satisfy $r_\tau = st$ and $s_\tau t = st_\tau$. We find $0 = \rho_x - zy = (r_\tau - st)\tau_x$, and so $r_\tau = st$, and $0 = z_x y - zy_x = \left(\frac{1}{2}(\tau_x)^{-\frac{1}{2}}\tau_{xx}s + (\tau_x)^{\frac{1}{2}}s_\tau\tau_x\right)(\tau_x)^{\frac{1}{2}}t - (\tau_x)^{\frac{1}{2}}s\left(\frac{1}{2}(\tau_x)^{-\frac{1}{2}}\tau_{xx}t + (\tau_x)^{\frac{1}{2}}t_\tau\tau_x\right) = (s_\tau t - st_\tau)(\tau_x)^2$, so $s_\tau t = st_\tau$. We may then identify $s^{-1}\widetilde{R}(r)t^{-1}$ in (2.14) as the matrix Schwarzian derivative $\widetilde{S}(r)$ of $r$ with respect to $\tau$ and so, since the coefficient of $\{\tau; x\}$ in (2.14) is then $s^{-1}r_\tau t^{-1} = s^{-1}(st)t^{-1} = I$, we obtain (2.10). □

**Remark 2.14.** Proposition 2.13 corresponds to Proposition 1, part ii), in [20] for the Lagrange Schwarzian derivative.

**Proposition 2.15.** *Let $\rho = r^T$. Then $S(\rho) = (S(r))^T$.*

*Proof.* We again take $S(\rho) = z^{-1}R(\rho)y^{-1}$, where $R(\rho)$ is given by (2.5). Substituting $\rho = r^T$ in (2.5) gives $R(\rho) = (r^T)_{xxx} - \frac{3}{2}(r^T)_{xx}((r^T)_x)^{-1}(r^T)_{xx} = (r_{xxx})^T - \frac{3}{2}(r_{xx})^T(r_x^{-1})^T(r_{xx})^T = (R(r))^T$ and so $S(\rho) = z^{-1}(R(r))^T y^{-1}$. Defining $s = y^T$ and $t = z^T$, we obtain $S(\rho) = (s^{-1}R(r)t^{-1})^T$. We now need to show that $s$ and $t$ satisfy $r_x = st$ and $s_x t = st_x$. First of all, we see $0 = (\rho_x - zy)^T = r_x - st$, and secondly, $0 = (z_x y - zy_x)^T = st_x - s_x t$. □

**Remark 2.16.** Proposition 2.15 has no counterpart in [20], as the Lagrange Schwarzian derivative is defined for symmetric matrices, and so is in fact invariant under transposition of $\rho$. As for the matrix Schwarzian derivative defined in [4], similarly to Proposition 2.10 (see Remark 2.12), this is not invariant, nor gives the transpose, under $\rho = r^T$, whereas the non-Abelian KdV singularity manifold equation, expressed in terms of this matrix Schwarzian derivative, is.

We now turn to the use of our matrix Schwarzian derivative in defining hierarchies of differential equations.

## 3. Schwarzian matrix differential equation hierarchies

In this section, we will define two new Schwarzian matrix differential equation hierarchies: a Schwarzian matrix second Painlevé hierarchy and a Schwarzian matrix KdV hierarchy. We will also define a new matrix second Painlevé hierarchy. For a comprehensive overview of Painlevé hierarchies, we refer to the recent review [8]. We begin here by recalling, in Definition 3.1, some basic facts about matrix KdV and mKdV hierarchies [1], [3], [5], [10], [18], [19], [24].

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**Definition 3.1** (Matrix KdV and mKdV hierarchies). The two Hamiltonian operators $\mathcal{B}_0[w]$ and $\mathcal{B}_1[w]$ of the matrix KdV hierarchy are given by

$$\mathcal{B}_0[w] = \partial_x, \qquad \mathcal{B}_1[w] = \partial_x^3 + A_w\partial_x + \partial_x A_w + C_w\partial_x^{-1}C_w, \tag{3.1}$$

wherein

$$A_w = L_w + R_w, \qquad C_w = L_w - R_w, \tag{3.2}$$

and the left and right multiplication operators $L_w$ and $R_w$ are given by

$$L_w(z) = wz, \qquad R_w(z) = zw. \tag{3.3}$$

The variational derivatives $M_k = \delta\mathcal{H}_k$ of the Hamiltonian densities $\mathcal{H}_k$ of the matrix KdV hierarchy are defined recursively via

$$M_0 = \frac{1}{2}I, \qquad \mathcal{B}_0[w]M_{j+1} = \mathcal{B}_1[w]M_j, \quad j = 0, 1, 2, \ldots \tag{3.4}$$

