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# THREE DIFFERENT FORMALISATIONS OF EINSTEIN'S RELATIVITY PRINCIPLE

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**Abstract.** We present three natural but distinct formalisations of Einstein's special principle of relativity, and demonstrate the relationships between them. In particular, we prove that they are logically distinct, but that they can be made equivalent by introducing a small number of additional, intuitively acceptable axioms.

**§1. Introduction.** The special principle of relativity (SPR), which states that the laws of physics should be the same in all inertial frames, has been foundational to physical thinking since the time of Galileo, and gained renewed prominence as Einstein's first postulate of relativity theory (Einstein, 1916, §1):

> If a system of coordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the *same* laws hold good in relation to any other system of coordinates K' moving in uniform translation relatively to K. This postulate we call the "special principle of relativity."

Despite its foundational status, the special principle of relativity remains problematic due to its inherent ambiguity (Szabó, 2004; Gömöri & Szabó, 2013a,b). What, after all, do we mean by "physical laws", and what does it mean to say that the "same laws" hold in two different frames?

These ambiguities often lead to misunderstandings and misinterpretations of the principle. See, e.g., Muller (1992) for the resolution of one such misinterpretation. We believe that formalisation is the best way to eliminate these ambiguities. In this paper we investigate the principle of relativity in an axiomatic framework of mathematical logic. However, we will introduce not one but three different naturally arising versions of the principle of relativity, not counting the parameters on which they depend, such as the formal language of the framework used.

It is not so surprising, when one tries to capture SPR formally, that more than one "natural" version offers itself – not only was Einstein's description of his principle given only informally, but its roots reach back to Galileo's even less formal "ship story" (Galileo, 1953, pp. 186–187).

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Since all three of the versions we investigate are "natural", and simply reflect different approaches to capturing the original idea, there is no point trying to decide which is the "authentic" formalisation. The best thing we can do is to investigate how the different formalisations are related to each other. Therefore, in this paper we investigate under which assumptions these formalisations become equivalent.

In a different framework but with similar motivations, Gömöri and Szabó also introduce several formalisations of Einstein's ideas (Szabó, 2004; Gömöri & Szabó, 2013a,b; Gömöri, 2015). Intuitively, what they refer to as "covariance" corresponds to our principles of relativity and what they call the principle of relativity is an even stronger assumption. However, justifying this intuition is beyond the scope of this paper, as it would require us to develop a joint framework in which both approaches can faithfully be interpreted.

***1.1. Contribution.*** In this paper we present three logical interpretations ($\mathsf{SPR}_{\mathfrak{M}}$, $\mathsf{SPR}^{+}$, and $\mathsf{SPR}_{B,IOb}$) of the relativity principle, and investigate the extent to which they are equivalent. We find that the three formalisations are logically distinct, although they can be rendered equivalent by the introduction of additional axioms. We prove rigorously the following relationships.

*1.1.1. Counter-examples and implications requiring no additional axioms* (*Fig. 1*).

- $\mathsf{SPR}^{+} \not\Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$ (Theorem 7.1)
- $\mathsf{SPR}_{B,IOb} \not\Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$ (Theorem 7.1)
- $\mathsf{SPR}_{B,IOb} \not\Longrightarrow \mathsf{SPR}^{+}$ (Theorem 7.1)
- $\mathsf{SPR}_{\mathfrak{M}} \Longrightarrow \mathsf{SPR}^{+}$ (Theorem 7.2)
- $\mathsf{SPR}_{\mathfrak{M}} \Longrightarrow \mathsf{SPR}_{B,IOb}$ (Theorem 7.3)

*1.1.2. Adding axioms to make the different formalisations equivalent* (*Fig. 2*).

- $\mathsf{SPR}^{+} \Longrightarrow \mathsf{SPR}_{B,IOb}$ assuming AxId, AxEv, AxIB, AxField (Theorem 7.4)
- $\mathsf{SPR}_{B,IOb} \Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$ assuming $\mathcal{L} = \mathcal{L}_0$, AxEv, AxExt (Theorem 7.7)
- $\mathsf{SPR}^{+} \Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$ assuming $\mathcal{L} = \mathcal{L}_0$, AxId, AxIB, AxField, AxEv, AxExt (Theorems 7.4, 7.7)
- $\mathsf{SPR}_{B,IOb} \Longrightarrow \mathsf{SPR}^{+}$ assuming $\mathcal{L} = \mathcal{L}_0$, AxEv, AxExt (Theorems 7.2, 7.7)

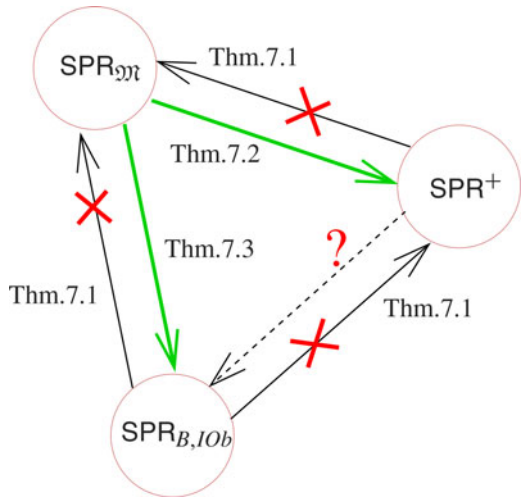



Fig. 1. Counter-examples and implications requiring no additional axioms.

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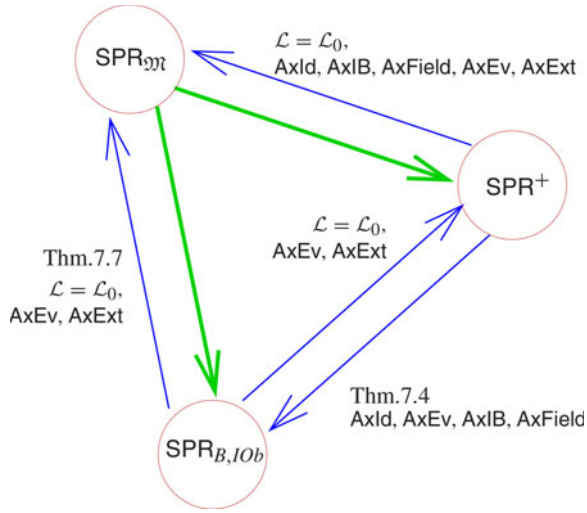



Fig. 2. Axioms required to make the different formalisations equivalent.

*1.1.3.* $\mathsf{SPR}_{\mathfrak{M}}$, $\mathsf{SPR}^+$ *and the decomposition of* $\mathsf{SPR}_{B,IOb}$ *into* $\mathsf{SPR}_{IOb}$ *and* $\mathsf{SPR}_B$ (*Fig. 3*).

- $\mathsf{SPR}^+ \Longrightarrow \mathsf{SPR}_{IOb}$ assuming AxId, AxEv (Theorem 7.5)
- $\mathsf{SPR}^+ \Longrightarrow \mathsf{SPR}_B$ assuming AxIB, AxField (Theorem 7.6)

**Outline of the paper.** We begin in §2 by characterising what we mean by a "law of nature" in our first-order logic framework. Rather than going into all the difficulties of defining what a law of nature *is*, we focus instead on the requirement that all inertial observers must agree as to the outcomes of experimental scenarios described by such a law. In §3 we give some examples, in §4 we demonstrate our three formalisations of SPR, in §5 we discuss the types of models our language admits, and in §6 we state the axioms that will be relevant to our results. These results are stated formally in §7. In §8 we discuss some alternative assumptions to axiom AxIB. The proofs of the theorems can be found in §9.

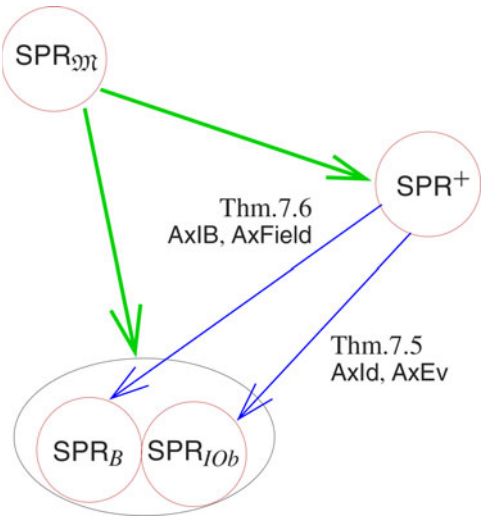



Fig. 3. $\mathsf{SPR}_{\mathfrak{M}}$, $\mathsf{SPR}^+$ and the decomposition of $\mathsf{SPR}_{B,IOb}$ into $\mathsf{SPR}_{IOb}$ and $\mathsf{SPR}_B$.

