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Subsequence principles for vector-valued random variables

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1. *Introduction.* Révész (8) has shown that if (f_n) is a sequence of random variables, bounded in L^2, there exists a subsequence (f_{n_k}) and a random variable f in L^2 such that $\Sigma_{k=1}^{\infty} a_k(f_{n_k}-f)$ converges almost surely whenever $\Sigma_{k=1}^{\infty} a_k^2 < \infty$. Komlós (5) has shown that if (f_n) is a sequence of random variables, bounded in L^1, then there is a subsequence (f_{n_k}) with the property that the Cesàro averages of any subsequence converge almost surely. Subsequently Chatterji (2) showed that if (f_n) is bounded in L^p (where $0 < p \leqslant 2$) then there is a subsequence $(g_k) = (f_{n_k})$ and f in L^p such that

$$l^{-1/p}\Sigma_{j=1}^{l}(g_{k_l}-f) \to 0$$

almost surely for every sub-subsequence. All of these results are examples of subsequence principles: a sequence of random variables, satisfying an appropriate moment condition, has a subsequence which satisfies some property enjoyed by sequences of independent identically distributed random variables. Recently Aldous (1), using tightness arguments, has shown that for a general class of properties such a subsequence principle holds: in particular, the results listed above are all special cases of Aldous' principal result.

What happens when the sequence of random variables takes values in a Banach space X? First, Aldous' methods cannot be applied, as tightness is lost. Secondly, the results depend upon the geometry of the Banach space X. Thus if f_n is constantly equal to e_n, the nth unit vector in l^p, then the Révész result holds if and only if $p \geqslant 2$. Similarly, by considering constant random variables, it is seen that the Komlós theorem holds for X-valued random variables only if X has the Banach–Saks property – that is, every bounded sequence in X has a subsequence whose Cesàro averages converge.

Of course Hilbert space has the best geometric properties of any Banach space, and recently Suchanek (9) has proved analogues of the Révész and Komlós theorems for sequences of random variables with values in Hilbert space. (In fact, her proof of the analogue of Komlós' theorem is incomplete, as were the original proofs of Theorems 5 and 6 below: I am most grateful to Professor S. D. Chatterji for pointing this out.) In this paper we shall obtain similar results for sequences of random variables with values in more general Banach spaces. It turns out that uniform smoothness is the property which is required – not only does it arise directly (in Lemma 1), but also it enables the results of Pisier (7) on the convergence of martingales to be used.

2. *Preliminaries.* Throughout this paper, (Ω, Σ, P) will be a probability space and X a real Banach space. If $1 \leqslant p < \infty$, $L^p(X)$ denotes the Banach space of all (equivalence classes of) p-th power Bochner integrable random variables taking values in X.

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We refer to the lecture notes of Diestel (3) for the definitions and basic properties of uniform smoothness and super-reflexivity (see also (4) and (7)). We say that X is *q-smooth* (for $1 < q \leqslant 2$) if there is a constant A such that the modulus of smoothness ρ_X satisfies $\rho_X(\tau) \leqslant A\tau^q$ for $\tau > 0$, and say that X is *q-smoothable* if it can be given an equivalent norm under which it is q-smooth. If X is q-smoothable we shall always suppose that it has been renormed in this way. The two fundamental results that we use are due to Pisier (7).

THEOREM A. *X is q-smoothable if and only if whenever (d_n) is a martingale difference sequence in $L^q(X)$ with $\Sigma_{n=1}^{\infty} E(\|d_n\|^q) < \infty$ then $\Sigma_{n=1}^{\infty} d_n$ converges in $L^q(X)$ and almost surely.*

THEOREM B. *X is super-reflexive if and only if X is q-smoothable for some $1 < q \leqslant 2$.*

We shall also require

THEOREM C. *If X is q-smoothable and $q \leqslant p < \infty$ then $L^p(X)$ is q-smoothable.*

This can be proved in the same ways as (4), proposition 19 (see the remarks on p. 134).

The proofs of the theorems involve careful truncation. Let us describe the terminology which we use. If f is a measurable X-valued function, $1 \leqslant p < \infty$ and l is an integer, we set

$$t_{p,l}(f)(\omega) = f(\omega) \quad \text{if} \quad \|f(\omega)\|^p \leqslant l$$
$$= 0 \quad \text{otherwise.}$$

3. *Subsequences of weakly null sequences.* The basic theorem of Komlós [(5), theorem 1] has been extended by Chatterji ((2), lemma 1). We shall extend it further: the proofs of (iii) and (iv) are very similar to those given by Komlós and Chatterji, but are included for completeness' sake.

