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α-Hölder linearization of hyperbolic diffeomorphisms with resonance

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Abstract. Concerning hyperbolic diffeomorphisms, one expects a better smoothness of linearization, but it may be confined by resonance among eigenvalues. Hartman gave a three-dimensional analytic mapping with resonance which cannot be linearized by a Lipschitz conjugacy. Since then, efforts have been made to give the α-Hölder continuity of the conjugacy and hope the exponent $\alpha < 1$ can be as large as possible. Recently, it was proved for some weakly resonant hyperbolic diffeomorphisms that α can be as large as we expect. In this paper we prove that this result holds for all C^∞ weakly resonant hyperbolic diffeomorphisms.

1. *Introduction*

The most fundamental linearization is C^0 linearization, that is, the existence of C^0 conjugacy between the diffeomorphism considered and its linear part. The well-known Hartman–Grobman theorem [**9**, Theorem (I)] indicates that a C^1 hyperbolic diffeomorphism can be linearized by a C^0 conjugacy. Concerning hyperbolic diffeomorphisms, one further expects a better smoothness of linearization and, therefore, C^r $(1 \le r \le \infty)$ linearization has been extensively investigated since the 1950s (see, for example, [**3**, **4**, **15**–**17**, **20**, **21**]). It was proved in [**3**] that all $C^{1,1}$ hyperbolic diffeomorphisms with linear part Λ at the fixed point can be C^1 linearized if the eigenvalues $r_1, \ldots, r_n \in \mathbb{C}$ of Λ satisfy

$$|r_i| \neq |r_{i_-}||r_{i_+}| \quad \text{for all } i, i_-, i_+ \in \{1, \ldots, n\}, \tag{1.1}$$

where $|r_{i_-}| < 1 < |r_{i_+}|$.

The smoothness of conjugacy is restricted by the resonance of those eigenvalues. For example, considering a three-dimensional analytic mapping

$$T : (x_1, x_2, x_3) \mapsto (r_1 x_1, r_2 x_2 + x_1 x_3, r_3 x_3), \tag{1.2}$$

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where $0 < r_1 < 1 < r_2 < r_3$ and $r_2 = r_1 r_3$ (a resonance condition), Hartman [**9**] proved that T cannot be linearized by a Lipschitz conjugacy, not mentioning a C^1-smooth one. This example suggests attention to *α-Hölder linearization* with $\alpha \in (0, 1)$, that is, the conjugacy Φ and its inverse Φ^{-1} satisfy

$$\|\Phi(x) - \Phi(y)\| \le L\|x - y\|^{\alpha}, \quad \|\Phi^{-1}(x) - \Phi^{-1}(y)\| \le L\|x - y\|^{\alpha},$$

with a constant $L > 0$ for all x, y near the fixed point. A result of the α-Hölder linearization for all C^1 hyperbolic diffeomorphisms in $\mathbb{R}^n$ was stated in [**7**, **18**] and the first published proof of the result can be found in [**1**]. Moreover, it was shown in [**1**] that there exits an $\alpha_0 \in (0, 1)$, which is determined by the spectrum $\sigma(\Lambda)$ of the linear part, such that α-Hölder linearization can be realized for any $\alpha \in (0, \alpha_0)$.

There has recently been some interest in the question: *Can the exponent α be an arbitrarily large one less than* 1*?* This question was answered positively by Rayskin [**12**] in 1998 for the mapping T defined in (1.2), a hyperbolic diffeomorphism with resonance. Later, Tan [**19**] generally considered a $C^{1,1}$ mapping (i.e. a C^1 mapping with Lipschitz continuous derivative) on $\mathbb{R}^n$, whose linear part has eigenvalues $r_1, \ldots, r_n \in \mathbb{C}$ such that

$$0 < |r_1| \le \cdots \le |r_p| < 1 < |r_{p+1}| \le \cdots \le |r_n| \tag{1.3}$$

with $p \in \{1, \ldots, n-1\}$ and

$$|r_n||r_p| \le |r_{p+1}|, \quad |r_1||r_{p+1}| \ge |r_p|, \tag{1.4}$$

and answered the question positively. It is clear that Tan's mapping includes the above T. Generally, as indicated in [**6**], eigenvalues $r_1, \ldots, r_n$ of Λ are *resonant* if there exists a $j \in \{1, \ldots, n\}$ such that

$$r_j = r_1^{k_1} \cdots r_n^{k_n} \tag{1.5}$$

for some non-negative integers $k_1, \ldots, k_n$ satisfying $\sum_{i=1}^{n} k_i \ge 2$. They are said to be *weakly resonant* if (1.5) only holds in modulus (see [**4**, **7**]), that is,

$$|r_j| = |r_1^{k_1}| \cdots |r_n^{k_n}|. \tag{1.6}$$

In what follows we are concerned with a special case of (1.6), namely, equalities

$$|r_i| = |r_{i_-}||r_{i_+}|, \tag{1.7}$$

for some $i, i_-, i_+ \in \{1, \ldots, n\}$ such that $|r_{i_-}| < 1 < |r_{i_+}|$. Clearly, conditions (1.1) and (1.7) are complementary and exclusive. Thus we only need to focus on (1.7) because it was proved in [**3**] that C^1 linearization, which implies α-Hölder linearization near the fixed point for any $\alpha \in (0, 1)$, can be realized under (1.1). Although Tan's condition (1.4) totally covers (1.7) in the three-dimensional case, it is easy to find an exceptional one in a higher-dimensional case. For example, consider a four-dimensional mapping

$$T_1 : (x_1, x_2, x_3, x_4) \mapsto (x_1/2, 2x_2 + x_1x_3, 4x_3 + x_1x_4, 8x_4), \tag{1.8}$$

where the eigenvalues $r_1 = \frac{1}{2}$, $r_2 = 2$, $r_3 = 4$ and $r_4 = 8$ satisfy $|r_3| = |r_1||r_4|$, possessing weak resonance, but do not satisfy (1.4) with $p = 1$ and $n = 4$.

Recall that an important step in [**19**] is to find α-Hölder continuous solutions $(q_m)_{m \ge 0}$ of the Lyapunov–Perron (LP) equation, where condition (1.4) is indispensable due to the exponential dichotomy of Λ. In this paper, condition (1.4) is relaxed and the question is answered positively in all cases of weak resonance (1.7), although stronger smoothness of the given mapping is required. Namely, we prove the following main result:

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THEOREM 1. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ diffeomorphism near its hyperbolic fixed point O (i.e. (1.3) holds), where O denotes the origin of $\mathbb{R}^n$. Then, for any constant $\alpha \in (0, 1)$, there is a neighborhood $U \subset \mathbb{R}^n$ of O such that F admits α-Hölder linearization in U.*

In order to prove our result, not using the weighted norm $\sup_{m\geq 0} \|\gamma^{-m} q_m\|$ with a constant $\gamma \in (|r_p|, |r_{p+1}|)$ as in [**19**], we need to employ every eigenvalue lying inside the unit circle together with γ to define another norm so that a more delicate characterization of asymptotic behaviors of $(q_m)_{m\geq 0}$ in various invariant subspaces corresponding to those eigenvalues can be given. This requires us to overcome the difficulties caused by the complicated nonlinear part. Our strategy is to consider C^1 normal forms of mappings, that is, polynomial mappings whose ith components of the nonlinear parts only possess quadratic terms $x_{i_-} x_{i_+}$ corresponding to condition (1.7).

Our paper is organized as follows. In §2 we give estimates for general iterates of the C^1 normal forms. Section 3 is devoted to the LP equation. A continuous solution with special asymptotic behaviors will be found. Based on this result, we further show that the solution is α-Hölder continuous in §4 and leave proofs of lemmas to §5. Finally, in §6, we prove our main result by employing invariant foliations, which are constructed by means of solutions of the LP equation.

2. *Iteration of C^1 normal forms*

As above we assume that Λ has eigenvalues $r_1, \ldots, r_n$ in $\mathbb{C}$ and the set $\{|r_1|, \ldots, |r_n|\}$ has exactly d $(1 \leq d \leq n)$ distinct values, denoted by $\lambda_1, \ldots, \lambda_d$. We further assume that

$$0 < \lambda_1 < \cdots < \lambda_\ell < 1 < \lambda_{\ell+1} < \cdots < \lambda_d, \tag{2.1}$$

where $\ell \in \{1, \ldots, d-1\}$. Then, corresponding to those λ_i, there exists an Λ-invariant decomposition $\mathbb{R}^n = X_1 \times \cdots \times X_d$, where the X_i are Λ-invariant subspaces. Accordingly,

$$\Lambda := \operatorname{diag}(\Lambda_1, \ldots, \Lambda_d),$$

where $\Lambda_i := \Lambda|_{X_i}$ for all $i = 1, \ldots, d$. Let $x := (x_1, \ldots, x_d) \in X_1 \times \cdots \times X_d$ and let π_i be the projection onto the subspace X_i, that is, $\pi_i x := x_i$. In what follows, we use $\|\cdot\|$ and $|\cdot|$ to denote the norms in $\mathbb{R}^n$ and its subspaces respectively and put

$$\|x\| := \max\{|x_1|, \ldots, |x_d|\} \quad \text{for all } x \in \mathbb{R}^n.$$

We also use $\|\cdot\|$ and $|\cdot|$ for the norms of linear mappings defined on the corresponding spaces, as there will be no possibility of confusion.

The following lemma, a direct corollary of [**4**, Theorem 10.1, p. 179], helps us to reduce F to its C^1 normal form.

LEMMA 1. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ diffeomorphism near its hyperbolic fixed point O. Then F is C^1 conjugate to*

$$G(x) := \Lambda x + g(x) \tag{2.2}$$

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near O such that $\pi_i g(x) := \sum_{(i_-,i_+)} C_{i,i_-,i_+}(x_{i_-}, x_{i_+})$ for all $i = 1, \dots, d$, where (i_-, i_+) in the sum takes all pairs of integers $i_- \in \{1, \dots, \ell\}$ and $i_+ \in \{\ell+1, \dots, d\}$ satisfying

$$\lambda_i = \lambda_{i_-}\lambda_{i_+}, \tag{2.3}$$

$x_{i_-} \in X_{i_-}$, $x_{i_+} \in X_{i_+}$, and all $C_{i,i_-,i_+} : X_{i_-} \times X_{i_+} \to X_i$ are bilinear mappings.

Note that $i_- < i < i_+$ in Lemma 1 since (2.1) and (2.3) imply that $\lambda_{i_-} < \lambda_i < \lambda_{i_+}$. In what follows, consider small neighborhoods U_i, V_i of the origin 0 in X_i such that $U_i \subset V_i \subset X_i$ for every $i \in \{1, \dots, d\}$. Let $\varrho_i : X_i \to \mathbb{R}$ be a smooth bump function, that is, a C^∞ function such that

$$\varrho_i(x) = 1 \quad \text{for all } x \in U_i \quad \text{and} \quad \varrho_i(x) = 0 \quad \text{for all } x \in X_i \setminus V_i. \tag{2.4}$$

Consider a modified mapping $\tilde{g}$ of g such that

$$\pi_i \tilde{g}(x) := \sum_{(i_-,i_+)} C_{i,i_-,i_+}(\varrho_{i_-}(x_{i_-})x_{i_-}, \varrho_{i_+}(x_{i_+})x_{i_+}) \quad \text{for all } i = 1, \dots, d.$$

Put

$$\widetilde{G}(x) := \Lambda x + \tilde{g}(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{2.5}$$

and $U := U_1 \times \cdots \times U_d$ and $V := V_1 \times \cdots \times V_d$. Then $\widetilde{G}(x) = G(x)$ for all $x \in U$. Let D be the differential operator and $\partial_{x_i} := \partial/\partial x_i$. Concerning some useful properties of the C_{i,i_-,i_+} and ϱ_i, we give the following lemma.

LEMMA 2. *Let C_{i,i_-,i_+} and ϱ_i be given above. Then, for all $i = 1, \dots, d$ and all pairs (i_-, i_+),*

$$\begin{aligned}
&|\partial_{x_{i_-}} C_{i,i_-,i_+}(\varrho_{i_-}(x_{i_-})x_{i_-}, \varrho_{i_+}(x_{i_+})x_{i_+})| \\
&\quad \le M|\varrho_{i_+}(x_{i_+})||x_{i_+}|(|D\varrho_{i_-}(x_{i_-})||x_{i_-}| + |\varrho_{i_-}(x_{i_-})|), \\
&|\partial_{x_{i_+}} C_{i,i_-,i_+}(\varrho_{i_-}(x_{i_-})x_{i_-}, \varrho_{i_+}(x_{i_+})x_{i_+})| \\
&\quad \le M|\varrho_{i_-}(x_{i_-})||x_{i_-}|(|D\varrho_{i_+}(x_{i_+})||x_{i_+}| + |\varrho_{i_+}(x_{i_+})|)
\end{aligned} \tag{2.6}$$

for all $x_{i_-} \in X_{i_-}$ and $x_{i_+} \in X_{i_+}$, where $M > 0$ is a constant. Moreover, for every $\mu > 0$, the neighborhoods U_i and V_i can be chosen so that

$$|\varrho_i(x_i)||x_i| \le \mu, \quad |D\varrho_i(x_i)||x_i| + |\varrho_i(x_i)| \le \mu + 1, \tag{2.7}$$

for all $i = 1, \dots, d$ and all $x_i \in X_i$.

