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Universal models for Lorenz maps

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Abstract. Lorenz maps are dynamical systems on the interval related to the Lorenz flow. Monotone Lorenz families will be introduced and it will be shown that all possible topological dynamical behavior of Lorenz maps is realized within every monotone Lorenz family. Moreover, the structure of the parameter space for such families is described.

1. *Introduction*

The aim of this paper is to describe some parameterized families of Lorenz flows that are topologically universal in the sense that, given any geometric Lorenz flow, its dynamics is essentially the same as the dynamics of some element of the family. Thus, these families play, in the context of Lorenz flows, the same role as the quadratic family in the context of unimodal interval maps.

Lorenz in [**L**] showed numerically the existence of some flows in three dimensions that have complicated recurrent behavior. What we now call a Lorenz flow has a singularity of saddle type with a one-dimensional unstable manifold and an infinite set of hyperbolic periodic orbits, whose closure contains the saddle point. More specifically, the closure of this set of periodic orbits is in general the global attractor of the flow.

To analyze the dynamics of such a flow we take a two-dimensional transversal section intersecting the local stable manifold in a line l and we look at the first return map to S. This map is not defined in the line l and in fact exhibits a discontinuity at l because orbits near l in opposite sides follow different branches of the unstable manifold. To describe the dynamics of such a flow, a new hypothesis was added in [**W**] (see also [**GW**]): the existence of a one-dimensional foliation in S that is invariant by the first return map has l as a leaf and is such that points in the same leaf are exponentially contracted under iteration by the first return map. A Lorenz flow with this extra structure we call a *geometric Lorenz flow*. Because of the exponential contraction on the leaves of the foliation, the dynamics of such a flow can be described by the action of the first return map on the space of leaves of

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the stable foliation. This space of leaves is an interval and the induced map has a unique discontinuity at the point corresponding to l. Such an interval map is called a Lorenz map. More precisely, we have the following.

Definition 1.1. Let $P < 0 < Q$ and $r \geq 1$. A C^r *Lorenz* map $f : [P, Q] \to [P, Q]$ is map described by a pair (f_-, f_+) where:

(1) $f_- : [P, 0] \to [P, Q]$ and $f_+ : [0, Q] \to [P, Q]$ are continuous and strictly increasing maps;

(2) $f(P) = P$ and $f(Q) = Q$ and f has no other fixed points in $[P, Q] \setminus \{0\}$;

(3) there exists $\rho > 0$, the exponent of f, such that

$$f_-(x) = \tilde{f}_-(|x|^\rho) \quad \text{and} \quad f_+(x) = \tilde{f}_+(|x|^\rho)$$

where $\tilde{f}_-$ and $\tilde{f}_+$, the coefficients of the Lorenz map, are C^r diffeomorphisms defined on appropriate closed intervals. This Lorenz map is denoted by (P, Q, f_-, f_+).

Notice that the triple $\{\rho, \tilde{f}_-, \tilde{f}_+\}$ is uniquely determined by f. If the map is associated to a Lorenz vector field then the exponent ρ is precisely the absolute value of the ratio between the unstable eigenvalue and the weak stable eigenvalue at the saddle point.

A Lorenz map is *non-trivial* iff $f([P, 0]) \supset [P, 0]$ and $f([0, Q]) \supset [0, Q]$. Otherwise f is trivial, any orbit of such a map is asymptotic to a fixed point.

Guckenheimer and Williams proved in [**GW**] that there exists an open set of vector fields in three-dimensional space that have a structure of geometric Lorenz flow with smooth associated Lorenz maps. In fact they only considered the situation where the exponent is smaller than one and the map expands with derivative everywhere bigger than $\sqrt{2}$. However, we can use the same arguments to construct open sets of vector fields having Lorenz maps with exponent bigger than one. As we will see soon, the Lorenz maps of exponent bigger than one present a much bigger variety of dynamical behavior due to the interplay of contractions and expansions. Compare this with the unimodal situation: the quadratic family exhibits more types of combinatorics than the expanding tent family. Before stating our results we need to discuss some combinatorial aspects of Lorenz maps.

A *branch of* f^n is a maximal closed interval J such that f^n is a diffeomorphism in the interior of J. An endpoint of J is either P, Q or a point in the backward orbit of 0. To each branch J of f^n we can associate a word $\omega = (\omega_0, \omega_1, \ldots, \omega_{n-1})$ where $\omega_i \in \{[P, 0], [0, Q]\}$ and $f^i(J) \subset \omega_i$. It is clear that given a word of length n, there exists at most one branch of f^n associated to it. The combinatorics of all possible words are determined by the kneading invariants of f, $K^-(f)$ and $K^+(f)$ defined as follows. The first n symbols of $K_-(f)$ are the symbols of the branch of f^{n+1} adjacent to 0 that is contained in $[P, 0]$. Similarly for $K^+(f)$. There are many papers describing the combinatorics of Lorenz maps, see, for example, [**P**] and specially [**HS**]. Using the results described there one can characterize all possible kneading invariants of Lorenz maps.

The intersection of all branches that contain a given point is either a point or a closed interval. If such an intersection is an interval, it is called a *homterval* of f. A *critical homterval* is a homterval that has zero as an endpoint. So there are at most two critical homtervals. The image of a homterval is always contained in another homterval and, if

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the homterval is not critical, it is onto. An orbit of homtervals is a sequence $J^0, J^1, \ldots$ of homtervals such that $f(J^n) \subset J^{n+1}$. There are three types of orbits of homtervals:

(1) J^0 is a *wandering interval* if its orbit contains infinitely many disjoint intervals;
(2) it is *periodic* of period n if $J^n \subset J^0$ and $J^i \neq J^0$ if $0 < i < n$;
(3) it is *eventually periodic* if it is not periodic but J^i is periodic for some i.

An homterval J is *essential* iff for some $k \geq 0$ we have that $f^k(J)$ is contained in a periodic homterval T with $0 \in \partial T$ (where ∂T denotes the boundary of T).

If a C^2 Lorenz f map has a critical homterval then all homtervals of f are eventually periodic and there is at most a finite number of periodic homtervals since otherwise we could, by making a modification of f with support in the critical homterval, obtain a smooth C^2 map with three non-flat critical points having either a wandering homterval or infinitely many periodic homtervals, which contradicts the main theorem of [**MMS**].

Definition 1.2. Let $f : [P, Q] \to [P, Q]$ be a Lorenz map. Two points $x, y \in [P, Q]$ are equivalent, $x \sim y$, if and only if the interval $[x, y]$ is contained in a union of non-essential homtervals.

Observe the following.

(1) The quotient map $\hat{f} = f\backslash\sim : [P, Q]\backslash\sim \to [P, Q]\backslash\sim$ is well defined. The map $\hat{f}$ is thought of as describing the essential part of the dynamics of f, including the periodic critical behavior.
(2) The kneading information of $\hat{f}$ can differ from that of f. If this is the case f has to have a non-essential homterval J attached to its critical point, say $0_+ \in \partial J$, and the orbit of 0_- passes eventually through J. The kneading sequences $K^-(f)$ and $K^+(f)$ are both non-periodic but $K^-(\hat{f})$ is periodic.
(3) If the Lorenz map f has negative Schwarzian derivative then the case described above cannot happen, $K^\pm(\hat{f}) = K^\pm(f)$.
(4) If f has at most one critical homterval then each equivalence class of the equivalence relation above consists of exactly one homterval.

In §3 we will discuss the notion of *quasi-conjugation* and show that maps $\hat{f}$ and $\hat{g}$ with $K^\pm(\hat{f}) = K^\pm(\hat{g})$ are quasi-conjugated.

Let $\mathcal{L}^r$ be the collection of C^r Lorenz maps. We endow $\mathcal{L}^r$ with a topology that takes care of the domain, of the exponents and of the coefficients: the coefficients are compared by the topology after a linear rescaling which makes the domains equal.

Definition 1.3. Let $\Lambda \subset \mathbb{R}^2$ be closed. A Lorenz family is a continuous map $F : \Lambda \to \mathcal{L}^r$,

$$F_\lambda = (P_\lambda, Q_\lambda, \phi_\lambda, \psi_\lambda).$$

A Lorenz family $\lambda \in \Lambda \mapsto F_\lambda$ is *full* if given any non-trivial Lorenz map $f \in \mathcal{L}^2$ there exists a parameter value λ such that $K^\pm(\hat{f}) = K^\pm(\hat{F}_\lambda)$.

A monotone Lorenz family is a C^3 Lorenz family such that:

(1) F_λ has negative Schwarzian derivative for all $\lambda \in \Lambda$;
(2) $\Lambda = [0, 1] \times [0, 1]$;
(3) $F : (s, t) \to (-1, 1, \phi_s, \psi_t)$ and $\rho_{s,t} = \rho > 1$;
(4) if $s_1 < s_2$ then $\phi_{s_1}(x) < \phi_{s_2}(x)$ for all $x \in (-1, 0]$ and if $t_1 < t_2$ then $\psi_{t_1}(x) < \psi_{t_2}(x)$ for all $x \in [0, 1)$;

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(5) $\phi_0(0) = 0$, $\phi_1(0) = 1$, $\psi_0(0) = -1$ and $\psi_1(0) = 0$;
(6) $DF_\lambda(\pm 1) > 1$ for $\lambda \in \Lambda$.

THEOREM A. *A monotone Lorenz family is a full family.*

Next we discuss the notion of renormalization for Lorenz maps. This will allow us to refine Theorem A.

Definition 1.4. A Lorenz map f is called renormalizable if there exist $P < p < 0 < q < Q$ such that the first return map to (p, q) is a non-trivial Lorenz map, say (p, q, f^a, f^b). This induced Lorenz map is called a renormalization of f. The interval $[p, q]$ is called a domain of renormalization. The orbits $\cup_{k\geq 0} f^k([p, 0))$ and $\cup_{k\geq 0} f^k((0, q])$ are called the cycles of renormalization, (a, b) are the periods of renormalization and the type of renormalization is the pair (α, β), where α (respectively β) is the word associated to the branch of f^a (respectively f^b) that contains $[p, 0]$ (respectively $[0, q]$).

Let $\mathcal{D}_{\alpha,\beta} \subset \mathcal{L}^r$ be the subset of Lorenz maps which have a renormalization of type (α, β). These sets are called *domains of renormalization*. In §4 it will be shown that:
(1) $\mathcal{D}_{\alpha,\beta}$ is closed and connected;
(2) the collection of sets $\mathcal{D}_{\alpha,\beta}$ is nested: if $\mathcal{D}_{\alpha,\beta} \cap \mathcal{D}_{\hat{\alpha},\hat{\beta}} \neq \emptyset$ then

$$\mathcal{D}_{\alpha,\beta} \subset \mathcal{D}_{\hat{\alpha},\hat{\beta}} \quad \text{or} \quad \mathcal{D}_{\hat{\alpha},\hat{\beta}} \subset \mathcal{D}_{\alpha,\beta}.$$

Let $\mathcal{D} \subset \mathcal{L}^r$ be the set of renormalizable maps. This set is the union of the nested collection consisting of the sets $\mathcal{D}_{\alpha,\beta}$.

Fix a Lorenz family $F : \Lambda \to \mathcal{L}^r$ with critical exponent $\rho > 1$. This family will intersect the domains $\mathcal{D}_{\alpha,\beta}$, giving rise to the following.

Definition 1.5. The archipelago of type (α, β) is the set of all parameter values $A_{\alpha,\beta}$ for which the corresponding Lorenz map is in $\mathcal{D}_{\alpha,\beta}$: $A_{\alpha,\beta} = F^{-1}(\mathcal{D}_{\alpha,\beta})$. An island in the archipelago $A_{\alpha,\beta}$ is a connected component of the interior of $A_{\alpha,\beta}$. To express the type of renormalization we are considering, we will speak about (α, β)-archipelagoes and their (α, β)-islands.

The archipelagoes inherit properties from the sets $\mathcal{D}_{\alpha,\beta}$, they are closed and nested. This implies that the closure of an island defines a Lorenz family, namely the family of the corresponding (α, β)-renormalizations. We call an island a *full island* if the induced family is a full Lorenz family.