THEOREM 1. *Suppose that X is reflexive, that $1 \leqslant p < q < 2$ and that (f_n) is a weakly null sequence in $L^p(X)$. Then there exists a subsequence $(g_k) = (f_{n_k})$ and a sequence (d_l) in $L^p(X)$ such that if $(h_l) = (g_{k_l})$ is any sub-subsequence we have the following:*

(i) *$t_{p,j}(h_l) \to d_j$ weakly in $L^2(X)$, for each j;*
(ii) *$\Sigma_{l=1}^{\infty} \|d_l - d_{l+1}\|_{L^1(X)} < \infty$, and, if $1 < p < 2$, $\Sigma_{l=1}^{\infty} l^{-1/p} \|dl\|_{L^1(X)} < \infty$;*
(iii) *$\Sigma_{l=1}^{\infty} P(h_l \neq t_{p,l}(h_l)) < \infty$;*
(iv) *$\Sigma_{l=1}^{\infty} (l^{-q/p}) \int \|t_{p,l}(h_l)\|^q \, dp < \infty$.*

Proof. Let $A = \sup_n \|f_n\|_{L^p(X)}$. Since X is reflexive, $L^2(X)$ is reflexive. For each j, the sequence $(t_{p,j}(f_n))$ is bounded in $L^2(X)$, so that, by extracting subsequences and using a diagonal argument, we may assume that, for each j, there exists d_j such that $t_{p,j}(f_n) \to d_j$ weakly in $L^2(X)$. Then (i) will certainly be satisfied.

Using another diagonal argument, there exists a subsequence $(g_k) = (f_{n_k})$ such that, if we set $\psi_k = \|g_k\|^p$,

$$\lim_{k \to \infty} P(j < \psi_k \leqslant j+1) = a_j$$

and

$$P(j < \psi_k \leqslant j+1) \leqslant a_j + 2^{-j} \quad \text{for} \quad k^2 \geqslant j.$$

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We shall show that (g_k) has the required properties. First note that

$$\Sigma_{j=1}^{\infty} j a_j \leqslant \liminf_{k\to\infty} \Sigma_{j=1}^{\infty} j P(j < \psi_k \leqslant j+1)$$
$$\leqslant \liminf_{k\to\infty} \|\psi_k\|_1 \leqslant A^p < \infty.$$

As $g_k \to 0$ weakly in $L^1(X)$, (g_k) is uniformly integrable: thus, given $\epsilon > 0$, there exists a positive integer K such that

$$\|g_k - t_{p,K}(g_k)\|_1 \leqslant \epsilon, \quad \text{for all} \quad k.$$

Now if $k^2 > K > j$,

$$\|t_{p,K}(g_k) - t_{p,j}(g_k)\|_1 \leqslant \Sigma_{n=j}^{K-1} (n+1)^{1/p} (a_n + 2^{-n}),$$

so that

$$\|g_k - t_{p,j}(g_k)\|_1 \leqslant \epsilon + \Sigma_{n=j}^{\infty} (n+1)^{1/p} (a_n + 2^{-n}),$$

for all j. But $g_k - t_{p,j}(g_k) \to -d_j$ weakly in $L^1(X)$ as $k \to \infty$, so that (bearing in mind that $\epsilon > 0$ is arbitrary)

$$\|d_j\|_1 \leqslant \Sigma_{n=j}^{\infty} (n+1)^{1/p} (a_n + 2^{-n}).$$

Thus if $p > 1$,

$$\Sigma_{l=1}^{\infty} l^{-1/p} \|d_l\|_1 \leqslant \Sigma_{l=1}^{\infty} l^{-1/p} (\Sigma_{n=l}^{\infty} (n+1)^{1/p} (a_n + 2^{-n}))$$
$$= \Sigma_{n=1}^{\infty} (\Sigma_{l=1}^{n} l^{-1/p}) (n+1)^{1/p} (a_n + 2^{-n})$$
$$\leqslant \frac{p}{p-1} \Sigma_{n=1}^{\infty} (n+1) (a_n + 2^{-n}) < \infty.$$

On the other hand, as

$$\|t_{1,j+1}(g_k) - t_{1,j}(g_k)\|_1 \leqslant (j+1)^{1/p} P(j < \psi_k \leqslant j+1)$$

and, as $t_{1,j+1}(g_k) - t_{1,j}(g_k) \to d_{j+1} - d_j$ as $k \to \infty$,

$$\|d_{j+1} - d_j\|_1 \leqslant (j+1)^{1/p} a_j \leqslant (j+1) a_j$$

so that $\Sigma_{j=1}^{\infty} \|d_{j+1} - d_j\|_1 \leqslant \Sigma_{j=1}^{\infty} (j+1) a_j < \infty$. Thus (ii) is satisfied.

If $(h_l) = (g_{k_l})$ is a sub-subsequence,

$$P(h_l \neq t_{p,l}(h_l)) = \Sigma_{j=l}^{l^2} P(j < \|h_l\|^p \leqslant j+1) + P(\|h_l\|^p \geqslant l^2)$$
$$\leqslant \Sigma_{j=l}^{\infty} (a_j + 2^{-j}) + \frac{A^p}{l^2},$$

by Chebychev's inequality, so that

$$\Sigma_{l=1}^{\infty} P(h_l \neq t_{p,l}(h_l)) \leqslant \Sigma_{l=1}^{\infty} \left(\Sigma_{j=l}^{\infty} (a_j + 2^{-j}) + \frac{A^p}{l^2} \right)$$
$$= \Sigma_{j=1}^{\infty} j (a_j + 2^{-j}) + \Sigma_{l=1}^{\infty} \frac{A^p}{l^2} < \infty.$$

Similarly,

$$\Sigma_{l=1}^{\infty} \frac{1}{l^{q/p}} \int \|t_{p,l}(h_l)\|^q \, dP \leqslant \Sigma_{l=1}^{\infty} \frac{1}{l^{q/p}} \Sigma_{k=0}^{l-1} (k+1)^{q/p} (a_k + 2^{-k})$$
$$= \Sigma_{k=0}^{\infty} \left(\Sigma_{l=k+1}^{\infty} \frac{1}{l^{q/p}} \right) (k+1)^{q/p} (a_k + 2^{-k})$$
$$\leqslant \frac{p}{q-p} \Sigma_{k=0}^{\infty} (k+1) (a_k + 2^{-k}) < \infty.$$

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This completes the proof of the theorem.