(we do not need here explicit expressions for the Hamiltonian densities themselves). Thus, for example,

$$M_1 = w, \qquad M_2 = w_{xx} + 3w^2, \qquad M_3 = w_{xxxx} + 5ww_{xx} + 5w_{xx}w + 5w_x^2 + 10w^3. \tag{3.5}$$

The matrix KdV hierarchy is related to the matrix mKdV hierarchy via the Miura map

$$w = M[u] = u_x - u^2. \tag{3.6}$$

Under this Miura map, the Hamiltonian operator $\mathcal{B}_1[w]$ factorizes as

$$\mathcal{B}_1[w]\Big|_{w=M[u]} = M'[u]\mathcal{B}[u](M'[u])^\dagger, \tag{3.7}$$

where

$$M'[u] = \partial_x - A_u \tag{3.8}$$

is the Fréchet derivative of $M[u]$,

$$(M'[u])^\dagger = -\partial_x - A_u \tag{3.9}$$

is the adjoint of this Fréchet derivative, and

$$\mathcal{B}[u] = -(\partial_x - C_u\partial_x^{-1}C_u) \tag{3.10}$$

is the Hamiltonian operator of the matrix mKdV hierarchy. Defining

$$\widetilde{K}[w] = M_n + \sum_{k=0}^{n-1} c_k M_k, \tag{3.11}$$

the matrix mKdV hierarchy may be written

$$u_t = \mathcal{B}[u](M'[u])^\dagger\widetilde{K}[M[u]], \tag{3.12}$$

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and the matrix KdV hierarchy as

$$w_t = \mathcal{B}_1[w]\widetilde{K}[w]. \tag{3.13}$$

We now define our new Schwarzian matrix hierarchies.

**Definition 3.2** (Schwarzian matrix second Painlevé hierarchy). Set

$$K[w] = \widetilde{K}[w] + E + g_{n-1}xI = M_n + \sum_{k=0}^{n-1} c_k M_k + E + g_{n-1}xI, \tag{3.14}$$

and consider the hierarchy

$$\mathcal{S}[\rho, z, y, E, C_1, C_2, C_3] \equiv K\left[\frac{1}{2}S(\rho)\right] - z^{-1}C_1\rho y^{-1} - z^{-1}\rho C_2 y^{-1} - z^{-1}\big(g_{n-1}\rho + C_3\big)y^{-1} = 0, \tag{3.15}$$

where $C_1$, $C_2$, and $C_3$ are arbitrary constant matrices, and $S(\rho)$, $z$, and $y$ are as in Definition 2.1. We define this hierarchy, in the case where $g_{n-1} \neq 0$, to be a Schwarzian matrix second Painlevé hierarchy.

**Definition 3.3** (Matrix second Painlevé hierarchy). Consider the hierarchy

$$Q[u, z, y, E, C_1, C_2] \equiv (\partial_x + A_u)K[M[u]] - z^{-1}C_1 z - yC_2 y^{-1} - g_{n-1}I = 0, \tag{3.16}$$

where $z_x = zu$ and $y_x = uy$. We define this hierarchy, in the case where $g_{n-1} \neq 0$, to be a matrix second Painlevé hierarchy.

**Remark 3.4.** The last term in (3.16) cancels with $g_{n-1}I$ arising from $(\partial_x + A_u)g_{n-1}xI = 2g_{n-1}xu + g_{n-1}I$. Thus, (3.16) may also be written (here each $M_k$ is evaluated at $w = M[u]$) as

$$(\partial_x + A_u)\left(M_n + \sum_{k=0}^{n-1} c_k M_k\right) + uE + Eu + 2g_{n-1}xu - z^{-1}C_1 z - yC_2 y^{-1} = 0. \tag{3.17}$$

It is the term $2g_{n-1}xu$ that motivates the requirement $g_{n-1} \neq 0$ when defining (3.16) to be a matrix second Painlevé hierarchy: this nonautonomous term is fundamental to scalar and matrix higher-order analogs of the second Painlevé equation of the form considered here, and, of course, to the second Painlevé equation itself. If $g_{n-1} \neq 0$ (all previously-known examples of) such equations have a Lax pair of the form $\psi_x = F\psi$, $\psi_\lambda = G\psi$.