Proof. Notice that

$$\partial_{x_{i_-}} C_{i,i_-,i_+}(x_{i_-}, x_{i_+}) = C_{i,i_-,i_+}(\cdot, x_{i_+}) \in \mathcal{L}(X_{i_-}, X_i), \tag{2.8}$$

where $C_{i,i_-,i_+}(\cdot, x_{i_+})x_{i_-} := C_{i,i_-,i_+}(x_{i_-}, x_{i_+})$ and $\mathcal{L}(X_{i_-}, X_i)$ is the set of all linear mappings from X_{i_-} into X_i. Being a linear mapping between finite-dimensional linear spaces, $x_{i_+} \mapsto C_{i,i_-,i_+}(\cdot, x_{i_+})$ is bounded, that is,

$$|C_{i,i_-,i_+}(\cdot, x_{i_+})| \le M|x_{i_+}| \quad \text{for all } x_{i_+} \in X_{i_+}, \tag{2.9}$$

for a constant $M > 0$. On the other hand, one sees that

$$D(\varrho_{i_-}(x_{i_-})x_{i_-}) = x_{i_-} D\varrho_{i_-}(x_{i_-}) + \varrho_{i_-}(x_{i_-})I_{X_{i_-}} \in \mathcal{L}(X_{i_-}, X_{i_-}), \tag{2.10}$$

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where $I_{X_{i_-}}$ denotes the identity mapping in X_{i_-} and $x_{i_-} D\varrho_{i_-}(x_{i_-})$ is the multiplication of the $i_- \times 1$ matrix x_{i_-} and the $1 \times i_-$ matrix $D\varrho_{i_-}(x_{i_-})$. It follows from (2.8)–(2.10) that

$$\begin{aligned}
&|\partial_{x_{i_-}} C_{i,i_-,i_+}(\varrho_{i_-}(x_{i_-})x_{i_-}, \varrho_{i_+}(x_{i_+})x_{i_+})| \\
&\quad = |C_{i,i_-,i_+}(\cdot, \varrho_{i_+}(x_{i_+})x_{i_+}) D(\varrho_{i_-}(x_{i_-})x_{i_-})| \\
&\quad \le M|\varrho_{i_+}(x_{i_+})||x_{i_+}|(|D\varrho_{i_-}(x_{i_-})||x_{i_-}| + |\varrho_{i_-}(x_{i_-})|).
\end{aligned}$$

The same arguments carry over to the partial derivative with respect to the variable x_{i_+}, which proves (2.6). Observing (2.4), we immediately get (2.7). The proof is complete. □

COROLLARY 1. *Let $\widetilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ be given in (2.5). Then, for any $\eta > 0$. there are neighborhoods U and V such that*

$$\|D\widetilde{G}(x) - \Lambda\| \le \eta \quad \textit{for all } x \in \mathbb{R}^n, \tag{2.11}$$

holds.

Proof. According to (2.6) and (2.7) given in Lemma 2, we have

$$\begin{aligned}
\|D(\pi_i \tilde{g})(x)\| &\le \sum_{(i_-,i_+)} \|DC_{i,i_-,i_+}(\varrho_{i_-}(x_{i_-})x_{i_-}, \varrho_{i_+}(x_{i_+})x_{i_+})\| \\
&\le M \sum_{(i_-,i_+)} \{|\varrho_{i_+}(x_{i_+})||x_{i_+}|(|D\varrho_{i_-}(x_{i_-})||x_{i_-}| + |\varrho_{i_-}(x_{i_-})|) \\
&\quad + |\varrho_{i_-}(x_{i_-})||x_{i_-}|(|D\varrho_{i_+}(x_{i_+})||x_{i_+}| + |\varrho_{i_+}(x_{i_+})|)\} \\
&\le \eta := 2d\mu(\mu+1)M \quad \text{for all } i = 1, \ldots, d.
\end{aligned}$$

Consequently,

$$\|D\widetilde{G}(x) - \Lambda\| = \|D\tilde{g}(x)\| = \max_{i=1,\ldots,d} \|D(\pi_i \tilde{g})(x)\| \le \eta \quad \text{for all } x \in \mathbb{R}^n.$$

The proof is complete. □

With the above preparation, we are ready to estimate iterates of $\widetilde{G}$.

LEMMA 3. *Let $\widetilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ be given in (2.5) and $\delta > 0$ be given arbitrarily. Then*

$$|\pi_i \widetilde{G}^k(x)| \le (\lambda_i + \delta)^k \|x\| \quad \textit{for all } k \in \mathbb{N}, \textit{ for all } x \in \mathbb{R}^n, \tag{2.12}$$

for all $i = 1, \ldots, d$. Moreover, there is a neighborhood $U_\delta \subset \mathbb{R}^n$ of O such that

$$\|D(\pi_i \widetilde{G}^k)(x)\| \le (\lambda_i + \delta)^k \quad \textit{for all } k \in \mathbb{N}, \textit{ for all } x \in U_\delta, \tag{2.13}$$

for all $i = 1, \ldots, d$.

Proof. Given any small constant $\epsilon > 0$, without loss of generality, assume that

$$|\Lambda_i| \le \lambda_i + \epsilon \quad \text{for all } i = 1, \ldots, d. \tag{2.14}$$

Otherwise, using another equivalent norm $|\cdot|_i$ defined by

$$|x_i|_i := \left\{ \sum_{k=1}^{\infty} ((\lambda_i - \epsilon)^k |\Lambda_i^{-k} x|)^2 + \sum_{k=0}^{\infty} ((\lambda_i + \epsilon)^{-k} |\Lambda_i^k x|)^2 \right\}^{1/2}$$

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in each X_i as suggested in [**10**, Proposition 2.8], we can verify that $|\Lambda_i x_i|_i \le (\lambda_i + \epsilon)|x_i|_i$. The convergence of the infinite sums appearing in the definition of $|\cdot|_i$ is guaranteed by the spectral radius theorem (cf. [**14**, Theorem 10.13, p. 253]).

In order to prove (2.12), for every $i \in \{1, \dots, d\}$, let

$$x_i^a := (x_1, \dots, x_i) \in X_1 \times \cdots \times X_i, \quad x_i^b := (x_{i+1}, \dots, x_d) \in X_{i+1} \times \cdots \times X_d,$$

and let P_i be the projection onto the subspace $X_1 \times \cdots \times X_i$, that is, $P_i x = x_i^a$. As noted just below Lemma 1, we have $i_- < i < i_+$ and therefore one checks that $P_i \widetilde{G}(0, x_i^b) = 0$ for all $x_i^b \in X_{i+1} \times \cdots \times X_d$. Then (2.11), obtained in Corollary 1, and (2.14) give

$$\begin{aligned} |P_i \widetilde{G}(x)| &= |P_i \widetilde{G}(x_i^a, x_i^b) - P_i \widetilde{G}(0, x_i^b)| \le \sup_{\xi_i \in P_i X} |\partial_{x_i^a}(P_i \widetilde{G})(\xi_i, x_i^b)||x_i^a| \\ &\le \Big(|\mathrm{diag}(\Lambda_1, \dots, \Lambda_i)| + \sup_{\xi_i \in P_i X} |\partial_{x_i^a}(P_i \widetilde{G})(\xi_i, x_i^b) - \mathrm{diag}(\Lambda_1, \dots, \Lambda_i)| \Big) |x_i^a| \\ &\le (\lambda_i + \epsilon + \eta)|P_i x| \quad \text{for all } x \in \mathbb{R}^n. \end{aligned} \tag{2.15}$$

It follows by induction that

$$|\pi_i \widetilde{G}^k(x)| \le |P_i \widetilde{G}^k(x)| \le (\lambda_i + \epsilon + \eta)^k |P_i x| \le (\lambda_i + \epsilon + \eta)^k \|x\| \quad \text{for all } k \in \mathbb{N}. \tag{2.16}$$

Thus, putting $\delta_1 := \epsilon + \eta$, we get

$$|\pi_i \widetilde{G}^k(x)| \le (\lambda_i + \delta_1)^k \|x\| \quad \text{for all } k \in \mathbb{N}, \text{ for all } x \in \mathbb{R}^n, \tag{2.17}$$

for all $i = 1, \dots, d$.

Concerning (2.13), we set

$$\delta_2 := \max\left\{ 2\epsilon, \max_{(i_-, i_+)} \frac{2\delta_1 \lambda_{i_+}}{1 - \lambda_{i_-} - \delta_1} \right\}. \tag{2.18}$$

It is clear that $\delta_2 > 0$ since $\lambda_{i_-} < 1$ and δ_1 $(= \epsilon + \eta)$ is small. Then one sees that the inequalities

$$\|D(\pi_i \widetilde{G}^k)(x)\| \le (\lambda_i + \delta_2)^k \quad \text{for all } x \in U_{\delta_2}, \text{ for all } i = 1, \dots, d, \tag{2.19}$$

hold when $k = 1$ because $\|D(\pi_i \widetilde{G}^k)(O)\| = |\Lambda_i|$ and (2.14) holds, where U_{δ_2} is a small neighborhood of O. Next, assume that (2.19) holds with $k = m - 1$ for an integer $m \ge 2$. In order to prove (2.19) with $k = m$, we put

$$D\widetilde{G}^{m-1}(x) := (A_{i,j}^{m-1}(x))_{i,j=1,\dots,d},$$

where $A_{i,j}^{m-1} := \partial_{x_j}(\pi_i \widetilde{G}^{m-1}) : \mathbb{R}^n \to \mathcal{L}(X_j, X_i)$. Since (2.19) holds with $k = m - 1$, we have

$$\sum_{j=1}^{d} |A_{i,j}^{m-1}(x)| = \|D(\pi_i \widetilde{G}^{m-1})(x)\| \le (\lambda_i + \delta_2)^{m-1} \quad \text{for all } x \in U_{\delta_2},$$

where the induced norm (corresponding to the vector norm $\|x\| = \max\{|x_1|, \dots, |x_d|\}$) is used for $D(\pi_i \widetilde{G}^{m-1})(x)$. Moreover, we understand that

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$$D(\pi_i\widetilde{G})(\tilde{G}^{m-1}(x))D(\widetilde{G}^{m-1})(x)$$
$$=\left(\sum_{\nu=1}^{d}\partial_{x_\nu}(\pi_i\widetilde{G})(\tilde{G}^{m-1}(x))A_{\nu,1}^{m-1}(x),\ldots,\sum_{\nu=1}^{d}\partial_{x_\nu}(\pi_i\widetilde{G})(\tilde{G}^{m-1}(x))A_{\nu,d}^{m-1}(x)\right)$$

and that $\sum_{(i_-,i_+)}$ means the sum taking over all pairs $(i_-,i_+)\in\{1,\ldots,\ell\}\times\{\ell+1,\ldots,d\}$ satisfying (2.3). Then (2.6), (2.7) and (2.17) yield

$$\begin{aligned}\|D(\pi_i\widetilde{G}^m)(x)\|&=\|D(\pi_i\widetilde{G})(\widetilde{G}^{m-1}(x))D(\widetilde{G}^{m-1})(x)\|\\&\le|\partial_{x_i}(\pi_i\widetilde{G})(\widetilde{G}^{m-1}(x))|\sum_{j=1}^{d}|A_{i,j}^{m-1}(x)|\\&\quad+\sum_{(i_-,i_+)}\Big(|\partial_{x_{i_-}}(\pi_i\widetilde{G})(\widetilde{G}^{m-1}(x))|\sum_{j=1}^{d}|A_{i_-,j}^{m-1}(x)|\\&\qquad+|\partial_{x_{i_+}}(\pi_i\widetilde{G})(\widetilde{G}^{m-1}(x))|\sum_{j=1}^{d}|A_{i_+,j}^{m-1}(x)|\Big)\\&\le|\Lambda_i|\sum_{j=1}^{d}|A_{i,j}^{m-1}(x)|+(1+\mu)M\\&\quad\cdot\sum_{(i_-,i_+)}\Big(|\varrho_{i_+}(\pi_{i_+}\widetilde{G}^{m-1}(x))||\pi_{i_+}\widetilde{G}^{m-1}(x)|\sum_{j=1}^{d}|A_{i_-,j}^{m-1}(x)|\\&\qquad+|\varrho_{i_-}(\pi_{i_-}\widetilde{G}^{m-1}(x))||\pi_{i_-}\widetilde{G}^{m-1}(x)|\sum_{j=1}^{d}|A_{i_+,j}^{m-1}(x)|\Big)\\&\le(\lambda_i+\epsilon)(\lambda_i+\delta_2)^{m-1}+(1+\mu)M\\&\quad\cdot\sum_{(i_-,i_+)}(\mu(\lambda_{i_-}+\delta_2)^{m-1}+|\pi_{i_-}\widetilde{G}^{m-1}(x)|(\lambda_{i_+}+\delta_2)^{m-1})\\&\le(\lambda_i+\epsilon)(\lambda_i+\delta_2)^{m-1}+(1+\mu)M\\&\quad\cdot\sum_{(i_-,i_+)}(\mu(\lambda_{i_-}+\delta_2)^{m-1}+\|x\|(\lambda_{i_-}+\delta_1)^{m-1}(\lambda_{i_+}+\delta_2)^{m-1})\\&\le\left(\frac{\lambda_i+\epsilon}{\lambda_i+\delta_2}+\lambda_1^{-1}(1+\mu)M(\mu\tau_1^m+\|x\|\tau_2^m)\right)(\lambda_i+\delta_2)^m,\end{aligned}\tag{2.20}$$

where

$$\tau_1:=\max_{i_-}\left\{\frac{\lambda_{i_-}+\delta_2}{\lambda_i+\delta_2}\right\},\quad\tau_2:=\max_{(i_-,i_+)}\left\{\frac{\lambda_{i_-}\lambda_{i_+}+\delta_2\lambda_{i_-}+\delta_1\lambda_{i_+}+\delta_1\delta_2}{\lambda_i+\delta_2}\right\}.$$

