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RELIABILITY PROPERTIES OF EXTENDED LINEAR FAILURE-RATE DISTRIBUTIONS

M. E. GHITANY
Department of Statistics and Operations Research
Faculty of Science
Kuwait University
Safat, Kuwait
E-mail: ghitany@kuc01.kuniv.edu.kw

SAMUEL KOTZ
Department of Engineering Management and Systems Engineering
The George Washington University
Washington DC 20052
E-mail: kotz@gwu.edu

This article investigates reliability properties of a flexible extended linear failure-rate family of distributions generated using the relation $\bar{F}(x) = \alpha\bar{F}_0(x)/[1 - \bar{\alpha}\bar{F}_0(x)]$, where $\alpha > 0$, $\bar{\alpha} = 1 - \alpha$, and $\bar{F}_0(x)$ is the reliability function of the linear failure-rate distribution.

1. INTRODUCTION

Adding parameters to a well-established family of distributions is a time-honored device for obtaining more flexible new families of distributions. Marshall and Olkin [7] introduced an ingeneous general method of adding a parameter to a family of distributions. Their method is as follows: If we have a family of distributions with reliability function (RF) $\bar{F}_0(x)$, then a new family of distributions can be constructed based on the relation

$$\bar{F}(x) = \frac{\alpha\bar{F}_0(x)}{1 - \bar{\alpha}\bar{F}_0(x)} \qquad -\infty < x < \infty, \alpha > 0, \bar{\alpha} = 1 - \alpha. \tag{1}$$

They discovered that the method has a stability property; namely if the method is applied twice, nothing new will be obtained the second time around. These authors

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have shown that the newly generated family of distributions with the RF (1) is *geometric extreme stable*; namely if $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables (RVs) with RF (1), independent from a RV N possessing a geometric distribution with the probability mass function

$$P(N = n) = (1 - p)^{n-1}p, \qquad n = 1, 2, \ldots, 0 < p < 1,$$

then the RVs

$$U = \min(X_1, X_2, \ldots, X_N), \qquad V = \max(X_1, X_2, \ldots, X_N)$$

also belong to the family of distributions with the RF (1).

In this article, we introduce a new variant of the Marshall–Olkin extended family of distributions by selecting in (1) the RF of the linear failure-rate distribution:

$$\bar{F}_0(x) = e^{-(\beta x + (\gamma/2)x^2)}, \qquad x > 0, \qquad \beta, \gamma > 0. \tag{2}$$

For more details about the statistical properties of the linear failure-rate distribution (also known as *linear exponential distribution*) and its applications, see Kodlin [5], Bain [2], Salvia [9], and Sen and Bhattacharyya [10].

Substituting (3) in (1), we obtain

$$\bar{F}(x) = \frac{\alpha e^{-(\beta x + (\gamma/2)x^2)}}{1 - \bar{\alpha} e^{-(\beta x + (\gamma/2)x^2)}}, \qquad x > 0. \tag{3}$$

We write $X \sim \text{ELFR}(\alpha, \beta, \gamma)$ to denote an absolutely continuous RV X possessing the Marshall–Olkin extended linear failure-rate distribution with parameters α, β, γ and RF given by (3).

The aim of this article is to reveal some reliability properties of the ELFR distribution. These properties include (i) expressing this proposed distribution as a compounding process with an exponential mixing model, (ii) shapes of the probability density function, (iii) moments and quantiles, (iv) shapes of the failure-rate function, (v) stochastic ordering, and (vi) limiting distributions of extreme order statistics.

2. COMPOUNDING

Let $\bar{F}(x|\theta)$, $-\infty < x < \infty$, $-\infty < \theta < \infty$, be the conditional RF of a continuous RV X given a continuous RV Θ. Let Θ follow a distribution with the probability density function (p.d.f.) $m(\theta)$. A distribution with the RF

$$\bar{F}(x) = \int_{-\infty}^{\infty} \bar{F}(x|\theta) m(\theta)\, d\theta, \qquad -\infty < x < \infty,$$

is called a *compound distribution with mixing density* $m(\theta)$. Compound distributions provide a tool for obtaining new parametric families of distributions in terms of existing ones. They represent heterogeneous models in which populations' items involve different risks.

