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Positional Games

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For Béla Bollobás on his 60th birthday

1. Informal introduction

This paper is partly an overview of the subject (Sections 1–4), and partly a research paper containing proofs for new results (Sections 5–7).

Every 'theory' of games concentrates on one particular aspect of games, and pretty much ignores the rest.

(1) Traditional *game theory* (J. von Neumann, J. Nash, *etc.*) focuses on the 'lack of complete information' (games like Poker). Its main result is a *minimax* theorem about *mixed* strategies ('random choice'), and it is basically linear algebra.
Games of 'complete information', like Chess, Go, Checkers, Nim, and Tic-Tac-Toe, are (almost completely) ignored by the traditional theory.

(2) One existing 'theory' for games of complete information is the 'theory of Nim-like games' (Berlekamp, Conway, Guy, *etc.*), which focuses on 'sum games', and it is basically algebra ('addition theory').

(3) In this paper I am talking about something completely different: I discuss a 'fake probability theory' of 'Tic-Tac-Toe-like games'. Note that 'Tic-Tac-Toe-like games' are officially called *positional games*.

The basic challenge of games of complete information is the 'combinatorial chaos'. To analyse a position (say in Chess), one has to examine all of its options, and all the options of these options, and all the options of the options of these options, and so on. The exhaustive search through the 'game tree' takes an *enormous* amount of time (usually more than the age of the universe). A (desperate) attempt to make up for the lack of time is to study the *random walk* on the game tree, that is, to study the *randomized game* where both players play randomly.

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The *extremely surprising part* is that the probabilistic analysis of the randomized game can often be *converted* into optimal strategies via potential arguments. It is basically a game-theoretic adaptation of the so-called *probabilistic method* in combinatorics ('Erdős theory') applied to hopelessly complicated Tic-Tac-Toe-like games, where the 'usual shortcuts' (pairing strategy, *etc.*) fail miserably.

I consider it a 'mathematical paradox' for the following reason. Game theory is about *perfect* players, and I find it shocking that a play between *random generators* ('dumb players') has anything to do with a play between perfect players! 'Poker and randomness' is a natural combination: *mixed* strategy (*i.e.*, random choice among deterministic strategies) is necessary to make up for the lack of complete information (see, *e.g.*, the role of 'bluffing' in Poker). On the other hand, 'Tic-Tac-Toe and randomness' sounds like a mismatch. To explain the connection between 'Tic-Tac-Toe' and 'randomness' requires a long and detailed analysis.

First note that the connection is not 'trivial', in the sense that an optimal strategy is *never* a 'random play'. (In fact, a 'random play' usually leads to a quick, catastrophic defeat.) For games of 'complete information' the optimal strategies are *always* deterministic ('pure'). The 'fake probability theory' is employed to *find* a deterministic optimal strategy. This is where the connection is: the 'fake probability theory' is *motivated* by traditional probability theory, but eventually it is *derandomized* by *potential arguments*. In other words, I eventually get rid of probability theory completely, but the intermediate 'probabilistic step' is an absolutely crucial, inevitable part of the understanding process.

The 'fake probability theory' consists of the following main chapters:

(i) game-theoretic first moment;
(ii) game-theoretic second and higher moments;
(iii) game-theoretic 'independence'.

By using these tools I could find the *exact* solution of infinitely many natural games, thought to be completely hopeless before, like some 'clique games' and some 'subspace' versions of 'multi-dimensional Tic-Tac-Toe'.

Before describing the results in detail, first I want to talk about positional games (*i.e.*, 'Tic-Tac-Toe-like games') in general.

Chess, Tic-Tac-Toe, and Hex are among the most well-known games of complete information with no chance move. What is common in these apparently very different games? Well, in each game that player wins who achieves a 'winning configuration' first. A 'winning configuration' in Tic-Tac-Toe is a '3-in-a-row', in Hex it is a 'connecting chain of hexagons', and in Chess it is a 'capture of the opponent's King' (called a *checkmate*).

The objective of other well-known games of complete information like Checkers and Go is more complicated. In Checkers the goal is to be the first player to either capture all of the opponent's pieces ('checkers') or make it impossible for the opponent to make a move. The capture of a single piece ('jumping over') is a 'mini-win configuration', and similarly, an arrangement where the opponent cannot make a move is a 'winning configuration'.

In Go the goal is to capture as many stones of the opponent as possible ('capturing' means to 'surround a set of opponent's stones by a connected set').

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These games are clearly very different, but the basic question is always the same: Which player can achieve a winning configuration *first*?

The bad news is that nobody knows *how* to achieve a winning configuration *first*, except by exhaustive case study. There is no general theorem whatsoever answering the question of 'how'. 'Strategy stealing' gives a partial answer to 'when', but doesn't say a word about 'how'. (Note that 'doing it first' means 'competition', a key characteristic of all game-playing.)

But if we ignore 'doing it first', then an even more fundamental question arises: What can a player achieve by his own moves against an adversary? Which configurations are achievable (but not necessarily first)? Or the (equivalent) 'complementary' question: What are the impossible configurations?

The main objective of 'positional game theory' is to 'draw the line' between achievable and impossible configurations in the class of 'Tic-Tac-Toe-like games'; for simplicity I ignore 'doing it first'.

Why 'Tic-Tac-Toe-like games'? Well, 'Tic-Tac-Toe-like games' are the *simplest case* in the sense that they are *static* games. Unlike Chess, Go, and Checkers, where the players repeatedly *relocate* or even *remove* pieces ('*dynamic* games') from the board, in Tic-Tac-Toe and Hex the players make permanent marks on the board, and 'relocating' or 'removing' is illegal. (Chess is particularly complicated. There are 6 types of pieces: King, Queen, Bishop, Knight, Rook, Pawn, and each one has its own set of rules of 'how to move the piece'. The 'instructions' for playing Tic-Tac-Toe are just a couple of lines, but the 'instructions' for playing Chess are several pages long.) The 'relative' simplicity of Tic-Tac-Toe-like games makes this class an ideal candidate for a mathematical theory.

What exactly are these 'Tic-Tac-Toe-like' games? Well, they are simply 'generalized Tic-Tac-Toe'. Nobody knows what 'generalized Chess' or 'generalized Go' are supposed to mean, but I think (almost) everybody would agree on what 'generalized Tic-Tac-Toe' should mean. In Tic-Tac-Toe the 'board' is a $3 \times 3 = 9$ element set, and there are 8 'winning triplets'. Similarly, one can play 'generalized Tic-Tac-Toe' on an arbitrary finite hypergraph, where the hyperedges are called 'winning sets', the union set is the 'board', the players alternately occupy elements of the 'board', and the winner is the player who occupies a whole 'winning set' first. This is what I call the *strong game*.

Unfortunately nobody knows *how* to win a strong game. For example, the $4 \times 4 \times 4 = 4^3$ Tic-Tac-Toe is a first-player's win, but the winning strategy is extremely complicated ('the size of a phone-book', according to O. Patashnik). The $5 \times 5 \times 5 = 5^3$ version is expected to be a draw, but no one can prove it. 'To occupy a winning line *first*', the common goal of these games, is an extremely difficult task. We know nothing beyond 'strategy stealing'. 'Strategy stealing' refers to the *existence argument* that 'if a draw is impossible, then the game is a first-player's win'. Strategy stealing does not give any hint about how to actually win. One way to do it is to search *all* strategies, but it is absurd: the total number is a doubly exponential function of the board size. The exhaustive search through all positions (searching the 'game tree', or the 'position graph'), called *backward labelling*, is more efficient, but still requires exponential time ('hard').

As I said before, nobody knows how to win a strong game. I had much more luck with *weak games*, the class of games where 'doing it first' is ignored. In a weak game,

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played on an arbitrary finite hypergraph, the two players are called 'Maker' and 'Breaker' (alternative names are 'Builder' and 'Blocker'). To win a strong game, a player has to 'build and block' at the *same* time. In a weak game these two jobs are separated, which makes the analysis somewhat *easier* (but not *easy*: Hex is clearly equivalent to a weak game, but it doesn't help to find an explicit first-player's winning strategy).

In a weak game, Maker's goal is to occupy a whole winning set, but not necessarily first. On the other hand, Breaker's goal is simply to prevent Maker from achieving his goal; Breaker does *not* want to build.

There is a trivial one-sided implication between strong and weak games (of course, played on the same hypergraph): if Breaker can mark every winning set (*i.e.*, wins the weak game), then the *same* strategy gives a draw in the strong game; I call it a *blocking draw*.

What I have been discussing so far was the 'achievement' version. The *reverse game* ('avoidance' version) is equally interesting (or even more interesting). In a reverse strong game the player who occupies a whole winning set first is the *loser*.

For example, consider the (K_n, K_q) clique game: the 'board' is a complete graph K_n on n vertices, the players alternately select edges from K_n, and the goal is a copy of K_q ('clique'). An almost trivial case study shows that the first player wins the strong (K_5, K_3) clique game in his 4th move, or before (the goal is a 'triangle'). The reverse 'triangle' game is much more exciting (and difficult). A very complicated case study (involving a computer) shows that the second player wins the (K_6, K_3) reverse strong game (the winning strategy is far from simple).

The general definition of the *reverse weak game* goes as follows. As usual, it is played on an arbitrary finite hypergraph. One player is a kind of 'anti-builder': he wants to avoid occupying a whole winning set – I call him *Avoider*. The other player is a kind of 'anti-blocker': he wants to force the reluctant Avoider to build a winning set – 'anti-builder' is officially called *Forcer*.

The best way to illustrate the striking connection between 'positional games' and 'random structures' is to study the weak and reverse weak versions of the (K_n, K_q) clique game. If $q = q(n)$ is 'very small' in terms of n, then Maker (or Forcer) can easily win. On the other hand, if $q = q(n)$ is 'not so small' in terms of n, then Breaker (or Avoider) can easily win. Where is the *game-theoretic breaking point*?

Well, for 'small' ns I don't know the answer, but for 'large' ns I know the *exact* value of the breaking point! Indeed, assume that n is sufficiently large, say $n > 10^{100}$. If we take the 'lower integral part'

$$q = \lfloor 2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 \rfloor$$

(base 2 logarithm), then Maker (or Forcer) wins. On the other hand, if we take the 'upper integral part'

$$q = \lceil 2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 \rceil,$$

then Breaker (or Avoider) wins.