Notice that $\tau_1,\tau_2\in(0,1)$ because $\lambda_{i_-}<\lambda_i$, $\lambda_i=\lambda_{i_-}\lambda_{i_+}$ and

$$\max_{(i_-,i_+)}\{\delta_2\lambda_{i_-}+\delta_1\lambda_{i_+}+\delta_1\delta_2\}<\delta_2$$

due to (2.18). Choose μ and U_{δ_2} to be small so that $\mu+\|x\|<\epsilon\lambda_1\{2M(\lambda_d+1)\}^{-1}$ for all $x\in U_{\delta_2}$. Then, by the fact that $\delta_2\ge2\epsilon$ as seen in (2.18), we have

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$$0 < \frac{\lambda_i + \epsilon}{\lambda_i + \delta_2} + \lambda_1^{-1}(1+\mu)M(\mu\tau_1^m + \|x\|\tau_2^m)$$
$$< 1 - \frac{\delta_2 - \epsilon}{\lambda_i + \delta_2} + 2\lambda_1^{-1}M(\mu + \|x\|) < 1 - \frac{\epsilon}{\lambda_i + 1} + \frac{\epsilon}{\lambda_i + 1} = 1$$

for all $m \in \mathbb{N}$, and therefore one sees from (2.20) that

$$\|D(\pi_i \widetilde{G}^m)(x)\| \le (\lambda_i + \delta_2)^m \quad \text{for all } x \in U_{\delta_2}, \text{ for all } i = 1, \ldots, d,$$

which proves (2.19) for all $k \in \mathbb{N}$ by induction. According to the definitions of the above δ_1 and δ_2, we can choose small ϵ and η such that $\max\{\delta_1, \delta_2\} \le \delta$. Hence, we obtain (2.12) and (2.13) in terms of (2.17) and (2.19), respectively. The proof is complete. □

3. *Asymptoticity for the LP equation*

Let

$$X_- = X_1 \times \cdots \times X_\ell, \qquad X_+ = X_{\ell+1} \times \cdots \times X_d,$$
$$\Lambda_- = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_\ell), \quad \Lambda_+ = \operatorname{diag}(\Lambda_{\ell+1}, \ldots, \Lambda_d)$$

and let π_- and π_+ denote the projections onto subspaces X_- and X_+, respectively. For convenience, identifying X_- and X_+ with $X_- \times \{0\}$ and $\{0\} \times X_+$ respectively, we can use the notation $\mathbb{R}^n = X_- \oplus X_+$ instead of the notation $\mathbb{R}^n = X_- \times X_+$. Thus, for $x \in \mathbb{R}^n$ we have $x := \pi_- x + \pi_+ x$.

Let $\widetilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ be given in (2.5). We will use the stable and unstable invariant foliations to decouple the mapping $\widetilde{G}$ into a contraction and an expansion. We recall from [**10**, **11**] that a *foliation* on a manifold is a collection of disjoint subsets, called *leaves*, which are injectively immersed connected submanifolds and decompose the manifold into a union of those leaves. The stable invariant foliation (cf. [**2**, **22**]) under $\widetilde{G}$ is a collection $\{\mathcal{M}_s(x)\}_{x\in\mathbb{R}^n}$, where each $\mathcal{M}_s(x)$ is a leaf on $\mathbb{R}^n$,

$$\widetilde{G}(\mathcal{M}_s(x)) \subset \mathcal{M}_s(\widetilde{G}(x)), \tag{3.1}$$

and $\mathcal{M}_s(x) = \{y \in \mathbb{R}^n \mid \sup_{m\ge 0}\{\gamma^{-m}\|\widetilde{G}^m(y) - \widetilde{G}^m(x)\|\} < \infty\}$ for a constant $\gamma \in (\lambda_\ell, 1)$; that is, any two orbits starting from the same leaf approach each other exponentially. An effective method of finding the stable invariant foliation is to study the LP equation

$$q_m(x, y_-) = \Lambda_-^m(y_- - \pi_- x) + \sum_{k=0}^{m-1} \Lambda_-^{m-k-1}\{\pi_-\tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_-\tilde{g}(\widetilde{G}^k(x))\}$$
$$- \sum_{k=m}^{\infty} \Lambda_+^{m-k-1}\{\pi_+\tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_+\tilde{g}(\widetilde{G}^k(x))\} \quad \text{for all } m \ge 0, \tag{3.2}$$

for the sequence of unknown $q_m : \mathbb{R}^n \times X_- \to \mathbb{R}^n$ so as to construct the stable invariant foliation with

$$\mathcal{M}_s(x) := \{x + q_0(x, y_-) \mid y_- \in X_-\}.$$

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In fact, one can check that a sequence $(q_m(x, y_-))_{m\geq 0}$ with the property

$$\sup_{m\geq 0}\{\gamma^{-m}\|q_m(x, y_-)\|\} < \infty \quad \text{for all } (x, y_-) \in \mathbb{R}^n \times X_-,$$

for a constant $\gamma \in (\lambda_\ell, 1)$ is a solution of equation (3.2) if and only if

$$\widetilde{G}^m(x + q_0(x, y_-)) - \widetilde{G}^m(x) = q_m(x, y_-) \quad \text{for all } m \geq 0.$$

Thus, if the above-mentioned solution $(q_m(x, y_-))_{m\geq 0}$ is unique then

$$\mathcal{M}_s(x) = \{x + q_0(x, y_-) \mid y_- \in X_-\} = \left\{ y \in \mathbb{R}^n \;\middle|\; \sup_{m\geq 0}\{\gamma^{-m}\|\widetilde{G}^m(y) - \widetilde{G}^m(x)\|\} < \infty \right\}.$$

Notice that the unstable invariant foliation can be obtained by considering the stable one of the inverse $\widetilde{G}^{-1}$.

The regularity of linearization is determined by the regularity of q_0. Hence, for the purpose of this paper we need to find α-Hölder solutions of equation (3.2) in all cases of weak resonance (1.7) for a constant $\alpha < 1$ as large as possible. As observed in our introduction, condition (1.4) is indispensable in Tan's paper [**19**] because with the weighted norm $\sup_{m\geq 0}\|\gamma^{-m}q_m(x, y_-)\|$ only the four eigenvalues r_1, r_p, r_{p+1} and r_n, which correspondingly concern the upper and lower bounds of $\|\Lambda_-\|$ and $\|\Lambda_+\|$, were used in estimation. In order to weaken the restriction (1.4), we will take all eigenvalues into account, which requires us to describe the asymptoticity of $(q_m)_{m\geq 0}$ more delicately.

According to [**5**, Theorem 2.1], one can find a solution

$$q(x, y_-) := (q_m(x, y_-))_{m\geq 0} \quad \text{for all } (x, y_-) \in \mathbb{R}^n \times X_-,$$

of equation (3.2) such that each mapping $q_m : \mathbb{R}^n \times X_- \to \mathbb{R}^n$ is continuous. In this section we mainly characterize the asymptoticity of projections of $(q_m)_{m\geq 0}$ onto various invariant subspaces with corresponding eigenvalues in the following lemma.

LEMMA 4. *Suppose that $q(x, y_-) = (q_m(x, y_-))_{m\geq 0}$ is as given above. Let $\gamma \in (\lambda_\ell, 1)$ be arbitrarily fixed and choose a constant $\delta > 0$ such that*

$$\delta < \min\{\lambda_{\ell+1} - \gamma, \gamma - \lambda_\ell\}.$$

Then, for any constant $\theta > 0$, there is a neighborhood $\Omega \subset \mathbb{R}^n \times X_-$ of the origin $\tilde{O} \in \mathbb{R}^n \times X_-$ such that

$$\sup_{(x,y_-)\in\Omega} \sup_{m\geq 0} \max\{(\lambda_1 + \delta)^{-m}|\pi_1 q_m(x, y_-)|, \ldots, (\lambda_\ell + \delta)^{-m}|\pi_\ell q_m(x, y_-)|, \gamma^{-m}|\pi_{\ell+1} q_m(x, y_-)|, \ldots, \gamma^{-m}|\pi_d q_m(x, y_-)|\} \leq \theta, \tag{3.3}$$

where the choice of Ω depends on δ and θ.

This lemma shows an asymptotic behavior of the sequence $(\pi_i q_m)$ as $m \to \infty$ because from (3.3) we see that $\limsup_{m\to\infty} r_i^{-m}|\pi_i q_m(x, y_-)| \leq \theta$ for each $i \in \{1, \ldots, d\}$, where r_i takes the corresponding value of either $(\lambda_i + \delta)$ or γ. Inequality (3.3) actually improves the inequality

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$$\sup_{m\ge 0}\{\gamma^{-m}\|q_m(x,\,y_-)\|\}$$
$$= \sup_{m\ge 0}\ \max\{\gamma^{-m}|\pi_1 q_m(x,\,y_-)|,\,\dots,\,\gamma^{-m}|\pi_\ell q_m(x,\,y_-)|,$$
$$\gamma^{-m}|\pi_{\ell+1} q_m(x,\,y_-)|,\,\dots,\,\gamma^{-m}|\pi_d q_m(x,\,y_-)|\} < \infty$$

employed in [**5**, **21**] for the weighted norm, because in (3.3) the first ℓ constants γ are replaced with smaller ones $(\lambda_i + \delta)$.

In order to prove the lemma, we need to introduce some notation. Let $C^0(\Omega, \mathbb{R}^n)$ be the Banach space of all C^0 mappings from the compact set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ equipped with the supremum norm, denoted by $\|\cdot\|_{C^0(\Omega,\mathbb{R}^n)}$. Moreover, for real constants $\delta > 0$ and $\gamma \in (\lambda_\ell, 1)$, let $E_{\delta,\gamma}$ consist of all sequences $h := (h_m)_{m\ge 0}$ of mappings $h_m \in C^0(\Omega, \mathbb{R}^n)$ such that

$$\|h\|_{E_{\delta,\gamma}} := \sup_{m\ge 0} \max\{(\lambda_1+\delta)^{-m}\|\pi_1 h_m\|_{C^0(\Omega,X_1)},\,\dots,\,(\lambda_\ell+\delta)^{-m}\|\pi_\ell h_m\|_{C^0(\Omega,X_\ell)},$$
$$\gamma^{-m}\|\pi_{\ell+1} h_m\|_{C^0(\Omega,X_{\ell+1})},\,\dots,\,\gamma^{-m}\|\pi_d h_m\|_{C^0(\Omega,X_d)}\} < \infty. \quad (3.4)$$

As shown in [**21**] before [**21**, Lemma 1], we can prove that $E_{\delta,\gamma}$ is a Banach space equipped with the norm $\|\cdot\|_{E_{\delta,\gamma}}$. Clearly, (3.3) can be represented as

$$\|q\|_{E_{\delta,\gamma}} \le \theta. \quad (3.5)$$

Proof of Lemma 4. For a small constant $\theta > 0$ (the upper bound of which will be determined later), put

$$E^\theta_{\delta,\gamma} := \{h \in E_{\delta,\gamma} : \|h\|_{E_{\delta,\gamma}} \le \theta\} \quad (3.6)$$

and define an operator $\mathcal{T} : E^\theta_{\delta,\gamma} \to E^\theta_{\delta,\gamma}$ by

$$(\mathcal{T}q)_m(x,\,y_-)$$
$$:= \Lambda_-^m(y_- - \pi_- x) + \sum_{k=0}^{m-1} \Lambda_-^{m-k-1}\{\pi_-\tilde g(q_k(x,\,y_-) + \widetilde G^k(x)) - \pi_-\tilde g(\widetilde G^k(x))\}$$
$$- \sum_{k=m}^{\infty} \Lambda_+^{m-k-1}\{\pi_+\tilde g(q_k(x,\,y_-) + \widetilde G^k(x)) - \pi_+\tilde g(\widetilde G^k(x))\}. \quad (3.7)$$

We claim that

$$\|\mathcal{T}q\|_{E_{\delta,\gamma}} \le \sup_{(x,y_-)\in\Omega}\ \max_{i=1,\dots,\ell}\{|y_i - x_i|\} + \tfrac{1}{2}\|q\|_{E_{\delta,\gamma}} \quad (3.8)$$

for all $q \in E_{\delta,\gamma}$ provided Ω is small enough. If the claim is true then it is clear that, for all $q \in E^\theta_{\delta,\gamma}$,

$$\|\mathcal{T}q\|_{E_{\delta,\gamma}} \le \sup_{(x,y_-)\in\Omega}\ \max_{i=1,\dots,\ell}\{|y_i| + |x_i|\} + \tfrac{1}{2}\|q\|_{E_{\delta,\gamma}} \le \theta/2 + \theta/2 = \theta$$

when Ω is small enough such that $\sup_{(x,y_-)\in\Omega}\max_{i=1,\dots,\ell}\{|y_i| + |x_i|\} < \theta/2$. This means that $\mathcal{T}$ given in (3.7) is well defined.