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The following theorem shows that the ELFR distribution can be expressed by the compounding argument.

THEOREM 1: *Let the conditional RF of a continuous RV X given* $\Theta = \theta$ *be expressed as*

$$\bar{F}(x|\theta) = \exp[-(e^{\beta x+(\gamma/2)x^2} - 1)\theta], \qquad x, \theta > 0, \qquad \beta, \gamma > 0.$$

Let Θ *follow an exponential distribution with the p.d.f.*

$$m(\theta) = \alpha e^{-\alpha\theta}, \qquad \theta > 0, \qquad \alpha > 0.$$

Then the compound distribution of X is the ELFR(α, β, γ) *distribution.*

PROOF: For all $x > 0$, α, β, $\gamma > 0$, the unconditional RF of X is given by

$$\begin{aligned}\bar{F}(x) &= \int_0^\infty \bar{F}(x|\theta)m(\theta)\, d\theta \\ &= \alpha \int_0^\infty \exp[-(e^{\beta x+(\gamma/2)x^2} - \bar{\alpha})\theta]\, d\theta \\ &= \frac{\alpha}{e^{\beta x+(\gamma/2)x^2} - \bar{\alpha}},\end{aligned}$$

which is, indeed, the RF (3) of the ELFR(α, β, γ) distribution. ■

Theorem 1 provides another way to interpret the ELFR distribution in addition to the geometric-extreme stability property given by Marshall and Olkin [7].

3. PROBABILITY DENSITY FUNCTION

The p.d.f. of the ELFR(α, β, γ) distribution with RF (3) is given by

$$f(x) = \frac{\alpha(\beta + \gamma x)e^{-(\beta x+(\gamma/2)x^2)}}{[1 - \bar{\alpha}e^{-(\beta x+(\gamma/2)x^2)}]^2}, \qquad x > 0. \tag{4}$$

The following theorem gives simple conditions under which the p.d.f. (4) is decreasing or unimodal.

THEOREM 2: *Let* $X \sim$ ELFR(α, β, γ); *then X has decreasing (unimodal) p.d.f. provided* $(\alpha - 2)\beta^2 + \alpha\gamma \leq 0$ (>0).

PROOF: The first derivative of $f(x)$ is given by

$$f'(x) = \frac{e^{-2(\beta x+(\gamma/2)x^2)}\psi(\beta x + (\gamma/2)x^2)}{[1 - \bar{\alpha}e^{-(\beta x+(\gamma/2)x^2)}]^3}, \qquad x > 0,$$

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where

$$\psi(z) = (\gamma - \beta^2 - 2\gamma z)e^z - \bar{\alpha}(\gamma + \beta^2 + 2\gamma z), \qquad z = \beta x + (\gamma/2)x^2 > 0.$$

The function $\psi(z)$ has no (one) zero on $(0, \infty)$ provided $\psi(0) = (\alpha - 2)\beta^2 + \alpha\gamma \leq 0$ (>0); that is, $f(x)$ has no (one) critical point provided $\psi(0) \leq 0$ (>0). Since $f(x)$ is nonnegative and $f(0) = \beta/\alpha$ and $f(\infty) = 0$, $f(x)$ is decreasing (unimodal) provided $\psi(0) \leq 0$ (>0). ■

Remarks:

1. When $\alpha \in [2, \infty)$, $f(x)$ is unimodal for all values of β, $\gamma > 0$. Otherwise, $f(x)$ might still exhibit a decreasing or unimodal behavior according to the conditions of Theorem 2.
2. As $\gamma \to 0$ (i.e., the extended exponential distribution), $f(x)$ is decreasing (unimodal) provided $0 < \alpha \leq 2$ $(\alpha > 2)$, as shown by Marshall and Olkin [7].
3. As $\beta \to 0$ (i.e., the extended Rayleigh distribution), $f(x)$ is unimodal, as shown by Ghitany, AL-Hussaini, and Al-Jarallah [4].

Figure 1 shows the p.d.f. of the ELFR distribution for selected values of the parameters α, β, and γ. Note that each selection of the parameters α, β, and γ satisfies one of the conditions of Theorem 2.

