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Functions with small *BMO* norm

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We characterize the functions with 'small' bounded mean oscillation (BMO) norm by establishing the precise connection between the space BMO and class A_∞ of Muckenhoupt weights. We prove that there exists a universal constant c_2^* such that $\|f\|_{BMO} < c_2^*$ if and only if $\exp f \in A_2$, where c_2^* is the sharp constant in the John and Nirenberg inequality. Similarly, in dimension one, we prove that $\|f\|_{BLO} < 1$ if and only if $\exp f \in A_1$. As application we introduce a structure of metric space in A_∞ and prove that the closed unit ball of A_∞ is a Banach space.

Keywords: Bounded Mean Oscillation; Bounded Lower Oscillation; Muckenhoupt weights; John and Nirenberg Inequality; Banach Spaces

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1. Introduction and motivations

A measurable function f defined over an open set $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is said to have *bounded mean oscillation* and to belong to the space denoted by $BMO(\Omega)$ if the quantity

$$\|f\|_{BMO} = \sup_Q \fint_Q |f(x) - f_Q| dx \tag{1.1}$$

is finite, where the supremum is taken over all cubes $Q \subseteq \Omega$ with sides parallel to the axes and f_Q stands for the integral mean of f over Q, i.e. $f_Q = \fint_Q f(x)dx$.

The quantity (1.1) defines a seminorm as the elements of bounded mean oscillation (BMO) are identified only up to an additive constant. Hence we can consider (1.1) a norm on the quotient space modulo constants.

An equivalent definition can be given over balls, rectangles, and more generally over standard basis of shapes enjoying certain covering properties.

The normed space BMO was introduced by John and Nirenberg in 1961 [6] and has a crucial relevance in harmonic analysis and partial differential equations. In fact, certain singular integral operators, which fail to be bounded from L_∞ to L_∞, result to be bounded from L_∞ to BMO.

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Of course essentially bounded functions have BMO, and for $f \in L^\infty$ it is $\|f\|_{BMO} \leq 2\|f\|_{L^\infty}$. On the other side, the inclusion $L^\infty \subset BMO$ is strict, and the function $\log|x|$ represents the standard example of unbounded function belonging to *BMO*.

A remarkable property of *BMO*, which is a Banach space once equipped with the norm (1.1), is that it is the dual of the Hardy Space $\mathcal{H}^1$ [3]:

$$(\mathcal{H}^1)^* = BMO.$$

This duality property has been shown to be connected to the fact that *BMO* functions can be regarded as logarithms of weights belonging to the *Muckenhoupt* class A_∞ (see §2 for the definitions and the basic properties of A_∞). Namely, we have that

$$w \in A_2 \implies \log w \in BMO. \tag{1.2}$$

An inverse implication also holds true but is less precise:

$$f \in BMO \implies \exp(\lambda f) \in A_2 \tag{1.3}$$

for some 'sufficiently small' $\lambda > 0$.

Inequalities (1.2) and (1.3) hold true for any A_p class with $p > 1$. We point out that (1.3) does not provide any information on the size of λ.

Aim of this article is to investigate under what conditions, for a given function $f \in BMO$, we have that $\exp f \in A_2$ (or, in other words, if and when we can choose $\lambda = 1$ in (1.3)). A counterexample shows that this is not true in general. In fact, the function $g = -\log|x|$ is a function in BMO but $\exp g = e^{-\log|x|} = |x|^{-1}$ does not belong to A_∞.

The interest in this question relies on the several applications where the mapping between *BMO* and A_2 is used by assuming 'λ sufficiently small' or considering 'functions with *BMO* norm sufficiently small', without specifying what 'sufficiently small' precisely means. In particular in PDEs, the coefficients or the initial data are sometimes required to have small BMO norm to ensure existence or to avoid singularities of the solutions.

We point out that while (1.2) is a simple consequence of the Jensen inequality, the proofs of the inverse implication (1.3) require the fundamental property enjoyed by BMO functions which is the *John–Nirenberg inequality* (JN in short): There exist positive constants c_1, c_2 independent on f such that

$$\sup_{Q\subset\Omega} \frac{1}{|Q|}\left|\{x \in Q : |f(x) - f_Q| > t\}\right| \leq c_1 e^{-\frac{c_2}{\|f\|_{BMO}}\, t} \tag{1.4}$$

for every $t > 0$.

In particular, the proof of (1.3) shows that if we pick $\lambda < c_2/\|f\|_{BMO}$ then λ is 'sufficiently small' to ensure that $e^{\lambda f}$ belongs to A_2.