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In order to prove claim (3.8), without loss of generality, assume that

$$|\Lambda_i| \le \lambda_i + \delta/2 \quad \text{and} \quad |\Lambda_i^{-1}| \le (\lambda_i - \delta)^{-1}$$

for each $i \in \{1, \dots, d\}$. This assumption is reasonable due to the arguments just below (2.14), where ϵ can be chosen to be smaller than $\delta/2$. Then from (3.4) and (3.7) we get

$$\begin{aligned}\|\mathcal{T}q\|_{E_{\delta,\gamma}} \le & \sup_{(x,y_-)\in\Omega} \max_{i=1,\dots,\ell}\{|y_i - x_i|\} \\ & + (\lambda_i + \delta/2)^{-1} \sup_{(x,y_-)\in\Omega} \sup_{m\ge 0} \max_{i=1,\dots,\ell} \left\{ \sum_{k=0}^{m-1} \left(\frac{\lambda_i + \delta/2}{\lambda_i + \delta} \right)^{m-k} \Upsilon_1(i,k) \right\} \\ & + (\lambda_i - \delta)^{-1} \sup_{(x,y_-)\in\Omega} \sup_{m\ge 0} \max_{i=\ell+1,\dots,d} \left\{ \sum_{k=m}^{\infty} \left(\frac{\gamma}{\lambda_i - \delta} \right)^{k-m} \Upsilon_2(i,k) \right\}, \quad (3.9)\end{aligned}$$

where

$$\begin{aligned}\Upsilon_1(i,k) &:= (\lambda_i + \delta)^{-k} |\pi_i \tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(\widetilde{G}^k(x))|, \\ &\qquad \text{for all } i = 1, \dots, \ell, \\ \Upsilon_2(i,k) &:= \gamma^{-k} |\pi_i \tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(\widetilde{G}^k(x))|, \\ &\qquad \text{for all } i = \ell + 1, \dots, d,\end{aligned}$$

for all integers $k \ge 0$. It is worth mentioning that in formula (3.9) the constant $\gamma < \lambda_{\ell+1} - \delta \le \lambda_i - \delta$ for all $i = \ell + 1, \dots, d$ by the choice of δ in the lemma. In order to estimate Υ_1, let $\hat{u}_k := q_k(x, y_-) + \widetilde{G}^k(x)$ and $u_k := \widetilde{G}^k(x)$ for convenience. Notice that

$$\begin{aligned}&|C_{i,i_-,i_+}(\hat{x}_{i_-}, \hat{x}_{i_+}) - C_{i,i_-,i_+}(x_{i_-}, x_{i_+})| \\ &\quad \le |C_{i,i_-,i_+}(\hat{x}_{i_-} - x_{i_-}, \hat{x}_{i_+})| + |C_{i,i_-,i_+}(x_{i_-}, \hat{x}_{i_+} - x_{i_+})| \\ &\quad \le |C_{i,i_-,i_+}(\cdot, \hat{x}_{i_+})||\hat{x}_{i_-} - x_{i_-}| + |C_{i,i_-,i_+}(x_{i_-}, \cdot)||\hat{x}_{i_+} - x_{i_+}| \\ &\quad \le M(|\hat{x}_{i_-} - x_{i_-}||\hat{x}_{i_+}| + |x_{i_-}||\hat{x}_{i_+} - x_{i_+}|)\end{aligned}$$

by (2.9) and that

$$\begin{aligned}&|\varrho_j(\pi_j\hat{u}_k)\pi_j\hat{u}_k - \varrho_j(\pi_j u_k)\pi_j u_k| \\ &\quad \le \sup_{\xi_j\in X_j} \{|D\varrho_j(\xi_j)||\xi_j| + |\varrho_j(\xi_j)|\}|\pi_j\hat{u}_k - \pi_j u_k| \\ &\quad \le (1+\mu)|\pi_j\hat{u}_k - \pi_j u_k| = (1+\mu)|\pi_j q_k(x, y_-)| \quad \text{for all } j = 1, \dots, d,\end{aligned}$$

by the mean value theorem and (2.7). It follows that

$$\begin{aligned}&|\pi_i \tilde{g}(\hat{u}_k) - \pi_i \tilde{g}(u_k)| \\ &\quad = \Bigg| \sum_{(i_-,i_+)} \{C_{i,i_-,i_+}(\varrho_{i_-}(\pi_{i_-}\hat{u}_k)\pi_{i_-}\hat{u}_k, \varrho_{i_+}(\pi_{i_+}\hat{u}_k)\pi_{i_+}\hat{u}_k) \\ &\qquad\qquad - C_{i,i_-,i_+}(\varrho_{i_-}(\pi_{i_-}u_k)\pi_{i_-}u_k, \varrho_{i_+}(\pi_{i_+}u_k)\pi_{i_+}u_k)\}\Bigg| \\ &\quad \le M \sum_{(i_-,i_+)} \{|\varrho_{i_-}(\pi_{i_-}\hat{u}_k)\pi_{i_-}\hat{u}_k - \varrho_{i_-}(\pi_{i_-}u_k)\pi_{i_-}u_k||\varrho_{i_+}(\pi_{i_+}\hat{u}_k)\pi_{i_+}\hat{u}_k| \\ &\qquad\qquad + |\varrho_{i_-}(\pi_{i_-}u_k)\pi_{i_-}u_k||\varrho_{i_+}(\pi_{i_+}\hat{u}_k)\pi_{i_+}\hat{u}_k - \varrho_{i_+}(\pi_{i_+}u_k)\pi_{i_+}u_k|\}\end{aligned}$$

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$$\begin{aligned}
&\le (1+\mu)M \sum_{(i_-,i_+)} \{|\varrho_{i_+}(\pi_{i_+}\hat{u}_k)||\pi_{i_+}\hat{u}_k||\pi_{i_-}q_k(x,\,y_-)| \\
&\qquad\qquad +|\varrho_{i_-}(\pi_{i_-}u_k)||\pi_{i_-}u_k||\pi_{i_+}q_k(x,\,y_-)|\} \\
&\le (1+\mu)M(\lambda_i+\delta)^k \\
&\quad\cdot \sum_{(i_-,i_+)} \left\{ \left(\frac{\lambda_{i_-}+\delta}{\lambda_i+\delta}\right)^k |\varrho_{i_+}(\pi_{i_+}\hat{u}_k)||\pi_{i_+}\hat{u}_k|(\lambda_{i_-}+\delta)^{-k}|\pi_{i_-}q_k(x,\,y_-)| \right. \\
&\qquad\qquad \left. +\left(\frac{(\lambda_{i_-}+\delta)\gamma}{\lambda_i+\delta}\right)^k (\lambda_{i_-}+\delta)^{-k}|\varrho_{i_-}(\pi_{i_-}u_k)||\pi_{i_-}u_k|\gamma^{-k}|\pi_{i_+}q_k(x,\,y_-)| \right\} \\
&\le (1+\mu)M(\lambda_i+\delta)^k \\
&\quad\cdot \sum_{(i_-,i_+)} \{|\varrho_{i_+}(\pi_{i_+}\hat{u}_k)||\pi_{i_+}\hat{u}_k| + (\lambda_{i_-}+\delta)^{-k}|\varrho_{i_-}(\pi_{i_-}u_k)||\pi_{i_-}u_k|\}\|q\|_{E_{\delta,\gamma}} \\
&\le (1+\mu)M(\lambda_i+\delta)^k \sum_{i_-} (\mu + (\lambda_{i_-}+\delta)^{-k}|\pi_{i_-}u_k|)\|q\|_{E_{\delta,\gamma}} \\
&\le d(1+\mu)M(\lambda_i+\delta)^k(\mu+\|x\|)\|q\|_{E_{\delta,\gamma}} \qquad (3.10)
\end{aligned}$$

because $\lambda_{i_-} < \lambda_i$, $\gamma < 1$ and $|\pi_{i_-}u_k| = |\pi_{i_-}\widetilde{G}^k(x)| \le (\lambda_{i_-}+\delta)^k\|x\|$ by (2.12), given in Lemma 3, where we recall that (i_-, i_+) in the sum takes all pairs $(i_-, i_+) \in \{1, \dots, \ell\} \times \{\ell+1, \dots, d\}$ satisfying (2.3). Thus, we obtain from (3.10) that

$$\Upsilon_1(i,k) \le d(1+\mu)M(\mu+\|x\|)\|q\|_{E_{\delta,\gamma}} \quad \text{for all } i = 1, \dots, \ell, \text{ for all } k \ge 0. \qquad (3.11)$$

In order to estimate Υ_2 given in (3.9), we see from (3.10) that

$$\begin{aligned}
&|\pi_i\tilde{g}(\hat{u}_k) - \pi_i\tilde{g}(u_k)| \\
&\quad\le (1+\mu)M \sum_{(i_-,i_+)} \{|\varrho_{i_+}(\pi_{i_+}\hat{u}_k)||\pi_{i_+}\hat{u}_k||\pi_{i_-}q_k(x,\,y_-)| \\
&\qquad\qquad + |\varrho_{i_-}(\pi_{i_-}u_k)||\pi_{i_-}u_k||\pi_{i_+}q_k(x,\,y_-)|\} \\
&\quad\le (1+\mu)M\gamma^k \sum_{(i_-,i_+)} \left\{ \left(\frac{\lambda_{i_-}+\delta}{\gamma}\right)^k |\varrho_{i_+}(\pi_{i_+}\hat{u}_k)||\pi_{i_+}\hat{u}_k|(\lambda_{i_-}+\delta)^{-k}|\pi_{i_-}q_k(x,\,y_-)| \right. \\
&\qquad\qquad \left. + |\varrho_{i_-}(\pi_{i_-}u_k)||\pi_{i_-}u_k|\gamma^{-k}|\pi_{i_+}q_k(x,\,y_-)| \right\} \\
&\quad\le 2d(1+\mu)M\gamma^k\mu\|q\|_{E_{\delta,\gamma}} \qquad (3.12)
\end{aligned}$$

because $\lambda_{i_-} + \delta \le \lambda_\ell + \delta < \gamma$ by the choice of δ in the lemma. Then (3.12) implies that

$$\Upsilon_2(i,k) \le 2d(1+\mu)M\mu\|q\|_{E_{\delta,\gamma}} \quad \text{for all } i = \ell+1, \dots, d, \text{ for all } k \ge 0. \qquad (3.13)$$

Hence, combining (3.9), (3.11) with (3.13), we get

$$\|\mathcal{T}q\|_{E_{\delta,\gamma}} \le \sup_{(x,y_-)\in\Omega} \max_{i=1,\dots,\ell} \{|y_i - x_i|\} + K_\delta(1+\mu)\left(2\mu + \sup_{(x,y_-)\in\Omega}\|x\|\right)\|q\|_{E_{\delta,\gamma}},$$

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where $K_\delta > 0$ depends on δ (independent of μ and (x, y_-)) and $\lim_{\delta\to 0} K_\delta = \infty$. Since $\mu > 0$ can be chosen arbitrarily small, as indicated just above (2.7), we let $\mu < \min\{1/(12K_\delta), 1\}$. Moreover, choosing Ω small enough such that $\sup_{(x,y_-)\in\Omega} \|x\| < 1/(12K_\delta)$, we can prove claim (3.8).

Next, we show that the operator defined in (3.7) is a contraction on $E^\theta_{\delta,\gamma}$, a complete metric space presented in (3.6). In fact, (3.7) yields

$$\begin{aligned}
&\|\mathcal{T}\hat{q} - \mathcal{T}q\|_{E_{\delta,\gamma}} \\
&\quad \le (\lambda_i + \delta/2)^{-1} \sup_{(x,y_-)\in\Omega} \sup_{m\ge 0} \max_{i=1,\dots,\ell} \left\{ \sum_{k=0}^{m-1} \left(\frac{\lambda_i + \delta/2}{\lambda_i + \delta} \right)^{m-k} \Upsilon_3(i,k) \right\} \\
&\qquad + (\lambda_i - \delta)^{-1} \sup_{(x,y_-)\in\Omega} \sup_{m\ge 0} \max_{i=\ell+1,\dots,d} \left\{ \sum_{k=m}^{\infty} \left(\frac{\gamma}{\lambda_i - \delta} \right)^{k-m} \Upsilon_4(i,k) \right\} \qquad (3.14)
\end{aligned}$$

for all $\hat{q}$, $q \in E^\theta_{\delta,\gamma}$, where

$$\begin{aligned}
\Upsilon_3(i,k) &:= (\lambda_i + \delta)^{-k} |\pi_i \tilde{g}(\hat{q}_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x))| \\
&\qquad \text{for all } i = 1, \dots, \ell, \\
\Upsilon_4(i,k) &:= \gamma^{-k} |\pi_i \tilde{g}(\hat{q}_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x))|, \\
&\qquad \text{for all } i = \ell + 1, \dots, d,
\end{aligned}$$

for all $k \ge 0$. For Υ_3, we let $\hat{v}_k := \hat{q}_k(x, y_-) + \widetilde{G}^k(x)$ and $v_k := q_k(x, y_-) + \widetilde{G}^k(x)$. Notice that

$$|\pi_j v_k| = |\pi_j q_k(x, y_-) + \pi_j \widetilde{G}^k(x)| \le (\theta + \|x\|)(\lambda_j + \delta)^k, \quad \text{for all } j = 1, \dots, \ell,$$

by (2.12) and the fact that $q \in E^\theta_{\delta,\gamma}$. Then the same arguments as (3.10) give

$$\begin{aligned}
&|\pi_i \tilde{g}(\hat{v}_k) - \pi_i \tilde{g}(v_k)| \\
&\quad \le (1 + \mu) M (\lambda_i + \delta)^k \sum_{i_-} (\mu + (\lambda_{i_-} + \delta)^{-k} |\pi_{i_-} v_k|) \|\hat{q} - q\|_{E_{\delta,\gamma}} \\
&\quad \le d(1 + \mu) M (\lambda_i + \delta)^k (\mu + \theta + \|x\|) \|\hat{q} - q\|_{E_{\delta,\gamma}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\Upsilon_3(i,k) &\le d(1 + \mu) M (\mu + \theta + \|x\|) \|\hat{q} - q\|_{E_{\delta,\gamma}} \quad \text{for all } i = 1, \dots, \ell, \\
&\qquad \text{for all } k \ge 0. \qquad (3.15)
\end{aligned}$$

The same arguments as (3.12) give

$$|\pi_i \tilde{g}(\hat{v}_k) - \pi_i \tilde{g}(v_k)| \le 2d(1 + \mu) M \gamma^k \mu \|\hat{q} - q\|_{E_{\delta,\gamma}},$$

which implies that

$$\Upsilon_4(i,k) \le 2d(1 + \mu) M \mu \|\hat{q} - q\|_{E_{\delta,\gamma}} \quad \text{for all } i = \ell + 1, \dots, d, \text{ for all } k \ge 0. \quad (3.16)$$

Hence, in view of (3.14)–(3.16), we have

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$$\|\mathcal{T}\hat{q} - \mathcal{T}q\|_{E_{\delta,\gamma}} \leq K_\delta(1+\mu)\left(2\mu + \theta + \sup_{(x,y_-)\in\Omega}\|x\|\right)\|\hat{q} - q\|_{E_{\delta,\gamma}} \quad \text{for all } \hat{q}, q \in E^\theta_{\delta,\gamma}.$$

On the other hand, since $K_\delta > 0$ is independent of μ, θ and (x, y_-), we may let $\mu < \min\{1/(16K_\delta), 1\}$ and θ, $\sup_{(x,y_-)\in\Omega}\|x\| < 1/(16K_\delta)$. Then

$$K_\delta(1+\mu)\left(2\mu + \theta + \sup_{(x,y_-)\in\Omega}\|x\|\right) < \frac{1}{2},$$

that is, the operator $\mathcal{T} : E^\theta_{\delta,\gamma} \to E^\theta_{\delta,\gamma}$ defined in (3.7) is a contraction. Banach's contraction principle implies that equation (3.2) has a unique solution $q \in E^\theta_{\delta,\gamma}$ such that (3.5) holds. This proves (3.3).