**Remark 3.5.** We justify Definition 3.3, that is, of (3.16) as a matrix second Painlevé hierarchy (when $g_{n-1} \neq 0$), in Remark 3.12 below. As we now see in Proposition 3.6, the relationship between our matrix second Painlevé hierarchy and its Schwarzian counterpart (Definition 3.2) mirrors that of the scalar case (see [13], and also [25] for the case of the second Painlevé equation itself), that is, solutions of the latter give rise to solutions of the former. The above Schwarzian matrix second Painlevé hierarchy is completely new. The above matrix second Painlevé hierarchy is also new, due to its greatly generalized parameter dependence—via the arbitrary matrices $C_1$ and $C_2$—when compared to the matrix second Painlevé hierarchy in [9], again as explained in Remark 3.12.

**Proposition 3.6.** *Solutions of the matrix hierarchy (3.15) give rise to solutions*

$$u = \frac{1}{2}z^{-1}\rho_{xx}y^{-1} \tag{3.18}$$

*of the matrix hierarchy (3.16).*

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*Proof.* We show that with the identification (3.18), equation (3.16) is obtained by acting on (3.15) with the operator $(\partial_x + A_u)$. From Definition 2.1, we see that

$$M[u] = u_x - u^2 = \frac{1}{2}S(\rho), \qquad \text{so} \qquad K[M[u]] = K\left[\frac{1}{2}S(\rho)\right], \tag{3.19}$$

and so the first term in (3.15) gives rise to the first term in (3.16). We are therefore left with showing that the action of $(\partial_x + A_u)$ on the remaining terms in (3.15) yields the remaining terms in (3.16). We begin by observing that, since $\rho_x = zy$ and $z_x y = z y_x$, we have

$$u = \frac{1}{2}z^{-1}\rho_{xx}y^{-1} = z^{-1}z_x = y_x y^{-1}. \tag{3.20}$$

It then follows that

$$\begin{aligned}(\partial_x + A_u)z^{-1}C_1\rho y^{-1} &= z^{-1}C_1\rho_x y^{-1} - (z^{-1}z_x - u)z^{-1}C_1\rho y^{-1} - z^{-1}C_1\rho y^{-1}(y_x y^{-1} - u) = z^{-1}C_1(zy)y^{-1}\\ &= z^{-1}C_1 z,\end{aligned} \tag{3.21}$$

$$\begin{aligned}(\partial_x + A_u)z^{-1}\rho C_2 y^{-1} &= z^{-1}\rho_x C_2 y^{-1} - (z^{-1}z_x - u)z^{-1}\rho C_2 y^{-1} - z^{-1}\rho C_2 y^{-1}(y_x y^{-1} - u) = z^{-1}(zy)C_2 y^{-1}\\ &= yC_2 y^{-1},\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}(\partial_x + A_u)z^{-1}\big(g_{n-1}\rho + C_3\big)y^{-1} &= g_{n-1}z^{-1}\rho_x y^{-1} - (z^{-1}z_x - u)z^{-1}\big(g_{n-1}\rho + C_3\big)y^{-1}\\ &\quad - z^{-1}\big(g_{n-1}\rho + C_3\big)y^{-1}(y_x y^{-1} - u)\\ &= g_{n-1}I.\end{aligned} \tag{3.23}$$

Thus, acting with $(\partial_x + A_u)$ on (3.15) gives (3.16). □

**Definition 3.7** (Schwarzian matrix KdV hierarchy). We define a Schwarzian matrix KdV hierarchy as

$$\mathcal{K}[\rho, z, y, E, C_1, C_2, C_3] \equiv -\frac{1}{2}z^{-1}\rho_t y^{-1} + K\left[\frac{1}{2}S(\rho)\right] - z^{-1}C_1\rho y^{-1} - z^{-1}\rho C_2 y^{-1} - z^{-1}\big(g_{n-1}\rho + C_3\big)y^{-1} = 0, \tag{3.24}$$

where $C_1$, $C_2$, $C_3$, and $g_{n-1}$, and also $E$ which, along with $g_{n-1}$, appears in $K[w]$ (3.14), are now functions of $t$.