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We conclude with a discussion of our findings in §10, where we also highlight questions requiring further investigation.

**§2. Laws of nature.** Before turning our attention towards formalising the principle of relativity, we need to present the framework in which our logical formalisms will be expressed. Following the approach described in (Andréka, Madarász, & Németi, 2007; Andréka, Madarász, Németi, & Székely, 2011), we will use the first-order logical (FOL) 3-sorted language

$$\mathcal{L}_0 = \{IOb, B, Q, 0, 1, +, \cdot, \mathsf{W}\}$$

as a core language for kinematics. In this language

- *IOb* is the sort of *inertial observers* (for labeling coordinate systems);
- *B* is the sort of *bodies*, i.e., things that move;
- *Q* is the sort of *quantities*, i.e., numbers, with constants 0 and 1, *addition* $(+)$ and *multiplication* $(\cdot)$;
- $\mathsf{W}$ is the *worldview relation*, a 6-ary relation of type $IOb \times B \times Q^4$.
  The statement $\mathsf{W}(k, b, p)$ represents the idea that "inertial observer $k$ coordinatises body $b$ to be at spacetime location $p$."

Throughout this paper we use $h$, $k$ and their variants to represent inertial observers (variables of sort *IOb*); we use $b$ and $c$ to represent bodies (variables of sort *B*); and $p$, $q$ and $r$ are variables of type $Q^4$. The sorts of other variables will be clear from context.

Given this foundation, various derived notions can be defined:

- The event observed by $k \in IOb$ as occurring at $p \in Q^4$ is the set of bodies that $k$ coordinatises to be at $p$:
$$\mathsf{ev}_k(p) \equiv \{b \in B : \mathsf{W}(k, b, p)\}.$$
- For each $k, h \in IOb$, the worldview transformation $\mathsf{w}_{kh}$ is a binary relation on $Q^4$ which captures the idea that $h$ coordinatises at $q \in Q^4$ the same event that $k$ coordinatises at $p \in Q^4$:
$$\mathsf{w}_{kh}(p, q) \equiv [\mathsf{ev}_k(p) = \mathsf{ev}_h(q)]. \qquad \text{(w.def)}$$
- The worldline of $b \in B$ as observed by $k \in IOb$ is the set of locations $p \in Q^4$ at which $k$ coordinatises $b$:
$$\mathsf{wline}_k(b) \equiv \{p : \mathsf{W}(k, b, p)\}.$$

We try to choose our primitive notions as simple and "observationally oriented" as possible, cf. Friedman (1983, p. 31). Therefore the set of events is not primitive, but rather a defined concept, i.e., an event is a set of bodies that an observer observes at a certain point of its coordinate system. Motivation for such a definition of event goes back to Einstein and can be found in Misner *et al.*, (1973, p. 6) and Einstein (1996, p. 153).

Since laws of nature stand or fall according to the outcomes of physical experiments, we next consider statements, $\phi$, which describe experimental claims. For example, $\phi$ might say "if this equipment has some specified configuration today, then it will have some expected new configuration tomorrow". This is very much a dynamic process-oriented description of experimentation, but since we are using the language of spacetime, the entire experiment can be described as a static four-dimensional configuration of matter in time and space. We therefore introduce the concept of *scenarios*, i.e., sentences describing both the initial conditions and the outcomes of experiments. Although our scenarios are primarily intended

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to capture experimental configurations and outcomes, they can also describe more complex situations, as illustrated by the examples in §3. One of our formalisations of SPR will be the assertion that *all* inertial observers *agree* as to whether or not certain situations are realizable. Our definition of scenarios is motivated by the desire to have a suitably large set of sentences describing these situations.

To introduce scenarios formally, let us fix a language $\mathcal{L}$ containing our core language $\mathcal{L}_0$. We will say that a formula

$$\phi \equiv \phi(k, \bar{x}) \equiv \phi(k, x_1, x_2, x_3, \dots)$$

of language $\mathcal{L}$ describes a *scenario* provided it has a single free variable $k$ of sort *IOb* (to allow us to evaluate the scenario for different observers), and none of sort $B$. The other free variables $x_1, \dots$ can be thought of as experimental parameters, allowing us to express such statements as $\phi(k, v) \equiv$ "$k$ can see some body $b$ moving with speed $v$". Notice that numerical variables (in this case $v$) can sensibly be included as free variables here, but bodies cannot – if we allow the use of specific individuals (*Thomas*, say) we can obtain formulae ("$k$ can see *Thomas* moving with speed $v$") which manifestly violate SPR, since we cannot expect *all* observers $k$ to agree on such an assertion. The truth values of certain formulas containing bodies as free variables can happen to be independent of inertial observers, for example $\nu_2$ in §3, but we prefer to treat these as exceptional cases to be proven from the principle of relativity and the rest of the axioms.

Thus $\phi \equiv \phi(k, \bar{x})$ represents a scenario provided

- $k$ is free in $\phi(k, \bar{x})$,
- $k$ is the *only* free variable of sort *IOb*,
- the free variables $x_i$ are of sort $Q$ (or any other sort of $\mathcal{L}$ representing mathematical objects), and
- there is no free variable of sort $B$ (or any other sort of $\mathcal{L}$ representing physical objects).

The set of all scenarios will be denoted by Scenarios.

Finally, for any formula $\phi(k, \bar{x})$ with free variables $k$ of sort *IOb* and $x_1, x_2, \dots$ of any sorts, the formula

$$\mathsf{AllAgree}\langle\phi\rangle \quad \equiv \quad (\forall k, h \in IOb)\big((\forall \bar{x})[\phi(k, \bar{x}) \leftrightarrow \phi(h, \bar{x})]\big)$$

captures the idea that for every evaluation of the free variables $\bar{x}$ all inertial observers agree on the truth value of $\phi$. Let us note that $\mathsf{AllAgree}\langle\rangle$ is defined not just for scenarios, e.g., it is defined for the nonscenario examples of §3, too.

In §4, one of the formalisations of SPR will be that $\mathsf{AllAgree}\langle\phi\rangle$ holds for every possible scenario $\phi$.

**§3. Examples.** Here we give examples for both scenarios and nonscenarios. To be able to show interesting examples beyond the core language used in this paper, let us expand our language with a unary relation Ph of *light signals* (*photons*) of type $B$ and a function $\mathsf{M} : B \to Q$ for *rest mass*, i.e., $M(b)$ is the rest mass of body $b$. For illustrative purposes we focus in particular on *inertial bodies*, i.e., bodies moving with uniform linear motion, and introduce the notations $\mathsf{speed}_k(b) = v$ and $\mathsf{vel}_k(b) = (v_1, v_2, v_3)$ to indicate that $b$ is an inertial body moving with speed $v \in Q$ and velocity $(v_1, v_2, v_3) \in Q^3$ according to inertial observer $k$. These notions can be easily defined assuming AxField introduced in §6, and their definitions can be found, e.g., in Andréka, Madarász, Németi, & Székely (2008), Madarász, Stannett, & Székely (2014).

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In the informal explanations of the examples below we freely use such modal expressions as "can set down" and "can send out", in place of "coordinatise", to make the experimental idea behind scenarios intuitively clearer, and to illustrate how the dynamical aspects of making experiments are captured in our static framework. See also (Molnár & Székely, 2015) for a framework where the distinction between actual and potential bodies is elaborated within first-order modal logic.

**Examples for scenarios:**

- Inertial observer $k$ can set down a body at spacetime location $(0, 0, 0, 0)$:

$$\phi_1(k) \equiv (\exists b)\mathsf{W}(k, b, 0, 0, 0, 0).$$

- Inertial observer $k$ can send out an inertial body with speed $v$:

$$\phi_2(k, v) \equiv (\exists b)\mathsf{speed}_k(b) = v.$$

- Inertial observer $k$ can send out an inertial body at location $(x_1, x_2, x_3, x_4)$ with velocity $(v_1, v_2, v_3)$ and rest mass $m$:

$$\phi_3(k, x_1, x_2, x_3, x_4, v_1, v_2, v_3, m) \equiv (\exists b)[W(k, b, x_1, x_2, x_3, x_4) \wedge \mathsf{vel}_k(b) = (v_1, v_2, v_3) \wedge \mathsf{M}(b) = m].$$

- The speed of every light signal is $v$ according to inertial observer $k$:

$$\phi_4(k, v) \equiv (\forall b)[\mathsf{Ph}(b) \rightarrow \mathsf{speed}_k(b) = v].$$

Let us consider scenario $\phi_4$ to illustrate that $\mathsf{AllAgree}\langle\phi\rangle$ means that all inertial observers agree on the truth value of $\phi$ for every evaluation of the free variables $\bar{x}$. Assume that the speed of light is 1 for every inertial observer, i.e., that $(\forall k)(\forall b)[\mathsf{Ph}(b) \rightarrow \mathsf{speed}_k(b) = 1] \wedge (\exists b)\mathsf{Ph}(b)$ holds. Then the truth value of $\phi_4(k, 1)$ is true for every inertial observer $k$, but the truth value of $\phi_4(k, a)$ is false for every inertial observer $k$ if $a \neq 1$. Thus $\mathsf{AllAgree}\langle\phi_4\rangle$ holds.