Moreover, the above-mentioned local solution $q(x, y_-) = (q_m(x, y_-))_{m\geq 0}$ can be extended from Ω to the whole space $\mathbb{R}^n \times X_-$. In fact, by [**5**, Theorem 2.1] we know that, for every point $(x, y_-) \in \mathbb{R}^n \times X_-$, equation (3.2) has a unique solution $\tilde{q}(x, y_-) = (\tilde{q}_m(x, y_-))_{m\geq 0} \subset \mathbb{R}^n$ such that

$$\sup_{m\geq 0} \gamma^{-m}\|\tilde{q}_m(x, y_-)\| < \infty. \tag{3.17}$$

Here each mapping $\tilde{q}_m : \mathbb{R}^n \times X_- \to \mathbb{R}^n$ is continuous. Noting that the local solution $(q_m(x, y_-))_{m\geq 0}$ also satisfies (3.17) for each $(x, y_-) \in \Omega$, as known from (3.3). Thus, by the uniqueness of $(\tilde{q}_m(x, y_-))_{m\geq 0}$, one easily obtains $\tilde{q}|_\Omega = q$ and we still use q to denote $\tilde{q}$ without loss of generality. The proof is complete. □

Remark 1. In the above proof, we stress that
(A1) $\lim_{\delta\to 0} K_\delta = \infty$,
(A2) $\theta < 1/(16K_\delta)$,
(A3) $\sup_{x\in\Omega}\|x\| < 1/(16K_\delta)$ and $\sup_{(x,y_-)\in\Omega}\max_{i=1,\ldots,\ell}\{|y_i| + |x_i|\} < \theta/2$.
In view of (A1)–(A3), one can conclude that

$$\lim_{\delta\to 0}\sup_{(x,y_-)\in\Omega}\|(x, y_-)\| = 0, \tag{3.18}$$

that is, the neighborhood Ω shrinks to the origin $\tilde{O}$ as $\delta \to 0$.

4. *Hölder continuity for the LP equation*

In this section, we further study α-Hölder continuity of the mapping $q_0(x, y_-)$ which is obtained in Lemma 4.

LEMMA 5. *For any constant $\alpha \in (0, 1)$, there is a neighborhood $\Omega \subset \mathbb{R}^n \times X_-$ of the origin $\tilde{O}$ such that $q_0 : \mathbb{R}^n \times X_- \to \mathbb{R}^n$ is α-Hölder in Ω.*

Given arbitrary $\gamma \in (\lambda_\ell, 1)$, we let $\alpha \in (0, 1)$ be an arbitrarily fixed constant satisfying

$$\alpha > \frac{\log\gamma}{\log\lambda_d} + 1 \in (-\infty, 1), \tag{4.1}$$

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where $\lambda_d > 1$ by (2.1). It follows from (4.1) that $\lambda_{i_+}^{1-\alpha}\gamma \le \lambda_d^{1-\alpha}\gamma < 1$, which implies that $\lambda_{i_+}^{\alpha} > \lambda_{i_+}\gamma$, and therefore

$$\lambda_{i_-}\lambda_{i_+}^{\alpha} > \lambda_{i_-}\lambda_{i_+}\gamma \quad \text{for all } i_- = 1, \dots, \ell, \text{ for all } i_+ = \ell+1, \dots, d. \tag{4.2}$$

On the other hand, we have

$$\lambda_i^{\alpha} > \lambda_i^{\alpha}\lambda_{i_-}^{1-\alpha} = (\lambda_{i_-}\lambda_{i_+})^{\alpha}\lambda_{i_-}^{1-\alpha} = \lambda_{i_-}\lambda_{i_+}^{\alpha}, \quad \lambda_i = \lambda_{i_-}\lambda_{i_+} > \lambda_{i_-}\lambda_{i_+}^{\alpha} \tag{4.3}$$

for every $i \in \{1, \dots, d\}$ and every pair $(i_-, i_+) \in \{1, \dots, \ell\} \times \{\ell+1, \dots, d\}$ satisfying (2.3) because $\lambda_{i_-} \in (0, 1)$ and $\lambda_{i_+} \in (1, \infty)$ by (2.1). In view of (4.2) and (4.3), there is a small constant $\delta > 0$ such that

$$\min\{(\lambda_i + \delta)^{\alpha}, \lambda_i\} > (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)^{\alpha} > (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)\gamma. \tag{4.4}$$

We further let

$$\delta < \min\{\lambda_{\ell+1} - \gamma, \gamma - \lambda_\ell\}. \tag{4.5}$$

According to the first inequality of (4.4), one can see that $\lambda_i - (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)^{\alpha} > 0$. Thus, using (4.4) again, we can choose an $\omega_i > 0$ such that

$$\lambda_i - (\lambda_i + \delta)^{\alpha} < \omega_i < \lambda_i - (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)^{\alpha} < \lambda_i - (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)\gamma. \tag{4.6}$$

Then, for every $i \in \{1, \dots, d\}$ and every pair (i_-, i_+) of integers, it is straightforward to verify that

$$(\lambda_{i_-} + \delta)^2(\lambda_{i_+} + \delta) < (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)\gamma < \lambda_i - \omega_i < \lambda_i + \delta \tag{4.7}$$

and

$$(\lambda_{i_-} + \delta)(\lambda_{i_+} - \omega_{i_+}) < (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)^{\alpha} < \lambda_i - \omega_i < \lambda_i + \delta \tag{4.8}$$

by (4.6). Moreover, formula (4.6) also gives

$$\omega_i < \lambda_i - (\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta)\gamma < \lambda_i - \gamma \quad \text{for all } i = \ell+1, \dots, d, \tag{4.9}$$

since $(\lambda_{i_-} + \delta)(\lambda_{i_+} + \delta) > \lambda_{i_-}\lambda_{i_+} = \lambda_i > 1$. Recalling that the conditions of Lemma 4 are well satisfied because $\gamma \in (\lambda_\ell, 1)$ and (4.5) holds, we can now give our proof of the lemma.

Proof of Lemma 5. First of all, it suffices to discuss the case of (4.1) because the fact that Φ is α-Hölder near O implies that Φ is also $\tilde{\alpha}$-Hölder near O for any $\tilde{\alpha} \in (0, \alpha)$. For the solution $q(x, y_-)$ of equation (3.2) obtained in Lemma 4, we actually need to prove that the operator $\mathcal{T}$, defined in (3.7), satisfies

$$\|\mathcal{T}q(\hat{x}, \hat{y}_-) - \mathcal{T}q(x, y_-)\|_{\delta,\omega} \le L\|(\hat{x}, \hat{y}_-) - (x, y_-)\|^{\alpha} + \tfrac{1}{2}\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} \tag{4.10}$$

for all $(\hat{x}, \hat{y}_-), (x, y_-) \in \Omega$, where $L > 0$ is a constant independent of $(\hat{x}, \hat{y}_-)$ and (x, y_-) and the norm $\|\cdot\|_{\delta,\omega}$ is defined by

$$\begin{aligned}\|s\|_{\delta,\omega} := \sup_{m \ge 0} \max\{&(\lambda_1 + \delta)^{-m}|\pi_1 s_m|, \dots, (\lambda_\ell + \delta)^{-m}|\pi_\ell s_m|, \\ &(\lambda_{\ell+1} - \omega_{\ell+1})^{-m}|\pi_{\ell+1} s_m|, \dots, (\lambda_d - \omega_d)^{-m}|\pi_d s_m|\}\end{aligned} \tag{4.11}$$

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for all sequences $s := (s_m)_{m\geq 0}$ of points in $\mathbb{R}^n$. By (4.9), $\gamma < \lambda_i - \omega_i$ for all $i = \ell+1, \ldots, d$. It follows from the definition (3.4) of $\|\cdot\|_{E_{\delta,\gamma}}$ and the definition (4.11) of $\|\cdot\|_{\delta,\omega}$ that

$$\sup_{(x,y_-)\in\Omega} \|q(x, y_-)\|_{\delta,\omega} < \|q\|_{E_{\delta,\gamma}} < \infty \quad \text{for } q \in E_{\delta,\gamma},$$

where we note that the same small γ in the terms $\pi_{\ell+1}|q_m(x, y)|, \ldots, \pi_d|q_m(x, y)|$ of the norm $\|\cdot\|_{E_{\delta,\gamma}}$ is replaced with the large $\lambda_{\ell+1} - \omega_{\ell+1}, \ldots, \lambda_d - \omega_d$ respectively in those terms of the norm $\|\cdot\|_{\delta,\omega}$. If (4.10) is true, then we see from equation (3.2) that

$$\|q_0(\hat{x}, \hat{y}_-) - q_0(x, y_-)\| \leq \|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} \leq 2L\|(\hat{x}, \hat{y}_-) - (x, y_-)\|^\alpha$$

for all $(\hat{x}, \hat{y}_-), (x, y_-) \in \Omega$. This proves the lemma.

In order to prove (4.10) complementarily, we notice that

$$\begin{aligned} &\pi_i\tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i\tilde{g}(\widetilde{G}^k(x)) \\ &\quad = \int_0^1 D(\pi_i\tilde{g})(\tau q_k(x, y_-) + \widetilde{G}^k(x))q_k(x, y_-)\, d\tau, \end{aligned} \tag{4.12}$$

which is a corollary of [**13**, Theorem 6.24, p. 135]. In fact, assume that $x = (\varsigma_1, \ldots, \varsigma_n) \in \mathbb{R}^n$ with $\varsigma_1, \ldots, \varsigma_n \in \mathbb{R}$ and that each subspace X_i is p_i-dimensional, where $i \in \{1, \ldots, d\}$ and $p_i \in \{1, \ldots, n\}$ such that $p_1 + \cdots + p_d = n$. Then

$$x_i = (\varsigma_{i_1}, \ldots, \varsigma_{i_{p_i}}) \quad \text{for all } x_i \in X_i,$$

with $i_1, \ldots, i_{p_i} \in \{1, \ldots, n\}$. Let

$$\pi_i\tilde{g}(x) = (\chi_{i_1}(x), \ldots, \chi_{i_{p_i}}(x)) \in X_i (= \mathbb{R}^{p_i}),$$

where $\chi_{i_j} : \mathbb{R}^n \to \mathbb{R}$ for each $j = 1, \ldots, p_i$. Clearly, $\tau \mapsto \chi_{i_j}(\tau x + y)$ is a one-dimensional function for any given $x, y \in \mathbb{R}^n$. Thus,

$$\begin{aligned} &\int_0^1 D(\pi_i\tilde{g})(\tau x + y)x\, d\tau \\ &= \begin{pmatrix} \displaystyle\int_0^1 \sum_{s=1}^n \partial_{\varsigma_s}\chi_{i_1}(\tau x + y)\varsigma_s\, d\tau \\ \vdots \\ \displaystyle\int_0^1 \sum_{s=1}^n \partial_{\varsigma_s}\chi_{i_{p_i}}(\tau x + y)\varsigma_s\, d\tau \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle\int_0^1 \frac{d\,\chi_{i_1}(\tau x + y)}{d\tau}\, d\tau \\ \vdots \\ \displaystyle\int_0^1 \frac{d\,\chi_{i_{p_i}}(\tau x + y)}{d\tau}\, d\tau \end{pmatrix} = \begin{pmatrix} \chi_{i_1}(\tau x + y) - \chi_{i_1}(y) \\ \vdots \\ \chi_{i_{p_i}}(\tau x + y) - \chi_{i_{p_i}}(y) \end{pmatrix} \\ &= \pi_i\tilde{g}(\tau x + y) - \pi_i\tilde{g}(y), \end{aligned}$$