On the other side, we observe that the JN inequality (1.4) remains true if we replace the constants with $c'_1 > c_1$ and $c'_2 < c_2$. Therefore, the condition $\lambda < c_2/\|f\|_{BMO}$ might provide information on the size of λ only if we consider the upper bound of the constant c_2 for which (1.4) holds true. As far as we know the

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sharp bounds c_1^* and c_2^* for the constants appearing in the JN inequality have been found only in dimension one. More precisely, $c_1^* = \frac{1}{2}e^{4/e}$ [11] and $c_2^* = 2/e$ [9].

The first main result of this article answers the previous question by establishing the following characterization of the class A_2:

THEOREM 1.1 (Characterization of A_2). *Let w be a weight defined on $\mathbb{R}^n$ and set $f := \log w$. Then*

$$w \in A_2 \iff f \in BMO \text{ with } \|f\|_{BMO} < c_2^* \tag{1.5}$$

where

$$c_2^* = \sup\{c_2 > 0 : \ (1.4) \text{ holds true for all } f \in BMO\}.$$

In essence, theorem 1.1 shows that the natural logarithm maps A_2 into open balls of BMO of radius smaller than c_2^*.

A corresponding mapping holds true if we restrict ourselves to the proper subsets $BLO \subset BMO$ and $A_1 \subset A_2$. We recall that a measurable function f defined over Ω is said to have *bounded lower oscillation*, and we write $f \in BLO(\Omega)$, if the quantity

$$\|f\|_{BLO} = \sup_Q (f_Q - \operatorname{ess\,inf}_Q f) \tag{1.6}$$

is finite.

The space BLO was introduced by Coifman and Rochberg in 1980 [1]. It comes out when we consider the action on BMO of the *Hardy–Littlewood maximal operator* defined by

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f|.$$

Namely, it results that the operator $M : BMO \to BLO$ is bounded.

The space BLO is strictly contained in BMO but it is not a linear subspace of BMO as the classical counterexample shows: $-\log|x| \in BLO$ but $\log|x| \notin BLO$. Nevertheless, we refer to (1.6) as the BLO *norm* of the function f, being this 'norm' in fact subadditive and positive homogeneous.

From the definitions, it is easy to observe that $BLO \cap -BLO = L_\infty$. Furthermore in [1], Coifman and Rochberg proved that

$$BMO = BLO - BLO.$$

Corresponding relationships to (1.3) and (1.2) hold true. Indeed we have

$$w \in A_1 \implies \log w \in BLO \tag{1.7}$$

and

$$f \in BLO \implies \exp(\lambda f) \in A_1 \tag{1.8}$$

for some $\lambda > 0$.

Therefore, the same question on the size of λ can be considered for the mapping $A_1 \to BLO$. It is also interesting because this mapping provides a connection which 'translates' some properties of the BMO space into properties of the class A_∞ and

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vice versa. As example, the statement $BMO = BLO - BLO$ is a property that can be regarded as the 'BMO version' of the Jones' factorization theorem $A_2 = A_1/A_1$.

This strict connection is also visible via the Hardy maximal operator, by observing that the mapping $BMO \to BLO$ is bounded as well as the mapping $A_\infty \to A_1$ is bounded.

Now, using the fact that the sharp bounds of the JN constants are known, it is possible to prove that in dimension one the natural logarithm maps A_1 into the unit ball of BLO. This is stated by the following theorem:

THEOREM 1.2 (Characterization of A_1). *Let w be weight defined on $\mathbb{R}$. Then*

$$w \in A_1 \iff \log w \in BLO \text{ with } \|\log w\|_{BLO} < 1. \tag{1.9}$$

As application we use theorem 1.1 to refine a result due to Pattakos and Volberg (see [15]) who made the class A_∞ a metric space by introducing a distance between two elements $u, v \in A_\infty$ defined as follows:

$$d_{A_\infty}(u, v) = \|\log u - \log v\|_{BMO}.$$

They proved that the closed balls are complete spaces provided the radius is assumed, once again, 'sufficiently small'. By using theorem 1.1, we can make this statement more precise by proving that the closed balls $B_{A_\infty}(\rho)$ are Banach spaces for any $\rho < c_2^*$ (see §5).

The connection between the space BMO and the class of Muckenhoupt's weights A_∞ has been largely investigated also in more general settings. As examples: The article by Kinnunen and Shukla studied the case of metric measure spaces under appropriate conditions (see [8]), the article by Dafni, Gibara, and Lavigne (see [2]) studies these properties when BMO is defined on a measure space with appropriate covering properties, and the article by Nielsen and Sikic (see [12]) consider the case when cubes are replaced by more general structured basis for $\mathbb{R}^n$ (including cubes, balls, rectangles, dyadic cubes). As future development, we consider of some interest to investigate under what conditions the results of this article could be proved also in these more general settings.