(I have to admit, I 'cheated' a little bit: if $2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3$ is 'very close' to an integer, then I don't know who wins. But for the overwhelming majority of

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ns the expression above is *not* too close to an integer. So for the overwhelming majority of ns I know exactly.)

First of all, I find it very surprising that the weak clique game and the *reverse* weak clique game have *exactly* the same breaking point. I feel this contradicts 'common sense'. I would expect that an eager Maker in the 'straight' game has a good chance of building a *bigger* clique than a reluctant Avoider in the reverse version, but this 'natural' expectation turned out to be wrong. I cannot give any *a priori* reason why the two breaking points coincide. All that I can say is that the highly technical proof of the 'straight' case (40 pages) can be easily adapted (*e.g.*, *maximum* is replaced by *minimum*) to yield the same breaking point for the reverse game, but this is hardly the answer we are looking for.

The reader is probably wondering about the mysterious expression

$$2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3.$$

What is this? Well, an expert in the theory of random graphs immediately recognizes that $2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3$ is exactly 2 less than the 'clique number of the symmetric random graph $\mathbf{G}(n, 1/2)$' ($1/2$ is the edge probability).

A combination of the first and second moment methods (standard probability theory) shows that the clique number $\omega(\mathbf{G}(n, 1/2))$ of the random graph has a very strong concentration. Typically it is concentrated on a *single* integer with probability $\to 1$ as $n \to \infty$ (and even in the worst case there are at most two values). Indeed, the expected number of q-cliques in $\mathbf{G}(n, 1/2)$ equals

$$f(q) = f_n(q) = \binom{n}{q} 2^{-\binom{q}{2}}.$$

The function $f(q)$ drops under 1 around $q \approx 2\log_2 n$. The 'real solution' of the equation $f(q) = 1$ is

$$q = 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 1 + o(1), \tag{1.1}$$

which is exactly 2 more than the game-theoretic breaking point

$$q = 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3 + o(1), \tag{1.2}$$

mentioned above.

The strong concentration of the clique number of the random graph is not as terribly surprising as it seems at first sight. Indeed, $f(q)$ is a very *rapidly* changing function:

$$\frac{f(q)}{f(q+1)} = \frac{q+1}{n-q} 2^q = n^{1+o(1)}$$

if $q \approx 2\log_2 n$. On an intuitive level it is explained by the fact that if q switches to $q+1$, then $\binom{q}{2}$ switches to $\binom{q+1}{2} = \binom{q}{2} + q$, which is a 'square root size increase'.

Is there a 'reasonable clique game' for which the breaking point is exactly (1.1), that is, the clique number of the random graph? The answer is 'yes', and the game is a *Picker–Chooser game*. To motivate the Picker–Chooser game, note that the *alternating* Tic-Tac-Toe-like play splits the board into two equal (or almost equal) parts. But there are many other ways to divide the board into two equal parts. The 'I-cut-you-choose' way (motivated by how a couple shares a single piece of cake after dinner) goes as follows: in

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each move, Picker picks two previously unselected points of the board, Chooser chooses one of them, and the other one goes back to Picker. In the *Picker–Chooser* game Picker is the 'builder' (*i.e.*, he wants to occupy a whole winning set) and Chooser is the 'blocker' (*i.e.*, his goal is to mark every winning set).

When Chooser is the 'builder' and Picker is the 'blocker', then I call it the *Chooser–Picker* game.

The proof of the theorem that (1.1) is the exact value of the breaking point for the (K_n, K_q) Picker–Chooser clique game is based on the concepts of

(a) game-theoretic first moment; and
(b) game-theoretic second moment.

The proof is far from trivial, but not so terribly difficult either (because Picker has so much control of the game). It is a perfect 'stepping stone' before conquering the much more challenging weak and reverse weak, and also the Chooser–Picker versions. These three clique games all have the *same* breaking point, namely (1.2). What is (1.2)?

Well, (1.2) is the 'real solution' of the equation $f(q) = \binom{n}{2}/2$. The intuitive meaning of the equation $f(q) = \binom{n}{2}/2$ is that the overwhelming majority of the edges of the random graph are covered by exactly one copy of K_q. In other words, the random graph may have a large number of copies of K_q, but they are well spread (un-crowded); in fact, there is room enough to be typically pairwise edge-disjoint. This suggests the following *heuristic argument*. Assume that we are at a 'last stage' of a clique game where Maker (in the weak game) has a large number of 'almost complete' K_qs: 'almost complete' in the sense that, in each one, all but *two edges* are occupied by Maker, and these 'extremely dangerous' K_qs are pairwise edge-disjoint. Then Breaker can still escape from losing: he can block these disjoint unoccupied edge-pairs by a simple *pairing strategy*! It is exactly the pairing strategy that distinguishes the Picker–Chooser game from the rest of the bunch. Indeed, in each of the weak, reverse weak, and Chooser–Picker games the 'blocker' can easily *win* the *disjoint game*, where the winning sets are disjoint, and contain at least two elements each, by employing a pairing strategy. In sharp contrast, Chooser always *loses* a 'sufficiently large' disjoint game in the Picker–Chooser version (more precisely: if there are at least 2^n disjoint n-element winning sets, then Picker wins the Picker–Chooser game).

This is the best *intuitive explanation* that I could come up with to understand breaking point (1.2). This intuition *requests* the 'random graph heuristic', that is, to (artificially) introduce a random structure in order to understand a deterministic game of complete information.

But the connection is much deeper than that. To *prove* that (1.2) is the exact value of the game-theoretic breaking point, one *requires* a *fake probability theory* (or at least this is how I could do it). The main steps of the proof are:

(1) game-theoretic first moment;
(2) game-theoretic higher moments ('sliding potentials'); and
(3) game-theoretic *independence*.

Developing (1)–(3) is a long and difficult task. The word 'fake' in *fake probability theory* refers to the fact that, when I actually define an optimal strategy, the 'probabilistic part'

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completely disappears. It is a 'metamorphosis': as a caterpillar turns into a butterfly, the 'probabilistic arguments' are converted into (deterministic) 'potential arguments'.

Note that 'potential arguments' are widely used in puzzles ('one-player games'). A well-known example is Conway's *Solitaire Army* puzzle: arrange men behind a line, and then, by playing 'jump and remove' horizontally or vertically, move a man as far across the line as possible. Conway's beautiful 'golden ratio' proof, a striking potential argument, shows that it is *impossible* to send a man forward 5 (4 is possible). Conway's result is from the early 1960s. It was shown very recently that if 'jumping a man diagonally' is permitted, then '5' is replaced by '9'; in other words, it is impossible to send a man forward 9, but 8 is possible. (The idea is the same, but the details are substantially more complicated.)

It is quite natural to use potential arguments to describe *impossible configurations* (like Conway did). It is more surprising that potential arguments are equally useful to describe *achievable configurations* ('Maker's win') as well. But the biggest surprise is that the 'Maker's win criteria' and the 'Breaker's win criteria' often coincide, yielding *exact* solutions of several seemingly hopeless games. However, there is a basic difference: Conway's argument works for small values like '5', but my arguments give sharp results for 'large values' of the parameters only; I call it a 'game-theoretic law of large numbers'.

These *exact solutions* all depend on the concept of 'game-theoretic independence'. What is 'game-theoretic independence'? Well, there is a 'trivial' and a 'nontrivial' interpretation of game-theoretic *independence*.

The 'trivial' (but still very useful) interpretation is about *disjoint* games. Consider a sequence of hypergraphs such that, in each one, Breaker (as the second player) can mark every winning set. If the hypergraphs are pairwise disjoint (in the strong sense that the 'boards' are disjoint), then of course Breaker can block the union hypergraph as well. Disjointness guarantees that, in any component, either player can play 'independently' from the rest of the components. For example, the concept of *pairing strategy* is based on this simple observation.

In the 'nontrivial' interpretation, the initial game does *not* fall apart into disjoint components. Instead Maker can *force* that eventually, in a much later stage of the play, the family of unblocked (yet) hyperedges *does* fall apart into much smaller (disjoint) components. This is how Maker can eventually finish the job blocking the whole initial hypergraph, namely 'blocking componentwise' in the 'small' components.

A convincing 'probabilistic intuition' behind the 'nontrivial' version is the well-known *Lovász Local Lemma* [12]. The LLL is a remarkable probabilistic *sieve* argument to prove the *existence* of certain very complicated structures that we are unable to construct directly (see [12] and [1]). To be precise, let $E_1, E_2, \ldots, E_s$ denote events in a probability space. In the applications, the E_is are 'bad' events, and we want to avoid all of them, that is, we wish to show that $\mathrm{Prob}\left(\cup_{i=1}^{s} E_i\right) < 1$. A trivial way to guarantee this is to assume $\sum_{i=1}^{s} \mathrm{Prob}(E_i) < 1$. A completely different way is when $E_1, E_2, \ldots, E_s$ are *mutually independent* and all $\mathrm{Prob}(E_i) < 1$. Indeed, then $\mathrm{Prob}\left(\cup_{i=1}^{s} E_i\right) = 1 - \prod_{i=1}^{s}(1 - \mathrm{Prob}(E_i)) < 1$. The LLL applies in the very important case when we don't have mutual independence, but 'independence dominates' in the following precise sense: each one of these events is *independent* of all but a *small number* of other events; if $\mathrm{Prob}(E_i) \leqslant p < 1$ holds uniformly, then a good choice for 'small number' is $(4p)^{-1}$.

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A typical application of the LLL goes as follows (see [12]).

Erdős–Lovász theorem. *Let $\mathscr{F} = \{A_1, A_2, A_3, \dots\}$ be an n-uniform hypergraph. Suppose that each A_i intersects at most 2^{n-3} other $A_j \in \mathscr{F}$ ('local size'). Then there is a 2-colouring of the 'board' $V = \bigcup_i A_i$ such that no $A_i \in \mathscr{F}$ is monochromatic.*

The conclusion means that there exists a *drawing terminal position* (well, *almost* true: in a drawing terminal position the two colour-classes are of *equal* size). The very surprising message of the Erdős–Lovász theorem is that the 'global size' of hypergraph $\mathscr{F}$ is irrelevant (it can even be infinite!): all that matters is the 'local size'.