4. MOMENTS AND QUANTILES

The rth moment of the ELFR distribution is given by

$$E(X^r) = r\int_0^\infty x^{r-1}\bar{F}(x)\,dx = r\alpha\int_0^\infty \frac{x^{r-1}}{e^{\beta x+(\gamma/2)x^2} - \bar{\alpha}}\,dx, \qquad r = 1, 2, \ldots.$$

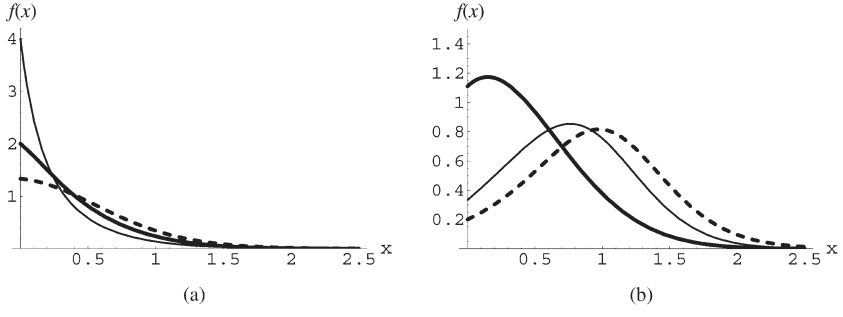


FIGURE 1. Plots of the p.d.f. $f(x)$ of the ELFR distribution: (a) $(\alpha, \beta, \gamma) \equiv (0.5, 1, 2)$ (bold), (0.5, 2, 1) (light), (1.5, 2, 1) (dashed); (b) $(\alpha, \beta, \gamma) \equiv (0.9, 1, 2)$ (bold), (3, 1, 2) (light), (10, 2, 1) (dashed).

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In general, the last integral cannot be given explicitly in terms of r, α, β, and γ; thus, even the first moment $E(X)$ must be calculated numerically. Also, for fixed r, β, and γ, we note that $E(X^r)$ is an increasing function in α.

The qth quantile of the ELFR distribution is given by

$$x_q = F^{-1}(q) = \frac{\beta}{\gamma}\left\{\sqrt{1+\frac{2\gamma}{\beta^2}\log\left(1+\frac{\alpha q}{1-q}\right)}-1\right\}, \qquad 0 \le q \le 1, \qquad \textbf{(5)}$$

where $F^{-1}(\cdot)$ is the inverse distribution function. For fixed r, β, and γ, we note that x_q is an increasing function in α.

The median of the ELFR distribution (i.e., $x_{0.5}$), is given by

$$\text{median}(X) = \frac{\beta}{\gamma}\left\{\sqrt{1+\frac{2\gamma}{\beta^2}\log(1+\alpha)}-1\right\}. \qquad \textbf{(6)}$$

Table 1 shows the measures of central tendency (mean, median, and mode) of the ELFR distribution for selected values of the parameters α, β, and γ, including those considered in Figure 1.

When $\text{mode}(X) = 0$ [i.e., when α, β, and γ satisfy $(\alpha - 2)\beta^2 + \alpha\gamma \le 0$ as shown by Theorem 2], Table 1 indicates that $E(X) > \text{median}(X)$. Otherwise [i.e., $\text{mode}(X) > 0$] this inequality is not necessarily true.

Table 1. Comparison Between the Measures of Central Tendency

α	β	γ	$E(X)$	median(X)	mode(X)
0.1	1	2	0.173	0.088	0
0.3	1	2	0.317	0.216	0
0.5	1	2	0.405	0.310	0
0.9	1	2	0.523	0.444	0.156
1.5	1	2	0.637	0.580	0.432
3.0	1	2	0.805	0.779	0.762
10.0	1	2	1.113	1.127	1.175
30.0	1	2	1.390	1.419	1.469
0.1	2	1	0.116	0.047	0
0.3	2	1	0.226	0.127	0
0.5	2	1	0.299	0.193	0
0.9	2	1	0.401	0.299	0
1.5	2	1	0.505	0.415	0
3.0	2	1	0.665	0.602	0.460
10.0	2	1	0.980	0.966	0.975
30.0	2	1	1.285	1.297	1.331

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5. FAILURE-RATE FUNCTION

The concept of the failure rate (FR) in engineering (also known as the *hazard rate* in medical science and the *force of mortality* in actuarial science) is the basic staple of modern reliability theory.