The article will proceed as follows: In §2, we provide definitions and some basic properties of A_∞ weights. In §3, we deepen the connection between BMO and A_2 in $\mathbb{R}^n$ and in §4 the connection between BLO and A_1 in $\mathbb{R}$. In §5, we briefly discuss the structure of A_∞ as metric space. Finally, §6 contains the proofs of theorem 1.1 and theorem 1.2.

2. The Muckenhoupt class of weights A_∞

A *weight function* is a non-negative and locally integrable function defined on $\Omega \subset \mathbb{R}^n$. For $p > 1$, we set

$$[w]_{A_p} := \sup_{Q \subseteq \Omega} \fint_Q w(x)dx \left(\fint_Q w^{\frac{1}{1-p}(x)} dx \right)^{p-1} \tag{2.1}$$

and the weight w is called a *Muckenhoupt A_p-weight* if $[w]_{A_p} < \infty$.

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In the limit case $p=1$, we set

$$[w]_{A_1} := \sup_{Q\subseteq\Omega} \fint_Q w(x)dx \ (\operatorname{ess\,inf}_{x\in Q} w(x))^{-1} \tag{2.2}$$

and w is said an *A_1-weight* if $[w]_{A_1} < \infty$.

Similarly, in the limit case $p=\infty$, definition (2.1) becomes

$$[w]_{A_\infty} := \sup_{Q\subseteq\Omega} \fint_Q w(x)dx \ \exp\left(\fint_Q \log\frac{1}{w(x)}dx\right) \tag{2.3}$$

and w is said an *A_∞-weight* if $[w]_{A_\infty} < \infty$.

We refer to the quantities $[w]_{A_p}$, $[w]_{A_1}$, and $[w]_{A_\infty}$, respectively, as the *A_p characteristic*, the *A_1-characteristic*, and the *Hruscev A_∞-characteristic* of the weight w. The last name is due to the fact that the class A_∞ can be also defined based on the *Fujii–Wilson A_∞-characteristic* given by (see Hytönen and Pérez [5]):

$$(w)_{A_\infty} := \sup_{Q\subset\Omega} \fint_Q M(w\mathbf{1}_Q) \tag{2.4}$$

where M stands for the usual *Hardy–Littlewood maximal operator*. The Hruscev characteristic $[w]_{A_\infty}$ and the Fujii–Wilson characteristic $(w)_{A_\infty}$ are equivalent and define the same class.

By the definitions and by an easy application of the Hölder inequality, it immediately follows that the application $p \to [w]_{A_p}$ is decreasing, i.e. $1 \le [w]_{A_\infty} \le [w]_{A_p} \le [w]_{A_1}$ and it is known that it is also continuous, i.e.

$$\lim_{p\to\infty}[w]_{A_p} = [w]_{A_\infty} \qquad\qquad \lim_{p\to 1^+}[w]_{A_p} = [w]_{A_1}$$

(see [16] and the references therein).

Recall that an equivalent definition of A_∞ is given by

$$A_\infty = \bigcup_{p>1} A_p$$

and if we replace the union of the A_p classes with their intersection we obtain the class

$$A_* = \bigcap_{p>1} A_p$$

which is strictly larger than A_1. This class shows up in some interesting problems and has applications in obtaining higher integrability of local solutions in non-linear elliptic PDEs. The class was introduced in [4] to characterize the closure of L^∞ in *BMO*:

$$f \in cl_{BMO}(L^\infty) \iff e^f, e^{-f} \in A_*.$$

More recently, it has been proved that in dimension one it plays a role in another connection between *BMO* and A_∞. Namely, if we consider the inner compositions

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of a BMO function f with an increasing homeomorphism h, a well-known theorem of Jones [7] states that, for $f \in BMO$,

$$f \circ h \in BMO \iff h' \in A_\infty$$

but it also holds true that, for $f \in BLO$,

$$f \circ h \in BLO \iff h' \in A_*$$

(see [17]).

The following proposition is a known characterization of A_2 which will be used in the following. We provide a little refinement of the statement and the proof for sake of completeness.

PROPOSITION 2.1. *Let f be a function in L^1 and define*

$$[f]_{\mathrm{EXP}} := \sup_Q \fint_Q e^{|f(x)-f_Q|}dx. \tag{2.5}$$

Then,

$$e^f \in A_2 \iff [f]_{\mathrm{EXP}} < \infty$$

and it is

$$\frac{1}{2}[f]_{\mathrm{EXP}} \le [e^f]_{A_2} \le [f]^2_{\mathrm{EXP}}. \tag{2.6}$$

Proof. Assume $e^f \in A_2$. We have

$$\begin{aligned} \fint_Q e^{|f-f_Q|} &= \tfrac{1}{|Q|}\textstyle\int_{\{x\in Q: f(x)>f_Q\}} e^{(f-f_Q)} + \tfrac{1}{|Q|}\int_{\{x\in Q: f(x)\le f_Q\}} e^{(f_Q-f)} \\ &= e^{-f_Q}\tfrac{1}{|Q|}\textstyle\int_{\{x\in Q: f(x)>f_Q\}} e^{f} + e^{f_Q}\tfrac{1}{|Q|}\int_{\{x\in Q: f(x)\le f_Q\}} e^{-f} \\ &\le e^{-f_Q}\fint_Q e^f + e^{f_Q}\fint_Q e^{-f} \le 2\fint_Q e^f \fint_Q e^{-f} \\ &\le 2[e^f]_{A_2} \end{aligned}$$

which gives, passing to the supremum over cubes, the left-hand side of (2.6).