THEOREM B. *Every archipelago of a monotone Lorenz family contains a full island.*

Observe that Theorem A is a special case of Theorem B. In §3 we will define for every Lorenz map f the Julia and Fatou set of $\hat{f}$. The Fatou set is the basin of the periodic attractors. We will distinguish Fatou sets of two different types. A quasi-conjugation between two Lorenz maps is a homeomorphism which conjugates the dynamics on the Julia sets and preserves the components of the Fatou set. In particular, quasi-conjugations preserve the type of the Fatou sets involved.

THEOREM C. *Let $f, g \in \mathcal{L}^r$, $r \geq 1$, $\rho > 1$ with $K^{\pm}(\hat{f}) = K^{\pm}(\hat{g})$. Then $\hat{f}$ and $\hat{g}$ are quasi-conjugated.*

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In particular, the family associated to a full island in a given archipelago realizes all possible quasi-conjugation classes.

Definition 1.6. A Lorenz map is called hyperbolic iff both critical orbits tend to hyperbolic periodic attractors and the complement of the basin of these periodic attractors is a hyperbolic set.

THEOREM D. *The hyperbolic elements in a monotone Lorenz family form an open and dense set in parameter space.*

1.1. *Notation.* Let $I \subset [-1, 1]$ be an interval with boundary points a and b, say $a < b$. Then $\partial I = \{a, b\}$, $\partial_- I = \{a\}$ and $\partial_+ I = \{b\}$. If $J \subset I \subset [-1, 1]$ are two intervals and $\partial_- J = \partial_- I$, we say $J \subset_l I$. If $\partial_+ J = \partial_+ I$ then we say $J \subset_r I$. In the case that $\partial J \cap \partial I = \emptyset$ we write $J \subset_{\text{int}} I$. The discontinuity of Lorenz maps in 0 causes such maps to have two *critical orbits*:

$$f^n(0_-) = \lim_{x \uparrow 0} f^n(x) \quad \text{and} \quad f^n(0_+) = \lim_{x \downarrow 0} f^n(x),$$

with $n \geq 0$.

2. *Realization of finite combinatorics*

In this section we will describe sufficient conditions for Lorenz families to be able to realize all possible finite kneading sequences. This is an adaption of similar results obtained in [**MS**] for continuous interval maps.

We will start by defining kneading sequences for Lorenz maps, similarly as was done in [**MT**] for continuous piecewise monotone interval maps. Clearly, to describe Lorenz maps we will need two kneading sequences, one for 0_- and one for 0_+. Fix a Lorenz map $f : [-1, 1] \to [-1, 1]$.

Let $\mathbb{B}_n(f)$ be the collection of *branches of* f^n. That is, the collection of maximal intervals on which f^n is monotone. If $I \in \mathbb{B}_n(f)$ then the word $\omega(I) \in \{[-1, 0], [0, 1]\}^n$ is such that

$$f^i(I) \subset \omega_i(I)$$

for $0 \leq i < n$. The *kneading sequences* are defined as

$$K_n^-(f) = \omega(I_-(n)) \quad \text{and} \quad K_n^+(f) = \omega(I_+(n))$$

where $I_-(n), I_+(n) \in \mathbb{B}_n(f)$, $n \geq 0$, with $\partial_+ I_-(n) = 0$ and $\partial_- I_+(n) = 0$. Let $K^-(f)$ and $K^+(f)$ be the limits of respectively $K_n^-(f)$ and $K_n^+(f)$, $n \to \infty$.

PROPOSITION 2.1. *Let* $V = [-1, 0] \times [0, 1]$ *and* $\Lambda \subset \mathbb{R}^2$ *be homeomorphic to* V. *Let*

$$\Lambda \ni \lambda \mapsto f_\lambda : [-1, 1] \to [-1, 1]$$

be a family of Lorenz maps satisfying the following properties:

(1) *the branches* $f_{\lambda,\pm}$ *are* C^1, $f_\lambda(\pm 1) = \pm 1$, *and there exists* $K > 0$ *such that* $0 < Df_\lambda(x) \leq K$ *for all* $x \neq 0$;

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(2) $Df_\lambda(x) \to 0$ *as* $x \to 0$;
(3) *let* $F : \Lambda \to V$ *be the continuous map defined by* $F(\lambda) = (f_\lambda(0+), f_\lambda(0-))$ *then* $F(\partial\Lambda) \subset \partial V$ *and the degree of the map* $F : \partial\Lambda \to \partial V$ *is different from zero.*

If $g \in \mathcal{L}^r$ *has finite critical orbits then there exists* λ *such that* $K^\pm(g) = K^\pm(f_\lambda)$.

Proof. Let $g \in \mathcal{L}^r$, $r \geq 0$ be a Lorenz map with finite critical orbits and $P(g) = \{z_1 < \cdots < z_l = 0 < z_{l+1} < \cdots < z_k\}$ be the post-critical set of g where $-1 \leq z_1 = g(0_+) \leq 0$ and $0 \leq z_k = g(0_-) \leq 1$ are the critical values of g. The order of these points and the mapping $c : \{1, 2, \ldots, l-1, l+1, \ldots, k\} \to \{1, \ldots, k\}$ defined by $g(z_i) = z_{c(i)}$ describe the combinatorics which we want to realize by some map of our family. Consider the $(k-1)$-dimensional simplex $P = \{x \in R^k \mid -1 = x_0 \leq x_1 \leq \cdots \leq x_l = 0 \leq x_{l+1} \leq \cdots \leq x_k \leq x_{k+1} = 1\}$ and let P_0 be the interior of P. Let $pr : P \to V$ be the projection $pr(x) = (x_1, x_k)$ and

$$Z = \{(\lambda, x) \in \Lambda \times P \mid pr(x) = F(\lambda)\}.$$

Let us consider the mapping $T : Z \to P$, $(\lambda, x) \mapsto y$, where $f_\lambda(y_i) = x_{c(i)}$ and y_i has the same sign as x_i. It is clear that T is well defined and continuous. To finish the proof we have to find $(\lambda, x) \in Z$ such that $T(\lambda, x) = x$.

Let $m(x) = \min\{|x_i - x_{i+1}|; i = 0, 1, \ldots, k\}$ and $d(x, y) = \max\{|x_i - y_i|; i = 1, \ldots, k\}$. We need the following.

LEMMA 2.2. *If* $(\lambda_n, x(n)) \in Z$ *is such that* $m(x(n)) \to 0$ *then*

$$\lim_{n\to\infty} \frac{d(T(\lambda_n, x(n)), x(n))}{m(x(n))} = \infty.$$

Proof. Suppose, by contradiction, that there exists $K_0 > 0$ and a sequence $(\lambda_n, x(n)) \in Z$ such that $m(x(n)) \to 0$ and

$$d(x(n), y(n)) \leq K_0 m(x(n))$$

where $y(n) = T(\lambda_n, x(n))$.

CLAIM 1. *There exists constant* $K_1 > 0$ *such that if* $x_i(n), x_j(n)$ *have the same sign then*

$$|x_{c(i)}(n) - x_{c(j)}(n)| \leq K|x_i(n) - x_j(n)| + K_1 m(x(n)).$$

Proof. Since $Df_\lambda(t) \leq K$, it follows from the mean value theorem that

$$\begin{aligned} |x_{c(i)}(n) - x_{c(j)}(n)| &\leq K|y_i(n) - y_j(n)| \\ &\leq K|x_i(n) - x_j(n)| + K|y_i(n) - x_i(n)| + K|y_j(n) - x_j(n)| \\ &\leq K|x_i(n) - x_j(n)| + 2K_0 m(x(n)) \end{aligned}$$

and the claim is proved. □

From Claim 1 we obtain, by induction, that for all s there exists a constant $C_s > 0$ such that if $x_{c^k(i)}(n)$ and $x_{c^k(j)}(n)$ have the same sign for $k < s$ then

$$|x_{c^s(i)}(n) - x_{c^s(j)}(n)| \leq C_s|x_i(n) - x_j(n)| + C_s m(x(n)).$$

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CLAIM 2. *There exists s_0 such that for all $l \in \{1, \dots, k\}$ there exists $s \le s_0$ with* $0 \in [x_{c^s(l)}(n), x_{c^s(l+1)}(n)]$.

Consider the set $\mathcal{P}$ of pairs (z_i, z_j) such that $0 \notin [z_i, z_j]$. If the claim is false then, since $\mathcal{P}$ is finite, there exists a periodic pair in $\mathcal{P}$. This implies the existence of a periodic homterval non-essential for g.

To finish the proof of the lemma let l be such that $m(x(n)) = |x_l(n) - x_{l+1}(n)|$ (l depends on n). Let $s \le s_0$ be such that $0 \in [x_{c^s(l)}(n), x_{c^s(l+1)}(n)]$ and $x_{c^k(l)}(n), x_{c^k(l+1)}(n)$ have the same sign for $k < s$. From Claim 2 we obtain some t depending on n such that $|x_t(n) - x_{t+1}(n)| \le C_s m(x(n))$ and either $x_t(n) = 0$ or $x_{t+1}(n) = 0$. Since $|f_{\lambda_n}([y_t(n), y_{t+1}(n)])| = |[x_{c(t)}(n), x_{c(t+1)}(n)]| \ge m(x(n))$ we have that $Df_{\lambda_n}(\theta_n)|y_t(n) - y_{t+1}(n)| \ge m(x(n))$ for some $\theta_n \in [y_t(n), y_{t+1}(n)]$.

Consider the case when $x_{t+1}(n) = 0$. Then also $y_{t+1}(n) = 0$. Therefore,

$$\begin{aligned} d(y(n), x(n)) &\ge |y_t(n) - x_t(n)| \\ &\ge \frac{1}{Df_{\lambda_n}(\theta_n)} \cdot m(x(n)) - C \cdot m(x(n)) \\ &\ge \left(\frac{1}{Df_{\lambda_n}(\theta_n)} - C \right) m(x(n)). \end{aligned}$$

On the other hand, $|y_j(n) - x_j(n)| \le d(y(n), x(n)) \le K_0 m(x(n)) \to 0$ as $n \to \infty$. This implies that $Df_{\lambda_n}(\theta_n) \to 0$. This contradicts $d(x(n), y(n)) \le K_0 m(x(n))$. □

Assume that there is no point in Z with $T(\lambda, x) = x$.

LEMMA 2.3. *There exists a continuous one-parameter family of continuous mappings $G_t : Z \to P$ such that:*

(1) $G_0(\lambda, x) = x$;
(2) $G_t(\lambda, x) = x$ *if* $x \in \partial P$;
(3) $G_1(Z) \subset \partial P$.

Proof. Let $D = \{x \in \mathbb{R}^{k-1} \mid \sum_{i=1}^{k-1} x_i^2 \le 1\}$ be the closed unit ball in $\mathbb{R}^{k-1}$ and let $\Phi : P \to D$ be a bi-Lipschitz homeomorphism. Let $\tilde{T} : Z \to D$ be the mapping $\tilde{T} = \Phi \circ T$. Since Φ is bi-Lipschitz and the distance from x to the boundary of P is equal to $\frac{1}{2}m(x)$, we have that if $(\lambda_n, x(n)) \in Z$ is such that $m(x(n)) \to 0$ then

$$\frac{d(\Phi(x_n), \tilde{T}(\lambda_n, x(n)))}{d(\Phi(x(n)), \partial D)} \to \infty.$$

Now define $\tilde{G}_1(\lambda, x)$ as the point in the boundary of D such that $\Phi(x)$ lies in the line segment bounded by $\tilde{T}(\lambda, x)$ and $\tilde{G}_1(\lambda, x)$. It follows that $\tilde{G}_1$ is continuous and $\tilde{G}_1(\lambda, x) = \Phi(x)$ for $x \in \partial P$. Define $\tilde{G}_t = t\tilde{G}_1 + (1-t)\Phi$ and $G_t = \Phi^{-1} \circ \tilde{G}_t$. This clearly satisfies the conditions of the lemma. □

LEMMA 2.4. *Let $h : V \to \Lambda$ be a homeomorphism such that $\pi_i(x) = 0 \Rightarrow \pi_i(F(h(x))) = 0$ for $i = 1, 2$, where $\pi_i : V \to \mathbb{R}$ is the projection in the ith coordinate. Then there exists a mapping $H : P \to Z$ with the following properties:*

(1) *H is continuous and the restriction of H to the interior of P is a homeomorphism;*

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(2) $\pi \circ H = h \circ pr$ *where* $\pi : Z \to \Lambda$ *is the projection* $(\lambda, x) \mapsto \lambda$*;*
(3) *the restriction of* H *to each fiber* $pr^{-1}(v) \cap \text{int}(P)$ *is a diffeomorphism onto the fiber* $\pi^{-1}(h(v)) \cap \text{int}(Z)$.