The FR function of the ELFR(α, β, γ) distribution is given by

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\beta + \gamma x}{1 - \bar{\alpha} e^{-(\beta x + (\gamma/2)x^2)}}, \qquad x > 0. \tag{7}$$

The following theorem presents a formal description of the possible shapes of the FR of the ELFR distribution in terms of the parameters α, β, and γ. The theorem shows that the FR is eventually increasing.

THEOREM 3: Let $\xi(y) = \gamma e^y - \bar{\alpha}(\beta^2 + \gamma + 2\gamma y)$, $y > 0$. If $X \sim$ ELFR(α, β, γ), *then X has the following properties*:

(i) *Increasing FR provided*
  (a) $0 < \alpha < 1/2$, $\xi(\ln(2\bar{\alpha})) \geq 0$, *or*
  (b) $1/2 \leq \alpha < 1$, $\xi(0) \geq 0$, *or*
  (c) $\alpha \geq 1$, β, $\gamma > 0$.

(ii) *Decreasing–increasing (bathtub) FR provided*
  (a) $0 < \alpha < 1/2$, $\xi(0) \leq 0$, $\xi(\ln(2\bar{\alpha})) < 0$, *or*
  (b) $1/2 \leq \alpha < 1$, $\xi(0) < 0$.

(iii) *Increasing–decreasing–increasing FR provided*
  $0 < \alpha < 1/2$, $\xi(0) > 0$, $\xi(\ln(2\bar{\alpha})) < 0$.

PROOF: The first derivative of $h(x)$ with respect to x is given by

$$h'(x) = \frac{e^{-(\beta x + (\gamma/2)x^2)} \xi(\beta x + (\gamma/2)x^2)}{[1 - \bar{\alpha} e^{-(\beta x + (\gamma/2)x^2)}]^2}, \qquad x > 0,$$

where

$$\xi(y) = \gamma e^y - \bar{\alpha}(\beta^2 + \gamma + 2\gamma y), \qquad y = \beta x + \frac{\gamma}{2} x^2 > 0.$$

Note that $\xi(0) = \gamma - \bar{\alpha}(\beta^2 + \gamma)$ and $\xi(\infty) = \infty$. Also, the first derivative of $\xi(y)$ is given by

$$\xi'(y) = \gamma e^y - 2\bar{\alpha}\gamma.$$

Under the stated conditions in (i)–(iii), $\xi(y)$ has no (one) (two) zeros. Hence, $h(x)$ has no (one) (two) critical points over $(0, \infty)$. Since $h(x)$ is non-negative and $h(0) = \alpha/\beta$, and $h(\infty) = \infty$, $h(x)$ is increasing (decreasing–increasing) (increasing–decreasing–increasing) in x. ■

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Remarks:

1. When $\alpha = 1$ (i.e., linear failure-rate distribution), $h(x)$ is increasing for all values of β, $\gamma > 0$. Otherwise, $h(x)$ might still exhibit increasing behavior, depending on α, β, and γ satisfying conditions (i)(a–c) of Theorem 3.
2. Increasing FR ELFR distribution implies other aging properties (e.g., the following chains of implications are well known:

$$\text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUE},$$

where IFR (IFRA) denotes increasing FR (increasing FR average) and NBU (NBUE) denotes new better than used (new better than used in expectation). For more details about the definitions of these aging concepts, see Barlow and Proschan [3, p. 159].

Figure 2 shows the FR of the ELFR distribution for selected values of the parameters α, β, and γ. Note that each selection of the parameters α, β, and γ satisfies one of the conditions of Theorem 3.