Of course, the existence of a single (or even many) drawing terminal position does *not* guarantee at all the existence of a *drawing strategy*. But perhaps it is still true that 'under the Erdős–Lovász condition (or under some slightly weaker local condition), Breaker (or Avoider, or Picker) has a *blocking draw strategy*, *i.e.*, he can block every winning set in the weak (or reverse weak, or Chooser–Picker) game on $\mathscr{F}$'.

This is a wonderful problem; I call it the 'game-theoretic LLL conjecture'. Unfortunately the conjecture is still open in general, in spite of all my efforts trying to prove it during the last 10 years.

Open Problem 1. *Prove the 'game-theoretic LLL conjecture'.*

I could prove, however, several partial results, which led to interesting applications. In fact, I have developed 3 different techniques, which all have their own range of applications.

A very important special case, when the conjecture is 'nearly proved', is the class of *almost disjoint hypergraphs*: where any two hyperedges have at most one common point. This is certainly the case for 'lines', the winning sets of the n^d Tic-Tac-Toe.

What can I prove about the n^d Tic-Tac-Toe? Well, I proved that it is a draw game even if the dimension d is as large as $d = c_1 n^2 / \log n$, that is, nearly quadratic in terms of (the winning size) n. What is more, the 'draw' is a stronger *blocking draw*: the second player can mark every winning line (if they play as long as the whole board is occupied).

Note that this bound is nearly best possible: if $d > c_2 n^2$ then the second player *cannot* force a blocking draw.

How come that for the clique game I could find the *exact* value of the breaking point, but for the multi-dimensional Tic-Tac-Toe I couldn't even find the asymptotic truth (because of a factor of $\log n$ in the denominator)? Well, the answer is somewhat technical. The winning lines in the multi-dimensional n^d Tic-Tac-Toe form an extremely *irregular* hypergraph: the maximum degree is *much* larger than the average degree. This is why I cannot apply my 'blocking criteria' directly to the 'n^d hypergraph'. First I have to employ a 'truncation procedure' to bring the maximum degree close to the average degree, and the price that I pay for the 'degree reduction' is the loss of a factor of $\log n$.

However, if I consider the n^d *torus* Tic-Tac-Toe, then the corresponding hypergraph becomes perfectly uniform (the torus is a *group*). For example, every point of the n^d torus Tic-Tac-Toe has $(3^d - 1)/2$ winning lines passing through it. For the n^d torus Tic-Tac-Toe

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I can prove asymptotically sharp thresholds – in fact, very sharp results in the quantitative sense that the 'error' is 'logarithmically small'.

A *winning line* in the n^d Tic-Tac-Toe is 'n points on a line forming an arithmetic progression'. This motivates the 'arithmetic progression game': the board is the interval $1, 2, \ldots, N$, and the goal is to build an n-term arithmetic progression. The corresponding hypergraph is 'nearly regular', and this is why I can prove asymptotically very sharp thresholds: the breaking point for n is $\log_2 N + O(\log_2 \log_2 N)$; again the 'error' is 'logarithmically small'.

Let us return to the n^d torus Tic-Tac-Toe. If 'winning line' is replaced by 'winning plane' (or 'winning subspace of dimension $\geqslant 2$' in general), then I can determine the *exact* value of the game-theoretic threshold, just like in the clique game. For example, a *winning plane* is an $n \times n$ lattice in the n^d torus. This is another 'rapidly changing structure': if n switches to $n+1$, then $n \times n$ switches to $(n+1) \times (n+1)$, which is again a 'square root size increase' just like in the case of the 'clique'. This formal similarity to the clique game explains why I had a chance of finding the *exact* value of the game-theoretic breaking point (the actual proof is rather different).

Let me summarize the meaning of 'game-theoretic independence'. It is about Tic-Tac-Toe-like games for which the 'local size' is much smaller than the 'global size'. Even if the game starts out as a coherent entity, either player can force it to develop into 'small' *composites* in the sense that every component-size is less than the 'local size'. A sort of intuitive explanation behind it is the LLL, which is a sophisticated application of *statistical* independence.

The exact solution of the clique game and the 'subspace game' of dimension $\geqslant 2$ played on the n^d torus are both difficult and long. They will be published somewhere else. In the rest of this paper I focus on some simple but still very interesting applications of the basic technique. (The only exception is Section 7, which is a little more complicated.) From now on I stop the informal style, and switch to a rigorous treatment.

2. Strong and weak games

I first recall a few concrete games which are won by the first player to complete some kind of winning configuration. These examples motivate the general definitions.

Tic-Tac-Toe (or Noughts and Crosses). The game board is a big square which is partitioned into $3 \times 3 = 9$ congruent small squares. Whoever moves first puts a nought in one of the nine small squares. The opponent puts a cross into any other small square, and then they alternate nought and cross in the remaining empty squares until one player wins by getting three of his own squares in a line. If neither player gets 3-in-a-row the play is a draw. There are eight winning triplets: three horizontal, three vertical and two diagonal lines (of slope 1 and -1).

Note that Tic-Tac-Toe is a '3-in-a-row' game on a 3×3 board. The 2-dimensional generalization is the 'n-in-a-row' game on an $n \times n$ board; we simply call it the n^2 game. The multi-dimensional generalization is n^d Tic-Tac-Toe, or simply the n^d game.

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n^d game. Two players alternately put their marks in the cells of a d-dimensional cube of size $n \times \cdots \times n = n^d$. The winner is the player who occupies a full-length-line *first*, *i.e.*, who has n of his marks in a line *first*. More precisely, the board V of the n^d game is the set of d-tuples

$$V = \{\mathbf{a} = (a_1, a_2, \ldots, a_d) : \ 1 \leqslant a_j \leqslant n \text{ for each } 1 \leqslant j \leqslant d\}.$$

The winning sets of the n^d game are the n-in-a-line sets, that is, the n-element sequences

$$\left(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}\right)$$

of the board V such that, for each j, the sequence $a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(n)}$ composed of the jth coordinates is either $1, 2, 3, \ldots, n$ ('increasing'), or $n, n-1, n-2, \ldots, 1$ ('decreasing'), or a *constant*. In other words, the winning sets are exactly the n-in-a-lines on the n^d board. If neither player gets n-in-a-line the play is a draw.

Note that in higher dimensions most of the winning lines are some kind of diagonal. The special case $n = 3, d = 2$ gives ordinary Tic-Tac-Toe.

The total number of winning lines in the n^d game is $\left((n+2)^d - n^d\right)/2$. Indeed, for each j, the sequence $a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(n)}$ composed of the jth coordinates is either *increasing* (*i.e.*, $1, 2, 3, \ldots, n$), or *decreasing* (*i.e.*, $n, n-1, n-2, \ldots, 1$), or a *constant* $c \in \{1, 2, \ldots, n\}$. Since for each coordinate we have $(n+2)$ possibilities, this gives $(n+2)^d$. But we have to subtract n^d since in a line at least one coordinate must change. Finally, we have to divide by 2 because every line has two orientations.

An alternative geometric way of getting $\left((n+2)^d - n^d\right)/2$ goes as follows. Imagine the board n^d is surrounded by an additional layer of cells, one cell thick. This new object is a cube

$$(n+2) \times (n+2) \times \cdots \times (n+2) = (n+2)^d.$$

It is easy to see that every winning line of the n^d-board extends to a uniquely determined *pair* of cells in the new surface layer. So the total number of lines is

$$\left((n+2)^d - n^d\right)/2. \tag{2.1}$$

A far-reaching generalization of n^d Tic-Tac-Toe is the class of *strong positional games* (see [9], [10], [13], [16]).

Strong positional games. Let $(V, \mathcal{F})$ be an arbitrary finite hypergraph. Here V is a finite set, called the *board* of the game, and $\mathcal{F}$ is an *arbitrary* family of subsets of V, called the family of *winning sets*. The two players, First Player and Second Player, alternately occupy previously unoccupied elements ('points') of board V. That player wins who occupies all the elements of some winning set $A \in \mathcal{F}$ *first*; otherwise the play ends in a draw.

Sometimes we just give the family $\mathcal{F}$ of winning sets. Then the board V is the union $\bigcup_{A \in \mathcal{F}} A$ of all winning sets.

First I recall Zermelo's classical theorem. Note that the old theorems are lettered and the new main theorems are numbered (see Sections 5–7).

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Theorem A. *Every finite perfect-information* 2*-player game is* determined*, which means that either*

(a) *First Player has a winning strategy, or*
(b) *Second Player has a winning strategy, or*
(c) *both of them have a drawing strategy.*

Remarks. These three alternatives are what one calls the three *outcomes of a game*. Of course every play has three possible outcomes (First Player wins, Second Player wins, or a draw), but the outcome of a particular play is not necessarily the 'same' as the *outcome of the game*. For example, Second Player can easily lose in ordinary Tic-Tac-Toe (if First Player opens in the centre, and Second Player replies on the side), even if Tic-Tac-Toe is a draw game (*i.e.*, belongs to case (c)).

Proof. The *existence* proof of Zermelo's theorem is a simple application of De Morgan's law. Indeed, there are three alternatives only: either

(a) First Player (= **I**) has a winning strategy: $\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots$ such that **I** wins; or
(b) Second Player (= **II**) has a winning strategy: $\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots$ such that **II** wins;
(c′) or the negation of (a)∨(b):

$$\neg\big((\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots \quad \mathbf{I} \text{ wins}) \vee (\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \quad \mathbf{II} \text{ wins})\big),$$

which, by De Morgan's law, is equivalent to

$$(\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \quad \mathbf{I} \text{ loses or draw}) \wedge (\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots \quad \mathbf{II} \text{ loses or draw}).$$

So the third alternative is that both players have a drawing strategy, *i.e.*, alternative (c). This completes the existence proof of Theorem A. □

Every game can be visualized as a 'tree of all possible plays' (called a game tree). The exhaustive search of the game tree gives a *constructive* proof of Zermelo's theorem. It provides an *explicit* winning (or drawing) strategy. The bad news is that exhaustive search is usually impractical. From a complexity point of view a better way to do the same thing is to work with the (usually smaller) position-graph; then 'exhaustive search' is called 'backward labelling'. Unfortunately even 'backward labelling' is impractical: it takes 'exponential time'.

Strategy stealing argument. It is well known that whoever plays first in a strong positional game can force at least a draw. In other words, for strong games alternative (b) cannot hold. Heuristically this is obvious because strong positional games are symmetric, and First Player has the advantage of the first move (note that an extra move in a strong game does not harm a player). The precise proof of this remarkable result is just a little bit more complicated.