**Examples for nonscenarios:**

- The speed of inertial body $b$ according to inertial observer $k$ is $v$:

$$\nu_1(k, v, b) \equiv \mathsf{speed}_k(b) = v.$$

Then $\mathsf{AllAgree}\langle\nu_1\rangle$ means that all inertial observers agree on the speed of each body. Obviously, we do not want such statements to hold.

Notice, incidentally, that it is possible for all observers to agree as to the truth value of a nonscenario, but this is generally something we need to prove, rather than assert a priori. For example, consider the nonscenario:

- The speed of light signal $b$ is $v$ according to inertial observer $k$:

$$\nu_2(k, v, b) \equiv \mathsf{Ph}(b) \rightarrow \mathsf{speed}_k(b) = v.$$

Then $\mathsf{AllAgree}\langle\nu_2\rangle$ means that all inertial observers agree on the speed of each light signal, and it happens to follow from $\mathsf{AllAgree}\langle\phi_4\rangle$, where scenario $\phi_4$ is given above. Therefore, $\mathsf{AllAgree}\langle\nu_2\rangle$ will follow from our formalisations of $\mathsf{SPR}$ which entail the truth of formula $\mathsf{AllAgree}\langle\phi_4\rangle$.

While different observers agree on the speed of any given photon in special relativity theory, they do not agree as to its direction of motion, which is captured by the following nonscenario:

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- The velocity of light signal $b$ is $(v_1, v_2, v_3)$ according to inertial observer $k$:

$$v_3(k, v_1, v_2, v_3, b) \equiv \mathsf{Ph}(b) \rightarrow \mathsf{vel}_k(b) = (v_1, v_2, v_3).$$

Then $\mathsf{AllAgree}\langle v_3\rangle$ means that all inertial observers agree on the velocity of each light signal, and once again we do not want such a formula to hold.

## §4. Three formalisations of SPR.

***4.1. First formalisation.*** A natural interpretation of the special principle is to identify a set $\mathcal{S}$ of scenarios on which all inertial observers should agree (i.e., those scenarios we consider to be experimentally relevant). If we now define

$$\mathsf{SPR}(\mathcal{S}) \equiv \{\mathsf{AllAgree}\langle\phi\rangle : \phi \in \mathcal{S}\}, \tag{1.$\mathcal{S}$}$$

the principle of relativity becomes the statement that every formula in $\mathsf{SPR}(\mathcal{S})$ holds. For example, if we assume that all inertial observers agree on *all* scenarios, and define

$$\mathsf{SPR}^{+} \equiv \{\mathsf{AllAgree}\langle\phi\rangle : \phi \in \mathsf{Scenarios}\}, \tag{1.+}$$

then we get a "strongest possible" version of $\mathsf{SPR}(\mathcal{S})$ formulated in the language $\mathcal{L}$.

It is important to note that the power of $\mathsf{SPR}(\mathcal{S})$ (and hence that of $\mathsf{SPR}^{+}$) strongly depends on which language $\mathcal{L}$ we use. It matters, for example, whether we can only use $\mathcal{L}$ to express scenarios related to kinematics, or whether we can also discuss particle dynamics, electrodynamics, etc. The more expressive $\mathcal{L}$ is, the stronger the corresponding principle becomes.

***4.2. Second formalisation.*** A natural indirect approach is to assume that the world-views of any two inertial observers are identical. In other words, given any model $\mathfrak{M}$ of our language $\mathcal{L}$, and given any observers $k$ and $h$, we can find an automorphism of the model which maps $k$ to $h$, while leaving all quantities (and elements of all the other sorts of $\mathcal{L}$ representing mathematical objects) fixed. That is, if the only sort of $\mathcal{L}$ representing mathematical objects is $Q$, we require the statement

$$\mathsf{SPR}_{\mathfrak{M}} \equiv (\forall k, h \in IOb)(\exists \alpha \in Aut(\mathfrak{M}))[\alpha(k) = h \wedge \alpha\restriction_Q = Id_Q] \tag{2}$$

to hold, where $Id_Q$ is the identity function on $Q$, and $\alpha\restriction_Q$ denotes the restriction of $\alpha$ to the quantity part of the model. If $\mathcal{L}$ has other sorts representing mathematical objects than $Q$, then in (2) we also require $\alpha\restriction_U = Id_U$ to hold for any such sort $U$.

***4.3. Third formalisation.*** Another way to characterise the special principle of relativity is to assume that all inertial observers agree as to how they stand in relation to bodies and each other (see Fig. 4). In other words, we require the formulae

$$\mathsf{SPR}_B \equiv (\forall k, k')(\forall b)(\exists b')[\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')] \tag{3.$B$}$$

and

$$\mathsf{SPR}_{IOb} \equiv (\forall k, k')(\forall h)(\exists h')[\mathsf{w}_{kh} = \mathsf{w}_{k'h'}] \tag{3.$IOb$}$$

to be satisfied. We will use the following notation:

$$\mathsf{SPR}_{B,IOb} = \{\mathsf{SPR}_B, \mathsf{SPR}_{IOb}\}.$$

$\mathsf{SPR}_{B,IOb}$ is only a "tiny kinematic slice" of $\mathsf{SPR}$. It says that two inertial observers are indistinguishable by possible world lines and by their relation to other observers.

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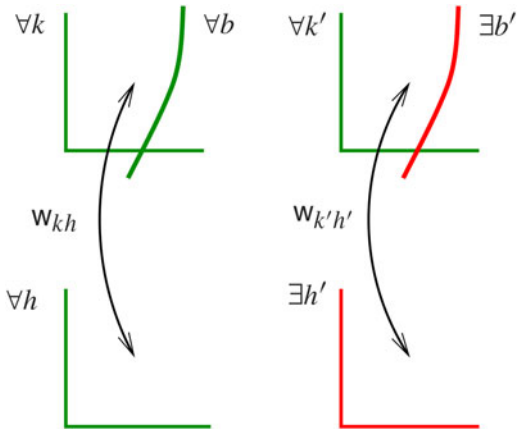



Fig. 4. Illustration for $\mathsf{SPR}_B$ and $\mathsf{SPR}_{IOb}$.

**§5. Models satisfying SPR.** According to Theorems 7.2 and 7.3 below, $\mathsf{SPR}_{\mathfrak{M}}$ is the strongest of our three formalisations of the relativity principle, since it implies the other two without any need for further assumptions.

The 'standard' model of special relativity satisfies $\mathsf{SPR}_{\mathfrak{M}}$ and therefore also $\mathsf{SPR}^{+}$ and $\mathsf{SPR}_{B,IOb}$, where by the 'standard' model we mean a model determined up to isomorphisms by the following properties: (*a*) the structure of quantities is isomorphic to that of real numbers; (*b*) all the worldview transformations are Poincaré transformations; (c) for every inertial observer $k$ and Poincaré transformation $P$, there is another observer $h$, such that $\mathsf{w}_{kh} = P$; (*d*) bodies can move exactly on the smooth timelike and lightlike curves; and (*e*) worldlines uniquely determine bodies and worldviews uniquely determine inertial observers.

In fact, there are several models satisfying $\mathsf{SPR}_{\mathfrak{M}}$ in the literature, and these models also ensure that the axioms used in this paper are mutually consistent. Indeed, in (Székely, 2013; Andréka, Madarász, Németi, Stannett, & Székely, 2014; Madarász & Székely, 2014) we have demonstrated several extensions of the 'standard' model of special relativity which satisfy $\mathsf{SPR}_{\mathfrak{M}}$. Applying the methods used in those papers, it is not difficult to show that $\mathsf{SPR}_{\mathfrak{M}}$ is also consistent with classical kinematics. Again, this is not surprising as there are several papers in the literature showing that certain formalisations of the principle of relativity cannot distinguish between classical and relativistic kinematics, and as Ignatowski (1910, 1911) has shown, when taken together with other assumptions, SPR implies that the group of transformations between inertial observers can only be the Poincaré group or the inhomogeneous Galilean group. For further developments of this theme, see (Lévy-Leblond, 1976; Borisov, 1978; Pal, 2003; Pelissetto & Testa, 2015).

