try $p = 2$. Writing y' for the imaginary part of $2a/b - \theta$ we calculate $y' = 2y - \sqrt{19}/2$ and then $\sqrt{3} - \sqrt{19}/2 \leqslant y' \leqslant 0$. In order to show that the distance from $2a/b$ to the nearest element of R is strictly less than 1, it suffices to show that $y' > -\sqrt{3}/2$. But $\sqrt{19} < \sqrt{27} = 3\sqrt{3}$, so $-\sqrt{3}/2 < \sqrt{3} - \sqrt{19}/2 < 0$, as required. The result now follows unless $2a/b \in R$, which can only happen when $y = \sqrt{19}/4$.

In this special case, adding ordinary integers to a/b as necessary, we may assume that $a/b = (\pm 1 + \sqrt{19})/4$, and by symmetry these two cases are essentially the same. We now choose $p = (\mp 1 + \sqrt{19})/2$, so that $ap/b = 5/2$ and if $q = 2$ then $|ap/b - q| = \frac{1}{4}$.

Acknowledgements

I thank Jérémy Blanc for his interest in this proof, and especially for pointing out that in an earlier version, among other deficiencies, I had failed to consider the final case $2a/b \in R$ in the proof of Lemma 1.

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10.1017/mag.2017.69

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101.16 When does $n!$ have n digits?

The problem in the title is a relatively old chestnut: in this Note, we look at what happens for different bases. Notice that $n = 1$ is always a trivial solution so we will focus on $n > 1$ below. Our main tool is the following crude version of Stirling's formula:

$$\left(\frac{n}{e}\right)^n < e\left(\frac{n}{e}\right)^n < n! < ne\left(\frac{n}{e}\right)^n \text{ for } n \geqslant 2. \tag{1}$$

A quick proof of these bounds uses lower and upper approximating rectangles for $\int_1^n \ln x\, dx$ to establish

$$\ln n! = \sum_{k=2}^{n} \ln k > \int_1^n \ln x\, dx = n\ln n - n + 1 \text{ or } n! > e\left(\frac{n}{e}\right)^n > \left(\frac{n}{e}\right)^n$$

and

$$\ln n! = \ln n + \sum_{k=1}^{n-1} \ln k < \ln n + \int_1^n \ln x\, dx = (n+1)\ln n - n + 1 \text{ or } n! < ne\left(\frac{n}{e}\right)^n.$$

https://doi.org/10.1017/mag.2017.70 Published online by Cambridge University Press

In base b, $n!$ has n digits if

$$b^{n-1} \leqslant n! < b^n \tag{2}$$

which can also be set in the form of the equation

$$\left\lfloor \frac{\ln n!}{\ln b} \right\rfloor = n - 1.$$

The Table below shows the non-trivial solutions in n for $2 \leqslant b \leqslant 10$.

Base (b)	Values of $n > 1$ for which $n!$ has n digits
2	2, 3
3	5, 6
4	7, 8
5	10, 11
6	12, 13
7	15, 16
8	17, 18, 19
9	20, 21
10	22, 23, 24

TABLE

In searching for these solutions, it is useful to observe that

$$b < n < eb \tag{3}$$

holds for any non-trivial solution when $b \geqslant 3$.

The upper bound comes from (1). For if $n > eb$, then $n! > \left(\frac{n}{e}\right)^n > b^n$ so that $n!$ has more than n digits.

The lower bound comes from the fact that the sequence $(n!)^{1/(n-1)}$ strictly increases from 2 for $n \geqslant 2$. Since $b! = 2 \times 3 \times \ldots \times b < b^{n-1}$ for $b \geqslant 3$, $(b!)^{1/(n-1)} < b$ so that $(n!)^{1/(n-1)} < b$ or $n! < b^{n-1}$ for $n \leqslant b$ so, for these values of n, $n!$ has fewer than n digits.

We now focus on three questions which are prompted by the data in the Table.

Question 1: *For each* $n \geqslant 15$, *is there a solution to* (2) *for* b?

Given n, solutions to (2) for b lie in the interval $(n!)^{1/n} < b \leqslant (n!)^{1/(n-1)}$ of length $f(n) = (n!)^{1/(n-1)} - (n!)^{1/n}$. If $f(n) > 1$, then there will be a solution for b. But

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$$\begin{aligned} f(n) &= (n!)^{1/n}\left[(n!)^{1/(n(n-1))} - 1\right] \\ &> \frac{n}{e}\left[\left(\frac{n}{e}\right)^{\frac{1}{n-1}} - 1\right] \qquad \text{by (1)} \\ &= \frac{n}{e}\left[e^{\frac{1}{n-1}\ln\left(\frac{n}{e}\right)} - 1\right] \\ &> \frac{n}{e}\cdot\frac{\ln\left(\frac{n}{e}\right)}{n-1} \qquad \text{since } e^x - 1 > x \text{ for all } x \neq 0 \\ &> \frac{\ln n - 1}{e}. \end{aligned}$$

Thus $f(n) > 1$ once $n \geqslant e^{e+1} > 41$ and it is easily checked that there are also solutions for $15 \leqslant n \leqslant 41$. So the omitted values of n in the Table (namely 4, 9, 14) are the only ones that do not appear and, although it looks as if the sets of values of n for each b are disjoint, because $f(n) \to \infty$ as $n \to \infty$ they eventually overlap more and more. But the growth rate is slow: $f(n)$ does not reach 2 until $n = 577$. (It is in fact the case that $f(n)$ is increasing with n and $f(22) > 1$, but I do not (yet) have a proof of the former suitable for sharing!)