On the other side, assuming $[f]_{\mathrm{EXP}} < \infty$, we have

$$\fint_Q e^f \fint_Q e^{-f} = \fint_Q e^{(f-f_Q)} \fint_Q e^{(f_Q-f)} \le \left(\fint e^{|f-f_Q|}\right)^2 \le [f]^2_{\mathrm{EXP}}$$

which provides the right-hand side of (2.6). □

As consequence of this property, we have the (strict) inclusions

$$L^\infty \subsetneq \log(A_2) \subsetneq BMO. \tag{2.7}$$

3. The relationship between A_2 and BMO

As mentioned logarithms of A_2 weights are functions in BMO. The following proposition provides a quantitative version of this property.

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PROPOSITION 3.1. *Let w be a weight defined on $\mathbb{R}^n$. Then*

$$\|\log w\|_{BMO} \leq \log(2\ [w]_{A_2}). \tag{3.1}$$

Proof. Fix $w \in A_2$ and set $A = [w]_{A_2}$ and $f = \log w$. By proposition 2.1, we know that $w \in A_2$ implies

$$\fint_Q e^{|f-f_Q|} \leq 2[w]_{A_2}. \tag{3.2}$$

Applying in turn the Jensen inequality and (3.2) we obtain:

$$\begin{aligned} e^{\|\log w\|_{BMO}} &= \exp\left(\sup_Q \fint_Q |f-f_Q|\right) \\ &= \sup_Q \exp\left(\fint_Q |f-f_Q|\right) \\ &\leq \sup_Q \fint_Q \exp\left(|f-f_Q|\right) \\ &= [f]_{\exp} \leq 2[w]_{A_2}. \end{aligned} \tag{3.3}$$

□

REMARK 3.1. Proposition 3.1 can be obtained as corollary of a stronger inequality obtained by Korey in [10] (with a completely different proof):

$$\|\log w\|_{BMO} \leq \log(2[w]_{A_\infty}) \tag{3.4}$$

simply recalling that for any $p>1$ is $[w]_{A_\infty} \leq [w]_{A_p}$. Worth to notice that this estimate is stronger than the one found by Hytönen and Pérez in [5]:

$$\|\log w\|_{BMO} \leq \log(2e[w]_{A_\infty}). \tag{3.5}$$

Furthermore, it is interesting to mention that for *flat weights* such as $[w]_{A_2} = 1+\delta < 2$ the inequality (3.1) can be improved. The first result in this direction has been proved by Korey [10]:

$$\|\log w\|_{BMO} \leq c\sqrt{\log([w]_{A_\infty})} \tag{3.6}$$

where c is a universal constant independent on w. The second one is due to Pattakos and Volberg [15] who proved that for flat A_2 weights

$$\|\log w\|_{BMO} \leq 2\sqrt{[w]_{A_2}-1} \tag{3.7}$$

and both results clearly improve (3.1) in case of flat weights.

As seen, the precise embedding of A_2 weights in BMO is obtained in theorem 1.1 which can in short be restated as

$$\log(A_2) = \{f \in BMO : \|f\|_{BMO} < c_2^*\}.$$

From the proof of theorem 1.1, we get two corollaries. The first one is a quantification of the known fact that functions in BMO are locally exponentially integrable.

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PROPOSITION 3.2. *Let $f \in BMO$. If $\lambda \in (0, c_2/\|f\|_{BMO})$ then*

$$[\lambda f]_{\exp} \leq 1 + \frac{c_1 \lambda \|f\|_{BMO}}{c_2 - \lambda \|f\|_{BMO}} \tag{3.8}$$

where c_1 and c_2 are the constants in the JN inequality (1.4).