**Remark 3.8.** The stationary (time-independent) reduction of the hierarchy (3.24) gives the hierarchy (3.15).

**Lemma 3.9.** *The following holds:* $\mathcal{B}[u](M'[u])^\dagger\left(\frac{1}{2}z^{-1}\rho_t y^{-1}\right) = (\partial_x - C_u\partial_x^{-1}C_u)(\partial_x + A_u)\left(\frac{1}{2}z^{-1}\rho_t y^{-1}\right) = u_t$.

*Proof.* First of all, we note that

$$\begin{aligned}(\partial_x + A_u)z^{-1}\rho_t y^{-1} &= z^{-1}\rho_{xt}y^{-1} - (z^{-1}z_x - u)z^{-1}\rho_t y^{-1} - z^{-1}\rho_t y^{-1}(y_x y^{-1} - u) = z^{-1}\rho_{xt}y^{-1}\\ &= z^{-1}(z_t y + z y_t)y^{-1} = z^{-1}z_t + y_t y^{-1}.\end{aligned} \tag{3.25}$$

Also,

$$\partial_x^{-1}C_u(z^{-1}z_t) = \partial_x^{-1}(z^{-1}z_x z^{-1}z_t - z^{-1}z_t z^{-1}z_x) = -z^{-1}z_t + \partial_x^{-1}(z^{-1}z_{xt} - z^{-1}z_t z^{-1}z_x) = -z^{-1}z_t + \partial_x^{-1}u_t, \tag{3.26}$$

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and

$$\partial_x^{-1}C_u(y_t y^{-1}) = \partial_x^{-1}(y_x y^{-1}y_t y^{-1} - y_t y^{-1}y_x y^{-1}) = \partial_x^{-1}(y_x y^{-1}y_t y^{-1} - y_{xt}y^{-1}) + y_t y^{-1} = -\partial_x^{-1}u_t + y_t y^{-1}, \tag{3.27}$$

so

$$\partial_x^{-1}C_u(z^{-1}z_t + y_t y^{-1}) = -z^{-1}z_t + y_t y^{-1} \tag{3.28}$$

(note the change in sign of the term $z^{-1}z_t$). Thus,

$$\begin{aligned}(\partial_x - C_u\partial_x^{-1}C_u)(\partial_x + A_u)z^{-1}\rho_t y^{-1} &= (\partial_x - C_u\partial_x^{-1}C_u)(z^{-1}z_t + y_t y^{-1})\\ &= \partial_x(z^{-1}z_t + y_t y^{-1}) - C_u(-z^{-1}z_t + y_t y^{-1})\\ &= -z^{-1}z_x z^{-1}z_t + z^{-1}z_{xt} + y_{xt}y^{-1} - y_t y^{-1}y_x y^{-1}\\ &\quad + z^{-1}z_x z^{-1}z_t - z^{-1}z_t z^{-1}z_x - y_x y^{-1}y_t y^{-1} + y_t y^{-1}y_x y^{-1}\\ &= z^{-1}z_{xt} - z^{-1}z_t z^{-1}z_x + y_{xt}y^{-1} - y_x y^{-1}y_t y^{-1}\\ &= (z^{-1}z_x + y_x y^{-1})_t\\ &= 2u_t.\end{aligned} \tag{3.29}$$

□

**Proposition 3.10.** *Solutions of the matrix hierarchy (3.24) give rise to solutions*

$$u = \frac{1}{2}z^{-1}\rho_{xx}y^{-1} \tag{3.30}$$

*of the matrix hierarchy*

$$\mathcal{M}[u,E] \equiv -u_t + \mathcal{B}[u](M'[u])^{\dagger}K[M[u]] = 0, \tag{3.31}$$

*wherein in $K[w]$, given by (3.14), both $E$ and $g_{n-1}$ are now functions of $t$ (as in equation (3.24)).*