Let us also mention the following result of Suchanek ((9), lemma 2·6), which is proved in a similar way:

THEOREM D. *If (f_n) is a bounded sequence in $L^1(X)$, there is a subsequence $(g_k) = (f_{n_k})$ such that if $(h_l) = (g_{k_l})$ is any sub-subsequence,*

(i) $\Sigma_{l=1}^{\infty} P(h_l \neq t_{1,l}(h_l)) < \infty$, *and*

(ii) *the sequence $(t_{1,l}(h_l))$ is uniformly integrable.*

4. *Weighted sums of conditional expectations.* This section contains the basic technical results which we need.

THEOREM 2. *Suppose that (f_n) is a uniformly integrable sequence of simple functions and that $1 \leqslant p \leqslant 2$. Then there exists a subsequence $(g_k) = (f_{n_k})$ such that if $(h_l) = (g_{k_l})$ is any sub-subsequence then*

$$\sup_l \|E(t_{p,l}(h_l)|h_1, \ldots, h_{l-1})(\omega)\| < \infty$$

and

$$\sup_l \|E(h_l|h_1, \ldots, h_{l-1})(\omega)\| < \infty$$

almost surely.

Proof. Let us set $u_n(\omega) = \|f_n(\omega)\|$ and, if $(g_k) = (f_{n_k})$ is a subsequence, set

$$v_k(\omega) = u_{n_k}(\omega).$$

Then since

$$\|E(t_{p,l}(h_l)|h_1, \ldots, h_{l-1})(\omega)\| \leqslant E(\|t_{p,l}(h_l)\| \,|h_1, \ldots, h_{l-1})(\omega)$$

and

$$\|E(h_l|h_1, \ldots, h_{l-1})(\omega)\| \leqslant E(\|h_l\| \,|h_1, \ldots, h_{l-1})(\omega)$$

almost surely (by Jensen's inequality) and since

$$\|t_{p,l}(g_{k_l})(\omega)\| = t_{p,l}(v_{k_l})(\omega),$$

it is sufficient to show that there is a subsequence (g_k) such that if (g_{k_l}) is any sub-subsequence then

$$\sup_l E(t_{p,l}(v_{k_l})|g_{k_1}, \ldots, g_{k_{l-1}})(\omega) < \infty$$

and

$$\sup_l E(v_{k_l}|g_{k_1}, \ldots, g_{k_{l-1}})(\omega) < \infty$$

almost surely. For each l, the sequence $(t_{p,l}(u_n))$ is uniformly integrable, and therefore weakly relatively compact in L^1, as is the sequence (u_n). By extracting subsequences, and using a diagonal argument, we may therefore suppose that, for each l, $t_{p,l}(u_n)$ converges weakly in L^1 to e_l, say, and that u_n converges weakly in L^1 to e. As in Theorem 1,

$$\Sigma_{l=1}^{\infty} \|e_l - e_{l+1}\|_1 < \infty.$$

We now construct $n_1 < n_2 < \ldots$ inductively such that setting $g_k = f_{n_k}$, if

$$k_1 < \ldots < k_l \leqslant k,$$

$$\left.\begin{aligned} &|E(t_{p,l}(v_{k_l}) - e_l|g_{k_1}, \ldots, g_{k_{l-1}})(\omega)| \leqslant 1 \\ \text{and} \quad &|E(v_{k_l} - e\,|g_{k_1}, \ldots, g_{k_{l-1}})(\omega)| \leqslant 1, \end{aligned}\right\} \qquad (*)$$

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for each ω in Ω. Set $n_1 = 1$, and suppose that $n_1, \ldots, n_k$ have been found. For each atom A in $\sigma(g_1, \ldots, g_k)$ and each $1 \leqslant l \leqslant k$,

$$\int_A (t_{p,l}(u_n) - e_l)\, dP \to 0$$

and

$$\int_A (u_n - e)\, dP \to 0$$

as $n \to \infty$. Since $\sigma(g_1, \ldots, g_k)$ is finite, this implies that

$$E(t_{p,l}(u_n) - e_l | g_1, \ldots, g_k)(\omega) \to 0$$

and

$$E(u_n - e | g_1, \ldots, g_k)(\omega) \to 0$$

uniformly in ω. Thus there exists $n_{k+1} > n_k$ such that

$$|E(t_{p,l}(u_n) - e_l | g_1, \ldots, g_k)(\omega)| \leqslant 1$$

and

$$|E(u_n - e | g_1, \ldots, g_k)(\omega)| \leqslant 1$$

for all ω. This clearly establishes the induction. Now let us set

$$g(\omega) = |e_1(\omega)| + \Sigma_{l=1}^{\infty} |e_l(\omega) - e_{l+1}(\omega)|.$$

Then by the monotone convergence theorem, $g \in L^1$.
If $\Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$ is *any* increasing sequence of σ-fields, for each l,

$$\sup_k E(e_l | \Sigma_k)(\omega) \leqslant \sup_k E(g | \Sigma_k)(\omega) < \infty$$

almost surely, and $\sup_k E(e | \Sigma_k)(\omega) < \infty$ almost surely, by the martingale convergence theorem (6). Combining this with (*), the theorem follows.