Proof. By the JN inequality using the same standard argument as in (6.1), we have

$$\int_Q e^{\lambda |f - f_Q|} \leq |Q| + |Q| \; c_1 \lambda \int_0^\infty e^{\left(\lambda - \frac{c_2}{\|f\|_{BMO}}\right) t} \, dt \tag{3.9}$$

which gives

$$[\lambda f]_{\exp} = \sup_Q \fint_Q e^{\lambda |f - f_Q|} \leq 1 + c_1 \lambda \int_0^\infty e^{\left(\lambda - \frac{c_2}{\|f\|_{BMO}}\right) t} \, dt.$$

Now the assumption $\lambda \in (0, c_2/\|f\|_{BMO})$ ensures that the integral on the right-hand side converges and by a direct calculation we obtain (3.8). □

PROPOSITION 3.3. *Let $w \in A_\infty$ a weight defined on $\mathbb{R}^n$. Then $\log w \in BMO$ and*

$$[w]_{A_\infty} \leq 1 + \frac{c_1^* \| \log w\|_{BMO}}{c_2^* - \| \log w\|_{BMO}} \tag{3.10}$$

where c_1^*, c_2^* are the sharp constants of the JN inequality.

Proof. By re-writing the condition A_∞ as follows

$$[w]_{A_\infty} = \sup_Q \fint_Q w \exp\left(\fint_Q \log \frac{1}{w}\right) = \sup_Q \fint_Q e^f \cdot e^{(-\log w)_Q} = \sup_Q \fint_Q e^{f - f_Q} \tag{3.11}$$

where $f = \log w$ we have that

$$\fint_Q e^{f - f_Q} \leq \fint_Q e^{|f - f_Q|} \leq 1 + \frac{c_1 \|f\|_{BMO}}{c_2 - \|f\|_{BMO}} \tag{3.12}$$

from which, passing to the supremum over cubes and over the infimum of the JN constants, the inequality (3.10) follows. □

4. The relationships between A_1 and BLO

A first relationship between these spaces is stated by the following theorem; it can be found in [14] where the proof leverages the commutation property for the natural maximal operator. We give here a new simpler proof which does not require the use of commutators.

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THEOREM 4.1. *Let w be an A_∞ weight. Then $w \in A_1$ if and only if $\log w \in BLO$ and*

$$e^{\|\log w\|_{BLO}} \le [w]_{A_1} \le [w]_{A_\infty} e^{\|\log w\|_{BLO}}. \tag{4.1}$$

REMARK 4.1. Of course the assumption $w \in A_\infty$ is necessary only in one direction of the equivalence because $f \in BLO$ does not imply $e^f \in A_\infty$. It is redundant in the other direction as $w \in A_1$ trivially implies $w \in A_\infty$.

Proof. The implication $w \in A_1 \implies \log w \in BLO$ is known and follows by the *Jensen inequality*. We report here for completeness:

$$\begin{aligned} w \in A_1 &\implies \fint_Q w \le [w]_{A_1} \operatorname*{ess\,inf}_Q w \\ &\implies \fint_Q e^{\log w} \le [w]_{A_1} \operatorname*{ess\,inf}_Q e^{\log w} \\ &\implies e^{(\log w)_Q} \cdot e^{-\operatorname{ess\,inf}_Q \log w} \le [w]_{A_1} \\ &\implies e^{\|\log w\|_{BLO}} \le [w]_{A_1}. \end{aligned} \tag{4.2}$$

This proves the first implication and the left-hand side of (4.1).

To prove the opposite implication assume $f = \log w \in BLO \cap \log(A_\infty)$ and prove the right-hand side of (4.1).

Let us start by observing that the assumption $w = e^f \in A_\infty$ can be written as a *reverse Jensen inequality*, i.e.

$$\fint_Q w \cdot \exp\left(\fint_Q \log \frac{1}{w}\right) \le [w]_{A_\infty} \iff \fint_Q w \le [w]_{A_\infty} e^{\log w_Q}.$$

By definition, the assumption $f \in BLO$ means

$$f_Q - \operatorname*{ess\,inf}_Q f \le \|f\|_{BLO}$$

thus from $f = \log w \in BLO \cap \log(A_\infty)$ follows that

$$\begin{aligned} \fint_Q e^f &\le [w]_{A_\infty}\ e^{f_Q} \le [w]_{A_\infty}\ e^{\|f\|_{BLO} + \operatorname{ess\,inf}_Q f} \\ &\le [w]_{A_\infty}\ e^{\|f\|_{BLO}} \cdot e^{\operatorname{ess\,inf}_Q f} \le [w]_{A_\infty}\ e^{\|f\|_{BLO}} \cdot \operatorname*{ess\,inf}_Q e^f. \end{aligned} \tag{4.3}$$

Hence we obtained

$$\fint_Q w \le [w]_{A_\infty}\ e^{\|\log w\|_{BLO}}\ \operatorname*{ess\,inf}_Q w$$

and therefore

$$[w]_{A_1} \le [w]_{A_\infty}\ e^{\|\log w\|_{BLO}}$$

which is the right-hand inequality of (4.1). □

The next theorem establishes a sharp estimate in dimension one holding between the *BMO* and the *BLO* norms which involves the sharp constant c_2^*.