*Proof.* We show that with the identification (3.30), equation (3.31) is obtained by acting on (3.24) with $\mathcal{B}[u](M'[u])^{\dagger} = (\partial_x - C_u\partial_x^{-1}C_u)(\partial_x + A_u)$. Taking into account Lemma 3.9 and the proof of Proposition 3.6, we see that it remains to be shown that $(\partial_x - C_u\partial_x^{-1}C_u)(z^{-1}C_1 z + yC_2y^{-1} + g_{n-1}I) = 0$. This follows from the fact that we may write $(\partial_x - C_u\partial_x^{-1}C_u) = (\partial_x - C_u)\partial_x^{-1}(\partial_x + C_u) = (\partial_x + C_u)\partial_x^{-1}(\partial_x - C_u)$, and that

$$(\partial_x + C_u)z^{-1}C_1 z = -(z^{-1}z_x - u)z^{-1}C_1 z + z^{-1}C_1(z_x - zu) = 0, \tag{3.32}$$

$$(\partial_x - C_u)yC_2y^{-1} = (y_x - uy)C_2y^{-1} - yC_2y^{-1}(y_x y^{-1} - u) = 0, \tag{3.33}$$

and

$$(\partial_x + C_u)I = (\partial_x - C_u)I = 0. \tag{3.34}$$

□

**Remark 3.11.** The hierarchy (3.31) is a generalization of the matrix mKdV hierarchy (3.12) (see (3.14)), being nonlocal (when $E \neq 0$) and nonisospectral (when $g_{n-1} \neq 0$). Acting with $M'[u] = (\partial_x - A_u)$ on (3.31) yields the result used in [9] that, just as in Definition 3.1 for the matrix mKdV and KdV hierarchies, the Miura map $w = M[u] = u_x - u^2$ maps from solutions of (3.31) to solutions of the nonlocal nonisospectral matrix KdV hierarchy $w_t = \mathcal{B}_1[w]K[w]$ (for the local non-isospectral matrix KdV hierarchy obtained by setting $E = 0$, we refer to [12]). The special case $E = 0$, $g_{n-1} = 0$, and

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$C_1 = C_2 = C_3 = 0$ of Proposition 3.10 corresponds to a result given in [17], but where the Lagrange Schwarzian derivative is used; a similar result can be found in [4].

**Remark 3.12.** It was suggested in [16] (see also [7]) that the stationary flows of nonisospectral integrable PDEs could be used to derive Painlevé equations. It was thus that in [9] we considered the stationary (time-independent) reduction of (3.31), that is, $\mathcal{B}[u](M'[u])^{\dagger}K[M[u]] = 0$, or

$$(\partial_x - C_u\partial_x^{-1}C_u)(\partial_x + A_u)K[M[u]] = 0, \tag{3.35}$$

wherein in $K[w]$, given by (3.14), $E$ and $g_{n-1}$ are now assumed constant. By observing that $(g_{n-1} + \alpha_n)I \in \mathrm{Ker}[\mathcal{B}[u]]$, we were led to the hierarchy

$$(\partial_x + A_u)K[M[u]] - (g_{n-1} + \alpha_n)I = 0, \tag{3.36}$$

which we defined, when $g_{n-1} \neq 0$, to be a matrix second Painlevé hierarchy. This result, as explained in [9], corresponds to identifying part (but not all) of $\mathrm{Ker}[\mathcal{B}[u]]$. In Proposition 3.13 below, we show that $\mathrm{Ker}[\mathcal{B}[u]]$ may be written $z^{-1}K_1z + yK_2y^{-1}$, where $K_1$ and $K_2$ are arbitrary square matrices. Equation (3.35) thus yields

$$(\partial_x + A_u)K[M[u]] - z^{-1}K_1z - yK_2y^{-1} = 0, \tag{3.37}$$

with equation (3.16) corresponding to setting, for example, $K_1 = C_1 + g_{n-1}I$ and $K_2 = C_2$; equation (3.36) may then be obtained by taking $C_1 = \alpha I$ and $C_2 = \beta I$, $\alpha$ and $\beta$ scalar constants, and then setting $\alpha + \beta = \alpha_n$. Proposition 3.13 is stronger than the reverse process, that is, the steps used above in the proof of Proposition 3.10, and allows us to claim that equation (3.16), for $g_{n-1} \neq 0$, defines a new, more general matrix second Painlevé hierarchy (when compared with (3.36)). We remark that the Schwarzian equation corresponding to (3.36) is given by $zK[\frac{1}{2}S(\rho)]y - (\alpha_n + g_{n-1})\rho - C_3 = 0$ (if $\alpha_n + g_{n-1} \neq 0$, we may set $C_3 = 0$ using a shift on $\rho$; see Lemma 2.9).