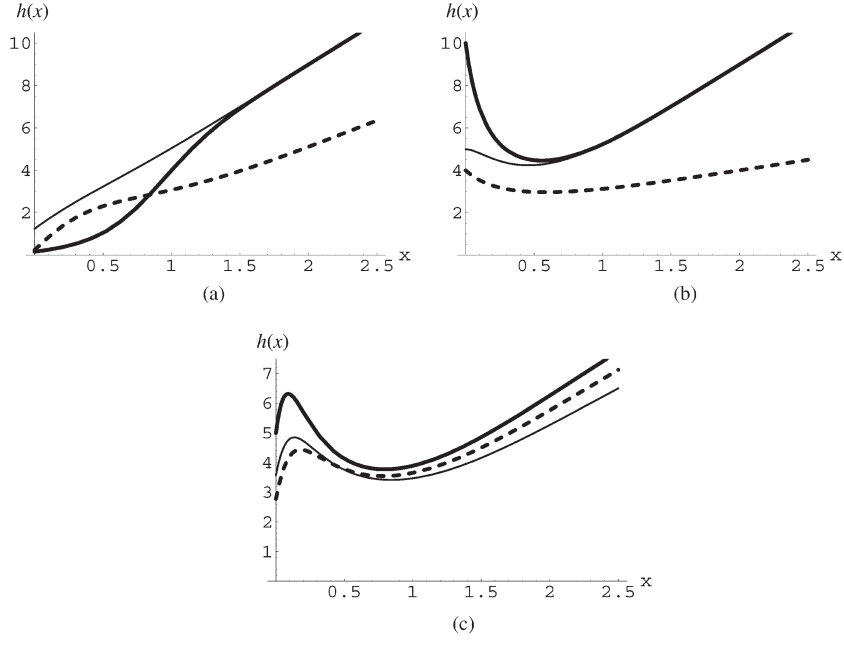


Figure 2. Plots of the FRF $h(x)$ of the ELFR distribution: (a) $(\alpha, \beta, \gamma) \equiv (6, 1, 4)$ (bold), (0.8, 1, 4) (light), (0.4, 0.1, 2.5) (dashed); (b) $(\alpha, \beta, \gamma) \equiv (0.1, 1, 4)$ (bold), (0.2, 1, 4) (light), (0.5, 2, 1) (dashed); (c) $(\alpha, \beta, \gamma) \equiv (0.05, 0.25, 3)$ (bold), (0.07, 0.25, 2.5) (light), (0.09, 0.25, 2.75) (dashed).

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6. STOCHASTIC ORDERINGS

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. We will recall some basic definitions.

A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order (denoted by $X \leq_{\text{st}} Y$) if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all x
(ii) hazard rate order (denoted by $X \leq_{\text{hr}} Y$) if $h_X(x) \geq h_Y(x)$ for all x
(iii) likelihood ratio order (denoted by $X \leq_{\text{lr}} Y$) if $f_X(x)/f_Y(x)$ decreases in x.

The following implications [see Ross (8, Chap. 9)] are well known:

$$X \leq_{\text{lr}} Y \Rightarrow X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{st}} Y. \tag{8}$$

The ELFR distributions are ordered with respect to the strongest "likelihood ratio" ordering, as shown in the following theorem.

THEOREM 4: *Let* $X \sim \text{ELFR}(\alpha_1, \beta, \gamma)$ *and* $Y \sim \text{ELFR}(\alpha_2, \beta, \gamma)$. *If* $\alpha_1 < \alpha_2$, *then*

$$X \leq_{\text{lr}} Y (X \leq_{\text{hr}} Y, X \leq_{\text{st}} Y).$$

PROOF: First note that

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1}{\alpha_2}\left[\frac{1-\bar{\alpha}_2 e^{-(\beta x+(\gamma/2)x^2)}}{1-\bar{\alpha}_1 e^{-(\beta x+(\gamma/2)x^2)}}\right]^2, \qquad x > 0.$$

Since, for $\alpha_1 < \alpha_2$,

$$\frac{d}{dx}\frac{f_X(x)}{f_Y(x)} = 2\frac{\alpha_1}{\alpha_2}(\alpha_1-\alpha_2)\frac{[1-\bar{\alpha}_2 e^{-(\beta x+(\gamma/2)x^2)}]^2}{[1-\bar{\alpha}_1 e^{-(\beta x+(\gamma/2)x^2)}]^3}\,(\beta+\gamma x)e^{-(\beta x+(\gamma/2)x^2)} < 0,$$