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which proves (4.12) by replacing x and y with $q_k(x, y_-)$ and $\widetilde{G}^k(x)$, respectively. Therefore, we are ready to prove (4.10). By the definition (3.7) of the operator $\mathcal{T}$, we have

$$\begin{aligned}
&|\pi_i(\mathcal{T}q)_m(\hat{x}, \hat{y}_-) - \pi_i(\mathcal{T}q)_m(x, y_-)| \\
&\quad \le |\Lambda_i^m|(|\hat{y}_i - y_i| + |\hat{x}_i - x_i|) + \sum_{k=0}^{m-1} |\Lambda_i^{m-k-1}| \\
&\qquad \cdot |\{\pi_i \tilde{g}(q_k(\hat{x}, \hat{y}_-) + \widetilde{G}^k(\hat{x})) - \pi_i \tilde{g}(\widetilde{G}^k(\hat{x}))\} \\
&\qquad\quad - \{\pi_i \tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(\widetilde{G}^k(x))\}| \\
&\quad \le (\lambda_i + \delta/2)^m(|\hat{y}_i - y_i| + |\hat{x}_i - x_i|) + \sum_{k=0}^{m-1} (\lambda_i + \delta/2)^{m-k-1} \\
&\qquad \cdot \Big\{ |\{\pi_i \tilde{g}(q_k(\hat{x}, \hat{y}_-) + \widetilde{G}^k(\hat{x})) - \pi_i \tilde{g}(\widetilde{G}^k(\hat{x}))\} \\
&\qquad\quad - \{\pi_i \tilde{g}(q_k(\hat{x}, \hat{y}_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(\widetilde{G}^k(x))\}| \\
&\qquad\quad + \Big| \{\pi_i \tilde{g}(q_k(\hat{x}, \hat{y}_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(\widetilde{G}^k(x))\} \\
&\qquad\qquad - \int_0^1 D(\pi_i \tilde{g})(\tau q_k(x, y_-) + \widetilde{G}^k(x)) q_k(\hat{x}, \hat{y}_-)\, d\tau \Big| \\
&\qquad\quad + \Big| \int_0^1 D(\pi_i \tilde{g})(\tau q_k(x, y_-) + \widetilde{G}^k(x)) q_k(\hat{x}, \hat{y}_-)\, d\tau \\
&\qquad\qquad - \{\pi_i \tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(\widetilde{G}^k(x))\} \Big| \Big\} \\
&\quad \le (\lambda_i + \delta)^m \left(\frac{\lambda_i + \delta/2}{\lambda_i + \delta} \right)^m (|\hat{y}_i - y_i| + |\hat{x}_i - x_i|) + (\lambda_i + \delta/2)^{-1}(\lambda_i + \delta)^m \\
&\qquad \cdot \sum_{k=0}^{m-1} \left(\frac{\lambda_i + \delta/2}{\lambda_i + \delta} \right)^{m-k} (\lambda_i + \delta)^{-k} (\Xi_1(i, k) + \Xi_2(i, k) + \Xi_3(i, k)) \qquad (4.13)
\end{aligned}$$

for all $i = 1, \ldots, \ell$. On the other hand, for all $i = \ell + 1, \ldots, d$, we also obtain from (3.7) that

$$\begin{aligned}
&|\pi_i(\mathcal{T}q)_m(\hat{x}, \hat{y}_-) - \pi_i(\mathcal{T}q)_m(x, y_-)| \\
&\quad \le \sum_{k=m}^{\infty} |\Lambda_i^{m-k-1}| |\{\pi_i \tilde{g}(q_k(\hat{x}, \hat{y}_-) + \widetilde{G}^k(\hat{x})) - \pi_i \tilde{g}(\widetilde{G}^k(\hat{x}))\} \\
&\qquad\qquad\qquad - \{\pi_i \tilde{g}(q_k(x, y_-) + \widetilde{G}^k(x)) - \pi_i \tilde{g}(\widetilde{G}^k(x))\}| \\
&\quad \le (\lambda_i - \omega_i/2)^{-1}(\lambda_i - \omega_i)^m \\
&\qquad \cdot \sum_{k=m}^{\infty} \left(\frac{\lambda_i - \omega_i}{\lambda_i - \omega_i/2} \right)^{k-m} (\lambda_i - \omega_i)^{-k} (\Xi_1(i, k) + \Xi_2(i, k) + \Xi_3(i, k)), \qquad (4.14)
\end{aligned}$$

where $\Xi_1(i, k)$, $\Xi_2(i, k)$ and $\Xi_3(i, k)$ appearing in (4.13) and (4.14) are defined by

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$$\Xi_1(i,k) := \left| \int_0^1 \{D(\pi_i \tilde{g})(\tau q_k(\hat{x},\hat{y}_-) + \widetilde{G}^k(\hat{x})) - D(\pi_i \tilde{g})(\tau q_k(\hat{x},\hat{y}_-) + \widetilde{G}^k(x))\} q_k(\hat{x},\hat{y}_-)\, d\tau \right|, \tag{4.15}$$

$$\Xi_2(i,k) := \left| \int_0^1 \{D(\pi_i \tilde{g})(\tau q_k(\hat{x},\hat{y}_-) + \widetilde{G}^k(x)) - D(\pi_i \tilde{g})(\tau q_k(x,y_-) + \widetilde{G}^k(x))\} q_k(\hat{x},\hat{y}_-)\, d\tau \right|, \tag{4.16}$$

$$\Xi_3(i,k) := \left| \int_0^1 D(\pi_i \tilde{g})(\tau q_k(x,y_-) + \widetilde{G}^k(x))(q_k(\hat{x},\hat{y}_-) - q_k(x,y_-))\, d\tau \right|, \tag{4.17}$$

for all $i = 1, \ldots, d$ and $k \geq 0$ due to (4.12).

LEMMA 6. *The above-defined $\Xi_j(i,k)$ $(j = 1, 2, 3)$, depending on $(\hat{x}, \hat{y}_-)$ and (x, y_-) in Ω, satisfy the following:*

$$\Xi_1(i,k) \leq \begin{cases} \varepsilon(\lambda_i + \delta)^k \|\hat{x} - x\|^\alpha & \text{for all } i = 1, \ldots, \ell, \\ \varepsilon(\lambda_i - \omega_i)^k \|\hat{x} - x\|^\alpha & \text{for all } i = \ell+1, \ldots, d, \end{cases} \tag{4.18}$$

$$\Xi_2(i,k) \leq \begin{cases} \varepsilon(\lambda_i + \delta)^k \|q(\hat{x},\hat{y}_-) - q(x,y_-)\|_{\delta,\omega} & \text{for all } i = 1, \ldots, \ell, \\ \varepsilon(\lambda_i - \omega_i)^k \|q(\hat{x},\hat{y}_-) - q(x,y_-)\|_{\delta,\omega} & \text{for all } i = \ell+1, \ldots, d, \end{cases} \tag{4.19}$$

$$\Xi_3(i,k) \leq \begin{cases} \varepsilon(\lambda_i + \delta)^k \|q(\hat{x},\hat{y}_-) - q(x,y_-)\|_{\delta,\omega} & \text{for all } i = 1, \ldots, \ell, \\ \varepsilon(\lambda_i - \omega_i)^k \|q(\hat{x},\hat{y}_-) - q(x,y_-)\|_{\delta,\omega} & \text{for all } i = \ell+1, \ldots, d, \end{cases} \tag{4.20}$$

for all $k \geq 0$, where $\varepsilon := \tilde{M}(\mu + \theta + \sup_{x \in \Omega \cap \mathbb{R}^n} \|x\|) > 0$ and $\tilde{M} > 0$ is a constant independent of μ, θ, $(\hat{x}, \hat{y}_-)$ and (x, y_-).

The proof of Lemma 6 is postponed to the next section. As indicated above (2.7) and by Lemma 4, one can choose small μ and neighborhood Ω such that

$$\varepsilon = \tilde{M}\Big(\mu + \theta + \sup_{x \in \Omega \cap \mathbb{R}^n} \|x\|\Big) < (8K_{\delta,\omega})^{-1},$$

where $K_{\delta,\omega} > 0$ is a constant independent of μ, θ, $(\hat{x}, \hat{y}_-)$ and (x, y_-). Hence,

$$\begin{aligned} \|\mathcal{T}q(\hat{x},\hat{y}_-) - \mathcal{T}q(x,y_-)\|_{\delta,\omega} &\leq \|\hat{x} - x\| + |\hat{y}_- - y_-| \\ &\quad + K_{\delta,\omega} \max_{i=1,\ldots,\ell} \sup_{k \geq 0} \{(\lambda_i + \delta)^{-k}(\Xi_1(i,k) + \Xi_2(i,k) + \Xi_3(i,k))\} \\ &\quad + K_{\delta,\omega} \max_{i=\ell+1,\ldots,d} \sup_{k \geq 0} \{(\lambda_i - \omega_i)^{-k}(\Xi_1(i,k) + \Xi_2(i,k) + \Xi_3(i,k))\} \\ &\leq \|\hat{x} - x\| + |\hat{y}_- - y_-| + 2\varepsilon K_{\delta,\omega} \|\hat{x} - x\|^\alpha + 4\varepsilon K_{\delta,\omega} \|q(\hat{x},\hat{y}_-) - q(x,y_-)\|_{\delta,\omega} \\ &\leq \tfrac{5}{4} \|(\hat{x},\hat{y}_-) - (x,y_-)\|^\alpha + \tfrac{1}{2} \|q(\hat{x},\hat{y}_-) - q(x,y_-)\|_{\delta,\omega} \end{aligned} \tag{4.21}$$

for all $(\hat{x}, \hat{y}_-), (x, y_-) \in \Omega$ by (4.13)–(4.14) and (4.18)–(4.20). This proves (4.10) with $L := \frac{5}{4}$ and the proof of the lemma is complete. □

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Remark 2. Notice that (4.3) is the only place where we use the inequality $\alpha < 1$ in the paper. Moreover, by (4.4) one sees that $\delta \to 0$ as $\alpha \to 1$. Thus, in view of (3.18), one can conclude that the neighborhood Ω given in Lemma 5 shrinks to the origin $\tilde{O}$ as $\alpha \to 1$.

5. *Proof of Lemma 6*

By the mean value theorem for integration, we have

$$\left| \int_0^1 D(\pi_i \tilde{g})(\tau q_k(x, y_-) + \widetilde{G}^k(x)) q_k(x, y_-)\, d\tau \right| \le \sup_{\tau \in (0,1)} |D(\pi_i \tilde{g})(\tau q_k(x, y_-) + \widetilde{G}^k(x)) q_k(x, y_-)|.$$

Thus, it suffices to estimate the integrands in the three formulae (4.15)–(4.17) with $\tau \in (0, 1)$. In order to prove (4.18), observing (4.15), we put

$$\hat{z}_k := \tau q_k(\hat{x}, \hat{y}_-) + \widetilde{G}^k(\hat{x}) \quad \text{and} \quad z_k := \tau q_k(\hat{x}, \hat{y}_-) + \widetilde{G}^k(x) \tag{5.1}$$

for all integers $k \ge 0$ and notice that

$$\begin{aligned} &|\{D(\pi_i \tilde{g})(\hat{z}_k) - D(\pi_i \tilde{g})(z_k)\} q_k(\hat{x}, \hat{y}_-)| \\ &\quad \le \sum_{(i_-, i_+)} (|\partial_{x_{i_-}}(\pi_i \tilde{g})(\hat{z}_k) - \partial_{x_{i_-}}(\pi_i \tilde{g})(z_k)||\pi_{i_-} q_k(\hat{x}, \hat{y}_-)| \\ &\qquad + |\partial_{x_{i_+}}(\pi_i \tilde{g})(\hat{z}_k) - \partial_{x_{i_+}}(\pi_i \tilde{g})(z_k)||\pi_{i_+} q_k(\hat{x}, \hat{y}_-)|) \end{aligned} \tag{5.2}$$

for every $i \in \{1, \dots, d\}$ and every pair $(i_-, i_+) \in \{1, \dots, \ell\} \times \{\ell + 1, \dots, d\}$ satisfying (2.3). Regarding the above term $|\partial_{x_{i_-}}(\pi_i \tilde{g})(\hat{z}_k) - \partial_{x_{i_-}}(\pi_i \tilde{g})(z_k)||\pi_{i_-} q_k(\hat{x}, \hat{y}_-)|$, we obtain from (2.6) that

$$\begin{aligned} &|\partial_{x_{i_-}}(\pi_i \tilde{g})(\hat{z}_k) - \partial_{x_{i_-}}(\pi_i \tilde{g})(z_k)||\pi_{i_-} q_k(\hat{x}, \hat{y}_-)| \\ &\quad = |C_{i,i_-,i_+}(\cdot, \varrho_{i_+}(\pi_{i_+} \hat{z}_k)\pi_{i_+} \hat{z}_k)(\pi_{i_-} \hat{z}_k D\varrho_{i_-}(\pi_{i_-} \hat{z}_k) + \varrho_{i_-}(\pi_{i_-} \hat{z}_k) I_{X_{i_-}}) \\ &\qquad - C_{i,i_-,i_+}(\cdot, \varrho_{i_+}(\pi_{i_+} z_k)\pi_{i_+} z_k)(\pi_{i_-} z_k D\varrho_{i_-}(\pi_{i_-} z_k) + \varrho_{i_-}(\pi_{i_-} z_k) I_{X_{i_-}})| \\ &\qquad \cdot |\pi_{i_-} q_k(\hat{x}, \hat{y}_-)| \\ &\quad \le \Pi_1^i(\hat{z}_k, z_k) + \Pi_2^i(\hat{z}_k, z_k), \end{aligned} \tag{5.3}$$

where we recall that $C_{i,i_-,i_+}(\cdot, x_{i_+}) \in \mathcal{L}(X_{i_-}, X_i)$ was defined just below (2.8) such that $C_{i,i_-,i_+}(\cdot, x_{i_+})x_{i_-} := C_{i,i_-,i_+}(x_{i_-}, x_{i_+})$ and

$$\begin{aligned} &\Pi_1^i(\hat{z}_k, z_k) \\ &\quad := |C_{i,i_-,i_+}(\cdot, \varrho_{i_+}(\pi_{i_+} \hat{z}_k)\pi_{i_+} \hat{z}_k)(\pi_{i_-} \hat{z}_k D\varrho_{i_-}(\pi_{i_-} \hat{z}_k)) \\ &\qquad - C_{i,i_-,i_+}(\cdot, \varrho_{i_+}(\pi_{i_+} z_k)\pi_{i_+} z_k)(\pi_{i_-} z_k D\varrho_{i_-}(\pi_{i_-} z_k))||\pi_{i_-} q_k(\hat{x}, \hat{y}_-)|, \\ &\Pi_2^i(\hat{z}_k, z_k) \\ &\quad := |C_{i,i_-,i_+}(\cdot, \varrho_{i_+}(\pi_{i_+} \hat{z}_k)\pi_{i_+} \hat{z}_k)(\varrho_{i_-}(\pi_{i_-} \hat{z}_k) I_{X_{i_-}}) \\ &\qquad - C_{i,i_-,i_+}(\cdot, \varrho_{i_+}(\pi_{i_+} z_k)\pi_{i_+} z_k)(\varrho_{i_-}(\pi_{i_-} z_k) I_{X_{i_-}})||\pi_{i_-} q_k(\hat{x}, \hat{y}_-)|. \end{aligned}$$