**Proposition 3.13.** $\mathrm{Ker}[\mathcal{B}[u]]$ *may be written* $z^{-1}K_1z + yK_2y^{-1}$, *where* $z$ *and* $y$ *are square matrix functions such that* $z_x = zu$ *and* $y_x = uy$, *and* $K_1$ *and* $K_2$ *are arbitrary square matrices.*

*Proof.* We observe that $-\mathcal{B}[u] = (\partial_x - C_u\partial_x^{-1}C_u) = (\partial_x + C_u)\partial_x^{-1}(\partial_x - C_u)$, and so consider the equation

$$(\partial_x + C_u)\partial_x^{-1}(\partial_x - C_u)L[u] = 0, \tag{3.38}$$

which we then write as

$$(\partial_x + C_u)\widetilde{L}[u] = 0, \quad \text{where} \quad \widetilde{L}[u] = \partial_x^{-1}(\partial_x - C_u)L[u]. \tag{3.39}$$

Since $z_x = zu$, we have

$$\left[z\widetilde{L}z^{-1}\right]_x = z\left[\widetilde{L}_x + u\widetilde{L} - \widetilde{L}u\right]z^{-1} = z\left[(\partial_x + C_u)\widetilde{L}\right]z^{-1}, \quad \text{and so} \quad \widetilde{L} = z^{-1}K_1z \tag{3.40}$$

for some constant matrix $K_1$. The second equation in (3.39) then gives

$$(\partial_x - C_u)L = \widetilde{L}_x = -z^{-1}z_xz^{-1}K_1z + z^{-1}K_1z_x = -uz^{-1}K_1z + z^{-1}K_1zu. \tag{3.41}$$

Since $y_x = uy$, we thus obtain

$$\left[y^{-1}Ly\right]_x = y^{-1}\left[L_x - uL + Lu\right]y = y^{-1}\left[(\partial_x - C_u)L\right]y = y^{-1}\left[-uz^{-1}K_1z + z^{-1}K_1zu\right]y = \frac{1}{2}\left[(zy)^{-1}K_1zy\right]_x, \tag{3.42}$$

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where in the last step, we have used the fact that $(zy)_x = z_x y + z y_x = zuy + zuy = 2zuy$ and similarly $((zy)^{-1})_x = -2y^{-1}uz^{-1}$. It then follows that

$$L = y\left(\frac{1}{2}y^{-1}z^{-1}K_1 zy + K_2\right)y^{-1} = \frac{1}{2}z^{-1}K_1 z + yK_2 y^{-1} \tag{3.43}$$

for some constant matrix $K_2$. Replacing $K_1$ by $2K_1$ gives the sought-after result. □

**Remark 3.14.** The procedure introduced in [22] can be used to invert $(\partial_x + A_u)$ and so obtain (3.15) from (3.16). This procedure is based on the observation that if $K$, $L$, and $M$ depend on $x$ and $u$ and its derivatives, and if $s$ and $t$ satisfy $s_x = sM$ and $t_x = Nt$, then $(sKt)_x = s\left[(\partial_x + L_M + R_N)K\right]t$. The first equality in (3.40) then corresponds to $K = \widetilde{L}$, $M = u$, and $N = -u$ where, for $z$ and $y$ satisfying $z_x = zu$ and $y_x = uy$, we have taken $s = z$ and $t = z^{-1}$ so $s_x = su$ and $t_x = -z^{-1}z_x z^{-1} = -ut$. Similarly, the first equality in (3.42) corresponds to $K = L$, $M = -u$, and $N = u$, where we have taken $s = y^{-1}$ and $t = y$ so $s_x = -y^{-1}y_x y^{-1} = -su$ and $t_x = ut$.