Theorem B (strategy stealing). *Let* $(V, \mathscr{F})$ *be an arbitrary finite hypergraph. Then, playing the strong positional game on* $(V, \mathscr{F})$*, First Player can force at least a draw, i.e., a draw or possibly a win.*

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Unfortunately the 'strategy stealing argument' doesn't supply any *explicit* strategy.

Open Problem 2. *Find an explicit First Player's drawing strategy in every n^d game.*

When a draw is impossible: Ramsey theory. When can First Player win in a strong positional game? A partial answer (sufficient condition) is the following: First Player has a *winning strategy* in a strong game when a draw is impossible.

Of course this condition is not necessary. For example, the full branches of a binary tree with n levels form an n-uniform family of 2^{n-1} winning sets such that First Player has an easy win (the players take the vertices of the tree). But this strong game has plenty of drawing terminal positions (*e.g.*, all vertices of degree one are red, and the rest of the vertices are all blue).

A drawing terminal position in a strong game $(V, \mathcal{F})$ means a 'halving' 2-colouring of the board V such that no winning set $A \in \mathcal{F}$ is monochromatic. A slightly more general concept is when we allow *arbitrary* 2-colourings (*i.e.*, the two colour classes may have very different sizes).

The *chromatic number* $\chi(\mathcal{F})$ of hypergraph $\mathcal{F}$ is the least integer $r \geqslant 2$ such that the elements of the board V can be coloured with r colours yielding no monochromatic $A \in \mathcal{F}$. If the chromatic number of $\mathcal{F}$ is bigger than 2, then a draw is impossible. In this case First Player's (at least) drawing strategy in Theorem B is actually a winning strategy (see, *e.g.*, [15]).

Theorem C (Ramsey criterion). *Suppose that the board V is finite, and the family $\mathcal{F}$ of winning sets has chromatic number at least three. Then First Player has a winning strategy in the strong positional game on $(V, \mathcal{F})$.*

Theorem C doesn't say a word about *how* to win. Unfortunately nobody knows how to win. The only general advice I know is 'try exhaustive search'! Theorem C describes a subclass of strong positional games with the remarkable property that one can easily diagnose the winner without being able to say how one wins.

A fundamental result in Ramsey theory is the Hales–Jewett theorem [16]. This is basically the combinatorial content of van der Waerden's famous theorem on arithmetic progressions. It states that there is a (least) *finite* threshold number $h(n)$ such that the family of all 'n-in-a-lines' in the n^d game (*i.e.*, the family of winning sets) has chromatic number $\geqslant 3$ if $d \geqslant h(n)$. We call $h(n)$ the Hales–Jewett threshold. By the Ramsey criterion (Theorem C), First Player has a winning strategy in the n^d game if $d \geqslant h(n)$.

Open Problem 3. *Find an explicit First Player's winning strategy in the n^d game when $d \geqslant h(n)$.*

Unfortunately our present knowledge on the Hales–Jewett threshold number $h(n)$ is rather poor. The best-known upper bound on $h(n)$ is a breakthrough result of Shelah [18] from 1988. It is a primitive recursive function (the *supertower* function), which is much-much better than the original van der Waerden–Hales–Jewett threshold (the original

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argument gave the 'notorious' Ackermann function: see, *e.g.*, [15]). Unfortunately Shelah's bound is still enormous for 'pedestrian combinatorics'. For example, it is an open problem whether $h(n)$ is less than the 'plain' *tower* function $\text{tower}_n(2)$, where $\text{tower}_k(x)$ denotes the k-fold iteration of the exponential function: $\text{tower}_1(x) = 2^x$ and for $k \geqslant 2$, $\text{tower}_k(x) = 2^{\text{tower}_{k-1}(x)}$ (so, *e.g.*, $\text{tower}_2(x) = 2^{2^x}$). To get an idea about the order of magnitude of Shelah's bound, the reader is recommended to study the *supertower* function defined as follows. Let $\text{tower}(k;x) = \text{tower}_k(x)$, and

$$\text{supertower}(n) = \text{tower}\Big(\text{tower}\big(\cdots \text{tower}\big(\text{tower}(2;2);2\big)\cdots;2\big);2\Big),$$

where the right-hand side is an n-fold iteration.

In the other direction, Hales and Jewett [16] proved the linear lower bound $h(n) \geqslant n$ by an explicit construction. A couple of years ago I could improve this linear bound to a nearly quadratic bound $h(n) > c_0 \cdot n^2/\log n$ where $c_0 > 0$ is an absolute constant (this was the first improvement since 1963). Of course there remains an enormous discrepancy between the (nearly) quadratic lower bound and Shelah's supertower upper bound, and we don't have the slightest idea which one is closer to the truth.

It seems highly unlikely that the game-theoretic breaking point for the n^d game is anywhere close to the Hales–Jewett threshold number $h(n)$, but no method is known for handling this problem.

I mention two 'plausible conjectures'.

Open Problem 4. *Is it true that if the n^d game is a First Player's win then the n^{d+1} game is also a win?*

A good reason why Open Problem 4 is so difficult is that one can construct a finite hypergraph (*i.e.*, a strong game) which is a First Player's win, but adding an extra winning set turns it into a draw game. An even more sophisticated 'extra set paradox' is the following. One can construct a finite hypergraph which is a draw, but it has an induced sub-hypergraph (*i.e.*, all winning sets in a subset of the board) which is a First Player's win. One can even construct an example where the hypergraph is uniform.

I challenge the reader to find such examples.

Open Problem 5. *Is it true that if the n^d game is a draw then the $(n+1)^d$ game is also a draw?*

Note that Golomb and Hales [14] proved the weaker result that 'if the n^d game is a draw then the $(n+2)^d$ game is also a draw'. I am sure the reader is wondering 'Why is it easier from n to $n+2$ than from n to $n+1$?' Well, a good reason is that the n^d game is the 'interior' of the $(n+2)^d$ game, explaining why it is more natural to switch from n to $n+2$. The lack of any simple connection between the n^d and $(n+1)^d$ games raises the possibility of an occasional *nonmonotonicity* here; that is, it may be, for some n and d, that the n^d game is a draw, the $(n+1)^d$ game is a First Player's win, and the $(n+2)^d$ game is a draw again.

I believe (but cannot prove) that *nonmonotonicity* never happens.

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Weak game: Maker–Breaker version. As we have seen in Theorem B, Second Player has no chance of winning a strong game against a perfect First Player. Then, why does he not just concentrate on preventing First Player from building a winning set, and simply ignore his own desire for building? We thus can name him Breaker, and the other one Maker. (I usually assume that Maker is the first player, and Breaker is the second player.)

In general, on the same hypergraph $(V, \mathcal{F})$, one can play the 'symmetric' strong game, and also the 'asymmetric' *weak game*: the *Maker–Breaker version*. Here Maker's aim is to claim every element of a winning set $A \in \mathcal{F}$ (but not necessarily first), and Breaker's aim is simply to prevent Maker from doing so (*i.e.*, Breaker's goal is to put his mark in every winning set). The winner is the one who achieves his goal; so a draw is impossible by definition.

If First Player can force a win in the strong game on $(V, \mathcal{F})$, then of course the *same* play gives him, as Maker, a win in the weak (Maker–Breaker) version (on the same hypergraph). But the converse is not true. It is possible that Maker, as the first player, has a winning strategy in the weak version, while Second Player can force a draw in the strong version. This happens for example in the standard 3^2 Tic-Tac-Toe: the strong 3^2 game is a draw, but the weak version is a win for Maker (as the first player). This is why one can call a Maker's winning strategy in the weak game a 'weak win strategy'. Breaker's winning strategy in the weak game can be called a 'blocking draw strategy'. I know a lot about the complementary concepts of 'weak win' and 'blocking draw', but I know (almost) nothing about a 'strong win' (*i.e.*, a win in the strong game) except Theorem C.

While playing the strong version on a hypergraph, both players have their own threats, and either of them, fending off the other's, may build his own winning set. Therefore, a play is a delicate balancing between threats and counter-threats and can be of very intricate structure even if the hypergraph itself is simple.

The weak version is usually somewhat simpler. Maker doesn't have to waste valuable moves fending off his opponent's threats. Maker can simply concentrate on his own goal of building. For example, the Maker–Breaker version of Open Problem 4 is trivial: Maker uses the winning strategy within, for example, a d-dimensional subcube of n^{d+1}.

However, 'easier' doesn't mean that Maker–Breaker games are 'easy'. For example, the well-known and notoriously difficult Hex is equivalent to a Maker–Breaker game, but it doesn't help to 'break' the game. In Hex the board is a rhombus of hexagons of size $n \times n$ (the standard size is $n = 11$). The two players, White (the first player) and Black (the second player) take the two pairs of opposite sides of the board. The players alternately put their pieces on unoccupied hexagons (White has white pieces, and Black has black pieces). White wins if his pieces connect his opposite sides of the board, and Black wins if his pieces connect the other pair. Observe that Hex is *not* a strong game: the winning sets for White and Black are mostly different (the only common winning sets are the chains connecting opposite corner points). Let Hex(+) denote the weak (Maker–Breaker) game in which the board is the $n \times n$ Hex board, Maker = White, Breaker = Black, and the winning sets are the connecting chains of White. We claim that Hex and Hex(+) are equivalent. First note that a draw in Hex is impossible. Indeed, in order to prevent the

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opponent from making a connecting chain, one must build a 'river', which itself contains a chain connecting the *other* pair of opposite sides (this fact seems plausible, but the precise proof is not completely trivial). This means that Breaker's goal in Hex(+) (*i.e.*, 'blocking') is equivalent to Black's goal in Hex (*i.e.*, 'building first'). Here 'equivalent' means that Breaker has a winning strategy in Hex(+) if and only if Black has a winning strategy in Hex. Since a draw is impossible, Hex and Hex(+) are equivalent.

By the strategy stealing argument White has a winning strategy in Hex. This remarkable result was discovered by J. Nash in the 1940s (perhaps the first application of strategy stealing). The fact that Hex is equivalent to a *weak game* doesn't make it 'easy': we don't have the slightest clue how to actually find an *explicit* winning strategy. It remains a long-standing open problem.

3. Potential functions

Positions with limited potential: Solitaire Army. *Solitaire Army* is a particular case of the class of *Solitaire* puzzles (*Peg Solitaire* in America). These puzzles are not 'games': a 'game' requires 2 players, and here there is only one player. The reason why I discuss this puzzle is to demonstrate the potential function technique on a simple (and very elegant) example.