$f_X(x)/f_Y(x)$ is decreasing in x; that is $X \leq_{\text{lr}} Y$. The remaining statements follow from the implications (8). ■

7. LIMITING DISTRIBUTIONS OF SAMPLE EXTREMES

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from an absolutely continuous distribution with p.d.f. $f(x)$ and cumulative distribution function (c.d.f.) $F(x) = 1 - \bar{F}(x)$. Limiting distributions of sample minima $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ and maxima $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ are a long-standing area in applications of probability and statistics (see, e.g. Kotz and Nadarajah [6]). First, we recall the following asymptotical results for $X_{1:n}$ and $X_{n:n}$ (see, e.g., Arnold, Balakrishnan, and Nagaraja [1, pp. 210–214]):

(i) For the minimal order statistic $X_{1:n}$, we have

$$\lim_{n\to\infty} P\{X_{1:n} \leq a_n^* + b_n^* x\} = 1 - \exp[-x^c], \qquad x > 0,\ c > 0 \tag{9}$$

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(of the Weibull type), where $a_n^* = F^{-1}(0)$ and $b_n^* = F^{-1}(1/n) - F^{-1}(0)$ if and only if $F^{-1}(0)$ is finite and, for all $x > 0$ and $c > 0$,

$$\lim_{\epsilon \to 0+} \frac{F(F^{-1}(0) + \epsilon t)}{F(F^{-1}(0) + \epsilon)} = x^c. \tag{10}$$

(ii) For the maximal order statistic $X_{n:n}$, we have

$$\lim_{n \to \infty} P\{X_{n:n} \le a_n + b_n t\} = \exp[-e^{-t}], \quad -\infty < t < \infty \tag{11}$$

(of the extreme value type), where $a_n = F^{-1}(1 - 1/n)$ and $b_n = 1/nf(a_n)$ if

$$\lim_{x \to \infty} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = 0. \tag{12}$$

The following theorem derives the limiting distributions of the smallest and largest order statistics from the ELFR distribution.

THEOREM 5: *Let $X_{1:n}$ and $X_{n:n}$ be, respectively, the smallest and largest order statistics from the* ELFR(α, β, γ) *distribution. Then the following hold*:

(i) $\lim_{n \to \infty} P\{X_{1:n} \le b_n^* x\} = 1 - \exp[-x]$, $x > 0$, where $b_n^* = F^{-1}(1/n)$ and $F^{-1}(\cdot)$ is given by (5).

(ii) $\lim_{n \to \infty} P\{X_{n:n} \le a_n + b_n t\} = \exp[-e^{-t}]$, $-\infty < t < \infty$, where $a_n = F^{-1}(1 - 1/n)$, $b_n = 1/nf(a_n)$, and $f(\cdot)$ and $F^{-1}(\cdot)$, respectively, are given by (4) and (5).

PROOF:

(i) For the ELFR distribution, $F^{-1}(0) = 0$ is finite and, using L'Hospital rule, we have

$$\lim_{\epsilon \to 0+} \frac{F(\epsilon t)}{F(\epsilon)} = t \lim_{\epsilon \to 0+} \frac{f(\epsilon t)}{f(\epsilon)} = t$$

since $f(0) = \beta/\alpha$. Hence, statement (i) follows from (9) and (10).

(ii) For the ELFR distribution, we have

$$\lim_{x \to \infty} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = \lim_{x \to \infty} \bar{\alpha} e^{-(\beta x + (\gamma/2)x^2)} - \gamma \frac{1 - \bar{\alpha} e^{-(\beta x + (\gamma/2)x^2)}}{(\beta + \gamma x)^2} = 0.$$

Hence, statement (ii) follows from (11) and (12). ■

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8. CONCLUSION

The proposed extended linear failure-rate distribution has a unimodal p.d.f. with no or one critical point and an eventually increasing hazard rate function with no, one, or two critical point(s). Such characteristics are useful for modeling continuous data from life-testing experiments. The extended linear failure-rate distributions are ordered with respect to the *likelihood ratio stochastic ordering*. The limiting distribution of the smallest (largest) order statistics of a random sample from the proposed distribution has an exponential (extreme value) distribution.

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