Proof. Let $v = (v_1, v_2)$ and $F(h(v)) = (\tilde{v_1}, \tilde{v_2})$. Let $H_v : \mathbb{R} \to \mathbb{R}$ be the Moebius transformation that maps v_1 to $\tilde{v_1}$, 0 to 0 and v_2 to $\tilde{v_2}$. For $x \in pr^{-1}(v) \cap \text{int}(P)$ define $H(x) = (h(v), (H_v(x_1), \ldots, H_v(x_k)))$. □

Choose a homeomorphism $h : V \to \Lambda$ with the property needed to apply Lemma 2.4 and let $H : P \to Z$ be the corresponding map from Lemma 2.4. Let $\hat{G}_t = G_t \circ H : P \to P$. We have that:
(1) $\hat{G}_t$ is continuous and depends continuously on t;
(2) $\hat{G}_t(\partial P) \subset \partial P$ and $\hat{G}_0(\text{int}(P)) \subset \text{int}(P)$;
(3) $\hat{G}_t(x) = G_0(H(x))$ for all $x \in \partial P$;
(4) $\hat{G}_1(P) \subset \partial P$;
(5) the degree of $\hat{G}_0$ is equal to the degree of F.

The only statement that needs a proof is (5). Notice that $pr \circ G_0 = F \circ \pi$ on Z, by definition of Z, whence

$$pr \circ \hat{G}_0 = pr \circ G_0 \circ H = F \circ \pi \circ H = (F \circ h) \circ pr$$

and the restriction of $\hat{G}_0$ to $pr^{-1}(v) \cap \text{int}(P)$ is a diffeomorphism onto $pr^{-1}(F(h(v)) \cap \text{int}(P))$. Hence the degree of $\hat{G}_0$ is equal to the degree of $F \circ h$, which is equal to the degree of F. Here we are using the following topological fact. Let $\Phi : D \to D$ be a continuous mapping that maps ∂D onto ∂D, $\text{int}(D)$ onto $\text{int}(D)$ and $\Phi|_{\text{int}(D)}$ is smooth. Then the degree of $\Phi|_{\partial D}$ equals the degree of $\Phi|\text{int}(D)$ (see [**D**, p. 67]). Contradiction.

It follows that there is a map f_λ in the family which also has periodic critical orbits. Moreover, the combinatorics of these critical orbits are the same as the combinatorics of the critical orbits of g. □

3. *Fatou sets and quasi-conjugations*

In this section we will study the behavior of essential homtervals.

Definition 3.1. A Cherry map is a Lorenz map $f : [P, Q] \to [P, Q]$ such that the restriction of f to the invariant interval $[f(0_+), f(0_-)]$ is semi-conjugated to a rotation of a circle, the restriction to $(f(0_+), f(0_-))$ is injective. A rational Cherry map is a Cherry map with rational rotation number. Compare [**MMMS**].

A simple Lorenz map $f : [P, Q] \to [P, Q]$ is a Lorenz map which is either trivial or a rational Cherry map. We will distinguish three types of simple Lorenz maps:
type 1: $f([P, 0)) \subset [P, 0)$ and $f(0_+) \in [P, 0)$ (or $f(0_-) \in (0, Q]$ and $f((0, Q]) \subset (0, Q]$);
type 2: $f([P, 0)) \subset [P, 0)$ and $f((0, Q]) \subset (0, Q]$;
type 3: f is a rational Cherry map.

Observe that all three types can be obtained by perturbations of a simple Lorenz map with $f(0_-) = f(0_+) = 0$.

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Fix a Lorenz map $f \in \mathcal{L}^2$ and its essential part $\hat{f} = f\backslash\sim$ and assume that $\hat{f}$ has a homterval. That means it has an essential homterval attached to 0.

LEMMA 3.2. *If $\hat{f}$ has only one essential homterval of the form $U = [p, 0]$ then it is periodic and p is a periodic point. If $\hat{f}$ has two essential homtervals, $[a, 0]$ and $[0, b]$ then $\hat{f}$ has a renormalization which is a simple Lorenz map whose domain U contains $[a, b]$.*

Proof. The case where $\hat{f}$ has only one essential homterval attached to zero can be dealt with as a direct consequence of the definition.

Assume that $\hat{f}$ has two essential homtervals $L = [a, 0]$ and $R = [0, b]$. A *chain* is an open interval which is a locally finite union of essential homtervals. Let C_- be the maximal chain in $[P, 0]$ containing L and C_+ the maximal chain in $[0, Q]$ containing R. There are three cases to be dealt with.

(i) $C_- = L$ and $C_+ = R$. The orbits of a and b cannot pass through the interior of $U = [a, b]$ because L and R are maximal chains. In particular, L and R are periodic essential intervals with periodic boundary points a and b. We observe that $\hat{f}$ admits a renormalization with domain U which is a simple Lorenz map of type 2.

(ii) $C_- \neq L$ and $C_+ = R$ (or $C_- = L$ and $C_+ \neq R$). Let $C_- = (p, 0]$. Again observe that the orbits of p and b cannot pass through the interior of $U = [p, b]$ because C_- and R are maximal chains. Because R is an essential homterval and the orbit of b does not hit 0, the homterval R itself has to be periodic. We have recovered the right branch of the first return map to U.

There exists a $k \geq 0$ such that $\hat{f}^k(L) \subset L \cup R$. Observe that $\hat{f}^k(C_-) \subset C_- \cup L \cup R$. If $\hat{f}^k(L) \subset L$ then $\hat{f}^k(C_- \setminus L) \subset C_- \setminus L$ and $C_- \setminus L$ would not consist of essential homtervals. So $\hat{f}^k(L) \subset R$ and, by maximality, $0 \in \partial \hat{f}^k(L)$. Now observe by maximality that $\hat{f}^k(C_-) = C_- \cup \hat{f}^k(L)$ and that p is a periodic point. We recover the left branch of the first return map to U. The map $\hat{f}$ admits a renormalization with domain U which is of type 1.

(iii) $C_- \neq L$ and $C_+ \neq R$. Let $C_- = (p, 0]$ and $C_+ = [0, q)$. As before observe that the maximality of C_- and C_+ implies that the orbits of p and q cannot pass through the interior of $U = \overline{C_- \cup C_+} = [p, q]$. There are minimal numbers $l, r > 0$ such that $0 \in \hat{f}^l(C_-)$ and $0 \in \hat{f}^r(C_+)$. This follows from the assumption that both C_- and C_+ are not homtervals, they differ from L and R respectively. It follows easily, by using the maximality of C_- and C_+, that $(p, q, \hat{f}^l, \hat{f}^r)$ is a renormalization with domain U.

It is left to show that this renormalization is a rational Cherry map. We claim that the interval $[\hat{f}^l(0_-), \hat{f}^r(0_+)]$ is contained in a finite union of maximal essential intervals. If C_- (and C_+) is a finite union of such intervals there is nothing to prove. If C_- is an infinite union of maximal essential homtervals then these intervals can only accumulate at p. Consequently, $\hat{f}^r(0_+)$ cannot be to close to p because $\hat{f}^r(R)$ is contained in a maximal essential homterval. The claim is proved.

Let

$$[\hat{f}^l(0_-), \hat{f}^r(0_+)] \subset \cup_{j=1}^{s} H_j.$$

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Assume H_j has its right endpoint in common with H_{j+1}. Observe that each H_j is mapped into some $H_{\pi(j)}$. This defines the map $\pi : \{0, \dots, s\} \to \{1, \dots, s\}$. This map π reflects in the following way that the renormalization is a Lorenz map and that the intervals H_j are maximal essential homtervals: there exists $t \in \{1, 2 \dots, s-1\}$ such that:

(1) $\pi : \{1, 2, \dots, t\} \to \{s-t, s-t+1, \dots, s\}$ is a bijection;

(2) $\pi : \{t+1, t+2, \dots, s\} \to \{1, 2, \dots, s-t\}$ is a bijection.

As a consequence we obtain that π itself is injective: the renormalization restricted to $(\hat{f}^l(0_-), \hat{f}^r(0_+))$ is injective. The renormalization is a Cherry map with a periodic attractor. It is a rational Cherry map. □

Observe, that rational Cherry maps have a renormalization of type 2.

COROLLARY 3.3. *If $\hat{f}$ has essential homtervals attached to both sides of the critical point then it admits a renormalization which is of type 1 or 2.*

In the case that $\hat{f}$ only has an essential homterval $U = [p, 0]$ (or $U = [0, p]$) on one side of the critical point we say that $\hat{f}$ admits a *renormalization of type 0 on U*. Observe that for each $\hat{f}$ which has essential homtervals there is a well defined smallest interval $U = [a, b]$ on which $\hat{f}$ admits a renormalization of type 0, 1 or 2. Let $U_{\hat{f}}$ be the interior of U in the case that $\hat{f}$ has essential homtervals. Otherwise let $U_{\hat{f}} = \emptyset$.

Definition 3.4. The Julia set of a Lorenz map $\hat{f} : [P, Q] \to [P, Q]$ is

$$\Lambda_{\hat{f}} = \{x \in [P, Q] \mid \mathrm{orb}(x) \cap U_{\hat{f}} = \emptyset\}$$

and the Fatou set is

$$F_{\hat{f}} = [P, Q] \setminus \Lambda_{\hat{f}}.$$

If $F_{\hat{f}} \neq \emptyset$ then the type of Fatou set is the type of renormalization with domain $U_{\hat{f}}$.

LEMMA 3.5. *If $U_{\hat{f}}$ is non-empty then $\Lambda_{\hat{f}}$ is a totally disconnected closed set and every connected component of $F_{\hat{f}}$ is eventually mapped monotonically onto $U_{\hat{f}}$. If $\hat{f}$ is C^2 and its periodic orbits are hyperbolic then $\Lambda_{\hat{f}}$ is hyperbolic.*

Proof. Clearly $\Lambda_{\hat{f}}$ cannot contain an interval because this interval would be an homterval which stays away from the critical point. The (quotient) map $\hat{f}$ does not have such homtervals.

Choose $x \in F_{\hat{f}}$ and let $k \geq 0$ be minimal such that $\hat{f}^k(x) \in U_{\hat{f}}$. We will show that there is an interval T containing x such that $\hat{f}^k : T \to U_{\hat{f}}$ is monotone and onto. In particular, T is a connected component of $F_{\hat{f}}$. Observe that the boundary points of $U_{\hat{f}}$ never pass through $U_{\hat{f}}$, so the connected component of $F_{\hat{f}}$ containing T cannot be bigger than T.

Let $(a, b) \ni x$ be the maximal interval on which $\hat{f}^k$ is monotone. Assume that $\hat{f}^k(a) \notin U_{\hat{f}}$. That means that $\hat{f}^k(a) = \hat{f}^{k-s}(0_+)$. Because $\hat{f}^s(x) \notin U_{\hat{f}} = (p, q)$ we obtain that $\hat{f}^{k-s}(q) \in \hat{f}^k((a, x]) \subset U_{\hat{f}}$. This contradicts the fact that the boundary points of $U_{\hat{f}}$ do not pass through $U_{\hat{f}}$. We have proved that $\hat{f}^k(M) \supset U_{\hat{f}}$. In particular, we have found the connected component of $F_{\hat{f}}$ containing x. It is mapped eventually onto $U_{\hat{f}}$.

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The hyperbolicity statement in the case that $\hat{f}$ is smooth and only has hyperbolic periodic orbits follows directly from Mane's lemma in [**M**]. □

In general, the Julia set of a Lorenz map with essential homtervals is not a Cantor set. It can have isolated points, for example, in the case when it admits a renormalization which is a rational Cherry map. However, the isolated points will always be preimages of periodic points which form the boundary of domains of renormalization.

The map $\hat{f}$ induces a map on the set of connected components of $F_{\hat{f}}$. It will be denoted by $\hat{f} : \mathcal{F}_{\hat{f}} \to \mathcal{F}_{\hat{f}}$.