For Π_1^i and Π_2^i, we know that (2.12), (3.3) and (5.1) yield

$$|\pi_j \hat{z}_k| \le (\theta + \|\hat{x}\|)(\lambda_j + \delta)^k \quad \text{and} \quad |\pi_j z_k| \le (\theta + \|x\|)(\lambda_j + \delta)^k$$

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for all $j = 1, \dots, d$ and that

$$\begin{aligned}
&|\pi_{i_-}\hat{z}_k D\varrho_{i_-}(\pi_{i_-}\hat{z}_k) - \pi_{i_-}z_k D\varrho_{i_-}(\pi_{i_-}z_k)| \\
&\quad \le \sup_{\xi_{i_-}\in X_{i_-}} \{|D^2\varrho(\xi_{i_-})||\xi_{i_-}| + |D\varrho(\xi_{i_-})|\}|\pi_{i_-}\hat{z}_k - \pi_{i_-}z_k| \\
&\quad \le K|\pi_{i_-}\hat{z}_k - \pi_{i_-}z_k|
\end{aligned}$$

with a constant $K > 0$. Moreover, recall that every mapping $C_{i,i_-,i_+} : X_{i_-} \times X_{i_+} \to X_i$ is bilinear and therefore $C_{i,i_-,i_+}(\cdot, \hat{x}_{i_+} - x_{i_+}) = C_{i,i_-,i_+}(\cdot, \hat{x}_{i_+}) - C_{i,i_-,i_+}(\cdot, x_{i_+})$. Then, by (2.13) and (3.3), given in Lemmas 3 and 4 separately, we have

$$\begin{aligned}
&\Pi_1^i(z_k, \hat{z}_k) \\
&\quad \le M(|\varrho_{i_+}(\pi_{i_+}\hat{z}_k)\pi_{i_+}\hat{z}_k - \varrho_{i_+}(\pi_{i_+}z_k)\pi_{i_+}z_k||D\varrho_{i_-}(\pi_{i_-}\hat{z}_k)||\pi_{i_-}\hat{z}_k| \\
&\qquad + |\varrho_{i_+}(\pi_{i_+}z_k)||\pi_{i_+}z_k||\pi_{i_-}\hat{z}_k D\varrho_{i_-}(\pi_{i_-}\hat{z}_k) - \pi_{i_-}z_k D\varrho_{i_-}(\pi_{i_-}z_k)|) \\
&\qquad \cdot |\pi_{i_-}q_k(\hat{x}, \hat{y}_-)| \\
&\quad \le M_1(|\pi_{i_-}\hat{z}_k||\pi_{i_+}\hat{z}_k - \pi_{i_+}z_k| + |\pi_{i_+}z_k||\pi_{i_-}\hat{z}_k - \pi_{i_-}z_k|)|\pi_{i_-}q_k(\hat{x}, \hat{y}_-)| \\
&\quad = M_1(|\pi_{i_-}\hat{z}_k||\pi_{i_+}\widetilde{G}^k(\hat{x}) - \pi_{i_+}\widetilde{G}^k(x)| + |\pi_{i_+}z_k||\pi_{i_-}\widetilde{G}^k(\hat{x}) - \pi_{i_-}\widetilde{G}^k(x)|) \\
&\qquad \cdot |\pi_{i_-}q_k(\hat{x}, \hat{y}_-)| \\
&\quad \le M_1 \sup_{\xi\in\Omega\cap\mathbb{R}^n} \{|\pi_{i_-}\hat{z}_k||D(\pi_{i_+}\widetilde{G}^k)(\xi)| + |\pi_{i_+}z_k||D(\pi_{i_-}\widetilde{G}^k)(\xi)|\}|\pi_{i_-}q_k(\hat{x}, \hat{y}_-)| \\
&\qquad \cdot \|\hat{x} - x\| \\
&\quad \le M_2\theta(\lambda_{i_-} + \delta)^k(\lambda_{i_+} + \delta)^k(\lambda_{i_-} + \delta)^k\|\hat{x} - x\| \\
&\quad \le \begin{cases} M_2\theta(\lambda_i + \delta)^k\|\hat{x} - x\| & \text{for all } i = 1, \dots, \ell, \\ M_2\theta(\lambda_i - \omega_i)^k\|\hat{x} - x\| & \text{for all } i = \ell + 1, \dots, d, \end{cases}
\end{aligned} \tag{5.4}$$

due to (4.7). Here the M_i are positive constants independent of μ, θ, $(\hat{x}, \hat{y}_-)$ and (x, y_-) because the four quantities μ, θ, $\|\hat{x}\|$ and $\|x\|$ involved in the computation of the M_i are small (i.e. less than 1). For Π_2^i, one needs to know that

$$|\varrho_{i_+}(\pi_{i_+}\hat{z}_k)\pi_{i_+}\hat{z}_k - \varrho_{i_+}(\pi_{i_+}z_k)\pi_{i_+}z_k| \le |\varrho_{i_+}(\pi_{i_+}\hat{z}_k)\pi_{i_+}\hat{z}_k - \varrho_{i_+}(\pi_{i_+}z_k)\pi_{i_+}z_k|^\alpha$$

because $\alpha \in (0, 1)$ and $|\varrho_{i_+}(\pi_{i_+}\hat{z}_k)\pi_{i_+}\hat{z}_k - \varrho_{i_+}(\pi_{i_+}z_k)\pi_{i_+}z_k| < 2\mu < 1$. It follows that

$$\begin{aligned}
&\Pi_2^i(z_k, \hat{z}_k) \\
&\quad \le M(|\varrho_{i_+}(\pi_{i_+}\hat{z}_k)\pi_{i_+}\hat{z}_k - \varrho_{i_+}(\pi_{i_+}z_k)\pi_{i_+}z_k||\varrho_{i_-}(\pi_{i_-}\hat{z}_k)| \\
&\qquad + |\varrho_{i_+}(\pi_{i_+}z_k)||\pi_{i_+}z_k||\varrho_{i_-}(\pi_{i_-}\hat{z}_k) - \varrho_{i_-}(\pi_{i_-}z_k)|)|\pi_{i_-}q_k(\hat{x}, \hat{y}_-)| \\
&\quad \le M_1(|\pi_{i_+}\hat{z}_k - \pi_{i_+}z_k|^\alpha + |\pi_{i_+}z_k||\pi_{i_-}\hat{z}_k - \pi_{i_-}z_k|)|\pi_{i_-}q_k(\hat{x}, \hat{y}_-)| \\
&\quad \le M_1(|\pi_{i_+}\widetilde{G}^k(\hat{x}) - \pi_{i_+}\widetilde{G}^k(x)|^\alpha + |\pi_{i_+}z_k||\pi_{i_-}\widetilde{G}^k(\hat{x}) - \pi_{i_-}\widetilde{G}^k(x)|)|\pi_{i_-}q_k(\hat{x}, \hat{y}_-)| \\
&\quad \le M_1 \sup_{\xi\in\Omega\cap\mathbb{R}^n} \{|D(\pi_{i_+}\widetilde{G}^k)(\xi)|^\alpha + |\pi_{i_+}z_k||D(\pi_{i_-}\widetilde{G}^k)(\xi)|\}|\pi_{i_-}q_k(\hat{x}, \hat{y}_-)|\|\hat{x} - x\|^\alpha \\
&\quad \le M_2\theta((\lambda_{i_+} + \delta)^{k\alpha} + (\lambda_{i_+} + \delta)^k(\lambda_{i_-} + \delta)^k)(\lambda_{i_-} + \delta)^k\|\hat{x} - x\|^\alpha \\
&\quad \le \begin{cases} M_2\theta(\lambda_i + \delta)^k\|\hat{x} - x\|^\alpha & \text{for all } i = 1, \dots, \ell, \\ M_2\theta(\lambda_i - \omega_i)^k\|\hat{x} - x\|^\alpha & \text{for all } i = \ell + 1, \dots, d, \end{cases}
\end{aligned} \tag{5.5}$$

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by (4.7) and (4.8). Hence, one knows from (5.3)–(5.5) that

$$\begin{aligned}
&|\partial_{x_{i_-}}(\pi_i\tilde{g})(\hat{z}_k) - \partial_{x_{i_-}}(\pi_i\tilde{g})(z_k)||\pi_{i_-}q_k(\hat{x},\,\hat{y}_-)| \\
&\quad\le \begin{cases} M_2\theta(\lambda_i+\delta)^k\|\hat{x}-x\|^\alpha & \text{for all } i=1,\dots,\ell, \\ M_2\theta(\lambda_i-\omega_i)^k\|\hat{x}-x\|^\alpha & \text{for all } i=\ell+1,\dots,d. \end{cases}
\end{aligned} \tag{5.6}$$

Exchanging i_- with i_+ in (5.4)–(5.5) and by similar arguments, we get

$$\begin{aligned}
&|\partial_{x_{i_+}}(\pi_i\tilde{g})(\hat{z}_k) - \partial_{x_{i_+}}(\pi_i\tilde{g})(z_k)||\pi_{i_+}q_k(\hat{x},\,\hat{y}_-)| \\
&\quad\le M_2\theta(\lambda_{i_+}+\delta)^k(\lambda_{i_-}+\delta)^k\gamma^k\|\hat{x}-x\| \\
&\qquad + M_2\theta((\lambda_{i_-}+\delta)^k + (\lambda_{i_-}+\delta)^k(\lambda_{i_+}+\delta)^k)\gamma^k\|\hat{x}-x\| \\
&\quad\le \begin{cases} M_2\theta(\lambda_i+\delta)^k\|\hat{x}-x\| & \text{for all } i=1,\dots,\ell, \\ M_2\theta(\lambda_i-\omega_i)^k\|\hat{x}-x\| & \text{for all } i=\ell+1,\dots,d. \end{cases}
\end{aligned} \tag{5.7}$$

Then, according to (4.15), (5.2) and (5.6)–(5.7), we prove the claimed result (4.18) with a large constant $\tilde{M}$ satisfying $\tilde{M} \ge dM_2$.

In order to prove (4.19), observing (4.16), we put

$$\hat{z}_k^* = \tau q_k(\hat{x},\,\hat{y}_-) + \widetilde{G}^k(x) \quad\text{and}\quad z_k^* = \tau q_k(x,\,y_-) + \widetilde{G}^k(x)$$

and therefore

$$|\pi_j\hat{z}_k^*| \le (\theta + \|x\|)(\lambda_j+\delta)^k \quad\text{and}\quad |\pi_j z_k^*| \le (\theta+\|x\|)(\lambda_j+\delta)^k$$

for all $j=1,\dots,d$. Similarly to (5.2) and (5.3), we obtain

$$\begin{aligned}
&|\{D(\pi_i\tilde{g})(\hat{z}_k^*) - D(\pi_i\tilde{g})(z_k^*)\}q_k(\hat{x},\,\hat{y}_-)| \\
&\quad\le \sum_{(i_-,i_+)} (|\partial_{x_{i_-}}(\pi_i\tilde{g})(\hat{z}_k^*) - \partial_{x_{i_-}}(\pi_i\tilde{g})(z_k^*)||\pi_{i_-}q_k(\hat{x},\,\hat{y}_-)| \\
&\qquad\qquad + |\partial_{x_{i_+}}(\pi_i\tilde{g})(\hat{z}_k^*) - \partial_{x_{i_+}}(\pi_i\tilde{g})(z_k^*)||\pi_{i_+}q_k(\hat{x},\,\hat{y}_-)|)
\end{aligned} \tag{5.8}$$

and

$$|\partial_{x_{i_-}}(\pi_i\tilde{g})(\hat{z}_k^*) - \partial_{x_{i_-}}(\pi_i\tilde{g})(z_k^*)||\pi_{i_-}q_k(\hat{x},\,\hat{y}_-)| \le \Pi_1^i(\hat{z}_k^*,\,z_k^*) + \Pi_2^i(\hat{z}_k^*,\,z_k^*), \tag{5.9}$$

where Π_1^i and Π_2^i are as given just below (5.3). Furthermore, the same arguments as (5.4) and (5.5) yield

$$\begin{aligned}
&\Pi_1^i(\hat{z}_k^*,\,z_k^*) \\
&\quad\le M_1(|\pi_{i_-}\hat{z}_k^*||\pi_{i_+}q_k(\hat{x},\,\hat{y}_-) - \pi_{i_+}q_k(x,\,y_-)| \\
&\qquad + |\pi_{i_+}z_k^*||\pi_{i_-}q_k(\hat{x},\,\hat{y}_-) - \pi_{i_-}q_k(x,\,y_-)|)|\pi_{i_-}q_k(\hat{x},\,\hat{y}_-)| \\
&\quad\le M_2\theta((\lambda_{i_-}+\delta)^k(\lambda_{i_+}-\omega_{i_+})^k + (\lambda_{i_+}+\delta)^k(\lambda_{i_-}+\delta)^k)(\lambda_{i_-}+\delta)^k \\
&\qquad\cdot \|q(\hat{x},\,\hat{y}_-) - q(x,\,y_-)\|_{\delta,\omega} \\
&\quad\le \begin{cases} M_2\theta(\lambda_i+\delta)^k\|q(\hat{x},\,\hat{y}_-) - q(x,\,y_-)\|_{\delta,\omega} & \text{for all } i=1,\dots,\ell, \\ M_2\theta(\lambda_i-\omega_i)^k\|q(\hat{x},\,\hat{y}_-) - q(x,\,y_-)\|_{\delta,\omega} & \text{for all } i=\ell+1,\dots,d \end{cases}
\end{aligned} \tag{5.10}$$