Next we establish an elementary geometrical lemma.

LEMMA 1. *Suppose that y is a non-zero element of X, that $\epsilon > 0$ and that (x_n) is a weakly null sequence in X. Then there exists n_0 such that*

$$(1-\epsilon)\|y\| \leqslant \|y + x_n\| \leqslant (1+\epsilon)\|\left(1 + 2\rho_X\left(\frac{\|x_n\|}{\|y\|}\right)\right)\|y\|$$

for $n \geqslant n_0$.

Proof. We may suppose that $\|y\| = 1$. Let $f \in X^*$ be such that $f(y) = \|f\| = 1$. There exists n_0 such that $|f(x_n)| \leqslant \epsilon$ for $n \geqslant n_0$. Thus if $n \geqslant n_0$, $f(y \pm x_n) \geqslant 1 - \epsilon$ and so $\|y \pm x_n\| \geqslant 1 - \epsilon$. Using the definition of ρ_X,

$$\|y + x_n\| + \|y - x_n\| \leqslant 2 + 2\rho_X(\|x_n\|)$$

so that

$$\begin{aligned} \|y + x_n\| &\leqslant 1 + \epsilon + 2\rho_X(\|x_n\|) \\ &\leqslant (1+\epsilon)(1 + 2\rho_X(\|x_n\|). \end{aligned}$$

THEOREM 3. *Suppose that X is reflexive and that (f_n) is a sequence of simple functions which converges weakly to 0 in $L^p(X)$ (where $1 \leqslant p \leqslant 2$). Then there exists a subsequence*

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$(g_k) = (f_{n_k})$ and a sequence (d_l) in $L^2(X)$ such that, if (β_l) is a sequence of scalars with $\Sigma_{l=1}^{\infty} \rho_X(|\beta_l|) < \infty$ and $(h_l) = (g_{k_l})$ is a sub-subsequence, then the series

$$\Sigma_{l=1}^{\infty} \beta_l E(t_{p,l}(h_l) - d_l | h_1, \ldots, h_{l-1})(\omega)$$

and

$$\Sigma_{l=1}^{\infty} \beta_l E(h_l | h_1, \ldots, h_{l-1})(\omega)$$

converge (in norm) almost surely.

Further if $1 < q < 2$, if $q \leqslant p < \infty$ and if X is q-smooth, then (g_k) can be chosen such that if $(\beta_l) \in l^q$ and (h_l) is a sub-subsequence then the series

$$\Sigma_{l=1}^{\infty} \beta_l E(t_{p,l}(h_l) - d_l | h_1, \ldots, h_{l-1})$$

and

$$\Sigma_{l=1}^{\infty} \beta_l E(h_l | h_1, \ldots, h_{l-1})$$

also converge in $L^p(X)$.

Proof. By Theorems 1 and 2, extracting a subsequence if necessary, there is a sequence (d_l) in $L^2(X)$ such that if $(g_k) = (f_{n_k})$ is any subsequence of (f_n) then

(i) $t_{p,j}(g_k) \to d_j$ weakly in $L^2(X)$, for each j,

(ii) $\Sigma_{l=1}^{\infty} \|d_l - d_{l+1}\|_{L^1(X)} < \infty$,

(iii) $\sup_k \|E(t_{p,k}(g_k) | g_1, \ldots, g_{k-1})(\omega)\| < \infty$, almost surely, and

(iv) $\sup_k \|E(g_k | g_1, \ldots, g_{k-1})(\omega)\| < \infty$ almost surely.

Let $s(\omega) = \|d_1(\omega)\| + \Sigma_{l=1}^{\infty} \|d_l(\omega) - d_{l+1}(\omega)\|$.

Then $s \in L^1$, by (ii), and $\|d_k\| \leqslant s$, so that

$$\begin{aligned} \|E(d_k | g_1 \ldots g_{k-1})(\omega)\| &\leqslant E(\|d_k\| \, | g_1, \ldots, g_{k-1})(\omega) \\ &\leqslant E(s | g_1, \ldots, g_{k-1})(\omega), \quad \text{almost surely.} \end{aligned}$$

Thus $\sup_k \|E(d_k | g_1, \ldots, g_{k-1})(\omega)\| < \infty$ almost surely, and so

(v) $\sup_k \|E(t_{p,k}(g_k) - d_k | g_1, \ldots, g_{k-1})(\omega)\| < \infty$, almost surely.