The simplest way to get to special relativity from $\mathsf{SPR}_{\mathfrak{M}}$ is to extend the language $\mathcal{L}_0$ with light signals and assume Einstein's light postulate, i.e., light signals move with the same speed in every direction with respect to *at least one* inertial observer. Then, by $\mathsf{SPR}_{\mathfrak{M}}$, AxPh follows, i.e., light signals move with the same speed in every direction according to *every* inertial observer. AxPh, even without any principle of relativity, implies (using only some trivial auxiliary assumptions such as AxEv (see p. 538)), that the transformations between inertial observers are Poincaré transformations; see, e.g., (Andréka *et al.*, 2011, Theorem 2.2).

It is worth noting that $\mathsf{SPR}_{\mathfrak{M}}$ also admits models which extend the 'standard' model of special relativity, for example models containing faster-than-light bodies which can interact

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dynamically with one another (Székely, 2013; Andréka *et al.*, 2014; Madarász & Székely, 2014).

**§6. Axioms.** We now define various auxiliary axioms. As we show below, whether or not two formalisations of SPR are equivalent depends to some extent on which of these axioms one considers to be valid.

In these axioms, the spacetime origin is the point $e_0 = (0, 0, 0, 0)$, and the unit points along each axis are defined by $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, and $e_4 = (0, 0, 0, 1)$. We call $e_0, \ldots, e_4$ the *principal locations*.

AxEv All observers agree as to what can be observed. If $k$ can observe an event somewhere, then $h$ must also be able to observe that event somewhere:

$$(\forall k, h)(\forall p)(\exists q)[\mathsf{ev}_k(p) = \mathsf{ev}_h(q)].$$

AxId If $k$ and $h$ agree as to what's happening at each of the 5 principal locations, then they agree as to what's happening everywhere (see Fig. 5):

$$(\forall k, h)\big((\forall i \in \{0, \ldots, 4\})[\mathsf{ev}_k(e_i) = \mathsf{ev}_h(e_i)] \to (\forall p)[\mathsf{ev}_k(p) = \mathsf{ev}_h(p)]\big).$$

We can think of this axiom as a generalised form of the assertion that all worldview transformations are affine transformations.

AxExtIOb If two inertial observers coordinatise exactly the same events at every possible location, they are actually the same observer:

$$(\forall k, h)\big((\forall p)[\mathsf{ev}_k(p) = \mathsf{ev}_h(p)] \to k = h\big).$$

AxExtB If two bodies have the same worldline (as observed by any observer $k$), then they are actually the same body:

$$(\forall k)(\forall b, b')[\mathsf{wline}_k(b) = \mathsf{wline}_k(b') \to b = b'].$$

AxExt We write this as shorthand for AxExtB $\wedge$ AxExtIOb.

AxField $(Q, 0, 1, +, \cdot)$ satisfies the most fundamental properties of $\mathbb{R}$, i.e., it is a field (in the sense of abstract algebra; see, e.g., (Stewart, 2009)).

Notice that we do not assume a priori that $Q$ is the field $\mathbb{R}$ of real numbers, because we do not know and cannot determine experimentally whether the structure of quantities in the real world is isomorphic to that of $\mathbb{R}$. Moreover, using arbitrary fields makes our findings more general.

AxIB All bodies (considered) are inertial, i.e., their worldlines are straight lines according to every inertial observer:

$$(\forall k)(\forall b)(\exists p, q)[q \neq e_0 \wedge \mathsf{wline}_k(b) = \{p + \lambda q : \lambda \in Q\}].$$

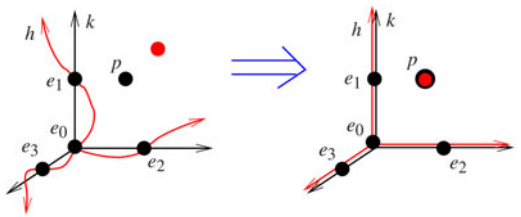



Fig. 5. Schematic representation of AxId: If $k$ and $h$ agree as to what's happening at each of the 5 principal locations, then they agree as to what's happening at every location $p$.

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AxIB is a strong assumption. In §8, we introduce generalisations of AxIB allowing accelerated bodies, too. We choose to include AxIB in our main theorems because it is arguably the simplest and clearest of these generalisations. The main generalisation $\mathsf{wIDef}_{\mathfrak{M}}$ of AxIB is a meta-assumption and the others are quite technical assertions which are easier to understand in relation to AxIB.

**§7. Results.** If $\mathfrak{M}$ is some model for a FOL language $\mathcal{L}$, and $\Sigma$ is some collection of logical formulae in that language, we write $\mathfrak{M} \models \Sigma$ to mean that every $\sigma \in \Sigma$ is valid when interpreted within $\mathfrak{M}$. If $\Sigma_1$, $\Sigma_2$ are both collections of formulae, we write $\Sigma_1 \models \Sigma_2$ to mean that

$$\mathfrak{M} \models \Sigma_2 \quad \text{whenever} \quad \mathfrak{M} \models \Sigma_1$$

holds for every model $\mathfrak{M}$ of $\mathcal{L}$. For a general introduction to logical models, see (Mendelson, 2015; Marker, 2002).

Theorem 7.1 demonstrates that our three formalisations of the principle of relativity are logically distinct. It is worth noting that based on the ideas used in the proof of Theorem 7.1 it is also easy to construct sophisticated counterexamples to their equivalence extending the 'standard' model of special relativity.

THEOREM 7.1. *The formalisations* $\mathsf{SPR}_{\mathfrak{M}}$, $\mathsf{SPR}^{+}$, *and* $\mathsf{SPR}_{B,IOb}$ *are logically distinct*:

- $\mathfrak{M} \models \mathsf{SPR}^{+} \not\Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$.
- $\mathfrak{M} \models \mathsf{SPR}_{B,IOb} \not\Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$.
- $\mathfrak{M} \models \mathsf{SPR}_{B,IOb} \not\Longrightarrow \mathfrak{M} \models \mathsf{SPR}^{+}$.

By Theorems 7.2 and 7.3, $\mathsf{SPR}_{\mathfrak{M}}$ is the strongest version of the three formalisations since it implies the other two without any extra assumptions.

THEOREM 7.2. $\mathsf{SPR}_{\mathfrak{M}} \Longrightarrow \mathfrak{M} \models \mathsf{SPR}^{+}$.

THEOREM 7.3. $\mathsf{SPR}_{\mathfrak{M}} \Longrightarrow \mathfrak{M} \models \mathsf{SPR}_{B,IOb}$.

Theorem 7.4 tells us that $\mathsf{SPR}^{+}$ can be made as powerful as $\mathsf{SPR}_{B,IOb}$ by adding additional axioms. This is an immediate consequence of Theorems 7.5 and 7.6.

THEOREM 7.4. $\mathsf{SPR}^{+} \cup \{\mathsf{AxId}, \mathsf{AxEv}, \mathsf{AxIB}, \mathsf{AxField}\} \models \mathsf{SPR}_{B,IOb}$.

THEOREM 7.5. *There exist scenarios* $\psi, \widetilde{\psi}$ *such that* $\mathsf{SPR}(\psi, \widetilde{\psi}) \cup \{\mathsf{AxId}, \mathsf{AxEv}\} \models \mathsf{SPR}_{IOb}$.

THEOREM 7.6. *There exists a scenario* $\xi$ *such that* $\mathsf{SPR}(\xi) \cup \{\mathsf{AxIB}, \mathsf{AxField}\} \models \mathsf{SPR}_{B}$.

Theorem 7.7 tells us that equipping $\mathsf{SPR}_{B,IOb}$ with additional axioms allows us to recapture the power of $\mathsf{SPR}_{\mathfrak{M}}$ (and hence, by Theorem 7.2, $\mathsf{SPR}^{+}$).

THEOREM 7.7. *Assume* $\mathcal{L} = \mathcal{L}_0$. *Then* $\mathfrak{M} \models \mathsf{SPR}_{B,IOb} \cup \{\mathsf{AxEv}, \mathsf{AxExt}\} \Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$.

Thus, although $\mathsf{SPR}_{\mathfrak{M}}$, $\mathsf{SPR}_{B,IOb}$, and $\mathsf{SPR}^{+}$ are logically distinct, they become equivalent in the presence of suitable auxiliary axioms.

**§8. Alternatives to AxIB.** In this section, we generalise AxIB to allow discussion of accelerated bodies.

For every model $\mathfrak{M}$ we formulate a property which says that world lines are parametrically definable subsets of $Q^4$, where the parameters can be chosen only from $Q$. For the definitions, cf. (Marker, 2002, §1.1.6, §1.2.1).