Definition 3.6. A quasi-conjugation from $\hat{f}$ to $\hat{g}$ is an order preserving homeomorphism such that:

(1) $h(\Lambda_{\hat{f}}) = \Lambda_{\hat{g}}$ and $h(F_{\hat{f}}) = F_{\hat{g}}$;
(2) $h : \Lambda_{\hat{f}} \to \Lambda_{\hat{g}}$ is a conjugation;
(3) h induces a conjugation between $\hat{f} : \mathcal{F}_{\hat{f}} \to \mathcal{F}_{\hat{f}}$ and $\hat{g} : \mathcal{F}_{\hat{g}} \to \mathcal{F}_{\hat{g}}$.

THEOREM C. *Let f, g be Lorenz maps with $\rho_f, \rho_g > 1$ and $K^{\pm}(\hat{f}) = K^{\pm}(\hat{g})$ then $\hat{f}$ and $\hat{g}$ are quasi-conjugated. Moreover, the Fatou sets $F_{\hat{f}}$ and $F_{\hat{g}}$ have the same type. If one Fatou set is empty the other is also empty and the maps $\hat{f}$, $\hat{g}$ are in fact conjugated.*

The proof of this theorem needs some preparation. For every branch $I \in \mathbb{B}_n(f)$, let $l_n(I, f)$ and $r_n(I, f)$ be minimal such that

$$0 \in \partial_+ f^{l_n(I,f)}(I) \quad \text{and} \quad 0 \in \partial_- f^{r_n(I,f)}(I),$$

the *cutting times* of the branch I. Observe that, except for the leftmost L_n and rightmost branch R_n, the cutting times are well defined. Let $l_n(L_n) = r_n(R_n) = \infty$. When it is clear which map is under consideration, we will suppress the symbol f in the above notation.

LEMMA 3.7. *Let $f : [-1, 1] \to [-1, 1]$ and $g : [-1, 1] \to [-1, 1]$ be Lorenz maps such that $K_n^{\pm}(f) = K_n^{\pm}(g)$ for some $n \geq 1$. There exists an orientation preserving piecewise affine homeomorphism $h : [-1, 1] \to [-1, 1]$ such that:*

(1) *h preserves branches and their words: for every $k \leq n$, $I \in \mathbb{B}_k(f)$*

$$h(I) \in \mathbb{B}_k(g) \quad \textit{and} \quad \omega(I, f) = \omega(h(I), g);$$

(2) *h preserves cutting times: if $I \in \mathbb{B}_k(f)$ then*

$$l_k(h(I), g) = l_k(I, f) \quad \textit{and} \quad r_k(h(I), g) = r_k(I, f).$$

Proof. The proof will be by induction. The lemma holds for $n = 1$. Assume the lemma holds for some $n \geq 1$ and let h_n be the homeomorphism given by the lemma.

CLAIM. *For every branch $I \in \mathbb{B}_n(f)$ we have*

$$I \in \mathbb{B}_{n+1}(f) \textit{ if and only if } h_n(I) \in \mathbb{B}_{n+1}(f).$$

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Proof. Assume $I \notin \mathbb{B}_{n+1}(f)$. So $f^n(I) \ni 0$. Observe that $f^n(I) = (f^{n-l_n(I)}(0_-), f^{n-r_n(I)}(0_+))$. In particular,

$$(K_n^+(f))_{n-l_n(I,f)} = [-1, 0] \quad \text{and} \quad (K_n^-(f))_{n-r_n(I,f)} = [0, 1].$$

Now use the fact that the cutting times of $h_n(I)$ coincide with the cutting times of I and that f and g have the same kneading sequence:

$$(K_n^+(g))_{n-l_n(I,g)} = [-1, 0] \quad \text{and} \quad (K_n^-(g))_{n-r_n(I,g)} = [0, 1].$$

This implies that the monotone image $g^n(h(I)$ contains 0. We proved that $h(I) \notin \mathbb{B}_{n+1}(g)$. The claim is proved by symmetry. □

Observe that the collection of branches $\mathbb{B}_k(f)$ with $k = 0, 1, 2, \ldots, n$ define a refining sequence of partitions of $[-1, 1]$. The homeomorphism h_n maps this sequence of partitions to the partitions formed by $\mathbb{B}_k(g)$ with $k \leq n$. To prove the lemma for $n + 1$ we have to construct a homeomorphism h_{n+1} which also preserves the above partitions and moreover maps $\mathbb{B}_{n+1}(f)$ to $\mathbb{B}_{n+1}(g)$. In particular, h_{n+1} is obtained by redefining h_n in the interior of the branches $\mathbb{B}_n(f)$.

Let $I \in \mathbb{B}_n(f)$. If $I \in \mathbb{B}_{n+1}(f)$ we do not change h_n : $h_{n+1}|I = h_n$ because $h_n(I) \in \mathbb{B}_{n+1}(g)$ (which follows from the claim). If $I \notin \mathbb{B}_{n+1}(f)$ then both I and $h_n(I)$ will have two branches of respectively $\mathbb{B}_{n+1}(f)$ and $\mathbb{B}_{n+1}(g)$ (apply the claim again). Define $H_{n+1}|I$ such that these two sub-branches of I are mapped onto the two sub-branches of $h_n(I)$.

Once this construction has been done for all branches $I \in \mathbb{B}_n(f)$ we will obtain a homeomorphism h_{n+1} which preserves the branches in $\mathbb{B}_{n+1}(f)$ and their types. Clearly, it will also preserve the cutting times. □

Proof of Theorem C. Assume that $\hat{f}$ and $\hat{g}$ have the same kneading sequences. Observe that the fact whether $\hat{f}$ has a non-empty Fatou set or not depends completely on its kneading information. Also the type of Fatou set is determined by the kneading information. For example, if $K^-(\hat{f}) = \alpha^\infty$ and $K^+(\hat{f}) = \beta^\infty$ then $\hat{f}$ has a non-empty Fatou set of type 2.

Let $\{h_n\}$ be the sequence of piecewise affine homeomorphisms determined by Lemma 3.7. We will show that the limit of this sequence exists and is a quasi-conjugation. Let $I_n \in \mathbb{B}_n(\hat{f})$ form a nested sequence of branches. If the Fatou set of $\hat{f}$ is empty then such sequences of nested branches will shrink down to a point. Using the fact that $\hat{g}$ has an empty Fatou set if and only if $\hat{f}$ has an empty Fatou set, it follows easily that h_n converges to a conjugation between $\hat{f}$ and $\hat{g}$ in the case of empty Fatou sets.

Assume the Fatou set of $\hat{f}$ (and $\hat{g}$) is not empty. We will characterize combinatorially the nested sequences of branches which do not shrink down to a point. Let $I_n \in \mathbb{B}_n(f)$ be a nested sequence of branches and define $\omega = \lim_{n\to\infty} \omega(I_n)$, an infinite symbol sequence. The map $\hat{f}$ (and $\hat{g}$) has maximal essential homtervals attached to the critical point. Let us consider the case when $\hat{f}$ has such homtervals on both sides of the critical point, say $[a, 0]$ and $[0, b]$. Let ω_- and ω_+ be the infinite symbol sequences describing the combinatorics of the orbits of $[a, 0]$ and $[0, b]$ respectively. Essential homtervals are eventually periodic. Hence

$$\omega_- = W_-\alpha^\infty \quad \text{and} \quad \omega_+ = W_+\beta^\infty,$$

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where $W_\pm$ are finite words and α^∞ (β^∞) is a periodic sequence obtained by concatenating the finite word α (β). Let σ denote the shift map on the space of one-sided infinite symbol sequences. The following claim is a straightforward consequence of the fact that every homterval of $\hat{f}$ is eventually mapped into $[a, 0]$ or $[0, b]$.

CLAIM. *$I = \cap I_n$ is a non-degenerate interval, a homterval, if and only if there is $n \geq 0$ such that $\sigma^n \omega = \alpha^\infty$ or β^∞.*

Hence, the nested sequence of branches $I_n \in \mathbb{B}_n(f)$ shrinks down to a point if and only if $h_n(I_n)$ shrinks down to a point. This implies that the limit of the sequence defines a conjugation from $\Lambda_{\hat{f}}$ to $\Lambda_{\hat{g}}$.

If the nested sequence of branches $I_n \in \mathbb{B}_n(f)$ does not shrink down to a point, it shrinks down to a connected component I of the Fatou set of $\hat{f}$. Moreover, $h_n(I_n)$ shrinks down to a connected component I' of the Fatou set of $\hat{g}$ with the same infinite symbol sequence. The restriction of h_n to I converges to the unique orientation preserving affine map from I to I'. We have shown that the sequence h_n converges to a quasi-conjugation. □

4. *Domains of renormalization*

In this section we will study some topological properties of the sets $\mathcal{D}_{\alpha,\beta}$, the Lorenz maps which have a non-trivial (α, β)-renormalization.

LEMMA 4.1. *Let $p < 0 < q$ be two periodic points of the Lorenz map f, say with period a and b respectively. Assume:*

(1) $(p, q) \cap (\mathrm{orb}(p) \cup \mathrm{orb}(q)) = \emptyset$;

(2) $f^a|[p, 0]$ *and* $f^b|[0, q]$ *are monotone;*

(3) $f^a([p, 0]) \supset [p, 0]$ *and* $f^b([0, q]) \supset [0, q]$.

Let $(l, 0)$ (respectively $(0, r)$) be the maximal interval on which f^a (respectively f^b) is monotone.

Then

$$f^a(l) \leq l \quad \text{and} \quad f^b(r) \geq r.$$

Furthermore, if $f^a(l) = l$ (respectively $f^b(r) = r$) then $f^b(0_+) = 0$ (respectively $f^a(0_-) = 0$). In particular, if the Lorenz map has negative Schwarzian derivative then the periodic points p and q are hyperbolic repellers.

Proof. Assume that $f^a(l) \geq l$. Let $L = (l, p)$. The assumption implies that $f^a(L) \subset L$. Consequently, we obtain that $f^n|_L$ is monotone for all $n \geq 0$, where L is a homterval.

The maximality of the interval $(l, 0)$ gives some $i < a$ such that $0 \in \partial f^i(L)$. From property (1), it follows that $f^i(L) \supset (0, q)$. In particular, $(0, q)$ is a homterval. This is only possible if $f^b(0_+) = 0$, where the point 0_+ is periodic. Hence $f^a(L) = L$ and therefore $f^a(l) = l$.

Let $L_- = L = (l, p)$ and $L_+ = (p, 0)$. We have $f^a(L_\pm) \supset L_\pm$. The maximal principle for maps with negative Schwarzian derivative, see [**MS**], implies that p is a repeller. □

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LEMMA 4.2. *If $f \in \mathcal{D}_{\alpha,\beta} \cap \mathcal{D}_{\hat{\alpha},\hat{\beta}}$ then $\hat{\alpha}$ and $\hat{\beta}$ are formed by concatenating the words α, β. In particular,*

$$\hat{\alpha} = \alpha\beta \ldots, \quad \hat{\beta} = \beta\alpha \ldots$$

and $|\hat{\alpha}|, |\hat{\beta}| \geq |\alpha| + |\beta|$. Vice versa, α and β can be expressed in terms of $\hat{\alpha}$ and $\hat{\beta}$.

Proof. Let $f \in \mathcal{D}_{\alpha,\beta} \cap \mathcal{D}_{\hat{\alpha},\hat{\beta}}$ with renormalizations (p, q, f^a, f^b) and $(\hat{p}, \hat{q}, f^{\hat{a}}, f^{\hat{b}})$ of type (α, β) and $(\hat{\alpha}, \hat{\beta})$ respectively. We will first show that

$$(\hat{p}, \hat{q}) \subset (p, q) \quad (\text{or } (p, q) \subset (\hat{p}, \hat{q})).$$

Without loss of generality, we may assume that (p, q) is not contained in $(\hat{p}, \hat{q})$, say $\hat{p} \in (p, 0)$. We are going to show that $\hat{q} \in (0, q)$, the intervals are nested.

Assume $\hat{q} > q$. Then

$$f^{\hat{b}}(q) \in f^{\hat{b}}((0, \hat{q})) \subset (\hat{p}, \hat{q}) \subset (p, \hat{q}).$$

However, the orbit of q never enters (p, q). So $f^{\hat{b}}(q) \in [q, \hat{q})$. Because the orbit of q is finite it is impossible that $f^{\hat{b}}(q) \in (q, \hat{q})$: the map $f^{\hat{b}} : [q, \hat{q}] \to [q, \hat{q}]$ is monotone and would have a periodic attractor. So $f^{\hat{b}}(q) = q$ and $\hat{b} = m \cdot b$. The interval $[q, \hat{q}]$ is periodic which is impossible because the orbits in the boundary are expanding, see Lemma 4.1. We proved $\hat{q} \leq q$.