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THEOREM 4.2. *If $f \in BLO(\mathbb{R})$ then*

$$\|f\|_{BMO} \leq c_2^* \, \|f\|_{BLO} \tag{4.4}$$

and the constant $c_2^* = 2/e$ is the best possible.

Proof of theorem 4.2. Fix $f \in BLO$ and let $I \subset \mathbb{R}$ be an interval. We recall that, for any $f \in BMO$, the quantity

$$\|f\|_{BMO_p} := \sup_I \left(\fint_I |f(x) - f_I|^p \right)^{\frac{1}{p}}$$

is finite for every $p \geq 1$ and $\|\cdot\|_{BMO_p}$ defines a norm equivalent to $\|\cdot\|_{BMO}$.

When f also belongs to BLO, it can be proven that (see [13], theorem 1.1)

$$\|f\|_{BMO_p} \leq K_p \, \|f\|_{BLO} \tag{4.5}$$

where the optimal value of K_p for all $p \geq 1$ is given by

$$K_p = \left(\int_{-1}^{\infty} |t|^p \, e^{-t-1} \, dt \right)^{\frac{1}{p}}.$$

The direct calculation of the integral shows that for $p = 1$ the sharp constant K_1 is equal to $2/e$ which coincides with the sharp constant c_2^* in the JN inequality, i.e. $K_1 = c_2^* = 2/e$. Therefore, the inequality (4.5) for $p = 1$ becomes

$$\|f\|_{BMO} \leq \frac{2}{e} \, \|f\|_{BLO}$$

where the constant is sharp in the sense that for the extremal function $f_0(x) = \ln \frac{1}{x} (x > 0)$ we have

$$\|f_0\|_{BMO} = \frac{2}{e} \qquad \|f_0\|_{BLO} = 1$$

so the constant cannot be decreased. □

Theorem 4.2, at least in dimension one, helps us make the link between BLO and A_1 more precise than the one between BMO and A_2. In fact, it will be used to prove theorem 1.2, which can be in short rephrased as follows

$$\log(A_1) = \{f \in BLO : \|f\|_{BLO} < 1\}.$$

It shows that the natural logarithm maps the Muckenhoupt class A_1 onto the open unit ball of BLO.

5. *BMO* and A_∞ as metric spaces

In [15], Pattakos and Volberg showed that A_∞ can be equipped with a structure of metric space. We start by observing that as BMO functions can be identified only

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up to an additive constant, we can establish a corresponding equivalence relation in A_∞ where functions are identified only up to a multiplicative constant. Namely for $u, v \in A_\infty$, we define

$$u \sim v \iff \exists\, c > 0 :\ u = cv \quad a.e.$$

and, for $p \in [1, +\infty]$, we can consider the quotient spaces

$$A_p / \sim .$$

For sake of simplicity, we will keep the same notation A_p for the quotient space. With this position, we can define a distance in A_∞ and in every A_p class. For any $u, v \in A_\infty$, we define

$$d_{A_\infty}(u, v) = \| \log u - \log v \|_{BMO}.$$

With these definitions, the closed unit ball of A_p of radius ρ is defined as the set of the A_p weights having distance from the constant weight 1 not greater than ρ:

$$B_{A_p}(\rho) = \{ w \in A_p : d_{A_\infty}(w, 1) \le \rho \}.$$

In [15], Pattakos and Volberg showed that none of the A_p spaces can be complete, but they proved that the closed unit ball $B_{A_p}(\rho)$ is actually complete for 'sufficiently small' radius ρ.

We want to make this statement more precise by using theorem 1.1 which also helps simplify the original proof. Namely, we prove the following:

THEOREM 5.1. *For every $p \in [1, +\infty]$, the closed ball $B_{A_p}(\rho)$ is complete in the metric space (A_p, d_{A_∞}) whenever $\rho < c_2^*$.*

Proof. Let (w_j) be a Cauchy sequence in $B_{A_p}(\rho)$ and set $f_j = \log w_j$. Hence, the sequence (f_j) belongs to BMO and is a Cauchy sequence in BMO which is a complete space. Therefore, there exists $f \in BMO$ such that $f_j \to f$ in the BMO norm, i.e.

$$\| f_j - f \|_{BMO} \to 0. \tag{5.1}$$

Observing that $w_j \in B_{A_p}(\rho)$ implies

$$d_{A_\infty}(w_j, 1) \le \rho \iff \| \log w_j - \log 1 \|_{BMO} \le \rho \iff \| f_j \|_{BMO} \le \rho$$

by theorem 1.1 we have that $w_j \in B_{A_2}(\rho)$. Moreover, this inequality together with (5.1) gives $\| f \|_{BMO} \le \rho < c_2^*$. Again by theorem 1.1, we have that the weight $w = e^f$ belongs to $B_{A_2}(\rho)$ and we have that

$$d_{A_\infty}(w_j, w) = \| \log w_j - \log w \|_{BMO} = \| f_j - f \|_{BMO} \to 0$$

which shows that the arbitrary Cauchy sequence w_j converges.