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$\mathsf{wlDef}_{\mathfrak{M}}$ For any $k \in IOb$ and $b \in B$ there is a formula $\varphi(y_1, y_2, y_3, y_4, x_1, \ldots, x_n)$, where all the free variables $y_1, y_2, y_3, y_4, x_1, \ldots, x_n$ of $\varphi$ are of sort $Q$, and there is $\bar{a} \in Q^n$ such that $\mathsf{wline}_k(b) \equiv \{q \in Q^4 : \mathfrak{M} \vDash \varphi(q, \bar{a})\}$.

We note that plenty of curves in $Q^4$ are definable in the sense above, e.g., curves which can be defined by polynomial functions, as well as the worldlines of uniformly accelerated bodies in both special relativity and Newtonian kinematics.

In general, not every accelerated worldline is definable – indeed, the set of curves which are definable depends both on the language and the model. For example, uniform circular motion is undefinable in many models; however, if we extend the language with the sine function as a primitive notion and assert its basic properties by including the appropriate axioms, then uniform circular motion becomes definable.

By Theorems 8.1 and 8.2, assumptions AxIB and AxField can be replaced by $\mathsf{wlDef}_{\mathfrak{M}}$ in Theorem 7.4 (Theorem 8.2 follows immediately from Theorems 7.5 and 8.1).

THEOREM 8.1. ($\mathsf{wlDef}_{\mathfrak{M}}$ and $\mathfrak{M} \vDash \mathsf{SPR}^+$) $\Longrightarrow$ $\mathfrak{M} \vDash \mathsf{SPR}_B$.

THEOREM 8.2. ($\mathsf{wlDef}_{\mathfrak{M}}$ and $\mathfrak{M} \vDash \mathsf{SPR}^+ \cup \{\mathsf{AxId}, \mathsf{AxEv}\}$) $\Longrightarrow$ $\mathfrak{M} \vDash \mathsf{SPR}_{B,IOb}$.

We note that $\mathfrak{M} \vDash \{\mathsf{AxIB}, \mathsf{AxField}\} \Longrightarrow \mathsf{wlDef}_{\mathfrak{M}}$. Moreover $\mathsf{wlDef}_{\mathfrak{M}}$ is more general than AxIB assuming AxField. The disadvantage of $\mathsf{wlDef}_{\mathfrak{M}}$ is that it is not an axiom, but a property of model $\mathfrak{M}$. We now introduce, for every natural number $n$, an axiom $\mathsf{AxWl}(n)$ which is more general than AxIB assuming AxField and $n \geq 3$, and stronger than $\mathsf{wlDef}_{\mathfrak{M}}$.

$\mathsf{AxWl}(n)$ Worldlines are determined by $n$ distinct locations, i.e., if two worldlines agree at $n$ distinct locations, then they coincide:

$$(\forall k, k')(\forall b, b')$$
$$[(\exists \text{ distinct } p^1, \ldots, p^n \in \mathsf{wline}_k(b) \cap \mathsf{wline}_{k'}(b')) \rightarrow \mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')].$$

We note that $\{\mathsf{AxIB}, \mathsf{AxField}\} \vDash \mathsf{AxWl}(2)$, and $\mathsf{AxWl}(n) \vDash \mathsf{AxWl}(i)$ if $i \geq n$.

Furthermore, for every $n$, we have $\mathfrak{M} \vDash \mathsf{AxWl}(n) \Longrightarrow \mathsf{wlDef}_{\mathfrak{M}}$. To see that this must be true, assume $\mathfrak{M} \vDash \mathsf{AxWl}(n)$, choose $k \in IOb$ and $b \in B$, let $p^1, \ldots, p^n \in \mathsf{wline}_k(b)$ be distinct locations, and define

$$\varphi(y_1, y_2, y_3, y_4, x_1^1, x_2^1, x_3^1, x_4^1, \ldots, x_1^n, x_2^n, x_3^n, x_4^n) \equiv$$
$$(\exists h)(\exists c)[(y_1, y_2, y_3, y_4), (x_1^1, x_2^1, x_3^1, x_4^1), \ldots, (x_1^n, x_2^n, x_3^n, x_4^n) \in \mathsf{wline}_h(c)].$$

Then it is easy to see that $\mathsf{wline}_k(b) \equiv \{q \in Q^4 : \mathfrak{M} \vDash \varphi(q, p^1, \ldots, p^n)\}$, whence $\mathsf{wlDef}_{\mathfrak{M}}$ holds, as claimed.

By Theorems 8.3 and 8.4, Theorems 7.4 and 7.6 remain true if we replace AxIB and AxField with $\mathsf{AxWl}(n)$.

THEOREM 8.3. $\mathsf{SPR}^+ \cup \{\mathsf{AxId}, \mathsf{AxEv}, \mathsf{AxWl}(n)\} \vDash \mathsf{SPR}_{B,IOb}$.

THEOREM 8.4. *There is a scenario $\xi$ such that* $\mathsf{SPR}(\xi) \cup \{\mathsf{AxWl}(n)\} \vDash \mathsf{SPR}_B$.

**§9. Proofs.** We begin by proving a simple lemma which allows us to identify when two observers are in fact the same observer. This lemma will prove useful in several places below.

LEMMA 9.1. $\{\mathsf{AxEv}, \mathsf{AxExtIOb}\} \vDash (\forall k, h, h')[(\mathsf{w}_{kh} = \mathsf{w}_{kh'}) \rightarrow h = h']$.

*Proof.* Suppose $\mathsf{w}_{kh} = \mathsf{w}_{kh'}$, and choose any $p \in Q^4$. Let $q \in Q^4$ satisfy $\mathsf{ev}_k(q) = \mathsf{ev}_h(p)$, so that $\mathsf{w}_{kh}(q, p)$ holds ($q$ exists by AxEv). Since $\mathsf{w}_{kh} = \mathsf{w}_{kh'}$, it follows that $\mathsf{w}_{kh'}(q, p)$ also holds, so that $\mathsf{ev}_h(p) = \mathsf{ev}_k(q) = \mathsf{ev}_{h'}(p)$. This shows that $h$ and

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$h'$ see the same events at every $p \in Q^4$, whence it follows by AxExtIOb that $h = h'$, as claimed. □

*Proof of Theorem 7.1.* $\mathsf{SPR}^+ \not\Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$, $\mathsf{SPR}_{B,IOb} \not\Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$, $\mathsf{SPR}_{B,IOb} \not\Longrightarrow \mathsf{SPR}^+$.

Constructing the required counterexamples in detail from scratch would be too lengthy and technical, and would obscure the key ideas explaining why such models exist. Accordingly, here we give only the 'recipes' on how to construct these models.

To prove that $\mathsf{SPR}^+$ does not imply $\mathsf{SPR}_{\mathfrak{M}}$, let $\mathfrak{M}^-$ be any model satisfying $\mathsf{SPR}_{\mathfrak{M}}$– and AxExt, and containing at least two inertial observers and a body $b$ such that the worldlines of $b$ are distinct according to the two observers. Such models exist, see, e.g., the ones constructed in Székely (2013). The use of AxExt ensures that distinct bodies have distinct worldlines. Let us now construct an extension $\mathfrak{M}$ of $\mathfrak{M}^-$ (violating AxExt) by adding uncountably-infinite many copies of body $b$, as well as countably-infinite many copies of every other body. Clearly, $\mathfrak{M}$ does not satisfy $\mathsf{SPR}_{\mathfrak{M}}$, since this would require the existence of an automorphism taking a body having uncountably many copies to one having only countably many copies. Nonetheless, $\mathfrak{M}$ satisfies $\mathsf{SPR}^+$ since it can be elementarily extended to an even larger model $\mathfrak{M}^+$ satisfying $\mathsf{SPR}_{\mathfrak{M}^+}$ (and hence, by Theorem 7.2, $\mathsf{SPR}^+$) by increasing the population of other bodies so that every body has an equal (uncountable) number of copies (see (Madarász, 2002, Theorem 2.8.20)). Thus $\mathfrak{M}$ satisfies $\mathsf{SPR}^+$ but not $\mathsf{SPR}_{\mathfrak{M}}$. In more detail: Let $\mathfrak{M}^+$ be an extension of $\mathfrak{M}$ obtained by increasing the population of bodies so that every body has an equal (uncountable) number of copies. We will use the Tarski–Vaught test (Marker, 2002, Prop. 2.3.5, p. 45) to show that $\mathfrak{M}^+$ is an elementary extension of $\mathfrak{M}$. Let $\phi(v, w_1, \ldots, w_n)$ be a formula and suppose $a_1, \ldots, a_n$ in $\mathfrak{M}$ and $d^+$ in $\mathfrak{M}^+$ satisfy $\mathfrak{M}^+ \vDash \phi(d^+, a_1, \ldots, a_n)$. We have to find a $d$ in $\mathfrak{M}$ such that $\mathfrak{M}^+ \vDash \phi(d, a_1, \ldots, a_n)$. If $d^+$ is not a body, then $d^+$ is already in $\mathfrak{M}$ since we extended $\mathfrak{M}$ only by bodies. Assume, then, that $d^+$ is a body. Then $d^+$ has infinitely many copies in $\mathfrak{M}$, so we can choose $d$, a copy of $d^+$ in $\mathfrak{M}$, such that $d \notin \{a_1, \ldots, a_n\}$. Let $\alpha$ be any automorphism of $\mathfrak{M}^+$ which interchanges $d$ and $d^+$ and leaves every other element fixed. Then $\alpha(a_1) = a_1, \ldots, \alpha(a_n) = a_n$ and $\alpha(d^+) = d$. By $\mathfrak{M}^+ \vDash \phi(d^+, a_1, \ldots, a_n)$, we have that $\mathfrak{M}^+ \vDash \phi(\alpha(d^+), \alpha(a_1), \ldots, \alpha(a_n))$. Thus $\mathfrak{M}^+ \vDash \phi(d, a_1, \ldots, a_n)$ as required.