**Lemma 3.15.** *Define $\mathcal{V}[w, E] \equiv K[w]$ as given by (3.14). Then $\mathcal{V}[w^T, E^T] = \mathcal{V}[w, E]^T$.*

*Proof.* This follows from the symmetric form of the Hamiltonian operator $\mathcal{B}_1[w] = \partial_x^3 + A_w\partial_x + \partial_x A_w + C_w\partial_x^{-1}C_w$. First of all, we note that any function $\mathcal{Z}[w]$ of the matrix $w$ such that $\mathcal{Z}[w^T] = \mathcal{Z}[w]^T$ satisfies $\mathcal{B}_1[w^T]\mathcal{Z}[w^T] = (\mathcal{B}_1[w]\mathcal{Z}[w])^T$. It then follows that, since $\mathcal{Z}[w] = M_0 = \frac{1}{2}I$ satisfies this condition on $\mathcal{Z}[w]$, then so does every member of the sequence $\mathcal{Z}[w] = M_k = \partial_x^{-1}\mathcal{B}_1[w]M_{k-1}$, $k = 1, 2, 3, \ldots, n$. Thus, we see that $\mathcal{V}[w^T, E^T] = \mathcal{V}[w, E]^T$. □

**Proposition 3.16.** *The Schwarzian matrix second Painlevé hierarchy, the matrix second Painlevé hierarchy, and the Schwarzian matrix KdV hierarchy, as defined in Definitions 3.2, 3.3, and 3.7, satisfy*

$$\mathcal{S}[\rho^T, z^T, y^T, E^T, C_1^T, C_2^T, C_3^T] = \mathcal{S}[\rho, y, z, E, C_2, C_1, C_3]^T, \tag{3.44}$$

$$\mathcal{Q}[u^T, z^T, y^T, E^T, C_1^T, C_2^T] = \mathcal{Q}[u, y, z, E, C_2, C_1]^T, \tag{3.45}$$

$$\mathcal{K}[\rho^T, z^T, y^T, E^T, C_1^T, C_2^T, C_3^T] = \mathcal{K}[\rho, y, z, E, C_2, C_1, C_3]^T, \tag{3.46}$$

*respectively. (Note the interchanges between $z$ and $y$, and $C_1$ and $C_2$.)*

*Proof.* This follows from Lemma 3.15 together with Proposition 2.15 and the fact that $M[u^T] = M[u]^T$. □

**Remark 3.17.** When $E = 0$, the Schwarzian matrix second Painlevé hierarchy and the Schwarzian matrix KdV hierarchy, that is,

$$zK\left[\frac{1}{2}S(\rho)\right]y - C_1\rho - \rho C_2 - g_{n-1}\rho - C_3 = 0, \tag{3.47}$$

and

$$\frac{1}{2}\rho_t = zK\left[\frac{1}{2}S(\rho)\right]y - C_1\rho - \rho C_2 - g_{n-1}\rho - C_3, \tag{3.48}$$

respectively, do not depend on $z$ and $y$. In order to see this, we note that for any matrix function $A$,

$$(z^{-1}Ay^{-1})_x = -z^{-1}z_x z^{-1}Ay^{-1} + z^{-1}A_x y^{-1} - z^{-1}Ay^{-1}y_x y^{-1} = z^{-1}\left(-\frac{1}{2}\rho_{xx}\rho_x^{-1}A + A_x - \frac{1}{2}A\rho_x^{-1}\rho_{xx}\right)y^{-1}. \tag{3.49}$$

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Since $S(\rho)$ has the form $z^{-1}Ay^{-1}$ then so do all its derivatives, and it then follows that all terms resulting from the substitution $w = \frac{1}{2}S(\rho)$ in any $M_k$, $k \geq 1$, in $K[w]$ are products of quantities of the form $z^{-1}Ay^{-1}$. In any such product, we may replace the combination $y^{-1}z^{-1}$, which arises between adjacent quantities, by $\rho_x^{-1}$. Thus, the result is a sum of terms which only depend on $x$-derivatives of $\rho$, and having a left factor $z^{-1}$ and a right factor $y^{-1}$, which are then canceled when constructing the combination $zK\left[\frac{1}{2}S(\rho)\right]y$ in equations (3.47) and (3.48). Finally, the terms $M_0[w] + g_{n-1}xI$ in $K[w]$ gives rise to terms $\frac{1}{2}c_0\rho_x + g_{n-1}x\rho_x$ in these equations. A consequence of the above equations not depending on $z$ and $y$ is that when applying Propositions 3.6 and 3.10, we are free to choose $z$ and $y$. Any alternative choice of $s$ and $t$ just yields an alternative variable $v = \frac{1}{2}s^{-1}\rho_{xx}t^{-1}$ for the matrix second Painlevé hierarchy or extended matrix mKdV hierarchy, respectively, this variable being related to $u$ in Propositions 3.6 and 3.10 via $v = PuP^{-1}$, where $P$ is the constant matrix given in Proposition 2.4 (and the hierarchies satisfy $Q[v, s, t, 0, C_1, C_2] = PQ[u, z, y, 0, C_1, C_2]P^{-1}$ and $\mathcal{M}[v, 0] = P\mathcal{M}[u, 0]P^{-1}$).