Now if $\Sigma_{l=1}^{\infty} \rho_X(|\beta_l|) < \infty$, then $\beta_l \to 0$, so that it is sufficient to consider the case where $|\beta_l| \leqslant 1$ for all l. Further if $(\beta_l) \in l^1$ then the sums under consideration converge almost surely. Thus it is sufficient to consider the case where $(\beta_l) \in D$, where

$$D = \{(p_l/2^l)_{l=1}^{\infty} : p_l \in \mathbb{Z}, -2^l \leqslant p_l \leqslant 2^l\}.$$

Let

$$D_m = \{(p_l/2^l)_{l=1}^{m}, p_l \in \mathbb{Z}, -2^l \leqslant p_l \leqslant 2^l \text{ for } 1 \leqslant l \leqslant m\}.$$

Let (ϵ_l) be a sequence of positive numbers such that

$$\Pi_{l=1}^{\infty}(1 + \epsilon_l) \leqslant 2 \quad \text{and} \quad \Pi_{l=1}^{\infty}(1 - \epsilon_l) \geqslant \tfrac{1}{2}.$$

Let $n_1 = 1$. Suppose we have found $n_1 < \ldots < n_k$ such that for each $1 \leqslant r < k$, if $l_1 < \ldots < l_{r+1} = k$ and $(\beta_1, \ldots, \beta_{r+1}) \in D_{r+1}$, then, setting

$$g_i = f_{n_i}, \quad h_j = g_{l_j}, \quad k_j = E(t_{p,j}(h_j) - d_j | h_1, \ldots, h_{j-1}) \quad \text{and} \quad k_j' = E(h_j | h_1, \ldots, h_{j-1}),$$

$$\begin{aligned} (1 - \epsilon_r) \|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\| &\leqslant \|\Sigma_{j=s}^{r+1} \beta_j k_j(\omega)\| \\ &\leqslant (1 + \epsilon_r) \|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\| \left(1 + 2\rho_X\left(\frac{|\beta_{r+1}| \, \|k_{r+1}(\omega)\|}{\|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\|}\right)\right) \end{aligned} \tag{†}$$

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for each ω in Ω and each $1 \leqslant s \leqslant r$ for which $\Sigma_{j=s}^{r} \beta_j k_j(\omega) \neq 0$, and similar inequalities hold for $(k'_1, \ldots, k'_{r+1})$. Then since

$$\int_A (t_{p,j}(f_n) - d_j)\, dP \to 0$$

and

$$\int_A f_n\, dP \to 0$$

weakly in X as $n \to \infty$, for each j and each measurable A, applying Lemma 1 to each atom of each finite σ-field $\sigma(f_{n_{l_1}}, \ldots, f_{n_{l_r}})$, where $l_1 < l_2 < \ldots < l_r \leqslant k$ and each $(\beta_1, \ldots, \beta_r) \in D_r$, we can find n_{k+1} for which (†) is satisfied by $(k_1, \ldots, k_{r+1})$ and $(k'_1, \ldots, k'_{r+1})$. Thus, inductively, we obtain a sequence (n_k). Let $g_k = f_{n_k}$, for $k = 1, 2, \ldots$.

Now suppose that $(\beta_l) \in D$, that $\Sigma_{l=1}^{\infty} \rho_X(|\beta_l|) < \infty$ and that $(h_l) = (g_{k_l})$ is a sub-subsequence. By (iii) and (iv) there is a finite measurable function C such that, setting

$$k_l = E(t_{p,l}(h_l) - d_l | h_1, \ldots, h_{l-1})$$

and

$$k'_l = E(h_l | h_1, \ldots, h_{l-1}),$$

we have

$$\sup_l (\max \|k_l(\omega)\|, \|k'_l(\omega)\|) \leqslant C(\omega)$$

except on a null set N. Suppose that $\omega \notin N$ and that $\epsilon > 0$. Since ρ_X satisfies the Δ_2-condition at 0 ((4), lemma 8), there exists $\tau_0 > 0$ and a constant K such that

$$\rho_X\left(\frac{2C(\omega)\tau}{\epsilon}\right) \leqslant K\rho_X(\tau) \quad \text{for} \quad 0 < \tau \leqslant \tau_0.$$

Since $\beta_l \to 0$ as $l \to \infty$, there exists l_0 such that

(i) $|\beta_l|\, C(\omega) \leqslant \epsilon/2$ for $l \geqslant l_0$,

(ii) $|\beta_l| \leqslant \tau_0$ for $l \geqslant l_0$, and

(iii) $\Sigma_{j=l_0}^{\infty} \rho_X(|\beta_j|) \leqslant 1/2K$.