To prove that $\mathsf{SPR}_{B,IOb}$ does not imply $\mathsf{SPR}_{\mathfrak{M}}$ or $\mathsf{SPR}^+$, let $\mathfrak{M}$ be any model of $\mathsf{SPR}_{B,IOb}$ and AxExt containing at least two inertial observers $k$ and $h$ and a body $b$ for which $\mathsf{wline}_k(b) = \{(0, 0, 0, t) : t \in Q\} \neq \mathsf{wline}_h(b)$. Such models exist, see, e.g., (Székely, 2013). Duplicating body $b$ leads to a model in which $\mathsf{SPR}_{B,IOb}$ is still satisfied since duplicating a body does not change the possible worldlines but it violates both $\mathsf{SPR}_{\mathfrak{M}}$ (the automorphism taking one inertial observer to another cannot take a body having only one copy to one having two) and $\mathsf{SPR}^+$ (since scenario $\varphi(m) \equiv (\forall b, c)[\mathsf{wline}_m(b) = \mathsf{wline}_m(c) = \{(0, 0, 0, t) : t \in Q\} \rightarrow b = c]$ holds for $h$ but does not hold for $k$). □

*Proof of Theorem 7.2.* $\mathsf{SPR}_{\mathfrak{M}} \Longrightarrow \mathfrak{M} \vDash \mathsf{SPR}^+$.

Suppose $\mathsf{SPR}_{\mathfrak{M}}$, so that for any observers $k$ and $h$ there is an automorphism $\alpha \in Aut(\mathfrak{M})$ such that $\alpha(k) = h$ and $\alpha$ leaves elements of all sorts of $\mathcal{L}$ representing mathematical objects fixed.

We will prove that $\mathsf{AllAgree}\langle\phi\rangle$ holds for all $\phi \in \mathsf{Scenarios}$, i.e., given any observers $k$ and $h$, any scenario $\phi$, and any set $\bar{x}$ of parameters for $\phi$, we have

$$\phi(k, \bar{x}) \leftrightarrow \phi(h, \bar{x}). \tag{1}$$

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To prove this, choose some $\alpha \in Aut(\mathfrak{M})$ which fixes $Q$ and all the other sorts representing mathematical objects, and satisfies $\alpha(k) = h$.

Suppose $\mathfrak{M} \models \phi(k, \bar{x})$. Since $\alpha$ is an automorphism, $\phi(\alpha(k), \alpha(\bar{x}))$ also holds in $\mathfrak{M}$. But $\alpha(k) = h$ and $\alpha(\bar{x}) = \bar{x}$, so this says that $\mathfrak{M} \models \phi(h, \bar{x})$. Conversely, if $\phi(h, \bar{x})$ holds in $\mathfrak{M}$, then so does $\phi(k, \bar{x})$, by symmetry. □

*Proof of Theorem 7.3.* $\mathsf{SPR}_{\mathfrak{M}} \Longrightarrow \mathfrak{M} \models \mathsf{SPR}_{B,IOb}$.

Suppose $\mathsf{SPR}_{\mathfrak{M}}$. Then $(\forall k, h)(\exists \alpha \in Aut(\mathfrak{M}))[\alpha(k) = h \wedge \alpha\restriction_Q = Id_Q]$. We wish to prove that $\mathfrak{M}$ satisfies

$$\mathsf{SPR}_{IOb} \equiv (\forall k, k')(\forall h)(\exists h')[\mathsf{w}_{kh} = \mathsf{w}_{k'h'}],$$
$$\mathsf{SPR}_{B} \equiv (\forall k, k')(\forall b)(\exists b')[\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')].$$

Recall that whenever $\alpha$ is an automorphism of $\mathfrak{M}$ and $R$ is a defined $n$-ary relation on $\mathfrak{M}$, we have

$$R(v_1, \ldots, v_n) \Longleftrightarrow R(\alpha(v_1), \ldots, \alpha(v_n)). \tag{2}$$

*Proof of* $\mathsf{SPR}_{IOb}$. Choose any $k, k', h$. We need to find $h'$ such that $\mathsf{w}_{kh} = \mathsf{w}_{k'h'}$, so let $\alpha \in Aut(\mathfrak{M})$ be some automorphism taking $k$ to $k'$, and define $h' = \alpha(h)$. Now it's enough to note that $\mathsf{w}_{kh}(p, q)$ is a defined 10-ary relation on $\mathfrak{M}$ (one parameter each for $k$ and $h$, 4 each for $p$ and $q$), so that

$$\mathsf{w}_{kh}(p, q) \Longleftrightarrow \mathsf{w}_{\alpha(k)\alpha(h)}(\alpha(p), \alpha(q))$$

holds for all $p, q$, by (2). Substituting $\alpha(k) = k'$, $\alpha(h) = h'$, and noting that $\alpha$ leaves all spacetime coordinates fixed (because $\alpha\restriction_Q = Id_Q$) now gives

$$\mathsf{w}_{kh}(p, q) \Longleftrightarrow \mathsf{w}_{k'h'}(p, q)$$

as required.

*Proof of* $\mathsf{SPR}_{B}$. Choose any $k, k'$ and $b$. We need to find $b'$ such that $\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')$. As before, let $\alpha \in Aut(\mathfrak{M})$ be some automorphism taking $k$ to $k'$, define $b' = \alpha(b)$, and note that "$p \in \mathsf{wline}_k(b)$" is a defined 6-ary relation on $\mathfrak{M}$. Applying (2) now tells us that

$$\mathsf{wline}_k(b) = \mathsf{wline}_{\alpha(k)}(\alpha(b)) = \mathsf{wline}_{k'}(b')$$

as required. □

*Proof of Theorem 7.4.* $\mathsf{SPR}^{+} \cup \{\mathsf{AxId}, \mathsf{AxEv}, \mathsf{AxIB}, \mathsf{AxField}\} \models \mathsf{SPR}_{B,IOb}$.

This is an immediate consequence of Theorems 7.5 and 7.6. □

*Proof of Theorem 7.5.* $\mathsf{SPR}(\psi, \widetilde{\psi}) \cup \{\mathsf{AxId}, \mathsf{AxEv}\} \models \mathsf{SPR}_{IOb}$ for some $\psi, \widetilde{\psi}$.

Given a 5-tuple of locations $\vec{x}_i = (x_0, x_1, x_2, x_3, x_4)$, let $\mathsf{match}$ (see Fig. 6) be the relation

$$\mathsf{match}(h, k, \vec{x}_i) \equiv (\forall i \in \{0, \ldots, 4\})(\mathsf{ev}_h(e_i) = \mathsf{ev}_k(x_i))$$

and define $\psi, \widetilde{\psi} \in \mathsf{Scenarios}$ by

$$\psi(k, p, q, \vec{x}_i) \equiv (\exists h)[(\mathsf{ev}_h(p) = \mathsf{ev}_k(q)) \wedge \mathsf{match}(h, k, \vec{x}_i)],$$
$$\widetilde{\psi}(k, p, q, \vec{x}_i) \equiv (\exists h)[(\mathsf{ev}_h(p) \neq \mathsf{ev}_k(q)) \wedge \mathsf{match}(h, k, \vec{x}_i)].$$

Choose any $k, h, k'$. In order to establish $\mathsf{SPR}_{IOb}$ we need to demonstrate some $h'$ such that $\mathsf{w}_{kh} = \mathsf{w}_{k'h'}$. To do this, let $x_i$ be such that $\mathsf{ev}_k(x_i) = \mathsf{ev}_h(e_i)$ for all $i = 0, \ldots, 4$ (these exist by $\mathsf{AxEv}$).