The common feature of the Solitaire puzzles is that each one is played with a board and men (pegs), the board contains a number of holes each of which can hold one man. Each move consists of a jump by one man over one or more other men, the men jumped over being removed from the board. Each move therefore reduces the number of men on the board.

The Solitaire Army is played on the infinite plane and the holes are in the lattice points. (see pp. 715–7 in Vol. 2 of *Winning Ways* [11]). The permitted move is to jump a man horizontally or vertically but not diagonally. Let us draw a horizontal line across the infinite board and start with all men behind this line. Assume this line is the horizontal axis, so all men are in the lower half-plane. How many men do we need to send one man forward 1, 2, 3, 4, or 5 holes into the upper half-plane?

Obviously two men are needed to send a man forward one hole. After playing around for a few minutes everyone can see how four men are enough to send a man forward two holes, and how eight men are enough to send a man forward three holes. It takes a couple of hours to discover how twenty men are enough to send a man forward four holes.

But the really surprising result is the case of five holes: it is *impossible* to send a man forward five holes into the upper half-plane. This striking result was discovered by Conway in 1961.

Conway's 'potential proof' goes as follows. We assign a weight to each hole subject to the condition that if H1, H2, H3 are *any* three consecutive holes in a row or in a column, and $w(\mathrm{H1})$, $w(\mathrm{H2})$, $w(\mathrm{H3})$ are the corresponding weights, then $w(\mathrm{H1}) + w(\mathrm{H2}) \geqslant w(\mathrm{H3})$. We can evaluate a position by the sum of the weights of those holes which are occupied by men – this sum is called the *value* of the position,

The meaning of inequality $w(\mathrm{H1}) + w(\mathrm{H2}) \geqslant w(\mathrm{H3})$ is very simple. The effect of a move where a man in H1 jumps over another man in H2 and arrives at H3 is that

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we replace men with weights $w(\mathrm{H1})$ and $w(\mathrm{H2})$ by a man with weight $w(\mathrm{H3})$. Since $w(\mathrm{H1})+w(\mathrm{H2}) \geqslant w(\mathrm{H3})$, this change cannot be an increase in the value of the new position.

Inequality $w(\mathrm{H1})+w(\mathrm{H2}) \geqslant w(\mathrm{H3})$ guarantees that *no play is possible from an initial position to a target position if the target position has a higher value.*

Let w be a positive number which satisfies $w+w^2=1$: w equals the golden ratio $\frac{\sqrt{5}-1}{2}$. Now Conway's potential argument goes as follows. Assume that one succeeded in sending a man 5 holes forward into the upper half-plane by starting from a configuration of a finite number of men in the lower half-plane. Write 1 where the man stands 5 holes forward into the upper half-plane, and extend it in the following way:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & & w & & & & & \\ & & & & & w^2 & & & & & \\ & & & & & w^3 & & & & & \\ & & & & & w^4 & & & & & \\ \cdots & w^9 & w^8 & w^7 & w^6 & w^5 & w^6 & w^7 & w^8 & w^9 & \cdots \\ \cdots & w^{10} & w^9 & w^8 & w^7 & w^6 & w^7 & w^8 & w^9 & w^{10} & \cdots \\ \cdots & w^{11} & w^{10} & w^9 & w^8 & w^7 & w^8 & w^9 & w^{10} & w^{11} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

The value of the top line of the lower half-plane is

$$w^5+2w^6+2w^7+2w^8+\cdots = w^5+2\frac{w^6}{1-w} = w^5+2\frac{w^6}{w^2} = w^5+2w^4 = w^3+w^4 = w^2.$$

So the value of the whole lower half-plane is

$$w^2(1+w+w^2+w^3+\cdots) = w^2\frac{1}{1-w} = w^2\frac{1}{w^2} = 1,$$

which is exactly the value of the target position. So no finite number of men in the lower half-plane will suffice to send a man forward five holes into the upper half-plane.

One can even show that eight men are in fact *needed* to send a man forward three holes, and similarly, twenty men are *needed* to send a man forward four holes (see Beasley's excellent little book [3]). Note that the 'doubling pattern' breaks for four holes; this is the warning sign for the big surprise coming in the case of 'five holes forward'.

We challenge the reader to solve the following exercise.

Exercise. We generalize Solitaire Army in such a way that 'to jump a man diagonally' is permitted. Show that it is impossible to send a man forward 9 holes.

Erdős–Selfridge theorem. The Erdős–Selfridge theorem [13] is the pioneering application of the 'potential technique' for 2-*player games*. It gives a sufficient condition for a blocking draw. The result was proved in 1973, and since then it has continued to play a central role in the developments of positional games. It changed the outlook of the subject: it shifted the emphasis from Ramsey theory and matching theory to the probabilistic method.

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Theorem D (Erdős–Selfridge theorem). *Let $\mathscr{F}$ be an n-uniform hypergraph, and assume that $|\mathscr{F}| + \mathrm{MaxDeg}(\mathscr{F}) < 2^n$, where $\mathrm{MaxDeg}(\mathscr{F})$ denotes the maximum degree of hypergraph $\mathscr{F}$. Then playing on $\mathscr{F}$ Second Player can force a blocking draw.*

Remark. A well-known result of Erdős states that, under the slightly weaker condition $|\mathscr{F}| < 2^{n-1}$,

(a) there is a proper 2-colouring; and what is slightly more,
(b) there is a *drawing terminal position.*

Both (a) and (b) can be proved by the 'counting argument'. Of course, the Erdős–Selfridge theorem implies (b) (and (b) implies (a)). The reason why I include the proofs of (a) and (b) is to demonstrate a key idea: the dual roles of 'randomization' and 'derandomization' in positional game theory.

The proof of (a) goes as follows. Let N denote the size of the union set (board) V of hypergraph $\mathscr{F}$. A simple counting shows that under the condition $|\mathscr{F}| < 2^{n-1}$ there exists a proper 2-colouring. Indeed, there are 2^N 2-colourings of board V, and for every single winning set $A \in \mathscr{F}$ there exist 2^{N-n+1} 'bad' 2-colourings which are monochromatic on A. By hypothesis $2^N - |\mathscr{F}|2^{N-n+1} > 0$, which implies that throwing out all 'bad' 2-colourings, there must remain at least one proper 2-colouring (*i.e.*, where no $A \in \mathscr{F}$ is monochromatic).

To prove (b) we have to find a drawing terminal position (*i.e.*, a 2-colouring of the board by 'colours' X and O such that the two colour classes have the same size, and each winning set contains both marks). For notational simplicity assume that N is even. The idea is exactly the same as that of (a), except that we restrict ourselves to the $\binom{N}{N/2}$ terminal positions (*i.e.*, 'halving' 2-colourings), instead of the 2^N (arbitrary) 2-colourings. The analogue of $2^N - |\mathscr{F}|2^{N-n+1} > 0$ is the following requirement: $\binom{N}{N/2} - 2|\mathscr{F}|\binom{N-n}{N/2} > 0$. This holds because

$$\frac{\binom{N-n}{N/2}}{\binom{N}{N/2}} = \frac{N/2}{N}\frac{(N/2)-1}{N-1}\frac{(N/2)-2}{N-2}\cdots\frac{(N/2)-n+1}{N-n+1} \leqslant 2^{-n},$$

and (b) follows.

The previous argument can be stated in the following slightly different form: the *average number* ('expected value') of winning sets completely occupied by either player is precisely

$$2|\mathscr{F}|\frac{\binom{N-n}{N/2}}{\binom{N}{N/2}}, \quad \text{which is less than 1.}$$

Since the *minimum* is less or equal to the *average*, *there must exist* at least one *drawing* terminal position (*i.e.*, no player owns a whole winning set). This kind of 'counting argument', due to Erdős, was the forerunner of the probabilistic method (see, *e.g.*, [1]).

The Erdős–Selfridge theorem can be viewed as a 'derandomization' of the first moment method. Indeed, if a drawing position exists, then there is a (slight) chance of a drawing

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strategy. The message of the Erdős–Selfridge theorem is that the existing drawing terminal position, proved by the first moment method, can be *upgraded* to a drawing strategy (in fact to a *blocking draw strategy*).

Finally, we prove the theorem itself.

Proof of Theorem D. Let $\mathscr{F} = \{A_1, A_2, \dots, A_M\}$ where $M < 2^{n-1}$. Assume we are at the stage of the play where First Player already occupied $x_1, x_2, \dots, x_i$, and Second Player occupied $y_1, y_2, \dots, y_{i-1}$. The question is how to choose Second Player's next point y_i. Those winning sets which contain at least one y_j ($j \leqslant i-1$) are 'harmless' – we call them 'dead sets'. The winning sets which are not 'dead' are called 'survivors'. The survivors have a chance of being completely occupied by First Player at the end of the play, so they each represent some 'danger'. What is the total danger of the *whole* position? We evaluate the given position by the following expression, called the 'danger function': $D_i = \sum_{s \in S_i} 2^{-u_s}$ where u_s is the number of unoccupied elements of the 'survivor' A_s ($s \in S_i$ = 'index set of the survivors'), and index i indicates that we are at the stage of choosing Second Player's ith point y_i. A natural choice for y_i is to minimize the danger D_{i+1} at the next stage. How to do that? Well, the simple linear structure of the danger function D_i gives an easy answer to this question. Indeed, let y_i and x_{i+1} denote the next two moves. What is the effect of these two points on D_i? How do we get D_{i+1} from D_i? Clearly y_i 'kills' all the survivors $A_s \ni y_i$, which means we have to subtract the sum

$$\sum_{s \in S_i:\ y_i \in A_s} 2^{-u_s}$$

from D_i. On the other hand, x_{i+1} doubles the danger of each survivor $A_s \ni x_{i+1}$, that is, we have to add the sum $\sum_{s \in S_i:\ x_{i+1} \in A_s} 2^{-u_s}$ back to D_i. Warning: if some survivor A_s contains both y_i and x_{i+1}, then we do not have to give the corresponding term 2^{-u_s} back because that A_s was previously 'killed' by y_i.