This completes the proof. □

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6. Proofs of theorems 1.1 and 1.2

Proof of theorem 1.1. Let us start by assuming $f \in BMO$ with $\|f\|_{BMO} < c_2^*$. Pick c_2 such that $\|f\|_{BMO} < c_2 < c_2^*$. By the JN inequality using standard arguments, we obtain:

$$\begin{aligned} \int_Q e^{|f-f_Q|} &= \int_0^\infty |\{x \in Q : e^{|f(x)-f_Q|} > s\}|\, ds \\ &= \int_0^1 |\{x \in Q : e^{|f(x)-f_Q|} > s\}|\, ds \\ &\quad + \int_1^\infty |\{x \in Q : e^{|f(x)-f_Q|} > s\}|\, ds \\ &= |Q| + \int_0^\infty e^t\, |\{x \in Q : |f(x) - f_Q| > t\}|\, dt \\ &\leq |Q| + \int_0^\infty e^t c_1 e^{-\frac{c_2}{\|f\|_{BMO}} t}\, dt \cdot |Q| \\ &= |Q| + |Q|\, c_1 \int_0^\infty e^{\left(1 - \frac{c_2}{\|f\|_{BMO}}\right) t}\, dt \end{aligned} \tag{6.1}$$

where the integral in the second terms of (6.1) we made the change of variable $s = e^t$. The assumption $\|f\|_{BMO} < c_2$ implies that the integral on the right-hand side converges and the direct calculation gives

$$\fint_Q e^{|f-f_Q|} \leq 1 + \frac{c_1 \|f\|_{BMO}}{c_2 - \|f\|_{BMO}}.$$

Passing to the supremum over cubes on the left-hand side and to the infimum over the constants on the right-hand side, we have

$$[f]_{\mathrm{EXP}} \leq 1 + \frac{c_1^* \|f\|_{BMO}}{c_2^* - \|f\|_{BMO}} < \infty. \tag{6.2}$$

Therefore, by proposition 2.1, from (6.2) and the assumption $\|f\|_{BMO} < c_2^*$, it follows that $e^f \in A_2$.

To prove the opposite implication, let $w \in A_2$ and set $f = \log w$. First we recall that $w \in A_2$ implies $f \in BMO$. In fact by the characterization of A_2 weights stated in proposition 2.1, we have

$$\|f\|_{BMO} = \sup_Q \fint_Q |f(x) - f_Q| dx \leq \sup_Q \fint_Q e^{|f(x)-f_Q|} dx = [f]_{EXP} < \infty.$$

Furthermore, by using the same standard arguments used in (6.1), for any $\lambda < \frac{c_2}{\|f\|_{BMO}}$, we have that $e^{\lambda f} \in A_2$. Therefore, if we define

$$\lambda_0(f) = \sup\{\lambda > 0 \ : \ e^{\lambda f} \in A_2\}$$

we have $\frac{c_2}{\|f\|_{BMO}} \leq \lambda_0(f)$ and, considering that c_2 does not depend on f, passing to the supremum we have

$$\frac{c_2^*}{\|f\|_{BMO}} \leq \lambda_0(f). \tag{6.3}$$

Observe that if $e^f \notin A_2$ then $\lambda_0(f) \leq 1$, thus from (6.3), it follows that

$$f \in BMO \setminus \log A_2 \implies \|f\|_{BMO} \geq c_2^*.$$

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It remains to prove that

$$f \in \log A_2 \implies \|f\|_{BMO} < c_2^*. \tag{6.4}$$

Assume that there exists $f \in \log A_2$ such that $\|f\|_{BMO} \geq c_2^*$. We consider two cases.

(i) First we consider the case when the equality $\|f\|_{BMO} = c_2^*$ holds, i.e. $\frac{c_2^*}{\|f\|_{BMO}} = 1$. Pick σ such that $\sigma > \lambda_0(f)$. As when $f \in \log A_2$ is strictly $\lambda_0(f) > 1$, clearly is $\sigma > 1$. Note that a basic property of A_2 weights gives $e^f \in A_2 \implies e^{\frac{f}{\sigma}} \in A_2$ for every $\sigma > 1$. This means that $\frac{f}{\sigma} \in \log A_2$ and applying (6.3) to the function $\frac{f}{\sigma}$ we obtain

$$\sigma = \sigma \frac{c_2^*}{\|f\|_{BMO}} = \frac{c_2^*}{\|\frac{f}{\sigma}\|_{BMO}} \leq \lambda_0(f)$$

against the assumptions $\sigma > \lambda_0(f)$. From the contradiction, we have that the equality $\|f\|_{BMO} = c_2^*$ cannot hold true.