The aim is to prove that $\hat{q} < q$. Assume $\hat{q} = q$. Then both renormalizations will have $f^b|_{(0,q)}$ as right branch. In particular, $f^b(0_+) > \hat{p}$.

Let $(x, 0) \subset (p, 0)$ be the maximal monotone interval on which $f^{\hat{a}}$ is monotone. The map f admits a renormalization of type $(\hat{\alpha}, \hat{\beta}) = (\hat{\alpha}, \beta)$, so we have:

(1) $\hat{p} \in (x, 0)$;

(2) $f^{\hat{a}}((x, 0)) \supset (x, 0)$ (by Lemma 4.1).

Observe that $f^a((x, 0)) \subset (0, q)$. The next images of this interval will always stay on the right-hand side of $f^b(0_+)$. We have

$$f^b(0_+) > \hat{p} > x,$$

so the images of $(x, 0)$ will never be able to cover $(x, 0)$ completely, a contradiction. We have proved $\hat{q} < q$ and, in particular, that the $(\hat{\alpha}, \hat{\beta})$ renormalization is a first return map of the (α, β) renormalization. □

In the proof of the next proposition we will use the usual lexicographic order on the 0,1-words. The length of a word ω will be denoted by $|\omega|$. Moreover, if $j \leq \min\{|\omega_1|, |\omega_2|\}$ then

$$\omega_1 <_j \omega_2$$

means that $\omega_1 < \omega_2$ and the words differ in the first j symbols.

PROPOSITION 4.3. *The domains of renormalization have the following properties:*

(1) *$\mathcal{D}_{\alpha,\beta}$ is closed and connected;*

(2) *the collection of sets $\mathcal{D}_{\alpha,\beta}$ is nested. If $\mathcal{D}_{\alpha,\beta} \cap \mathcal{D}_{\hat{\alpha},\hat{\beta}} \neq \emptyset$ then*

$$\mathcal{D}_{\alpha,\beta} \subset \mathcal{D}_{\hat{\alpha},\hat{\beta}} \quad \text{or} \quad \mathcal{D}_{\hat{\alpha},\hat{\beta}} \subset \mathcal{D}_{\alpha,\beta}.$$

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Proof. The sets $\mathcal{D}_{\alpha,\beta}$ are closed because of Lemma 4.1. Next we will show that $\mathcal{D}_{\alpha,\beta}$ is path connected. Let $f_0, f_1 \in \mathcal{D}_{\alpha,\beta}$. We can conjugate f_0 to a map $f_{1/2} \in \mathcal{D}_{\alpha,\beta}$ such that the cycles of $f_{1/2}$ coincide with the cycles of f_1. In particular, by deforming the conjugation, we can construct a path from f_0 to f_1 within $\mathcal{D}_{\alpha,\beta}$. Consider the restrictions of $f_{1/2}$ and f_1 to their cycles, which are the same for both maps. Clearly we can deform one to the other, for example linearly. This deformation extends to a deformation of $f_{1/2}$ to f_1 which preserves the cycles. We have constructed a deformation from $f_{1/2}$ to f_1 within $\mathcal{D}_{\alpha,\beta}$.

Let $f, g \in \mathcal{D}_{\hat{\alpha},\hat{\beta}}$ and assume that $f \in \mathcal{D}_{\alpha,\beta}$ but $\mathcal{D}_{\alpha,\beta}$ is not a subset of $\mathcal{D}_{\hat{\alpha},\hat{\beta}}$. To prove the second part of the proposition we have to show that also $g \in \mathcal{D}_{\alpha,\beta}$. Observe that, by Lemma 4.2, the kneading sequences of f and g are equal up to at least $|\alpha| + |\beta| = a + b$.

Let (p, q, f^a, f^b) be the (α, β) renormalization of f and $(0, y)$ the maximal interval on which f^b is monotone. Lemma 4.1 states that $f^b((0, y)) \supset (0, y)$. In particular, $f^b(y) > y$.

There exists a unique $j = r_b((0, y), f) < b$, which by Lemma 3.7 depends only on α and β, such that $0 = \partial_+ f^j((0, y))$. The kneading sequence of $f^b(y)$ equals $\sigma^{b-j}(\hat{\alpha})$ and satisfies

$$\sigma^{b-j}(\hat{\alpha}) >_j \beta,$$

which is the combinatorial formulation of $f^b(y) > y$. Observe that

$$|\sigma^{b-j}(\hat{\alpha})| = \hat{a} - (b - j) \geq a + b - (b - j) > j,$$

so the above relation between $\sigma^{b-j}(\hat{\alpha})$ and β is well defined.

The map g has a $(\hat{\alpha}, \hat{\beta})$ renormalization. Lemma 4.2 states that $\hat{\beta} = \beta\alpha \ldots$. The map g has a branch $(0, y')$ of g^b of type β. We are going to prove that

$$g^b((0, y')) \supset (0, y').$$

First, because $\hat{\beta} = \beta\alpha$ we have $g^b(0_+) < 0$. The branch $(0, y')$ is also going to be cut at moment $j = r_b((0, y'), g) = r_b((0, y), f)$. So the kneading sequence of $g^b(y')$ equals

$$\sigma^{b-j}(\hat{\alpha}) >_j \beta.$$

In particular, $g^b(y') > y'$, otherwise the kneading sequence $\sigma^{b-j}(\hat{\alpha})$ would start with the word β. This contradicts the kneading information obtained from f. We have proved that $g^b((0, y')) \supset (0, y')$ and hence the existence of a periodic point $q' \in (0, y')$ with $g^b(q') = q'$.

Let $(x', 0)$ be the maximal interval on which g^a is monotone, i.e. the branch of type α. In a similar way as above we show that $f^a((x', 0)) \supset (x', 0)$ and the existence of a periodic point $p' \in (x', 0)$ with $g^a(p') = p'$.

It is left to show that (p', q', g^a, g^b) is an (α, β) renormalization. First we will show that the orbits of p' and q' never enter (p', q'). The kneading sequences of the periodic orbits of p' and q' are α^∞ and β^∞ respectively. The periodic orbits of p and q of f never enter the interval (p, q) and also have kneading sequences α^∞ and β^∞ respectively. This implies the following kneading information: for every $k \geq 0$ it is impossible that

$$\alpha^\infty < \sigma^k(\beta^\infty), \sigma^k(\alpha^\infty) < \beta^\infty.$$

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This kneading information implies that the orbits of p' and q' never enter (p', q').

We now show that $g^b((0, q')) \subset [p', q']$ (and $g^a((p', 0)) \subset [p', q']$). Because g has an $(\hat{\alpha}, \hat{\beta})$ renormalization there exists a branch $\hat{q} \in (0, z) \subset (0, q')$ with kneading sequence $\hat{\beta} = \beta\alpha \ldots$. If $g^b(0_+)$ is to the left of p' then also $g^b((0, z))$ is to the left of p' (the orbit of p' never enters (p', q')). According to the word $\hat{\beta} = \beta\alpha\alpha \ldots$ we have to apply the branch $g^a|(x', 0)$ a few times. Then $g^{b+i\cdot a}((0, z))$ will still be to the left of p'. This is impossible. Either $b + i \cdot a = \hat{b}$, in which case $g^{b+i\cdot a}(\hat{q}) = \hat{q}$, but $g^{b+i\cdot a}(\hat{q}) < p'$; or $b + i \cdot a < \hat{b}$, in which case we have to apply the branch $g^b|(0, y)$, but $g^{b+i\cdot a}((0, z)) \cap (0, y) = \emptyset$.

We have proved that (p', q', g^a, g^b) is a (α, β) renormalization. □

5. *Density of hyperbolicity in monotone families*

Example. Let $\phi_\lambda : [-1, 1] \to [-1, 1]$, $\lambda \in [0, 1]$, be a one-parameter family of unimodal maps. Say, $\phi_\lambda(\pm 1) = -1$ for $\lambda \in [0, 1]$, and $\phi_0(0) = 0$, $\phi_1(0) = 1$. Moreover, assume that this unimodal family is monotone: whenever $\lambda_1 < \lambda_2$ we have $\phi_{\lambda_1}(t) < \phi_{\lambda_2}(t)$ for all $t \in (-1, 1)$.

Consider the Lorenz family $f_{(x,y)} : [-1, 1] \to [-1, 1]$ with $(x, y) \in [0, 1] \times [0, 1]$ and

$$f_{(x,y)}(t) = \phi_x(t), \quad \text{if } t < 0$$
$$f_{(x,y)}(t) = \phi_{1-y}(t), \quad \text{if } t > 0.$$

Notice that for $(x, y) \in U = \{(t, s) \in [0, 1] \times [0, 1] \mid s = 1 - t\}$ the Lorenz map $f_{(x,y)}$ behaves exactly like the *unimodal* map ϕ_x. Namely

$$f^n_{(x,y)}(t) = \pm\phi^n_x(t),$$

for $n \geq 0$ and $t \in [-1, 1]$.

This example inspires us to change the coordinates of the parameter space of monotone Lorenz families. Fix a monotone Lorenz family $F : [0, 1] \times [0, 1] \to \mathcal{L}$. We will identify the parameters with the corresponding Lorenz maps. The usual coordinates x and y for $[0, 1] \times [0, 1]$ are not appropriate to explore the monotonicity of the family. We will use

$$U = \{(x, y) \mid y = 1 - x\} \quad M = \{(x, y) \mid y = x\}$$

as coordinate axes for $[0, 1] \times [0, 1]$ giving rise to the coordinates u and m on $[0, 1] \times [0, 1]$, $u = x - y$, $m = x + y - 1$. The axis U is called the unimodal axis and the axis M is called the monotonicity axis.

For $u \in U$ let $M_u = \{(u, m) \mid m \in M\} = \{(x, y) \mid x - y = u\}$ and for $(x, y) \in [0, 1] \times [0, 1]$ let

$$C^+_{(x,y)} = \{(t, s) \mid t \geq x, s \geq y\},$$
$$C^-_{(x,y)} = \{(t, s) \mid t \leq x, s \leq y\},$$
$$B_{(x,y)} = [0, 1] \times [0, 1] \setminus (C^+_{(x,y)} \cup C^-_{(x,y)}).$$

These cones will play a crucial role in the study of the parameter space of monotone Lorenz families. The C^+-cones describe deformations in which both branches *move up*.

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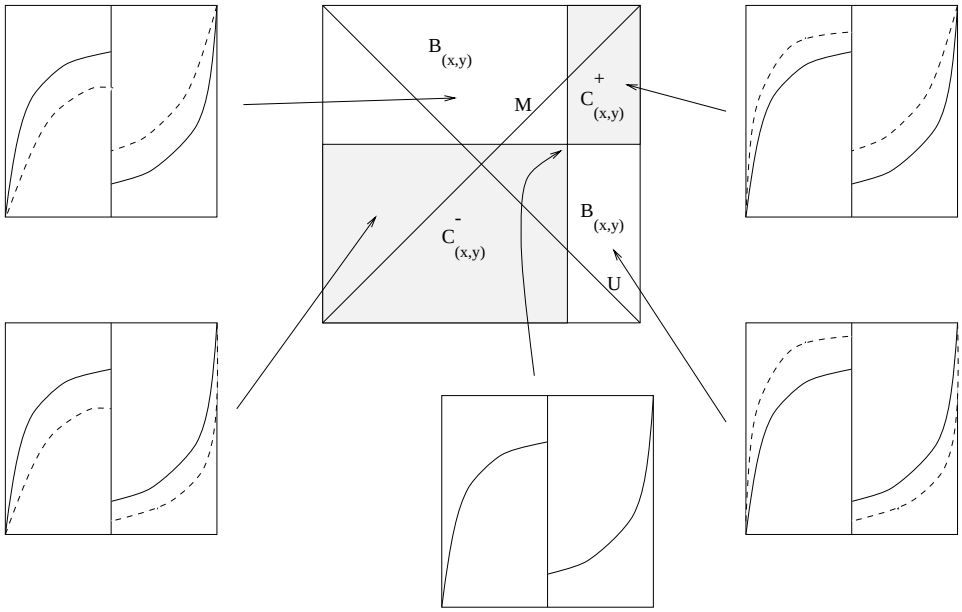


FIGURE 1. The u, m coordinates.