The natural choice for y_i is the unoccupied z for which $\sum_{s \in S_i:\ z \in A_s} 2^{-u_s}$ attains its maximum. Then what we subtract is at least as large as what we add back:

$$\begin{aligned} D_{i+1} &\leqslant D_i - \sum_{s \in S_i:\ y_i \in A_s} 2^{-u_s} + \sum_{s \in S_i:\ x_{i+1} \in A_s} 2^{-u_s} \\ &\leqslant D_i - \sum_{s \in S_i:\ y_i \in A_s} 2^{-u_s} + \sum_{s \in S_i:\ y_i \in A_s} 2^{-u_s} = D_i. \end{aligned}$$

In other words, Second Player can force the *decreasing property* $D_1 \geqslant D_2 \geqslant \cdots \geqslant D_{\text{last}}$ of the danger function.

Second Player's ultimate goal is to prevent First Player from completely occupying some $A_j \in \mathscr{F}$, that is, to avoid $u_j = 0$. If $u_j = 0$ for some j then $D_{\text{last}} \geqslant 2^{-u_j} = 1$. By hypothesis

$$D_{\text{start}} = D_1 = \sum_{A:\ x_1 \in A \in \mathscr{F}} 2^{-n+1} + \sum_{A:\ x_1 \notin A \in \mathscr{F}} 2^{-n} \leqslant (|\mathscr{F}| + \text{MaxDeg}(\mathscr{F}))2^{-n} < 1,$$

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so by the *decreasing property* of the danger function, $D_{\text{last}} < 1$. This is exactly what we were talking about in the Solitaire Army puzzle: *no play is possible from an initial position if the target position has a higher value than the initial position.* This completes the proof of the Erdős–Selfridge theorem. □

Remarks. **(1)** Exactly the same proof gives the following blocking criterion in the general, not necessarily uniform, case. If

$$\sum_{A\in\mathscr{F}} 2^{-|A|} + \max_{x\in V} \sum_{A\in\mathscr{F}:\ x\in A} 2^{-|A|} < 1,$$

then Second Player can block every winning set $A \in \mathscr{F}$.

(2) If $\mathscr{F}$ is n-uniform, then multiplying the 'danger' 2^{-u_s} of a 'survivor' by 2^n, the renormalized danger becomes 2^{n-u_s}. The exponent, $n - u_s$, is the number of First Player's marks in a survivor (*i.e.*, Second Player-free) set (u_s denotes the number of unoccupied points). This means the following *power-of-two scoring system.* A winning set containing an O (Second Player's mark) scores zero, a blank winning set scores 1, a set with a single X (First Player's mark) and no O scores 2, a set with two Xs and no O scores 4, a set with three Xs and no O scores 8, and so on (*i.e.*, the 'values' are integers rather than small fractions). Occupying a whole n-element winning set scores 2^n, *i.e.*, due to the renormalization, the 'target value' becomes 2^n (instead of 1).

It is just a matter of taste which scoring system one prefers: the first one, where the scores were negative powers of 2 and the target value was 1, or the second one, where the scores were positive powers of 2 and the target value was 2^n.

(3) Theorem D is sharp. The full branches of a binary tree with n levels form an n-uniform family of 2^{n-1} winning sets such that First Player can occupy a full branch in n moves (the players occupy vertices of the binary tree).

Another *extremal system* is the following. The board V is a $(2n-1)$-element set $\{w, x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{n-1}\}$. The family of winning sets consists of all possible n-element subsets A of V with the following two properties: (1) $w \in A$, (2) A contains exactly one point from each pair $\{x_i, y_i\}$, $i = 1, 2, \ldots, n-1$. The number of winning sets is 2^{n-1}, and again First Player can occupy a winning set in n moves.

Note that very recently my PhD students, Klay Kruczek and Eric Sundberg, could find, for every n, an 'extremal system' for Theorem D (*i.e.*, an n-uniform hypergraph of size 2^{n-1}, where First Player wins) such that it takes *more* than n moves for First Player to occupy a whole winning set (if Second Player plays rationally). In fact, Second Player can postpone his inevitable loss as long as $2n-3$ moves. They named their tricky construction a 'tumbleweed'.

We are very far from being able to classify all 'extremal systems' for Theorem D. (Why is it so difficult? I don't know!)

Theorem D gives a quick proof of the fact that the 4^2 game is a draw, in fact Second Player can force a blocking draw. Indeed, in the 4^2 game there are 10 winning sets, the

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maximum degree is 3, and $3 + 10 < 2^4 = 16$, so Theorem D applies. The same argument works for all n^2-games with $n \geqslant 4$.

A very clever and complicated 'pairing' of S. Golomb shows that the 8^3 game has a pairing strategy draw (see *Winning Ways* [11], Vol. 2, pp. 677–8). By using Theorem D one can easily show that the 8^3 game has a blocking draw. Indeed, in the 8^3 game there are $(10^3 - 8^3)/2 = 244$ winning sets, the maximum degree is $2^3 - 1 = 7$, and $244 + 7 < 2^8 = 256$, so Theorem D applies. The same argument works for all n^3 games with $n \geqslant 8$. (Of course a blocking draw is not necessarily a pairing strategy draw.)

Hales–Jewett conjecture. The n^d-hypergraph is *almost disjoint*: the winning sets are 'lines' (more precisely: finite subsets of straight lines), and any two straight lines have at most one point in common. In general, a hypergraph is called *almost disjoint* if any two winning sets have at most one point in common. If an almost disjoint hypergraph has a pairing strategy draw, then there are at least twice as many points as winning sets (indeed, two sets cannot share the same point-pair). Hales and Jewett conjectured [16] that for the subclass of n^d-games the 'converse' holds. They conjectured that the n^d-game has a draw, in fact a pairing strategy draw, if there are at least twice as many points (*i.e.*, 'cells') as winning lines. This is the *Hales–Jewett conjecture*. It has two natural interpretations. The stronger form is as follows.

Open Problem 6. *Is it true that if the n^d game has at least twice as many points as lines, then it is a pairing strategy draw?*

In view of the well-known Marriage Lemma (Hall's theorem) a *necessary and sufficient* condition for the existence of a draw-forcing pairing is that for every subfamily of winning lines the union set has at least twice as many points as the number of lines in the subfamily. (We actually use the Bigamy Corollary of the Marriage Lemma: every man needs 2 wives.) Consequently, what Open Problem 6 really says is that *the point/line ratio attains its minimum for the family of all lines, and for any proper subfamily the ratio is greater or equal*. This *Ratio Conjecture* is very compelling, not only when a pairing strategy exists, but in general for arbitrary n^d-game. The Ratio Conjecture was formulated in Patashnik [17].

Erdős–Selfridge [13] and *Winning Ways* [11] both gave the following broader interpretation of the Hales–Jewett conjecture; I call it the 'weaker form'.

Open Problem 7. *Is it true that if the n^d game has at least twice as many points as lines, then it is a draw (but not necessarily a pairing strategy draw)?*

Note that Open Problem 7 is nearly solved: I proved that it holds true for all dimensions $d \geqslant 33$. More precisely, for all $d \geqslant 16$ with n even, and for all $d \geqslant 33$ with n odd. Asymptotically I can prove much more: the n^d game can have far fewer points than lines, in fact the point/line ratio can tend to zero, and Second Player can still force a blocking draw.

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By formula (2.1) the number of winning lines is $((n+2)^d - n^d)/2$, so the stronger (weaker) Hales–Jewett conjecture states that the n^d game has a pairing strategy draw (has a draw) if $n^d \geqslant (n+2)^d - n^d$, or equivalently, if $2 \geqslant ((n+2)/n)^d$. Since $((n+2)/n)^d = (1+\frac{2}{n})^d \approx e^{2d/n}$ if n is large, the conjecture is *asymptotically* equivalent to the threshold

$$n \geqslant \frac{2d}{\log 2} = 2.8854d. \tag{3.1}$$

We can get pretty close to this threshold: in an unpublished paper R. Schroeppel (Arizona) proved that the n^d game has a pairing strategy draw if $n \geqslant 3d$, $d \geqslant 2$ *even*, and if $n \geqslant 3d-1$, $d \geqslant 3$ *odd*. Schroeppel's new idea is to involve the concept of 'fractional matching'.

Is Theorem D powerful enough to settle the weaker Hales–Jewett conjecture? Unfortunately the answer is 'no'. Indeed, the Erdős–Selfridge theorem applies to the n^d-hypergraph if

$$\frac{(n+2)^d - n^d}{2} + \{(3^d-1)/2 \quad \text{or} \quad 2^d - 1\} < 2^n,$$

depending on the parity of n. This means Second Player can force a blocking draw if $n > \text{const} \cdot d \cdot \log d$. This 'superlinear' result clearly falls short of the 'linear' Hales–Jewett conjecture (see (3.1)).

Unfortunately a pairing strategy 'beats' the Erdős–Selfridge theorem for all n^d-games (except for the single case of 4^2). If the Erdős–Selfridge theorem yields mediocre results for the n^d games, then what was the point of introducing it? Well, the importance of the Erdős–Selfridge theorem is five-fold.

(1) *Combining* the Erdős–Selfridge power-of-two scoring system with a pairing strategy I could nearly solve the weaker Hales–Jewett conjecture, and could in fact go far beyond that. I call the technique developed for this purpose 'BigGame SmallGame Decomposition'.
(2) By using the Erdős–Selfridge scoring system in *both* the 'BigGame' and 'SmallGame', I can get an asymptotically *better* blocking draw criterion.
(3) A straightforward adaptation of the Erdős–Selfridge argument gives a weak win criterion, which is the best-known result for almost disjoint hypergraphs.
(4) By developing a 'higher moment version' of the 'linear' (4), I proved the best-known weak win criterion for arbitrary (not necessarily almost disjoint) hypergraphs.
(5) Finally, for the *clique game* the Erdős–Selfridge theorem gives an excellent upper bound, very close to the 'truth' (the 'gap' is at most 2).

Here I discuss (5).

Blocking draws in the clique game. (5) is about the 'straight' clique game (K_n, K_q): the board is K_n, *i.e.*, the players claim *edges* of K_n and the goal is to have a copy of K_q first. The well-known Erdős–Szekeres theorem ('Ramsey theory') states that, given any 2-colouring of the edges of the complete graph K_n with $n \geqslant \binom{2q-2}{q-1}$ vertices, there is always a monochromatic copy of K_q. Since every play has a winner, strategy stealing applies (see Theorem C), and yields a First Player's winning strategy. In other words, if $n = \binom{2q-2}{q-1} \approx 4^{q-1}/\sqrt{\pi q}$, or equivalently, if $q = (\frac{1}{2} + o(1)) \log_2 n$, then First Player can force

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a win in the 'straight' clique game (K_n, K_q). Of course, it does *not* say a word about what First Player's winning strategy actually looks like. To find an explicit winning strategy remains a big open problem.