(ii) If now assume that for some $f \in \log A_2$, $\|f\|_{BMO} > c_2^*$, for any $\epsilon < 1$, we also have that $\epsilon f \in \log A_2$. Hence, we can pick $\epsilon_0 = \frac{c_2^*}{\|f\|_{BMO}} < 1$ so that $\epsilon_0 f \in \log A_2$ and $\|\epsilon_0 f\|_{BMO} = c_2^*$. But we have already seen in (i) that the equality cannot hold true.

Therefore in both cases, the assumption $\|f\|_{BMO} \geq c_2^*$ gives a contradiction. This proves that $\|f\|_{BMO} < c_2^*$ and completes the proof. □

Proof of theorem 1.2. Set $f = \log w$ and assume first $\|f\|_{BLO} < 1$. By theorem 4.2 for any $f \in BLO$, we have that

$$\|f\|_{BMO} \leq c_2^* \|f\|_{BLO} \tag{6.5}$$

so that the assumption $\|f\|_{BLO} < 1$ implies $\|f\|_{BMO} < c_2^*$. Hence by theorem 1.1, we have that $w = e^f \in A_2 \subset A_\infty$. Finally, theorem 4.1 gives $w \in A_1$ with the norm control

$$e^{\|f\|_{BLO}} \leq [w]_{A_1} \leq [w]_{A_\infty} e^{\|f\|_{BLO}}. \tag{6.6}$$

To prove the opposite implication assume $w \in A_1$, thus is also $w \in A_2$ and $f = \log w \in BMO$. By proposition 2.1, we have

$$[f]_{\mathrm{EXP}} = \sup_Q \fint_Q e^{|f(x) - f_Q|} dx < \infty.$$

Now we can estimate the constant $[f]_{\mathrm{EXP}}$. Following the same steps as in theorem 1.1, the JN inequality gives

$$\fint_Q e^{|f - f_Q|} = 1 + \; c_1 \int_0^\infty e^{\left(1 - \frac{c_2}{\|f\|_{BMO}}\right) t} \, dt. \tag{6.7}$$

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By theorem 1.1, we have that $w \in A_2 \implies \|f\|_{BMO} < c_2^*$. Therefore, for any c_2 such that $\|f\|_{BMO} < c_2 < c_2^*$, the integral converges and we have

$$\fint_Q e^{|f-f_Q|} \leq 1 + \frac{c_1\|f\|_{BMO}}{c_2 - \|f\|_{BMO}}. \tag{6.8}$$

As c_1 and c_2 do not depend on f, passing to the sharp constant we define

$$C(f) := 1 + \frac{c_1^*\|f\|_{BMO}}{c_2^* - \|f\|_{BMO}}$$

and from (6.8), for every cube Q we have

$$\fint_Q e^{f-f_Q} \leq \fint_Q e^{|f-f_Q|} \leq C(f). \tag{6.9}$$

Now recall that $w \in A_1$ implies $f = \log w \in BLO$ (theorem 4.1), i.e.

$$\fint_Q f - \operatorname{ess\,inf}_Q f \leq \|f\|_{BLO}. \tag{6.10}$$

Combining (6.9) and (6.10) follows that

$$\begin{aligned} \fint_Q e^f \quad &\leq C(f)\ e^{f_Q} \leq C(f)\ e^{\|f\|_{BLO}+\operatorname{ess\,inf}_Q f} \\ &\leq C(f)\ e^{\|f\|_{BLO}} \cdot e^{\operatorname{ess\,inf}_Q f} \leq C(f)\ e^{\|f\|_{BLO}} \cdot \operatorname*{ess\,inf}_Q e^f \end{aligned} \tag{6.11}$$

and using the estimate $e^{\|\log w\|_{BLO}} \leq [w]_{A_1}$ of theorem 4.1 we land to

$$\fint_Q w \leq C(f)\ [w]_{A_1}\ \operatorname*{ess\,inf}_Q w.$$

Therefore, by definition of A_1 characteristic, $C(f) \geq 1$ which means

$$\frac{c_1^*\|f\|_{BMO}}{c_2^* - \|f\|_{BMO}} \geq 0. \tag{6.12}$$

Now apply again the sharp estimate (6.5) to obtain

$$0 \leq \frac{c_1^*\|f\|_{BMO}}{c_2^* - \|f\|_{BMO}} \leq \frac{c_1^*c_2^*\|f\|_{BLO}}{c_2^* - c_2^*\|f\|_{BLO}} = \frac{c_1^*\|f\|_{BLO}}{1 - \|f\|_{BLO}} \tag{6.13}$$

that necessarily implies $\|f\|_{BLO} < 1$. The proof is complete. □

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