The C^--cones describe deformations in which both branches *move down*. The B-cone contains the maps for which one branch moves up and the other down. The situation is illustrated in Figure 1. The deformations into the B-cones are the ones which are difficult to understand. We will explore the deformations into the $C^{\pm}$-cones.

For example, the monotonicity of the family implies immediately the monotonicity in kneading information: let $K^{\pm}(z) = K^{\pm}_{F_z}$ then

$$K^{\pm}(z') \geq K^{\pm}(z), \quad \text{for } z' \in C^+(z)$$

and

$$K^{\pm}(z') \leq K^{\pm}(z), \quad \text{for } z' \in C^-(z),$$

where we used the usual lexicographic order on the $L - R$-sequences ($L < R$).

Proof of Theorem D. Given a map f in a monotone Lorenz family, we will construct arbitrarily close a map in the family whose critical orbits tend to hyperbolic periodic attractors. Then general arguments [**M**] will show that this perturbation is actually a hyperbolic Lorenz map.

Assume that both critical orbits do not tend to a periodic attractor. We may also assume that the map f is not in the boundary of parameter space: $C^-_f \cap ([0,1] \times [0,1]) \neq \emptyset$. The first step will be to find a map in C^-_f close to f such that 0_- is periodic.

Take $g_1 \in C^-_f$ close to f. Assume that there is some minimal $n_0 > 0$ such that $g_1^{n_0}(0_-) < 0 < f^{n_0}(0_-)$. If we move back along the straight line from g_1 to f then $g^{n_0}(0_-)$ will increase up to $f^{n_0}(0_-)$. Hence along this straight line there is a map g with $g^{n_0}(0_-) = 0$.

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The second case is that for all $n > 0$ we have that $g_1^n(0_-)$ and $f^n(0_-)$ are on the same side of 0. In particular, $g_1^n(0_-) < f^n(0_-)$ for all $n > 0$. Consider the intervals

$$J_n = [g_1^n(0_-), f^n(0_-)]$$

with $n \geq 0$. Observe $f(J_n) \subset J_{n+1}$. The intervals J_n have to accumulate at 0_+. If not the interval J_1 would be a homterval of f not accumulating on 0_+. By redefining f on a small interval $(0, a]$, we can define a continuous map which is C^2 away from 0 and which is around 0 of the form $h(|x|^\rho)$ with h smooth, having J_1 as a homterval: by [**MMS**] we know that J_1 tends to a periodic attractor of f. In particular, the orbit $f^n(0_-)$ tends to a periodic attractor. This is a contradiction because we have assumed that both critical orbits do not accumulate at periodic attractors.

We have proved that the critical orbit $g_1^n(0_-)$ accumulates at 0_+. For each perturbation $g_2 \in C^-_{g_1}$ there will be some $n_0 > 0$ such that $g_2^{n_0}(0_-) < 0 < g_1^{n_0}(0_-)$. (If not the points $g_2^k(0_-)$ and $g_1^k(0_-)$ would be on the same side of 0 for all $k \geq 0$. Consequently, $g_2^k(0_-) < g^k(0_-) - \epsilon$, for some $\epsilon > 0$ depending on g_2. This contradicts the fact that the critical orbit $g_1^n(0_-)$ accumulates at 0_+.) Again we can move back slightly to find a map $g \in C^-_f$ close to f such that 0_- is periodic for g. We have proved the following.

CLAIM. *If both critical orbits of f are not accumulating at a periodic attractor then arbitrarily close there is a map g such that 0_- is periodic. In particular, arbitrarily close there is a map such that 0_- is attracted to a hyperbolic periodic attractor but is not itself a periodic attractor.*

Assume that one critical orbit is attracted to a periodic attractor but is not itself periodic. We may assume that this orbit is that of the critical point 0_-. The boundary points -1 and 1 are expanding fixed points. Because the map has negative Schwarzian derivative the periodic attractor is of one of the following types.

(1) There exist an interval $(0, p]$ and $n > 0$ such that $f^n((0, p)) \subset (0, p)$ and $f^n(p) = p$. By taking $g \in C^-_f$ close to f we may assume that the periodic attractor is hyperbolic and also still attracts 0_-. We have obtained a situation in which both critical orbits are attracted to a hyperbolic periodic attractor. Theorem D is proved.

(2) There exists an interval $[p, 0)$ and $n > 0$ such that $f^n([p, 0)) \subset [p, 0)$. Again by taking $g \in C^+_f$ we may assume that the periodic orbit of p is a hyperbolic attractor. This situation will persist in a small neighborhood of g.

It is left only to deform g such that 0_+ also is attracted towards a hyperbolic attractor. Assume that 0_+ is not attracted towards a hyperbolic attractor. It could be attracted towards a neutral periodic attractor. Take $g_1 \in C^+_g$ close to g. Consider the two orbits $g^n(0_+)$ and $g_1^n(0_+)$, $n \geq 1$. If at some moment these points are separated then we can move back slightly towards g and make 0_+ periodic and by moving back slightly more we can make it to be attracted towards the periodic attractor which already attracted 0_-.

If these points are never separated, we show, as before, that they accumulate at 0_+ or are attracted towards a periodic attractor. As before we can make 0_+ to be attracted towards a hyperbolic attractor. □

The proof of Theorem D also shows that the hyperbolic Lorenz maps are dense in the whole space $\mathcal{L}^r$ of Lorenz maps (this can also be obtained from the arguments of [**R**]).

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Notice that this space contains a closed subspace $\mathcal{L}_s$ of symmetric Lorenz maps. These are Lorenz maps f with branches f_- and f_+ such that $f_-(x) = -f_+(-x)$. To each symmetric Lorenz map we can associate a unimodal map ϕ_f, namely

$$\phi_f(x) = f_-(x), \quad \text{for } x < 0$$
$$\phi_f(x) = -f_+(x), \quad \text{for } x \geq 0.$$

Observe that $\phi^n(x) = \pm f^n(x)$, for $n \geq 0$. The Lorenz map f can be seen as an orientation preserving 'lift' of the unimodal map ϕ_f. It follows that the density of the hyperbolic Lorenz maps in $\mathcal{L}_s$ would imply the density of hyperbolic maps in the space of unimodal maps. This is an important open question.

One should compare this with the situation of flows on non-orientable surfaces. These flows are covered by flows on the orientable cover of the manifold. Theorem D should be compared with the Peixoto theorem for flows on orientable surfaces.

6. *Archipelagoes in the parameter plane*

In this section we will study the parameter plane of a given monotone Lorenz family $F : [0, 1] \times [0, 1] \to \mathcal{L}^r$, $r \geq 0$. The main object of our study is to understand the topological structure of archipelagoes. The next proposition states that the boundaries of islands of an archipelago are formed by graphs of Lipschitz functions $\gamma_\pm : U \to M$. The fact that these boundaries are graphs of continuous functions is the central ingredient in the study of archipelagoes.

Let us concentrate on the archipelago $A = A_{\alpha,\beta}$. For every $\lambda \in A$ the domain of an (α, β)-renormalization is denoted by (p_λ, q_λ). Furthermore, let $a = |\alpha|$ and $b = |\beta|$.

PROPOSITION 6.1. (Island structure) *Let I be an island of the archipelago A. Then we have the following.*

(1) *For all $u \in U$, $M_u \cap A_{\alpha,\beta} = M_u \cap I$.*

(2) *There exists an interval $[u_1, u_2] \subset U$ and Lipschitz functions $\partial_+, \partial_- : [u_1, u_2] \to M$ with Lipschitz constant 1 and $\partial_-(x) < \partial_+(x)$, $x \in (u_1, u_2)$, such that*

$$I = \{(u, m) \mid u \in (u_1, u_2) \text{ and } \partial_-(u) < m < \partial_+(u)\}.$$

Furthermore, $\partial_-(u_1) = \partial_+(u_1)$ and $\partial_-(u_2) = \partial_+(u_2)$.

(3) *The points $\lambda = (u_1, \partial_-(u_1))$ and $\lambda = (u_2, \partial_-(u_2))$ are called* extremal *points of the island. There are two possible types of extremal points, either*

$$f_\lambda^a(0_-) = 0 \quad \text{and} \quad f_\lambda^b(0_+) = 0$$

or

$$f_\lambda^a(0_-) = q_\lambda \quad \text{and} \quad f_\lambda^b(0_+) = p_\lambda.$$

The first possibility is called a trivial extremal point, *the second a* full-branch extremal point.

(4) *For all $\lambda \in \partial_+$*

$$f_\lambda^a(0_-) = q_\lambda \quad \text{or} \quad f_\lambda^b(0_+) = 0.$$

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(5) *For all* $\lambda \in \partial_-$

$$f^a_\lambda(0_-) = 0 \quad \text{or} \quad f^b_\lambda(0_+) = p_\lambda.$$

Proof. Fix $u \in U$ such that $I \cap M_u \neq \emptyset$, where $M_u = \{u\} \times M$. Let $(p_{u,m}, q_{u,m}, f^a_{u,m}, f^b_{u,m})$ be the renormalization of $f_m \in I \cap M_u$. Observe that $m \mapsto q_{u,m}$ and $m \mapsto p_{u,m}$ are strictly monotone decreasing and $m \mapsto f^a_{u,m}(0_-)$ and $m \mapsto f^b_{u,m}(0_+)$ are strictly monotone increasing. This implies that, once we arrive at the boundary of $I \cap M_u$, by increasing m up to m_+ we are in the situation

$$f^a_{u,m}(0_-) = q_m \quad \text{or} \quad f^b_{u,m}(0_+) = 0.$$

In particular, for any $m' > m_+$ we have

$$K^-(f_{u,m'}) > \alpha\beta^\infty \quad \text{or} \quad K^+(f_{u,m'}) \geq \beta^\infty,$$

and hence $m' \notin A$. Similarly, we study what happens when m decreases up to the boundary of $I \cap M_u$. We have proved that

$$A \cap M_u = I \cap M_u$$

and that this intersection is an interval $\{u\} \times (\partial_-(u), \partial_+(u))$. The boundary of $I \cap M_u$ consists of points $(u, \partial_-(u))$ and points $(u, \partial_+(u))$: the boundary of I consists of two parts, $\partial_- I$ and $\partial_+ I$. Moreover,

$$f^b_{u,\partial_-(u)}(0_+) = p_{u,\partial_-(u)} \quad \text{or} \quad f^a_{u,\partial_-(u)}(0_-) = 0$$

and

$$f^a_{u,\partial_+(u)}(0_-) = q_{u,\partial_+(u)} \quad \text{or} \quad f^b_{u,\partial_+(u)}(0_+) = 0.$$

The island I is open and connected. Hence the set of values u for which $M_u \cap I \neq \emptyset$ is an open interval, say (u_1, u_2). Now observe that for any $z \in \partial_\pm I$ we have

$$\partial_\pm I \subset B_z,$$

where B_z is the complement of the cones C^+_z, C^-_z defined in §5. In particular, the functions $\partial_\pm : (u_1, u_2) \to M$ are Lipschitz and extend uniquely to continuous functions $\partial_\pm : [u_1, u_2] \to M$.

It is left for us to explain the behavior of the boundary points $\partial_\pm(u)$ when u tends to u_1 or u_2. For every $u \in (u_1, u_2)$ we have $\partial_-(u) < \partial_+(u)$. Let us assume that also $\partial_-(u_1) < \partial_+(u_1)$. The archipelago A is closed. Hence for any (u_1, m) with $\partial_-(u_1) < m < \partial_+(u_1)$ there is a renormalization $(p_{u_1,m}, q_{u_1,m}, f^a_{u_1,m}, f^b_{u_1,m})$. Because $(u_1, m) \in C_-((u_1, \partial_+(u_1))$ and $(u_1, m) \in C_+((u_1, \partial_-(u_1))$ we have $f^a_{u_1,m}(0_-) < q_{u_1,m}$ and $f^b(0_+) > p_{u_1,m}$. Because of Lemma 4.1 the point (u_1, m) is in the interior of I. This is a contradiction, proving that $\partial_-(u_1) = \partial_+(u_1)$. □

We have to distinguish special points on the boundary of islands. One type of special point are the extremal points, discussed in the above proposition. The other type of special

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point are vertices. Let $\partial_+ I$ be the upper part of the boundary of the island I. A point $\lambda \in \partial_+ I$ in the upper boundary is called a *vertex* if

$$f_\lambda^a(0_-) = q_\lambda \quad \text{and} \quad f_\lambda^b(0_+) = 0.$$

A vertex in the lower boundary is defined similarly. In Lemmas 6.3 and 6.4 we will show that each of the conditions defining a vertex holds on a set of parameter values in $U \times M$ which forms the graph of a Lipschitz function from U to M. Hence, the set of vertices is formed by intersecting these two graphs. In particular, it does not necessarily consist of isolated points.