**Remark 3.18.** As simple examples, let us present the first members of the hierarchies (3.15), (3.16), and (3.24). Our new Schwarzian matrix second Painlevé equation is

$$\frac{1}{2}\left(\rho_{xxx} - \frac{3}{2}\rho_{xx}\rho_x^{-1}\rho_{xx}\right) + \frac{1}{2}c_0\rho_x + zEy + g_{n-1}x\rho_x - C_1\rho - \rho C_2 - g_{n-1}\rho - C_3 = 0, \tag{3.50}$$

where $zy = \rho_x$ and $z_x y = zy_x$ (note the $zEy$ term). Our corresponding new matrix second Painlevé equation is

$$u_{xx} - 2u^3 + c_0 u + uE + Eu + 2g_{n-1}xu - z^{-1}C_1 z - yC_2 y^{-1} = 0, \tag{3.51}$$

where $z_x = zu$ and $y_x = uy$. The first of these is more general than the Schwarzian matrix second Painlevé equation given in [22], and the second is more general than the matrix second Painlevé equation obtained in [9], which in turn includes, as a further special case, the original version presented in [2], [18]. As explained in Remark 3.12, these previous results may be recovered by taking, for example, $C_1 = \alpha I$ and $C_2 = \beta I$, where $\alpha$ and $\beta$ are two scalar constants satisfying $\alpha + \beta = \alpha_n$. Finally, our new Schwarzian matrix KdV equation is given by

$$\frac{1}{2}\rho_t = \frac{1}{2}\left(\rho_{xxx} - \frac{3}{2}\rho_{xx}\rho_x^{-1}\rho_{xx}\right) + \frac{1}{2}c_0\rho_x + zEy + g_{n-1}x\rho_x - C_1\rho - \rho C_2 - g_{n-1}\rho - C_3, \tag{3.52}$$

where $zy = \rho_x$ and $z_x y = zy_x$ (again, note the $zEy$ term). This equation is a strong generalization of the Lagrangian Schwarzian KdV equation presented in [17], defined via the Lagrange Schwarzian derivative introduced in [20], and also of the (equivalent, but differently-defined) non-Abelian KdV singularity manifold equation given in [4]. The equations in [4], [17] may be recovered, for example, by setting $E = 0$, $g_{n-1} = 0$, $C_1 = C_2 = C_3 = 0$, and $c_0 = 0$.

## 4. Conclusions

Let us summarize the results obtained in this article:

- We have given a new definition of matrix Schwarzian derivative.
- We have seen that some basic properties to be expected of a matrix Schwarzian derivative are satisfied.
- We have used this matrix Schwarzian derivative to define two new Schwarzian matrix hierarchies:
  (a) a Schwarzian matrix second Painlevé hierarchy (3.15);
  (b) a Schwarzian matrix KdV hierarchy (3.24).
- In addition, we have defined a new matrix second Painlevé hierarchy (3.16).
- It is also of interest to note that most of the results in this article can, in fact, be phrased as auto-Bäcklund and Bäcklund transformations. This is true not only (and obviously) in Section 3 but also

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in Section 2: the proofs of all of the lemmas and propositions in Section 2 involve defining mappings between the dependent and independent variables used in defining matrix Schwarzian derivatives.

The results obtained in this article have implications for the future study of Painlevé equations and hierarchies, as well as for that of integrable systems and ODEs and PDEs more generally. The inversion of Hamiltonian operators, as in Proposition 3.13, which has led to the inclusion of the terms $z^{-1}C_1z + yC_2y^{-1}$ rather than just $\alpha_n I$ in (3.17), has implications for the construction of other Painlevé equations and hierarchies; similarly, the inversion of operators, Hamiltonian or otherwise, for other ODEs and PDEs and hierarchies, whether integrable or not. The properties of our new matrix second Painlevé hierarchy (3.16), such as Lax pairs and auto-Bäcklund transformations, amongst others, will be explored in later papers.

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