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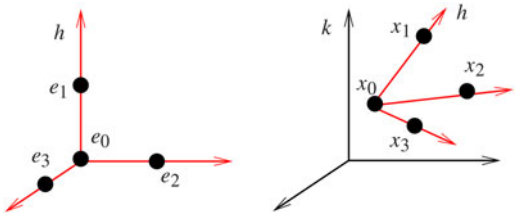



Fig. 6. Schematic showing the behaviour of function match, which tells us which locations in $k$'s worldview correspond to the principle locations in $h$'s worldview.

Then, in particular, $\psi(k, e_0, x_0, \vec{x}_i) \equiv (\exists h)[(\mathsf{ev}_h(e_0) = \mathsf{ev}_k(x_0)) \wedge \mathsf{match}(h, k, \vec{x}_i)]$ holds. Since $\mathsf{SPR}(\psi, \widetilde{\psi})$, it follows that $\psi(k', e_0, x_0, \vec{x}_i)$ also holds, so there is some $h'$ satisfying

$$\mathsf{match}(h', k', \vec{x}_i). \tag{3}$$

Now choose any $p$ and $q$. We will show that $\mathsf{ev}_h(p) = \mathsf{ev}_k(q)$ if and only if $\mathsf{ev}_{h'}(p) = \mathsf{ev}_{k'}(q)$, whence $\mathsf{w}_{kh} = \mathsf{w}_{k'h'}$, as claimed.

*Case 1.* Suppose $\mathsf{ev}_h(p) = \mathsf{ev}_k(q)$. In this case, $\psi(k, p, q, \vec{x}_i)$ holds, and we need to prove that $\mathsf{ev}_{h'}(p) = \mathsf{ev}_{k'}(q)$. It follows from $\mathsf{SPR}(\psi, \widetilde{\psi})$ that $\psi(k', p, q, \vec{x}_i)$ also holds, i.e., there exists $h''$ satisfying

$$(\mathsf{ev}_{h''}(p) = \mathsf{ev}_{k'}(q)) \wedge \mathsf{match}(h'', k', \vec{x}_i). \tag{4}$$

It follows from (3) that

$$\mathsf{match}(h', k', \vec{x}_i) \wedge \mathsf{match}(h'', k', \vec{x}_i)$$

holds, i.e.,

$$\mathsf{ev}_{h'}(e_i) = \mathsf{ev}_{k'}(x_i) = \mathsf{ev}_{h''}(e_i)$$

holds for all $i = 0, \ldots, 4$. By AxId it follows that $(\forall r)(\mathsf{ev}_{h'}(r) = \mathsf{ev}_{h''}(r))$.

It now follows from (4) that

$$\mathsf{ev}_{h'}(p) = \mathsf{ev}_{h''}(p) = \mathsf{ev}_{k'}(q),$$

i.e., $\mathsf{ev}_{h'}(p) = \mathsf{ev}_{k'}(q)$, as required.

*Case 2.* Suppose $\mathsf{ev}_h(p) \neq \mathsf{ev}_k(q)$. In this case, $\widetilde{\psi}(k, p, q, \vec{x}_i)$ holds, and we need to prove that $\mathsf{ev}_{h'}(p) \neq \mathsf{ev}_{k'}(q)$. It follows from $\mathsf{SPR}(\psi, \widetilde{\psi})$ that $\widetilde{\psi}(k', p, q, \vec{x}_i)$ also holds, i.e., there exists $h''$ satisfying

$$(\mathsf{ev}_{h''}(p) \neq \mathsf{ev}_{k'}(q)) \wedge \mathsf{match}(h'', k', \vec{x}_i). \tag{5}$$

As before, it follows from (3) that $\mathsf{ev}_{h'}(e_i) = \mathsf{ev}_{k'}(x_i) = \mathsf{ev}_{h''}(e_i)$ holds for all $i = 0, \ldots, 4$, and hence by AxId that $(\forall r)(\mathsf{ev}_{h'}(r) = \mathsf{ev}_{h''}(r))$.

It now follows from (5) that

$$\mathsf{ev}_{h'}(p) = \mathsf{ev}_{h''}(p) \neq \mathsf{ev}_{k'}(q),$$

i.e., $\mathsf{ev}_{h'}(p) \neq \mathsf{ev}_{k'}(q)$, as required. □

*Proof of Theorem 7.6.* $\mathsf{SPR}(\xi) \cup \{\mathsf{AxIB}, \mathsf{AxField}\} \models \mathsf{SPR}_B$ for some $\xi$.

We define $\xi \in \mathsf{Scenarios}$ by $\xi(k, p, q) \equiv (\exists b)(p, q \in \mathsf{wline}_k(b))$.

To see that this satisfies the theorem, suppose that AxIB and AxField both hold, and choose any $k, k' \in IOb$ and any $b \in B$. We will demonstrate a body $b'$ satisfying $\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')$, whence $\mathsf{SPR}_B$ holds, as claimed.

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According to AxIB, there exist points $p_k, q_k \in Q^4$, where $q_k \neq e_0$, and

$$\mathsf{wline}_k(b) = \{p_k + \lambda q_k : \lambda \in Q\}.$$

In particular, therefore, the points $p = p_k$ and $q = p_k + q_k$ are distinct elements of the straight line $\mathsf{wline}_k(b)$.

This choice of $p$ and $q$ ensures that the statement $\xi(k, p, q)$ holds. By $\mathsf{SPR}(\xi)$, it follows that $\xi(k', p, q)$ also holds; i.e., there is some body $b'$ such that $p, q \in \mathsf{wline}_{k'}(b')$. By AxIB, $\mathsf{wline}_{k'}(b')$ is also a straight line.

It follows that $\mathsf{wline}_k(b)$ and $\mathsf{wline}_{k'}(b')$ are both straight lines containing the same two distinct points $p$ and $q$. Since there can be at most one such line (by AxField) it follows that $\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')$, as claimed. □

*Proof of Theorem 7.7.* If $\mathcal{L} = \mathcal{L}_0$, then $\mathfrak{M} \vDash \mathsf{SPR}_{B,IOb} \cup \{\mathsf{AxEv}, \mathsf{AxExt}\} \implies \mathsf{SPR}_{\mathfrak{M}}$.

Choose any $k, k'$. We need to demonstrate an automorphism $\alpha \in Aut(\mathfrak{M})$ which is the identity on $Q$ and satisfies $\alpha(k) = k'$.

- *Action of $\alpha$ on IOb*: Suppose $h \in IOb$. According to $\mathsf{SPR}_{IOb}$, there exists some $h'$ such that $\mathsf{w}_{kh} = \mathsf{w}_{k'h'}$. By Lemma 9.1, this $h'$ is uniquely defined (since we would otherwise have distinct $h', h''$ satisfying $\mathsf{w}_{k'h'} = \mathsf{w}_{k'h''}$). Define $\alpha(h) = h'$.
- *Action of $\alpha$ on B*: Suppose $b \in B$. According to $\mathsf{SPR}_B$, there exists some $b'$ such that $\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')$. By AxExtB, this $b'$ is uniquely defined. Define $\alpha(b) = b'$.
- *Action on Q*: Define $\alpha\restriction_Q = Id_Q$. Notice that this forces $\alpha(p) = p$ for all $p \in Q^4$.

We already know that $\alpha$ fixes $Q$. It remains only to prove that $\alpha \in Aut(\mathfrak{M})$, and that $\alpha(k) = k'$.

*Proof that $\alpha(k) = k'$*: Recall first that for any equivalence relation $R$, it is the case that $R = R \circ R^{-1}$, and that given any $k, k', k''$, we have

- $\mathsf{w}_{kk}$ is an equivalence relation;
- $\mathsf{w}_{kk'}^{-1} = \mathsf{w}_{k'k}$; and
- $\mathsf{w}_{kk'} \circ \mathsf{w}_{k'k''} = \mathsf{w}_{kk''}$ (by AxEv and (w.def)).

By construction, we have $\mathsf{w}_{kk} = \mathsf{w}_{k'\alpha(k)}$, whence $\mathsf{w}_{k'\alpha(k)}$ is an equivalence relation. It now follows that

$$\begin{aligned}\mathsf{w}_{k'\alpha(k)} &= \mathsf{w}_{k'\alpha(k)} \circ \mathsf{w}_{k'\alpha(k)}^{-1} \\ &= \mathsf{w}_{k'\alpha(k)} \circ \mathsf{w}_{\alpha(k)k'} \\ &= \mathsf{w}_{k'k'}\end{aligned}$$

and hence, by Lemma 9.1, that $\alpha(k) = k'$, as required.