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and

$$\begin{aligned}
&\Pi_2^i(\hat{z}_k^*, z_k^*)\\
&\quad\le M_1(|\pi_{i_+}q_k(\hat{x}, \hat{y}_-) - \pi_{i_+}q_k(x, y_-)|\\
&\qquad\quad + |\pi_{i_+}z_k^*||\pi_{i_-}q_k(\hat{x}, \hat{y}_-) - \pi_{i_-}q_k(x, y_-)|)|\pi_{i_-}q_k(\hat{x}, \hat{y}_-)|\\
&\quad\le M_2\theta((\lambda_{i_+} - \omega_{i_+})^k + (\lambda_{i_+} + \delta)^k(\lambda_{i_-} + \delta)^k)(\lambda_{i_-} + \delta)^k\\
&\qquad\cdot \|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega}\\
&\quad\le \begin{cases} M_2\theta(\lambda_i + \delta)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = 1, \dots, \ell,\\ M_2\theta(\lambda_i - \omega_i)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = \ell+1, \dots, d, \end{cases}
\end{aligned} \tag{5.11}$$

by (4.7) and (4.8). Thus, (5.9)–(5.11) give

$$\begin{aligned}
&|\partial_{x_{i_-}}(\pi_i\tilde{g})(\hat{z}_k^*) - \partial_{x_{i_-}}(\pi_i\tilde{g})(z_k^*)||\pi_{i_-}q_k(\hat{x}, \hat{y}_-)|\\
&\quad\le \begin{cases} M_2\theta(\lambda_i + \delta)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = 1, \dots, \ell,\\ M_2\theta(\lambda_i - \omega_i)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = \ell+1, \dots, d. \end{cases}
\end{aligned} \tag{5.12}$$

Exchanging i_+ with i_- in (5.10)–(5.11) and by similar arguments, we get

$$\begin{aligned}
&|\partial_{x_{i_+}}(\pi_i\tilde{g})(\hat{z}_k^*) - \partial_{x_{i_+}}(\pi_i\tilde{g})(z_k^*)||\pi_{i_+}q_k(\hat{x}, \hat{y}_-)|\\
&\quad\le M_2\theta((\lambda_{i_+} + \delta)^k(\lambda_{i_-} + \delta)^k + (\lambda_{i_-} + \delta)^k(\lambda_{i_+} - \omega_{i_+})^k)\gamma^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega}\\
&\qquad + M_2\theta((\lambda_{i_-} + \delta)^k + (\lambda_{i_-} + \delta)^k(\lambda_{i_+} - \omega_{i_+})^k)\gamma^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega}\\
&\quad\le \begin{cases} M_2\theta(\lambda_i + \delta)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = 1, \dots, \ell,\\ M_2\theta(\lambda_i - \omega_i)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = \ell+1, \dots, d, \end{cases}
\end{aligned} \tag{5.13}$$

since $\gamma < 1$. Combining (4.16), (5.8) with (5.12)–(5.13), we obtain (4.19) with a large constant $\tilde{M} > 0$ satisfying $\tilde{M} \ge dM_2$.

In order to prove (4.20), due to (4.17), we give the following estimates:

$$\begin{aligned}
&|D(\pi_i\tilde{g})(z_k^*)(q_k(\hat{x}, \hat{y}_-) - q_k(x, y_-))|\\
&\quad\le M\sum_{(i_-,i_+)}\{|\varrho_{i_+}(\pi_{i_+}z_k^*)||\pi_{i_+}z_k^*|(|D\varrho_{i_-}(\pi_{i_-}z_k^*)||\pi_{i_-}z_k^*| + |\varrho_{i_-}(\pi_{i_-}z_k^*)|)\\
&\qquad\qquad\cdot|\pi_{i_-}q_k(\hat{x}, \hat{y}_-) - \pi_{i_-}q_k(x, y_-)|\\
&\qquad\qquad + |\varrho_{i_-}(\pi_{i_-}z_k^*)||\pi_{i_-}z_k^*|(|D\varrho_{i_+}(\pi_{i_+}z_k^*)||\pi_{i_+}z_k^*| + |\varrho_{i_+}(\pi_{i_+}z_k^*)|)\\
&\qquad\qquad\cdot|\pi_{i_+}q_k(\hat{x}, \hat{y}_-) - \pi_{i_+}q_k(x, y_-)|\}\\
&\quad\le M_1\sum_{(i_-,i_+)}(\mu|\pi_{i_-}q_k(\hat{x}, \hat{y}_-) - \pi_{i_-}q_k(x, y_-)| + |\pi_{i_-}z_k^*||\pi_{i_+}q_k(\hat{x}, \hat{y}_-)\\
&\qquad\qquad - \pi_{i_+}q_k(x, y_-)|)\\
&\quad\le M_1\sum_{(i_-,i_+)}(\mu(\lambda_{i_-} + \delta)^k + (\theta + \|x\|)(\lambda_{i_-} + \delta)^k(\lambda_{i_+} - \omega_{i_+})^k)\\
&\qquad\qquad\cdot\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega}\\
&\quad\le \begin{cases} dM_1(\mu + \theta + \|x\|)(\lambda_i + \delta)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = 1, \dots, \ell,\\ dM_1(\mu + \theta + \|x\|)(\lambda_i - \omega_i)^k\|q(\hat{x}, \hat{y}_-) - q(x, y_-)\|_{\delta,\omega} & \text{for all } i = \ell+1, \dots, d, \end{cases}
\end{aligned}$$

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by (4.8). This proves (4.17) with a large constant $\tilde{M} > 0$ satisfying $\tilde{M} \geq dM_1$. The proof of Lemma 6 is complete. □

6. *α-Hölder linearization*

In this section we prove our main result of this paper.

Proof of Theorem 1. According to Lemma 1, it suffices to study Hölder linearization of the mapping $\widetilde{G}$ given in (2.5). Once we have done this, the Hölder linearization of F will be naturally derived because the conjugacy between F and $\widetilde{G}$ is of class C^1 near O. In what follows, our strategy is to employ invariant foliations as coordinate systems to decouple the mapping $\widetilde{G}$ so that the problem can be deduced to linearization of a contraction and linearization of an expansion separately. For this purpose, we let $q_0 : \mathbb{R}^n \times X_- \to \mathbb{R}^n$ be the local α-Hölder mapping given in Lemma 5. Then define

$$\mathcal{M}_s(x) := \{x + q_0(x, y_-) \mid y_- \in X_-\} \quad \text{for all } x \in \mathbb{R}^n,$$

which forms a local α-Hölder stable invariant foliation, as mentioned before Lemma 4. Similarly, considering the inverse $\widetilde{G}^{-1}$, we also obtain a local α-Hölder unstable invariant foliation $\{\mathcal{M}_u(x) \mid x \in \mathbb{R}^n\}$. Moreover, let $w_s : X_- \to X_+$ and $w_u : X_+ \to X_-$ whose graphs are the stable and unstable manifolds of $\widetilde{G}$, respectively. Then it is clear that $\mathrm{Lip}(w_s) = \mathrm{Lip}(w_u) = 0$, where Lip means the Lipschitz constant, because $\widetilde{G}$ has straightened stable manifold (i.e. X_-) and unstable manifold (i.e. X_+).

Hence, the same arguments as in [**21**, §8] show that [**19**, Theorem 3.1] can be applied. Namely, there are homeomorphisms $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ and its inverse $\Psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, which are obtained by means of the invariant foliations and are both α-Hölder near O, such that the equality

$$\Psi \circ \widetilde{G} = \widetilde{G}_- \circ \pi_- \Psi + \widetilde{G}_+ \circ \pi_+ \Psi \tag{6.1}$$

holds, where the mappings $\widetilde{G}_- : X_- \to X_-$ and $\widetilde{G}_+ : X_+ \to X_+$ are defined by

$$\begin{aligned} \widetilde{G}_- &:= \pi_- \widetilde{G} \circ (\mathrm{id}_- + w_s) = \pi_- \widetilde{G} \circ \mathrm{id}_-, \\ \widetilde{G}_+ &:= \pi_+ \widetilde{G} \circ (\mathrm{id}_+ + w_u) = \pi_+ \widetilde{G} \circ \mathrm{id}_+ \end{aligned}$$

and id_- and id_+ denote the identity mappings in X_- and X_+, respectively. Obviously, $D\widetilde{G}_-(O) = \Lambda_-$ and $D\widetilde{G}_+(O) = \Lambda_+$, implying that $\widetilde{G}_-$ is a C^∞ contraction and $\widetilde{G}_+$ is a C^∞ expansion. Then, in view of [**8**], there exist neighborhoods $U_- \subset X_-$ and $U_+ \subset X_+$ of the origin and C^1 diffeomorphisms $\psi_- : U_- \to X_-$ and $\psi_+ : U_+ \to X_+$ such that

$$\psi_- \circ \widetilde{G}_- = \Lambda_- \circ \psi_-, \quad \psi_+ \circ \widetilde{G}_+ = \Lambda_+ \circ \psi_+. \tag{6.2}$$

This enables us to define a local α-Hölder mapping Φ together with its local α-Hölder inverse Φ^{-1} by

$$\Phi := (\psi_- \circ \pi_- + \psi_+ \circ \pi_+) \circ \Psi, \quad \Phi^{-1} := \Psi^{-1} \circ (\psi_-^{-1} \circ \pi_- + \psi_+^{-1} \circ \pi_+).$$

Hence (6.1) and (6.2) yield

$$\begin{aligned} \Phi \circ \widetilde{G} &= (\psi_- \circ \pi_- + \psi_+ \circ \pi_+) \circ (\widetilde{G}_- \circ \pi_- \Psi + \widetilde{G}_+ \circ \pi_+ \Psi) \\ &= \psi_- \circ \widetilde{G}_- \circ \pi_- \Psi + \psi_+ \circ \widetilde{G}_+ \circ \pi_+ \Psi \end{aligned}$$

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$$\begin{aligned}&= (\Lambda_- \circ \psi_- \circ \pi_- + \Lambda_+ \circ \psi_+ \circ \pi_+) \circ \Psi\\ &= \Lambda \circ (\psi_- \circ \pi_- + \psi_+ \circ \pi_+) \circ \Psi\\ &= \Lambda \circ \Phi.\end{aligned}$$

The proof is complete. □

Under (1.4), a stronger condition than ours, ref. [**19**] only needs $C^{1,1}$ smoothness of the given mapping F, a weaker smoothness than our C^∞. In fact, if F is $C^{1,1}$ then, replacing $\widetilde{G}$ and $\tilde{g}$ with F and $F - \Lambda$ respectively in (4.15), one sees that

$$\begin{aligned}\Xi_1(i, k) &\le K\|F^k(\hat{x}) - F^k(x)\|^\alpha \|q_k(x, y_-)\|\\ &\le K\gamma^k\Theta^k\|q(x, y_-)\|_{\gamma_0}\|\hat{x} - x\|^\alpha,\end{aligned} \tag{6.3}$$

where $\Theta := (|r_n| + \eta)^\alpha\gamma_0/\gamma$, $\alpha \in (0, 1)$, $\eta > 0$ is small, and $\|q(x, y_-)\|_{\gamma_0} := \sup_{k\ge 0}\{\gamma_0^{-k}\|q_k(x, y_-)\|\} \le \theta$ for any fixed $\gamma_0 \in (|r_p|, |r_{p+1}|)$. Under condition (1.4), we have $\Theta < 1$ with an appropriate choice of constants $\gamma_0, \gamma \in (|r_p|, |r_{p+1}|)$ because $|r_n||r_p|/|r_{p+1}| \le 1$. Then (6.3) yields that

$$\Xi_1(i, k) \le \varepsilon\gamma^k\|\hat{x} - x\|^\alpha,$$

which (corresponding to our (4.18)) is important in [**19**]. However, in weakly resonant cases, (1.4) does not hold (see the example of (1.8)) and therefore Θ may be greater than 1. For this reason we consider C^1 normal forms for more delicate estimates. A result of C^1 normal forms was given in [**4**, Theorem 10.1] under the assumption that F is C^k with a certain $k \in \mathbb{N}$ depending on $\sigma(\Lambda)$, the set of all eigenvalues of Λ. As shown in the proof of [**4**], by Sternberg's theorem (cf. [**16**, **17**]), one can use a C^1 diffeomorphism to transform a C^k mapping into a polynomial normal form of order k, and the polynomial normal form can be further reduced to the simpler one (2.2) by another C^1 diffeomorphism. Moreover, k is unbounded as the ratio $\log|r_1|/\log|r_p|$ or $\log|r_n|/\log|r_{p+1}|$ increases. Therefore, in our Lemma 1 we assume that F is C^∞ because C^k smoothness with any fixed $k \in \mathbb{N}$ cannot guarantee [**4**, Theorem 10.1] to be true in all cases of weak resonance (1.7).

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