Suppose that $s \geqslant l_0$. Then $\|\beta_s k_s(\omega)\| \leqslant \epsilon/2$, by (i). Either $\|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\| \leqslant \epsilon$ for all $r > s$, or there exists a least r_1 for which $\|\Sigma_{j=s}^{r_1} \beta_j k_j(\omega)\| > \epsilon$. Note that, in this case,

$$\begin{aligned} \|\Sigma_{j=s}^{r_1} \beta_j k_j(\omega)\| &\leqslant \|\Sigma_{j=s}^{r_1-1} \beta_j k_j(\omega)\| + \|\beta_{r_1} k_{r_1}(\omega)\| \\ &\leqslant 3\epsilon/2, \end{aligned}$$

and that $\|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\| \geqslant \epsilon/2$ for all $r \geqslant r_1$, by repeated application of the left-hand inequality of (†). Thus, using the right-hand inequality of (†), if $r \geqslant r_1$,

$$\begin{aligned} \|\Sigma_{j=s}^{r+1} \beta_j k_j(\omega)\| &\leqslant (1+\epsilon_r) \|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\| \left(1 + 2\rho_X\left(\frac{2|\beta_{r+1}|\, C(\omega)}{\epsilon}\right)\right) \\ &\leqslant (1+\epsilon_r) \|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\| (1 + 2K\rho_X(|\beta_{r+1}|)). \end{aligned}$$

Thus if $r \geqslant r_1$,

$$\|\Sigma_{j=s}^{r} \beta_j k_j(\omega)\| \leqslant 3\epsilon \Pi_{j=r_1+1}^{r} (1 + 2K\rho_X(|\beta_{r+1}|)) \leqslant 3e\epsilon.$$

Thus $\Sigma_{l=1}^{\infty} \beta_l (E(t_{p,l}(h_l) - d_l | h_1, \ldots, h_{l-1})\,(\omega)$ converges almost surely; in the same way we see that $\Sigma_{l=1}^{\infty} \beta_l E(h_l | h_1, \ldots, h_{l-1})\,(\omega)$ does, too.

Now suppose that the additional conditions of the second part of the theorem are satisfied. If $(\beta_l) \in l^q$ then $\Sigma_{l=1}^{\infty} \rho_X(|\beta_l|) < \infty$ so that by what we have already proved we

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may assume that all subsequences of (f_n) have the required properties for almost sure convergence. Likewise, it is sufficient to prove the result for $(\beta_l) \in D \cap l^q$.

As $L^p(X)$ is q-smooth, there is a constant A such that

$$\rho_{L^p(X)}(\tau) \leqslant A\tau^q \quad \text{for} \quad 0 \leqslant \tau < \infty.$$

We now proceed inductively much as before. Let $n_1 = 1$. Suppose we have found $n_1 < \ldots < n_k$ such that for each $1 \leqslant r < k$, if $l_1 < \ldots < l_{r+1} = k$ and $(\beta_1, \ldots, \beta_{r+1}) \in D_{r+1}$, then, setting $g_i = f_{n_i}$, $h_j = g_{l_j}$, $k_j = E(t_{p,j}(h_j) - d_j | h_1, \ldots, h_{j-1})$ and $k'_j = E(h_j | h_1, \ldots, h_{j-1})$,

$$(1-\epsilon_r)\|\Sigma_{j=s}^r \beta_j k_j\|_{L^p(X)} \leqslant \|\Sigma_{j=s}^{r+1} \beta_j k_j\|_{L^p(X)}$$
$$\leqslant (1+\epsilon_r)\|\Sigma_{j=s}^r \beta_j k_j\|_{L^p(X)} \left(1 + \frac{2A\,|\beta_{r+1}|^q \|k_{r+1}\|^q_{L^p(X)}}{\|\Sigma_{j=s}^r \beta_j k_j\|^q_{L^p(X)}}\right) \quad (\dagger\dagger)$$

for each $1 \leqslant s \leqslant r$ for which $\Sigma_{j=s}^r \beta_j k_j \neq 0$, and similar inequalities hold for $(k'_1, \ldots, k'_{r+1})$. Then since

$$E(t_{p,j}(f_n) - d_j | \Sigma_0) \to 0$$

and

$$E(f_n | \Sigma_0) \to 0$$

weakly in $L^p(X)$ for each j and each sub-σ-field Σ_0, we can apply Lemma 1 to find a suitable n_{k+1}. The proof then concludes as above.

Considering constant functions in X, we obtain

COROLLARY. *Suppose that (x_n) is a weakly null sequence in X. Then there is a subsequence $(y_k) = (x_{n_k})$ such that, if $\Sigma_{k=1}^\infty \rho_X(|\beta_k|) < \infty$, then $\Sigma_{k=1}^\infty \beta_k y_k$ converges in norm.*

5. *Some subsequence principles.* We now come to the main results of this paper. We begin with an analogue of Révész' theorem.

THEOREM 4. *Suppose that X is q-smoothable (where $1 < q \leqslant 2$) and that (f_n) is a bounded sequence in $L^q(X)$. Then there exists a subsequence (f_{n_k}) and $f \in L^q(X)$ such that if $(\beta_k) \in l^q$ then*

$$\Sigma_{k=1}^\infty \beta_k (f_{n_k} - f)$$

converges almost surely and in $L^q(X)$.

Proof. X is reflexive, and so therefore is $L^q(X)$. By extracting a subsequence if necessary, we may suppose that (f_n) converges weakly to $f \in L^q(X)$. We shall show that f satisfies the requirements of the theorem.