Question 2: *For each base b, is there a solution to* (2) *for n with* $n > 1$?

For any base b we show that there is indeed an $n > 1$ for which $n!$ has n digits. From the Table, we may suppose that $b \geqslant 3$. Since $(n!)^{1/(n-1)}$ strictly increases to infinity, there is a largest N for which $(N-1)!^{1/(N-2)} < b$ or $(N-1)! < b^{N-2}$. We claim that $N!$ has N digits. For $N!^{1/(N-1)} \geqslant b$ means that $N!$ has at least N digits. If it were to have more than N digits, then $N! \geqslant b^N$ so that $N = \dfrac{N!}{(N-1)!} > \dfrac{b^N}{b^{N-2}} = b^2$ which, for $b \geqslant 3$, contradicts the bound $N - 1 < eb$ from (3).

Question 3: *How many solutions are there to Questions* 1 *and* 2?

For this, we need to use the stronger version of Stirling's approximation, $n! \approx \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$. Then

$$(n!)^{\frac{1}{n}} \approx \frac{n}{e}\exp\left(\frac{\ln 2\pi n}{2n}\right) = \frac{n}{e}\left[1 + \frac{\ln n}{2n} + \ldots\right]$$

and

$$(n!)^{\frac{1}{n-1}} \approx \frac{n}{e}\left(\frac{\sqrt{2\pi}\,n^{3/2}}{e}\right)^{\frac{1}{n-1}} = \frac{n}{e}\exp\left[\frac{1}{n-1}\left(\ln\frac{\sqrt{2\pi}}{e} + \frac{3}{2}\ln n\right)\right] = \frac{n}{e}\left[1 + \frac{3\ln n}{2n} + \ldots\right].$$

Thus, in Question 1, $f(n) = (n!)^{1/(n-1)} - (n!)^{1/n} = \dfrac{\ln n}{e} + \ldots$, so for large n, there are approximately $\dfrac{\ln n}{e}$ solutions for b.

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Similarly, in Question 2, the condition $b^{n-1} \leqslant n! < b^n$ or $(n!)^{1/n} < b \leqslant (n!)^{1/(n-1)}$ becomes, for large b, $\frac{n}{e} + \frac{\ln n}{2e} < b \leqslant \frac{n}{e} + \frac{3\ln n}{2e}$ or

$$eb - \frac{3\ln n}{2} \leqslant n < eb - \frac{\ln n}{2}. \tag{4}$$

There are thus about $\ln n \approx \ln(eb)$ solutions for n near (but less than) eb.

In support of these calculations, the referee kindly supplied the following data. For $n = 10^6$, we expect about $\frac{\ln(10^6)}{e} \approx 5.08$ solutions for b: in fact they lie in the interval $367883 \leqslant b \leqslant 367887$. And, for $b = 10^6$, we expect about $\ln(eb) \approx 14.82$ solutions for n: in fact they lie in the interval $2718260 \leqslant n \leqslant 2718273$. These latter digits are reminiscent of e and we end by observing that this problem does indeed give rise to an unusual (and unusually inefficient!) method of calculating e. For each base $b \geqslant 2$, let x_b be a specific choice of one of the values of $n > 1$ for which $n!$ has n digits in base b. Then, as is immediate from (4), $\frac{x_b}{b} \to e$ as $b \to \infty$. (Actually, the full force of (4) is not needed here: the limit can be deduced from the bounds in (1) and (3).)

Acknowledgement

I am very grateful for the detailed and helpful comments of the referee which have enabled me to provide a much more satisfactory analysis of the problem.

10.1017/mag.2017.70

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101.17 Goats and birds

A goatherd wishes to tether a goat to a point on the edge of a circular field, in such a way that it has access to exactly half the area of the field. How long should the rope be? This is usually called the 'goat problem', although the creature in question could equally well be a dog, a bull or an alpaca.

Let us take the radius of the field to be 1 unit: it comprises a circle which we denote by C_1, with centre O. The goat is constrained to a circle C_2 of radius r (where r is to be found), with centre at a point P on the circumference of C_1. We need an expression for the area $A(r)$ of the intersection of the two circles. Of course, we take $r \leqslant 2$ (otherwise C_2 includes the whole of C_1).

In Figure 1, let θ_0 be the angle POR_1 and θ_1 the angle OPR_1. Then $OP = OR_1 = 1$ and $PR_1 = r$. Clearly, $2\theta_1 = \pi - \theta_0$, and from the triangle OPT we see that

$$\tfrac{1}{2}r = \cos\theta_1 = \sin\tfrac{1}{2}\theta_0. \tag{1}$$

so that

$$\theta_1 = \cos^{-1}\frac{r}{2}, \qquad \theta_0 = 2\sin^{-1}\frac{r}{2}. \tag{2}$$

https://doi.org/10.1017/mag.2017.70 Published online by Cambridge University Press