In the other direction, the Erdős–Selfridge theorem implies a blocking draw if

$$\binom{n}{q} < 2^{\binom{q}{2}-1}. \tag{3.2}$$

This is equivalent to

$$q \geqslant 2\log_2 n - 2\log_2\log_2 n + c_0 + o(1) \quad \text{where} \quad c_0 = 2\log_2 e - 1. \tag{3.3}$$

The deduction of (3.3) from (3.2) is a routine calculation by using Stirling's formula and taking binary logarithm of both sides.

The reason why I gave the very precise form (3.3) instead of simply writing $(2 + o(1))\log_2 n$ was that (3.3) comes *very close* to the truth! By using a more sophisticated argument I can show that the *exact* threshold for a blocking draw is actually 2 less than (3.3): if

$$q > 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3 + o(1), \tag{3.4}$$

then Second Player can still block every K_q; on the other hand, if

$$q < 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3 + o(1), \tag{3.5}$$

then First Player can completely occupy a K_q (but not necessarily first – this is called a weak win), implying that a blocking draw is impossible.

To prove (3.5) I needed an 'advanced weak win criterion'. The precise form of this criterion is rather complicated. Instead I just give an oversimplified heuristic form as follows: *If $\mathcal{F}$ is m-uniform, the average degree is 'substantially larger' than 2^m, and the average degree is 'substantially larger' than the 'typical pair-degree', then First Player can force a weak win.*

The clique-hypergraph $\mathcal{F}_{n,q}$ is degree-regular, so the maximum degree equals the average degree. For many degree-regular hypergraphs, including the clique-hypergraph, the weak win criterion and the game-theoretic LLL Conjecture (see Open Problem 1) nearly complement each other, which leads to asymptotically very good results.

A common feature of the proofs of the 'advanced criteria' is to define an appropriate auxiliary hypergraph of 'big sets'. The 'advanced weak win criterion' is a 'game-theoretic higher moment method' ('higher moment' means 'big sets'), and the proof uses the technique of 'sliding potentials' (see [7]). 'Sliding potentials' is a technique to divided the whole play into several phases, and in each phase one uses a new potential function.

The best blocking draw criteria are proved by using the 'BigGame SmallGame Decomposition' (see [4], [5], [8]). It means dividing the board into two non-interacting subgames (with disjoint subboards), and use the *big game* (played on a family of big sets) to control the *small game*: a family of 'dangerous' winning sets which are almost completely occupied by the opponent. In the simpler version, the small game is handled by a pairing strategy; a nice illustration of this is Section 7. In the more advanced versions, the small game is handled by an Erdős–Selfridge-type scoring system.

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The Erdős–Selfridge theorem itself hardly ever gives the best quantitative result. To get the exact solution of an interesting game like the 'clique game' (or the 'subspace game') one adapts the proof technique ('power-of-two scoring system') rather than using the Erdős–Selfridge theorem itself.

4. Weak win criteria

The primary objective of game-playing is to win ('winning is everything' as they say in the USA). Unfortunately we don't know how to force a strong win (a 'win in a strong game'). The next best thing is a weak win (a 'Maker's win in a weak game'). In this section I discuss weak win criteria, and in the next section I show several applications.

The Ramsey criterion (Theorem C) for a strong win was a pure *existence* argument (via strategy stealing), which left the question of 'How to actually win?' completely unanswered. Weak games are different: there is an *explicit* version of the Ramsey criterion, which is a simple 'copycat' pairing strategy.

Theorem E (weak win by Ramsey). *Let $(W, \mathcal{G})$ be a hypergraph such that it contains two disjoint copies of a hypergraph with chromatic number $\geqslant 3$: let $(V, \mathcal{F})$ be a finite hypergraph of chromatic number at least 3, let $(V', \mathcal{F}')$ be a disjoint copy of $(V, \mathcal{F})$, let $W \supseteq V \cup V'$ and $\mathcal{G} \supseteq \mathcal{F} \cup \mathcal{F}'$. Then Maker has an explicit winning strategy in the weak game on $(W, \mathcal{G})$, no matter whether Maker is the first or second player.*

Proof. Let $f : V \to V'$ be the isomorphism between $(V, \mathcal{F})$ and $(V', \mathcal{F}')$. Maker wins by using the following copycat pairing strategy: if Breaker's last move was $x \in V$ or $x' \in V'$, then Maker's next move is $f(x) \in V'$ or $f^{-1}(x') \in V$ (unless it was occupied by Maker before; then Maker's next move is arbitrary). Since the chromatic number of $(V, \mathcal{F})$ is larger than two, one of the two players will completely occupy a winning set. If this player is Maker, we are done. If Breaker occupies some $A \in \mathcal{F}$, then Maker occupies $f(A) \in \mathcal{F}'$, and we are done again. □

Theorem E is *folklore*; for an interesting infinite version, see [2].

The first application of Theorem E is the weak n^d-game where $d \geqslant h(n) + 1$. The condition guarantees that the board contains two disjoint copies of $n^{h(n)}$. The copycat pairing of Theorem E provides an *explicit* winning strategy for Maker (whether Maker is the first or second player).

Next consider the weak clique game (K_n, K_q) where $n \geqslant 2\binom{2q-2}{q-1}$. The condition guarantees that the board contains two disjoint copies of K_m with $m = \binom{2q-2}{q-1}$ (Erdős–Szekeres threshold). The copycat pairing of Theorem E supplies an *explicit* winning strategy for Maker if $q = \frac{1}{2}\log_2 n$ (and again whether Maker is the first or second player).

Theorem E was a Ramsey-type weak win criterion. The next criterion is completely different: it is a 'weak win counterpart' to the Erdős–Selfridge theorem.

Theorem F (Erdős–Selfridge-type weak win criterion). *Assume that $\mathcal{F}$ is n-uniform. Let V denote the union set (board), and assume that, fixing any two distinct points of V, there*

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are no more than $\Delta_2 = \Delta_2(\mathscr{F})$ *winning sets* $A \in \mathscr{F}$ *containing both points; we call* $\Delta_2(\mathscr{F})$ *the pair-degree. If*

$$\frac{|\mathscr{F}|}{|V|} > 2^{n-3} \cdot \Delta_2,$$

then First Player has a weak win in $(V, \mathscr{F})$.

In particular, for almost disjoint $\mathscr{F}$, *the condition simplifies to* $|\mathscr{F}| > 2^{n-3}|V|$.

Remark. If $\mathscr{F}$ is n-uniform, then $\frac{|\mathscr{F}|}{|V|}$ is $\frac{1}{n}$ times the *average degree*. In other words, the hypothesis of Theorem F is a simple 'density condition'.

Proof of Theorem F. I basically repeat the proof of the Erdős–Selfridge theorem. Assume we are at the stage of the play where First Player already occupied $x_1, x_2, \ldots, x_i$, and Second Player occupied $y_1, y_2, \ldots, y_i$. The question is how to choose First Player's next point x_{i+1}. Those winning sets which contain at least one y_j $(j \leqslant i)$ are 'useless' for First Player. We call them 'dead sets'. The winning sets which are not 'dead' (yet) are called 'survivors'. The survivors have a chance of being completely occupied by First Player ('weak win'). What is the total 'chance' of the position? We evaluate the given position by the following 'chance function': $C_i = \sum_{s \in S_i} 2^{n-u_s}$ where u_s is the number of unoccupied points of the survivor A_s ($s \in S_i$ ='index set of the survivors', and index i indicates that we are at the stage of choosing the $(i+1)$st point x_{i+1} of First Player. Note that the 'chance function' can be much greater than 1 (it is not a *probability*), but it is always nonnegative.

A natural choice for x_{i+1} is to maximize the 'chance' C_{i+1} at the next stage. Let x_{i+1} and y_{i+1} denote the next moves of the two players. What is their effect on C_{i+1}? Well, first x_{i+1} doubles the chances for each survivor $A_s \ni x_{i+1}$, that is, we have to add the sum $\sum_{s \in S_i:\ x_{i+1} \in A_s} 2^{n-u_s}$ to C_i.

On the other hand, y_{i+1} 'kills' all the survivors $A_s \ni y_{i+1}$, which means we have to subtract the sum

$$\sum_{s \in S_i:\ y_{i+1} \in A_s} 2^{n-u_s}$$

from C_i.

Warning: we have to make a correction to those survivors A_s which contain both x_{i+1} and y_{i+1}. These survivors A_s were 'doubled' first and killed second. So what we have subtract from C_i is not

$$\sum_{s \in S_i:\ \{x_{i+1}, y_{i+1}\} \subset A_s} 2^{n-u_s}$$

but the twice-as-large

$$\sum_{s \in S_i:\ \{x_{i+1}, y_{i+1}\} \subset A_s} 2^{n-u_s+1}.$$

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It follows that

$$C_{i+1} = C_i + \sum_{s\in S_i:\ x_{i+1}\in A_s} 2^{n-u_s} - \sum_{s\in S_i:\ y_{i+1}\in A_s} 2^{n-u_s} - \sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s}. \tag{4.1}$$

Now the natural choice for x_{i+1} is the unoccupied z for which $\sum_{s\in S_i:\ z\in A_s} 2^{n-u_s}$ attains its maximum. Then clearly

$$C_{i+1} \geqslant C_i - \sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s}.$$

We trivially have

$$\sum_{s\in S_i:\ \{x_{i+1},y_{i+1}\}\subset A_s} 2^{n-u_s} \leqslant \Delta_2 2^{n-2}.$$

Indeed, there are at most Δ_2 winning sets A_s containing the given two points $\{x_{i+1}, y_{i+1}\}$, and $2^{n-u_s} \leqslant 2^{n-2}$, since x_{i+1} and y_{i+1} were definitely unoccupied points at the previous stage.

Therefore,

$$C_{i+1} \geqslant C_i - \Delta_2 2^{n-2}.$$

What happens at the end? Let ℓ denote the number of stages, *i.e.*, the ℓth stage is the last one. Clearly $\ell = |V|/2$. Inequality $C_\ell = C_{\text{last}} > 0$ means that Second Player could not 'kill' (block) all the winning sets. Indeed, at the last stage all points are occupied, so $C_\ell = C_{\text{last}} > 0$ means that First Player was able to completely occupy a winning set, meaning a weak win.