For each n, there exists a simple function g_n such that

$$\|f_n - f - g_n\|_{L^q(X)} \leqslant 4^{-n}.$$

Then

$$P\{\omega : \|f_n(\omega) - f(\omega) - g_n(\omega)\| \geqslant 2^{-n}\} \leqslant 2^{-nq}$$

so that if $(\beta_k) \in l^q$, $\Sigma_{k=1}^\infty \beta_k (f_{n_k} - f)$ converges almost surely and in $L^q(X)$ if and only if $\Sigma_{k=1}^\infty \beta_k g_{n_k}$ does. Further $g_n \to 0$ weakly in $L^q(X)$, and so $g_n \to 0$ weakly in $L^1(X)$. By Theorem 3 there is a subsequence (g_{n_k}) such that, if $(\beta_k) \in l^q$, then

$$\Sigma_{k=1}^\infty \beta_k E(g_{n_k} | g_{n_1}, \ldots, g_{n_{k-1}})(\omega)$$

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converges almost surely and in $L^q(X)$. On the other hand, if we set

$$d_1 = \beta_1(g_{n_1} - E(g_{n_1})\chi_\Omega)$$

and

$$d_k = \beta_k(g_{n_k} - E(g_{n_1}|g_{n_1}, \ldots, g_{n_{k-1}})) \quad \text{for} \quad k > 1,$$

then (d_k) is a martingale difference sequence, and

$$\Sigma_{k=1}^{\infty} E(\|d_k\|^q) \leqslant 2^q(\Sigma_{k=1}^{\infty} |\beta_k|^q) \sup_k \|g_{n_k}\|^q_{L^q(X)}$$

so that, by Theorem A, $\Sigma_{k=1}^{\infty} d_k$ converges almost surely and in $L^q(X)$. Thus $\Sigma_{k=1}^{\infty} \beta_k g_{n_k}$ converges almost surely and in $L^q(X)$.

Applying Kronecker's lemma we get

COROLLARY. *If $(\beta_k) \in l^q$ is non-negative decreasing,*

$$\frac{1}{\beta_k} \Sigma_{j=1}^{k} (f_{n_j} - f) \to 0$$

almost surely and in $L^q(X)$.

Next we turn to the analogue of Chatterji's theorem.

THEOREM 5. *Suppose that X is q-smoothable (where $1 < q \leqslant 2$) and that (f_n) is a bounded sequence in $L^p(X)$ (where $1 < p < q$). Then there exists a subsequence $(g_k) = (f_{n_k})$ and $f \in L^p(X)$ such that if $(\beta_l) = O(l^{-1/p})$ then $\Sigma_{l=1}^{\infty} \beta_l (g_{k_l} - f)(\omega)$ converges almost surely, for each sub-sequence (g_{k_l}).*

Proof. As $L^p(X)$ is reflexive, we may assume, by extracting a subsequence if necessary, that $f_n \to f \in L^p(X)$ weakly in $L^p(X)$. We shall show that f satisfies the requirement of the theorem: in other words we shall assume that $f_n \to 0$ weakly in $L^p(X)$, and show that there is a subsequence $(g_k) = (f_{n_k})$ such that if $\beta_l = O(l^{-1/p})$, then $\Sigma_{l=1}^{\infty} \beta_l g_{k_l}$ converges almost surely for each subsequence (g_{k_l}). By the standard approximation argument used in the preceding theorem, we may also assume that each f_n is simple.

If $(\beta_l) = O(l^{-1/p})$, $(\beta_l) \in l^q$, so that by Theorems 1 and 3 there exist a subsequence $(g_k) = (f_{n_k})$ and a sequence (d_l) in $L^1(X)$ such that if $(h_l) = (g_{k_l})$ is any sub-subsequence and if $(\beta_l) = O(l^{-1/p})$, then

(i) $\Sigma_{l=1}^{\infty} \beta_l \|d_l\|_{L^1(X)} < \infty$,

(ii) $\Sigma_{l=1}^{\infty} P(h_l \neq t_{p,l}(h_l)) < \infty$,

(iii) $\Sigma_{l=1}^{\infty} \frac{1}{l^{q/p}} \int \|t_{p,l}(h_l)\|^q \, dP < \infty$ and

(iv) $\Sigma_{l=1}^{\infty} \beta_l E(t_{p,l}(h_l) - d_l | h_1, \ldots, h_{l-1})(\omega)$

converges almost surely. By (i),

$$\Sigma_{l=1}^{\infty} \beta_l \|E(d_l|h_1, \ldots, h_{l-1})\|_{L^1(X)} < \infty$$

so that

$$\Sigma_{l=1}^{\infty} \beta_l \|E(d_l|h_1, \ldots, h_{l-1})(\omega)\|$$

converges almost surely, and so

$$\Sigma_{l=1}^{\infty} \beta_l E(d_l|h_1, \ldots, h_{l-1})(\omega)$$

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converges almost surely. Thus by (iv), $\Sigma_{l=1}^{\infty}\beta_l E(t_{p,l}(h_l)|h_1, \ldots, h_{l-1})(\omega)$ converges almost surely. On the other hand, by (iii),