LEMMA 6.2. *Every archipelago contains only finitely many islands with a vertex.*

Proof. Suppose the archipelago $A_{\alpha,\beta}$ has infinitely many islands I_n which have a vertex $v_n \in \partial_+ I_n$. Here $\partial_+ I_n$ denotes the upper boundary of I_n. We may assume that $v_n \to v$. Since archipelagoes are closed sets, $v \in A_{\alpha,\beta}$.

The topological type of the maps in the vertices are all the same and it follows easily that the map corresponding to the parameter v also inherits this type. So

$$f_v^a(0_-) = q_v \quad \text{and} \quad f_v^b(0_+) = 0.$$

By decreasing both parameters a little, we observe that v is actually a vertex of some island I in the same archipelago $A_{\alpha,\beta}$. In particular, v is not an extremal point of I. From Proposition 6.1 we obtain a neighborhood $(a, b) \times M$ of I such that the archipelago intersects this neighborhood only in I, and there are no other islands intersecting this neighborhood. But the vertex v of I lies in this neighborhood, and it cannot be accumulated by the islands I_n. Contradiction. □

LEMMA 6.3. *Let λ be a parameter for which 0_- is periodic, say with period $a = |\alpha|$, where α is the word describing the combinatorics of the orbit of 0_-. Denote the set of parameters for which 0_- is periodic with the same combinatorics α by α_-.*

Then there exists an interval $[u_1, u_2] \subset U$ and a Lipschitz function $\gamma : [u_1, u_2] \to M$ with Lipschitz constant 1 such that $[u_1, u_2] \times M$ is a neighborhood of λ and

$$\text{graph}(\gamma) \cap ([u_1, u_2] \times M) = \alpha_- \cap ([u_1, u_2] \times M).$$

Proof. Let $z = (u, m) \in \alpha_-$. The monotonicity of the family implies

$$\alpha_- \subset B_z$$

and

$$K^-(z') < \alpha, \quad z' \in \partial C_z^- \setminus \{z\}$$
$$K^-(z') > \alpha, \quad z' \in \partial C_z^+ \setminus \{z\}.$$

For a small enough neighborhood $[u_1, u_2] \times N$ of $z \in \alpha_-$ the orbit of 0_- will have the combinatorics given by α except for the last symbol. For each $u' \in [u_1, u_2]$, let $m'_\pm$ be such that $(u', m'_\pm) \in \partial C^\pm(z)$. According to the above observation we have for $u' \neq u$

$$K^-((u', m'_-)) < \alpha, \quad K^-((u', m'_+)) > \alpha.$$

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In particular, for each $u' \in [u_1, u_2]$ there exists a unique $m'(u') \in (m'_-, m'_+) \subset M_{u'}$ with

$$(u', m'(u')) \in \alpha_-$$

where the set $\alpha_- \cap ([u_1, u_2] \times M)$ is the graph of the function $m' : [u_1, u_2] \to M$. The first property discussed in this proof, namely $\alpha_- \subset B_z$ for each $z \in \alpha_-$, implies that this function m' is 1-Lipschitz. □

LEMMA 6.4. *Let λ be a parameter for which the map f_λ has two periodic points $p < 0 < q$, say with period $a = |\alpha|$ and $b = |\beta|$, respectively. Furthermore, assume*

(1) $(p, q) \cap (\text{orb}(p) \cup \text{orb}(q)) = \emptyset$;
(2) $f^a|[p, 0]$ *and* $f^b|[0, q]$ *are monotone;*
(3) $Df^a(p) > 1$ *and* $Df^b(q) > 1$;
(4) $f^a([p, 0]) = [p, q]$.

The word $\alpha\beta^\infty$ describes the combinatorics of the orbit of 0_-. Denote the set of parameters for which the corresponding map has the properties (1), (2), (3) and (4) by $\alpha\beta^\infty_-$.

Then there exists an interval $[u_1, u_2] \subset U$ and a Lipschitz function $\gamma : [u_1, u_2] \to M$ with Lipschitz constant 1 *such that $[u_1, u_2] \times M$ is a neighborhood of λ and*

$$\text{graph}(\gamma) \cap ([u_1, u_2] \times M) = \alpha\beta^\infty_- \cap ([u_1, u_2] \times M).$$

Proof. Let $z \in \alpha\beta^\infty_-$. The properties (1), (2) and (3) hold in a neighborhood of z. The proof continues in this neighborhood as the proof of Lemma 6.3. □

Similar statements hold for the combinatorics of the orbit of 0_+. The corresponding sets of parameters will be denoted by β_+ and $\beta\alpha^\infty_+$ respectively. The uniqueness part of the above lemma allows us to consider maximal arcs: the connected components of the sets α_- and $\alpha\beta^\infty_-$ are graphs of Lipschitz functions with uniform Lipschitz constant.

LEMMA 6.5. *Let $\gamma : (u_0, u_1) \to M$ be the Lipschitz function whose graph is a component of $\alpha\beta^\infty_-$. Then:*

(1) *this function can be extended to a Lipschitz function on $[u_0, u_1]$;*
(2) *there exists a function $\gamma_2 : [u_0, u_1] \to M$ whose graph is contained in β_+;*
(3) *if there exists $\lambda = (u, m) \in (\text{graph}(\gamma) \cap A_{\alpha,\beta}) \setminus \beta_+$, then β_+ intersects each component of $graph(\gamma) \setminus \{\lambda\}$ unless such a component terminates in the boundary of the parameter domain. Moreover, $\gamma_2(u) > \gamma(u)$, and the β_+ curve lies above the point (u, m) in the $\alpha\beta^\infty_-$ curve.*

In particular, a component of $\alpha\beta^\infty_- \setminus \{(u, m)\}$ which does not terminate in the boundary of the parameter domain contains a vertex in the upper boundary of an island of the archipelago $A_{\alpha,\beta}$.

Proof. The function γ is defined on an open interval. However, every Lipschitz function can be extended to a Lipschitz function on a closed interval. To prove (2) it is enough to show that $\beta_+ \cap M_u \neq \emptyset$ for each $u \in (u_0, u_1)$. Let $u \in (u_0, u_1)$ and $\{\lambda\} = \alpha\beta^\infty_- \cap M_u$.

Case I: $f^b_\lambda(0_+) > 0$. Consider the maps in M_u with the properties:

(a) there exists a branch $T = (0, x)$ of type β;

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(b) there exists $x_0 \in T$ with $f^b_{u,m}(x_0) < x_0$;
(c) $f^b_{u,m}(0_+) > 0$.

Observe that properties (b) and (c) imply the existence of a periodic attractor in T which attracts the orbit of 0_+. In particular, the orbit of 0_+ is infinite. This implies that $0 \notin \partial_- f^i(T)$ for each $i \leq b$. Consequently, if (u, m) is a point with these properties than there is a small interval $\{u\} \times (m - \epsilon, m]$ of maps with the three properties.

Observe that the map in $\lambda = (u, m_0)$ has the three properties. Let $H = \{t\} \times (m_1, m_0]$ be the maximal interval of maps with the above properties. We are going to show that in (t, m_1) the map is in β_+.

Observe that by decreasing $m \in (m_1, m_0]$ we see that the interval T is increasing in length. Because the family is monotone, the same point x_0 in T which was moved to the left persists to be in T and will be moved to the left. As before we see that the orbit of the interval T will never hit 0_+.

We have shown that properties (a) and (b) also hold in (u, m_1). Hence, in the boundary of H property (c) has to be violated. Otherwise we could decrease m slightly more.

Case II: ${f_\lambda}^b(0_+) < 0$. Consider the maps in M_u with the properties:
(a) there exists a branch $T = (0, x)$ of type β;
(b) $f^b_{u,m}(x) > x$;
(c) $f^b_{u,m}(0_+) < 0$.

Properties (b) and (c) imply that there is a hyperbolic periodic point $q \in T$ of type β. Consequently, the branch of type β will persist under small perturbations. The collection of the above maps is open.

Observe that the map in λ satisfies the above properties. We have $f^b_\lambda(x) > x$. Otherwise the orbit of 0_- could not be trapped in the orbit of type β, which is the case in $\lambda \in \alpha\beta^\infty_-$. Let $H = \{u\} \times [m_0, m_1)$ be the maximal interval of maps with the above properties. We are going to prove that $(u, m_1) \in \beta_+$.

Observe that by increasing $m \in [m_0, m_1)$ the branch T will be decreasing, a consequence of the fact that the family is monotone. However, the periodic orbit persists and there has to be a periodic orbit at (u, m_1). This may no longer be hyperbolic. In particular, there is a branch $T = (0, x')$ of type β in (u, m_1).

The monotonicity of the family implies

$$f^b_{u,m_1}(x') = f^j_{u,m_1}(0_-) > f^j_\lambda(0) = f^b_\lambda(x_\lambda) > x_\lambda,$$

where x_λ is the boundary point of the branch of f_λ mentioned in property (a). The branch mentioned in property (a) decreases when m increases. Hence, $x' < x_\lambda$. Therefore

$$f^b_{u,m_1}(x') > x'.$$

The only property which can be violated in (u, m_1) is property (c): $f^b_{u,m_1}(0_+) = 0$ and $(u, m_1) \in \beta_+$.

To prove (3) let $(u, m) = \lambda \in (\text{graph}(\gamma) \cap A_{\alpha,\beta}) \setminus \beta_+$. Then M_u contains a point in β_+. Because $(u, m) \in A_{\alpha,\beta} \setminus \beta_+$, there is a renormalization, so we obtain that $\gamma_2(u) > \gamma(u)$. To finish the proof of (3) we have to show that $f^b_{u_1,\gamma(u_1)}(0_+) > 0$ (and $f^b_{u_2,\gamma(u_2)}(0_+) > 0$).

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Because $(u_1, \gamma(u_1))$ is a boundary point we obtain that $Df^b_{u_1,\gamma(u_1)}(q) = 1$. Otherwise we could extend $\alpha\beta^\infty_-$. The map under consideration has negative Schwarzian derivative. Hence, the neutral periodic point attracts a critical point with an infinite orbit. The orbit of 0_- is trapped in the orbit of q. Only the orbit of 0_+ can be attracted towards q. In particular, $f^b_{u_1,\gamma(u_1)}(0_+) > 0$. □

A similar lemma holds for the combinatorics of 0_+. Moving up or down has well understood consequences on the combinatorics of the map. On the other hand moving one branch up and the other down are the deformations which are difficult to understand, with the deformations in directions parallel to U. The next lemma indicates a difference between such movements to the right and to the left.

LEMMA 6.6. *Let $\lambda \in \alpha\beta^\infty_- \cap \beta\alpha^\infty_+$, say $\lambda = (u, m)$. Then there exists a parameter $\mu = (t, m')$ such that:*

(1) *$t \le u$;*
(2) *μ is the right extremal point of an island I of the archipelago $A_{\alpha,\beta}$, moreover it is a full branch extremal point;*
(3) *the island I has a vertex in its upper (and lower) boundary.*

Proof. Let $\gamma_+ : (t_+, u] \to M$ be the function whose graph is the component of $\alpha\beta^\infty_- \setminus \{\lambda\}$ which touches λ from the left-hand side. Similarly let $\gamma_- : (t_-, u] \to M$ be the function whose graph is the component of $\beta\alpha^\infty_+ \setminus \{\lambda\}$ which also touches λ from the left-hand side. Lemma 6.4 ensures that these functions and graphs are well defined.

Let $t \in [t_+, u] \cap [t_-, u]$ be minimal such that $\gamma_+(t) = \gamma_-(t)$, describing the left most intersection point of the components under consideration of $\alpha\beta^\infty_-$ with $\beta\alpha^\infty_+$. Observe that these components are graphs of functions: the leftmost intersection is well defined. Let μ be the intersection point. Observe that Lemma 6.4 can be applied in this point, giving us local extensions of the arcs $\alpha\beta^\infty_-$ and $\beta\alpha^\infty_+$. So μ is a point in the interior of both graphs.