*Proof that $\alpha \in Aut(\mathfrak{M})$*: We know that $\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(\alpha(b)) = \mathsf{wline}_{\alpha(k)}(\alpha(b))$, or in other words, given any $b$ and $q \in Q^4$,

$$b \in \mathsf{ev}_k(q) \iff \alpha(b) \in \mathsf{ev}_{\alpha(k)}(q). \tag{6}$$

We wish to prove $\mathsf{W}(h, b, p) \leftrightarrow \mathsf{W}(\alpha(h), \alpha(b), p)$ for all $h$, $b$ and $p$ (recall that $\alpha(p) = p$). This is equivalent to proving

$$b \in \mathsf{ev}_h(p) \iff \alpha(b) \in \mathsf{ev}_{\alpha(h)}(p). \tag{7}$$

Choose any $h, b, p$. Let $q \in Q^4$ satisfy $\mathsf{ev}_k(q) = \mathsf{ev}_h(p)$ – such a $q$ exists by AxEv. Then

$$\mathsf{ev}_{\alpha(k)}(q) = \mathsf{ev}_{\alpha(h)}(p) \tag{8}$$

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because $\mathsf{w}_{kh} = \mathsf{w}_{k'\alpha(h)} = \mathsf{w}_{\alpha(k)\alpha(h)}$ by the definition of $\alpha(h)$ and the fact, established above, that $k' = \alpha(k)$.

To prove (7), let us first assume that $b \in \mathsf{ev}_h(p)$. Then $b \in \mathsf{ev}_k(q)$, so $\alpha(b) \in \mathsf{ev}_{\alpha(k)}(q)$ by (6). Then, by (8) we have $\alpha(b) \in \mathsf{ev}_{\alpha(h)}(p)$. Next, choose $b \notin \mathsf{ev}_h(p) = \mathsf{ev}_k(q)$. Then $\alpha(b) \notin \mathsf{ev}_{\alpha(k)}(q) = \mathsf{ev}_{\alpha(h)}(p)$ by (6) and (8). It follows that $b \in \mathsf{ev}_h(p) \iff \alpha(b) \in \mathsf{ev}_{\alpha(k)}(q)$, as required.

It remains to show that $\alpha$ is a bijection.

*Proof of injection*:

- *Observers*: Suppose $\alpha(h) = \alpha(h')$. Then, by the definition of $\alpha$, we have
$$\mathsf{w}_{kh} = \mathsf{w}_{k'\alpha(h)} = \mathsf{w}_{k'\alpha(h')} = \mathsf{w}_{kh'}$$
and now $h = h'$ by Lemma 9.1.
- *Bodies*: Suppose $\alpha(b) = \alpha(c)$. By definition of $\alpha$ we have
$$\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(\alpha(b)) = \mathsf{wline}_{k'}(\alpha(c)) = \mathsf{wline}_k(c)$$
and now $b = c$ follows by AxExtB.

*Proof of surjection*: We need for every $h', b'$ that there are $h, b$ satisfying $\alpha(h) = h'$ and $\alpha(b) = b'$.

- *Observers*: Let $h' \in IOb$. By $\mathsf{SPR}_{IOb}$ there exists $h$ such that $\mathsf{w}_{k'h'} = \mathsf{w}_{kh}$, and now $h' = \alpha(h)$ for any such $h$.
- *Bodies*: Let $b' \in B$. By $\mathsf{SPR}_B$ there exists $b \in B$ such that $\mathsf{wline}_{k'}(b') = \mathsf{wline}_k(b)$, and now $b' = \alpha(b)$ for any such $b$.

This completes the proof. □

*Proof of Theorem 8.1.* ($\mathsf{wlDef}_{\mathfrak{M}}$ and $\mathfrak{M} \models \mathsf{SPR}^+$) $\Longrightarrow$ $\mathfrak{M} \models \mathsf{SPR}_B$.

Assume $\mathfrak{M} \models \mathsf{SPR}^+$ and $\mathsf{wlDef}_{\mathfrak{M}}$. Choose any $k, k' \in IOb$ and any $b \in B$. We will demonstrate a body $b'$ satisfying $\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')$.

Let $\varphi(\bar{y}, \bar{x}) \equiv \varphi(y_1, y_2, y_3, y_4, x_1, \ldots, x_n)$ be a formula such that all the free variables $\bar{y}, \bar{x}$ of $\varphi$ are of sort $Q$, and choose $\bar{a} \in Q^n$ such that
$$\mathsf{wline}_k(b) \equiv \{q \in Q^4 : \mathfrak{M} \models \varphi(q, \bar{a})\},$$
Such $\varphi$ and $\bar{a}$ exist by $\mathsf{wlDef}_{\mathfrak{M}}$.

We define $\psi \in \mathsf{Scenarios}$ by $\psi(h, \bar{x}) \equiv (\exists c)(\forall q)[q \in \mathsf{wline}_h(c) \leftrightarrow \varphi(q, \bar{x})]$. Clearly, $\mathfrak{M} \models \psi(k, \bar{a})$. Then, by $\mathsf{SPR}^+$, $\mathfrak{M} \models \psi(k', \bar{a})$. Thus, there is $b' \in B$ such that $\mathsf{wline}_{k'}(b') \equiv \{q \in Q^4 : \mathfrak{M} \models \varphi(q, \bar{a})\}$, and $\mathsf{wline}_k(b) = \mathsf{wline}_{k'}(b')$ for this $b'$. □

*Proof of Theorem 8.2.* ($\mathsf{wlDef}_{\mathfrak{M}}$ and $\mathfrak{M} \models \mathsf{SPR}^+ \cup \{\mathsf{AxId}, \mathsf{AxEv}\}$) $\Longrightarrow$ $\mathfrak{M} \models \mathsf{SPR}_{B,IOb}$.

This is an immediate consequence of Theorems 7.5 and 8.1. □

*Proof of Theorem 8.3.* $\mathsf{SPR}^+ \cup \{\mathsf{AxId}, \mathsf{AxEv}, \mathsf{AxWl}(n)\} \models \mathsf{SPR}_{B,IOb}$.

This is an immediate consequence of Theorems 7.5 and 8.4. □

*Proof of Theorem 8.4.* $\mathsf{SPR}(\xi) \cup \{\mathsf{AxWl}(n)\} \models \mathsf{SPR}_B$ for some $\xi$.

We define $\xi \in \mathsf{Scenarios}$ by $\xi(k, p^1, \ldots, p^n) \equiv (\exists b)(p^1, \ldots, p^n \in \mathsf{wline}_k(b))$. It is easy to check that $\xi$ satisfies the theorem, and we omit the details. □

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**§10. Discussion and conclusions.** In this paper we have shown formally that adopting different viewpoints can lead to different, but equally 'natural', formalisations of the special principle of relativity. The idea that different formalisations exist is, of course, not new, but the advantage of our approach is that we can investigate the formal relationships between different formalisations, and deduce the conditions under which equivalence can be restored.

We have shown, in particular, that the model-based interpretation of the principle, $\mathsf{SPR}_{\mathfrak{M}}$, is strictly stronger than the alternatives $\mathsf{SPR}^{+}$ and $\mathsf{SPR}_{B,IOb}$, and have identified various counterexamples to show that the three approaches are not, in general equivalent. On the other hand, equivalence is restored in the presence of various axioms. We note, however, that the following question remains open, since it is unclear whether $\mathsf{SPR}^{+}$ is enough, in its own right, to entail $\mathsf{SPR}_{B,IOb}$.

CONJECTURE 10.1. $\mathsf{SPR}^{+} \not\Longrightarrow \mathsf{SPR}_{B}$.

An interesting direction for future research would be to investigate the extent to which our existing results can be strengthened by removing auxiliary axioms. For example, our proof that $\mathsf{SPR}_{\mathfrak{M}}$ can be recovered from $\mathsf{SPR}^{+}$ currently relies on $\mathcal{L} = \mathcal{L}_0$, AxId, AxIB, AxField, AxEv, and AxExt. While we know that *some* additional axiom(s) must be required (since we have presented a counterexample showing that $\mathsf{SPR}^{+} \not\Longrightarrow \mathsf{SPR}_{\mathfrak{M}}$), the question remains whether we can develop a proof that works over *any* language, $\mathcal{L}$, or whether the constraint $\mathcal{L} = \mathcal{L}_0$ is required. Again, assuming we allow the same auxiliary axioms, how far can we minimise the set $\mathcal{S}$ of scenarios while still entailing the equivalence between $\mathsf{SPR}(\mathcal{S})$ and $\mathsf{SPR}_{\mathfrak{M}}$?

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