$$\begin{aligned}\Sigma_{l=1}^{\infty}\|\beta_l(t_{p,l}(h_l) - E(t_{p,l}(h_l)|h_1, \ldots, h_{l-1}))\|_{L^q(X)}^q \\ \leqslant 2^q\Sigma_{l=1}^{\infty}|\beta_l|^q\|t_{p,l}(h_l)\|_{L^q(X)}^q \\ \leqslant 2^q M^q\Sigma_{l=1}^{\infty}\frac{1}{l^{q/p}}\|t_{p,l}(h_l)\|_{L^q(X)}^q < \infty,\end{aligned}$$

where $M = \sup_l (l^{1/p}|\beta_l|)$. Thus, by Theorem A,

$$\Sigma_{l=1}^{\infty}\beta_l(t_{p,l}(h_l)(\omega) - E(t_{p,l}(h_l)|h_1, \ldots, h_{l-1})(\omega))$$

converges almost surely. Thus $\Sigma_{l=1}^{\infty}\beta_l t_{p,l}(h_l)(\omega)$ converges almost surely, and so therefore does $\Sigma_{l=1}^{\infty}\beta_l h_l(\omega)$, by (ii).

Applying Kronecker's lemma, we obtain

COROLLARY. *Suppose that X is q-smoothable (where $1 < q \leqslant 2$) and that (f_n) is a bounded sequence in $L^p(X)$ (where $1 < p < q$). Then there is a subsequence $(g_k) = (f_{n_k})$ and f in $L^p(X)$ such that*

$$\frac{1}{l^{1/p}}\Sigma_{j=1}^{l}(g_{k_j} - f)(\omega)$$

converges almost surely, for each subsequence (g_{k_l}).

Finally we come to an analogue of Komlós' theorem.

THEOREM 6. *Suppose that X is super-reflexive and that (f_n) is a bounded sequence in $L^1(X)$. Then there is a subsequence $(g_k) = (f_{n_k})$ and f in $L^1(X)$ such that*

$$(1/l)(\Sigma_{j=1}^{l}g_{k_j}(\omega)) \to f(\omega)$$

almost surely, for each subsequence (g_{k_l}).

Proof. There exists $1 < q \leqslant 2$ such that X is q-smoothable, by Theorem B. By Theorem D, extracting a subsequence if necessary, we may suppose that

$$\Sigma_{n=1}^{\infty}P(f_n \neq t_{1,n}(f_n)) < \infty$$

and that $(t_{1,n}(f_n))$ is uniformly integrable. It is therefore enough to prove the theorem for uniformly integrable sequences. As X is reflexive, such a sequence is weakly relatively compact in $L^1(X)$, and we may therefore suppose that $f_n \to f_0$ weakly in $L^1(X)$. We may suppose that $f_0 = 0$ and, as before, suppose that each f_n is simple. Then, by Theorems 1 and 3, there exist a subsequence $(g_k) = (f_{n_k})$ and a sequence (d_l) in $L^1(X)$ such that if $(h_l) = (g_{k_l})$ is any sub-subsequence then

(i) $t_{1,j}(h_l) \to d_j$ weakly in $L^2(X)$, for each j,

(ii) $\Sigma_{l=1}^{\infty}\|d_l - d_{l+1}\|_{L^1(X)} < \infty$,

(iii) $\Sigma_{l=1}^{\infty}P(h_l \neq t_{1,l}(h_l)) < \infty$,

(iv) $\Sigma_{l=1}^{\infty}\frac{1}{l^q}\int\|t_{1,l}(h_l)\|^q\,dP < \infty$ and

(v) $\Sigma_{l=1}^{\infty}\frac{1}{l}E(t_{1,l}(h_l) - d_l|h_1, \ldots, h_{l-1})(\omega)$

converges almost surely.

As in the preceding theorem,

$$\Sigma_{l=1}^{\infty}\frac{1}{l}(t_{1,l}(h_l)(\omega) - E(t_{1,l}(h_l)|h_1, \ldots, h_{l-1})(\omega))$$

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converges almost surely. Further by (i) and (iv),

$$\Sigma_{l=1}^{\infty} \frac{1}{l^q} \int \|d_l\|^q \, dP < \infty,$$

and so similarly

$$\Sigma_{l=1}^{\infty} \frac{1}{l} (d_l(\omega) - E(d_l | h_1, \ldots, h_{l-1})(\omega))$$

converges almost surely. Combining these results with (v),

$$\Sigma_{l=1}^{\infty} \frac{1}{l} (t_{1,l}(h_l)(\omega) - d_l(\omega))$$

converges almost surely. By (iii),

$$\Sigma_{l=1}^{\infty} \frac{1}{l} (h_l(\omega) - d_l(\omega))$$

converges almost surely, and so by Kronecker's lemma,

$$\frac{1}{l} \Sigma_{j=1}^{l} (h_j(\omega) - d_j(\omega)) \to 0$$

almost surely. But it follows from (ii) that $\Sigma_{l=1}^{\infty} \|d_l(\omega) - d_{l+1}(\omega)\| < \infty$ for almost all ω, and so there exists $d \in L^1(X)$ such that $d_l(\omega) \to d(\omega)$ almost surely. Then

$$\frac{1}{l} \Sigma_{j=1}^{l} d_j(\omega) \to d(\omega)$$

almost surely, and so the result follows.

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