In fact, $\mu \in A_{\alpha,\beta}$, the map is of full branch type. So we can apply Lemma 6.5: let $s \le t$ be maximal such that $\mu_1 = (s, \gamma_+(s)) \in \beta_+ \cap \alpha\beta^\infty_-$. Observe that $\alpha\beta^\infty_-$ cannot terminate to the left in the boundary of parameter space.

Clearly μ_1 is a vertex point in the upper boundary of some island $I \subset A_{\alpha,\beta}$. Observe that the island I intersects $[s, t] \times M$ only below the graph of $\gamma_+|[s, t]$, moving up a point in this graph would destroy the renormalization immediately.

The aim is to show that the graph of $\gamma_+|[s, t]$ is part of the upper boundary of the island I. From Lemma 6.5 we obtain that the arc β_+ lies above the point μ. Furthermore, μ_1 is the rightmost intersection of the graph of $\gamma_+|(t_+, t]$ with the β_+ arc. So the β_+ arc in the strip $(s, t) \times M$ lies above the graph of γ_+, which lies above the island I. So the boundary points of the island I in the strip $(s, t) \times M$ lie in the graph of γ_+, and the boundary points in the upper boundary can only be of two types $\alpha\beta^\infty_-$ or β_+, from Proposition 6.1.

We have shown that the boundary $\partial_+ I \cap ([s, t] \times M) \subset \text{graph}(\gamma_+|[s, t])$. We need to show equality. Clearly the piece of the boundary in the strip can no longer have vertices, for this one needs points in β_+. So the graph contains the right extremal point of the island. This extremal point cannot be trivial, for this one needs a point in β_+. So it is of full branch type. The only parameter of this type in the graph of $\gamma_+|[s, t]$ is μ. Clearly the island does not extend beyond the right of μ, and the archipelago intersects $\{t\} \times M$ only in μ. □

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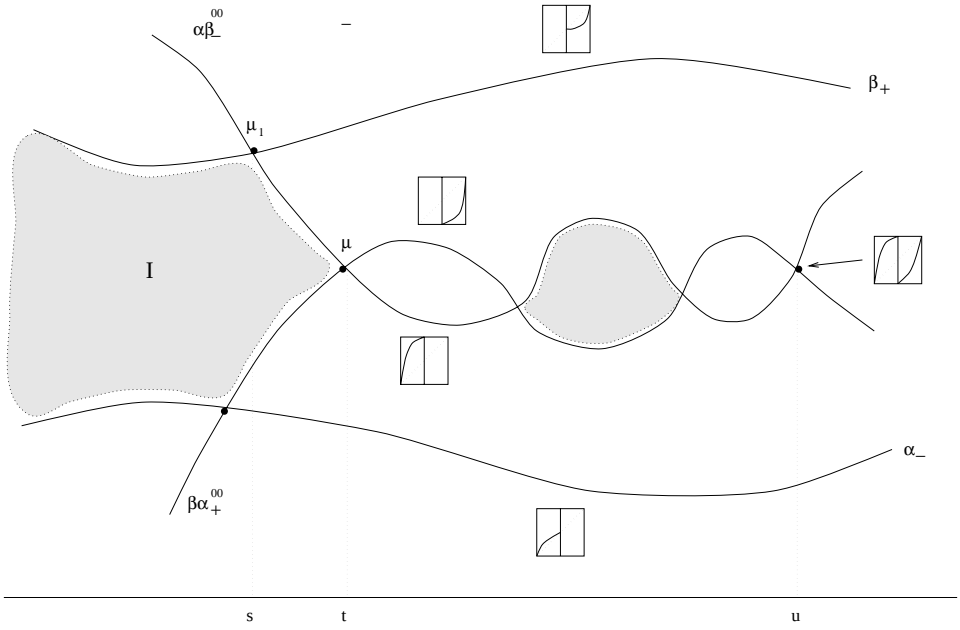


FIGURE 2. Illustration of the proof of Lemma 6.6.

7. *Proof of Theorem B*

Fix a monotone Lorenz family $F : [0,1] \times [0,1] \to \mathcal{L}^r$, $r \geq 3$. Let $A = A_{\alpha,\beta} \subset [0,1] \times [0,1]$ be on the α, β-archipelago.

PROPOSITION 7.1. *The archipelago A has an island $I \subset A$ which has a trivial and a full branch extremal point.*

Proof. By Proposition 2.1 there exists $\lambda \in \alpha\beta_-^\infty \cap \beta\alpha_+^\infty$. Lemma 6.6 gives an island which has a vertex and its right extremal point is of full branch type. Because there are only finitely many islands with a vertex we can take the leftmost island $I \subset A_{\alpha,\beta}$ which has a vertex and whose right extremal point is of full branch type.

We now only need to show that the left extremal point of this island is trivial. Suppose not and apply Lemma 6.6 again: we will find an island left of I which has a vertex and whose right extremal point is of full branch type. But I was the leftmost island with these properties. This is a contradiction. □

LEMMA 7.2. *Let $I \subset A_{\alpha,\beta}$ be an island which has a trivial and a full branch extremal point. The Lorenz family $G : I \to \mathcal{L}^r$ defined by*

$$I \ni \lambda \mapsto (p_\lambda, q_\lambda, f_\lambda^a, f_\lambda^b) \in \mathcal{L}^r$$

contains Cherry maps with arbitrary rotation number.

Proof. Let $\lambda = (u, m) \in I$ be the trivial extremal point. This point is the intersection of two graphs, say of the functions $\gamma_1, \gamma_2 : [a, b] \to M$ with

(i) $\text{graph}(\gamma_1) \subset \alpha_-^\infty$;

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(ii) $\text{graph}(\gamma_2) \subset \beta_+^\infty$;
(iii) $\lambda = (u, m)$, with $m = \gamma_1(u) = \gamma_2(u)$, $u \in [a, b]$;
(iv) an $\epsilon > 0$ such that $\gamma_1(t) < \gamma_2(t)$ for all $t \in (u, u + \epsilon)$.

The last property holds because of the following. The two extremal points are of different types: one is of full branch type and the other is trivial. Hence, both boundary components $\partial_+ I$ and $\partial_- I$ have a vertex. Let $v_\pm \in \partial_\pm I$ be vertices closest to the trivial extremal point. Observe that these are well defined. We use the fact that the boundaries $\partial_\pm$ are graphs. Then

$$v_+ = (u_+, \gamma_2(u_+)) \in \partial_+ I, \quad v_- = (u_-, \gamma_1(u_-)) \in \partial_- I,$$

with $u_\pm > u$. Hence, there is an $\epsilon > 0$ such that

$$\gamma_2(u + t) > \gamma_1(u + t),$$

for $t \in (0, \epsilon]$. We may take ϵ small enough such that for all $t \in [0, \epsilon]$ and $m \in [\gamma_1(u + t), \gamma_2(u + t)]$ the Lorenz map $g_{u+t,m}$ is a Cherry map. This is possible because $\rho > 1$.

Fix $t_1 \in (0, \epsilon]$ and consider the family of Cherry maps

$$m \mapsto g_{u+t_1,m}$$

with $m \in [\gamma_1(u+t_1), \gamma_2(u+t_1)]$. This is a continuous family and hence the rotation number will depend continuously on m. Observe that the rotation number of $g_{u+t_1,\gamma_1(u+t_1)}$ is zero and the rotation number of $g_{u+t_1,\gamma_2(u+t_1)}$ equals 1. The lemma follows, every rotation number will be realized. □

THEOREM B′. *Every Archipelago of a monotone Lorenz family $\{f_\lambda\}$ contains a full island. In particular, let $I \subset A_{\alpha,\beta}$ be an island which has a trivial and a full branch extremal point. The Lorenz family $G : I \to \mathcal{L}^r$ defined by*

$$I \ni \lambda \mapsto (p_\lambda, q_\lambda, f_\lambda^a, f_\lambda^b) \in \mathcal{L}^r$$

is a full family.

Proof. Observe, that Proposition 2.1 can be applied to the family G. Let $f \in \mathcal{L}^r$ with $\rho > 1$. In the previous Lemma 7.2 we showed that Cherry combinatorics are realized in this family. We may assume that f does not have kneading sequences which correspond to Cherry maps.

Case I: f admits infinitely many renormalizations. Say,

$$f \in \cdots \mathcal{D}_{\alpha_2,\beta_2} \subset \mathcal{D}_{\alpha_1,\beta_1} \subset \mathcal{D}_{\alpha_0,\beta_0} \subset I.$$

Proposition 2.1 implies that the family G has an archipelago A_{α_0,β_0} of type (α_0, β_0). By Proposition 7.1, applied to the original family, we find an island $I_{\alpha_0,\beta_0} \subset A_{\alpha_0,\beta_0}$ with a trivial and full branch extremal point. This argument can be repeated and we find a sequence of archipelagos and islands of the following types

$$A_{\alpha_0,\beta_0} \supset I_{\alpha_0,\beta_0} \supset A_{\alpha_1,\beta_1} \supset I_{\alpha_1,\beta_1} \supset A_{\alpha_2,\beta_2} \supset I_{\alpha_2,\beta_2} \ldots,$$

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the intersection of which contains maps g_λ with

$$K^{\pm}(\hat{g_\lambda}) = K^{\pm}(\hat{f}).$$

Case II: f admits only finitely many renormalizations. By performing the previous argument only finitely many times we will find an island I_{α_n,β_n} with a simple and a full branch extremal point such that the maps in this island are at least n times renormalizable and follow the same renormalization steps as f. The case II is reduced to case III or IV.

Case III: f admits no renormalizations and is simple. This means that the kneading sequences are

$$K^+(\hat{f}) = \alpha^\infty, \quad K^-(\hat{f}) = \beta^\infty$$

or

$$K^+(\hat{f}) = \alpha^\infty, \quad K^-(\hat{f}) = \beta\alpha^\infty$$

or the corresponding cases $(\alpha\beta^\infty, \beta^\infty)$. These cases can be represented by maps which have finite critical orbits; such maps are assured to exist by Proposition 2.1.

Case IV: f admits no renormalizations and is not simple. Let $K_n^{\pm}(\hat{f})$ be the truncated kneading sequences of $\hat{f}$. By a small perturbation of f we obtain a Lorenz map whose critical orbits are finite and the first n symbols of the kneading sequences coincide with those of $\hat{f}$. Proposition 2.1 assures the existence of a map $g_n \in G$ such that

$$K_n^{\pm}(\hat{f}) = K_n^{\pm}(\hat{g}_n).$$

The family G is compact, so we can take a limit $g_n \to g \in G$. The map $\hat{g}$ has the same kneading sequences as $\hat{f}$. To prove this we have to show that the sequences of $\hat{g}$ are not degenerate, that is they are not periodic. To do so we consider two cases.

Case IVa: $\hat{f}$ has one periodic kneading sequence (say $K^- = \alpha^\infty$). By construction we have that 0_- is periodic and $K^-(\hat{g}) = \alpha^\infty$. If $K^+(\hat{g})$ where also degenerate, that is periodic, then, by Lemma 3.2, $\hat{g}$ would be renormalizable. In particular, the kneading sequences of $\hat{f}$ would be of the form $K^-(\hat{f}) = \alpha^\infty$ and $K^+(\hat{f}) = \beta\alpha^\infty$. This kneading information implies that $\hat{f}$ has essential homtervals on both sides of 0. Lemma 3.2 implies that $\hat{f}$ is renormalizable. Contradiction.

Case IVb: $\hat{f}$ has no periodic kneading sequences. Suppose $\hat{g}$ has periodic kneading sequences. First, observe both sequences cannot be periodic. Otherwise $\hat{g}$ would be a trivial extremal point of the family and every map nearby would be a simple Lorenz map or an irrational Cherry map. This is impossible because $\hat{f}$ does not have Cherry combinatorics. If $K^+(\hat{g}) = \beta^\infty$ then

$$K^+(\hat{f}) = \beta K^-(\hat{f}).$$

This together with the fact that $K^-(\hat{f})$ is non-periodic implies that $\hat{f}$ has a non-periodic homterval attached to 0_-. By construction, $\hat{f}$ cannot have such a homterval. Contradiction. □

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