So all what we have to check is that $C_\ell = C_{\text{last}} > 0$. But this is trivial. Indeed, $C_{\text{start}} = C_0 = |\mathscr{F}|$, so we have

$$C_{\text{last}} \geqslant |\mathscr{F}| - \frac{|V|}{2}\Delta_2 2^{n-2}.$$

It follows that $C_{\text{last}} > 0$ if $|\mathscr{F}| > 2^{n-3}|V|\Delta_2$, which implies that at the end of the play First Player was able to completely occupy a winning set.

Finally note that if $\mathscr{F}$ is almost disjoint, then fixing any two points of the board V, there is at most one winning set containing both of them. So $\Delta_2 = \Delta_2(\mathscr{F}) = 1$. □

Next I apply the two 'first moment criteria' (the Erdős–Selfridge theorem and its 'weak win counterpart') to the three basic 'Ramsey-type games': n^d Tic-Tac-Toe, the clique game, and the van der Waerden game. One obtains the following quantitative results.

(1) n^d Tic-Tac-Toe. If $d < c_1 \cdot n/\log n$ then either player can block every winning line; on the other hand, if $d > (\log 2)n^2/2$ then First Player can occupy a whole winning line (weak win).

(2) (K_n, K_q) clique game. If $q > 2\log_2 n$ then either player can block every copy of K_q; on the other hand, if $q < c_2\sqrt{\log n}$ then First Player can occupy a whole K_q (weak win).

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(3) (N, n) van der Waerden game ('arithmetic progression game'). If $n > 2\log_2 N$ then either player can block every n-term AP; on the other hand, if $n = (1 - o(1))\log_2 N$ then First Player can completely occupy an n-term AP (weak win).

Observe that the 'first moment criteria' are not powerful enough to determine the true order of magnitude in either case. (To get the true order one needs 'higher moments'.)

First I prove (1), then (3), and finally (2).

Let us apply Theorem F to the n^d game. The corresponding hypergraph is almost disjoint, so First Player has a weak win in the n^d game if

$$\frac{(n+2)^d - n^d}{2} > 2^{n-3} n^d,$$

which is equivalent to

$$\left(1 + \frac{2}{n}\right)^d > 2^{n-2} + 1. \tag{4.2}$$

Inequality (4.2) holds for the 3^3, 4^4, 5^7, 6^{10}, 7^{14}, 8^{19}, 9^{25}, 10^{31}, … games, and guarantees a weak win. (Exhaustive search upgrades the games 3^3 and 4^3 to a strong win.)

The previous list is complemented by the following list of small blocking draw games: 4^2, 8^3, 11^4, 14^5, 18^6, 20^7, and so on. They are all pairing strategy draws with the exception of 4^2. (Potential function technique begins to 'beat' a pairing strategy only in higher dimensions, like $d \geqslant 16$.) There is a *big* gap between the two lists.

Note that inequality (4.2) holds for $n < \sqrt{2d/\log 2}$, or equivalently, if $d > (\log 2) \cdot n^2/2$. In the other direction, the Erdős–Selfridge theorem guarantees a blocking draw if $N > c_0 d \log d$, which is equivalent to $d < c_1 n/\log n$ (this is pretty weak: even a pairing strategy can beat it). Which one is closer to the truth: the 'quadratic' weak win result or the (nearly) 'linear' strong draw result (via E-S)? Well, the true order turns out to be 'quadratic'. I can prove, by roughly 'squaring' the Erdős–Selfridge bound, that if $d < c_2 n^2/\log n$ then the n^d game is a blocking draw.

This proves that, if n is a fixed large integer, then the game-theoretic breaking point between a weak win and a blocking draw in the n^d-game occurs when (the dimension) d is roughly around n^2 (there is an 'error factor' of $\log n$). Note that in this range the point/line ratio tends to zero, so a pairing strategy doesn't have a chance.

The second weak win criterion (Theorem F) gives a 'close-to-optimal' bound for a weak win in the n^d game (dimension d is 'around' *quadratic* in terms of n), but the right order of magnitude remains an unsolved problem. I don't know how to get rid of a factor of $\log n$.

A more impressive example is the 'weak van der Waerden game': the board is the interval $[N] = \{1, 2, \ldots, N\}$, and the players, Maker and Breaker, alternately take previously unselected integers from $[N]$. Maker wins if he can occupy a whole n-term arithmetic progression; otherwise Breaker wins.

An elementary calculation shows that the total number of arithmetic progressions of length n in $[N]$ is $\geqslant N(\lfloor \frac{N}{n-1} \rfloor - 1)/2$. The corresponding hypergraph is not almost disjoint, but the pair-degree is very small (in terms of N): the pair-degree is $\leqslant \binom{n}{2}$. Indeed, fixing

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two integers $1 \leqslant x < y \leqslant N$ and two indices $0 \leqslant i < j \leqslant n-1$, the equations $x = a + id$ and $y = a + jd$ uniquely determine the arithmetic progression $a, a+d, \ldots, a+(n-1)d$ of length n. Theorem F applies if $\lfloor \frac{N}{n-1} \rfloor > 2^{n-2}\binom{n}{2} + 1$. For $n = 4, 5, 6$ this gives the weak win thresholds 76, 325, 1206. Asymptotically, for large n, we get the weak win threshold $\approx 2^{n-3}n^3$. It follows that Maker wins if $n \leqslant (1 + o(1))\log_2 N$.

This result can be complemented by the following (see [5]): Breaker wins the weak van der Waerden game if $n \geqslant (1 + o(1))\log_2 N$. (This beats Erdős–Selfridge by a factor of two – see (3) above.) The game-theoretic breaking point of the weak van der Waerden game is therefore $(1 + o(1))\log_2 N$.

The van der Waerden game is a 'victory' for Theorem F, but how about the weak clique game (K_n, K_q)? How close can one get to the truth by applying Theorem F to the weak clique game (K_n, K_q)? Well, Theorem F implies a weak win if

$$\binom{n}{q} > 2^{\binom{q}{2}-3}\binom{n}{2}\Delta_2.$$

For this particular family of winning sets the pair-degree Δ_2 is at most $\binom{n}{q-3}$. Indeed, two distinct edges determine at least 3 different vertices. Therefore, it suffices to check the inequality

$$\binom{n}{q} > 2^{\binom{q}{2}-3}\binom{n}{2}\binom{n}{q-3},$$

which yields $n \geqslant 2^{q^2/2}$, or in terms of n, $q < \sqrt{2\log_2 n}$.

Well, this is a very *disappointing* result. It is asymptotically much weaker than the 'Ramsey-type bound' $q = \frac{1}{2}\log_2 n$ obtained by combining Theorem E and the Erdős–Szekeres upper bound. The reason why Theorem F gave such a poor result in the weak clique game is that the 'clique-hypergraph' is very far from being almost disjoint. Indeed, two different cliques can share a lot of edges; in fact two different K_qs can have as many as $\binom{q-1}{2}$ common edges.

On the other hand, the reason why Theorem F gave such a good result for the van der Waerden game was that the corresponding hypergraph is (1) 'nearly' almost disjoint; and (2) 'nearly' degree-regular.

My techniques work best for degree-regular hypergraphs. The n^d-hypergraph is highly degree-irregular: this is why I don't know the right order of magnitude of the game-theoretic breaking point.

Next I study the clique game for k-graphs where $k \geqslant 3$, which corresponds to *higher Ramsey numbers*. This means the straightforward generalization of complete graphs, where the board is a complete k-uniform hypergraph instead of K_n, and of course the players claim k-sets instead of edges. Even if the case $k = 2$ gave disappointing result, for $k \geqslant 4$ we will have a pleasant surprise: for the first time we will be able to separate a (higher) *Ramsey number* from the corresponding *weak game number*.

Clique games on k-graphs where $k \geqslant 3$. For every natural number n write $[n] = \{1, \ldots, n\}$. If S is a set, then let $\binom{S}{k}$ denote the family of all k-element subsets of S. Then $\binom{[n]}{2}$ can be interpreted as a complete graph with n vertices, *i.e.*, $\binom{[n]}{2} = K_n$.

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Let $2 \leqslant k \leqslant q < n$. The board of the game is $\binom{[n]}{k}$, and the family of winning sets consists of all possible $\binom{S}{k}$ where $S \in \binom{[n]}{q}$. The family of winning sets is a $\binom{q}{k}$-uniform hypergraph of size $\binom{n}{q}$. We consider the weak (*i.e.*, Maker–Breaker) game on this particular hypergraph, and call it the *weak* $(\binom{n}{k}, \binom{q}{k})$ *clique game.*

The general form of Ramsey's theorem states that for every $k \geqslant 2$ and for every $q \geqslant k$, there is a least finite threshold number $R_k(q)$ such that the family of winning sets of the $(\binom{n}{k}, \binom{q}{k})$ clique game has chromatic number at least 3 if $n \geqslant R_k(q)$. If $n \geqslant R_k(q)$ then by Theorem C First Player has a strong win in the $(\binom{n}{k}, \binom{q}{k})$ clique game (but we don't know what the winning strategy looks like), and if $n \geqslant 2R_k(q)$ then by Theorem E First Player has an explicit ('copycat') weak win in the $(\binom{n}{k}, \binom{q}{k})$ clique game.

What do we know about the size of the higher Ramsey number $R_k(q)$? Well, it is huge! Indeed, let $\text{tower}_k(x)$ denote the k-fold iteration of the exponential function: $\text{tower}_1(x) = 2^x$ and for $k \geqslant 2$, $\text{tower}_k(x) = 2^{\text{tower}_{k-1}(x)}$. So $\text{tower}_2(x) = 2^{2^x}$, $\text{tower}_3(x) = 2^{2^{2^x}}$, and so on. We call the index k in $\text{tower}_k(x)$ the *height* (it is in fact the height of the exponential expression).

For graphs (*i.e.*, $k = 2$) it is well known that

$$2^{q/2} < R_2(q) < 4^q.$$

For k-graphs with $k \geqslant 3$,

$$2^{q^2/6} < R_3(q) < 2^{2^{4q}}, \tag{4.3}$$

$$2^{2^{q^2/6}} < R_4(q) < 2^{2^{2^{4q}}}, \tag{4.4}$$

and in general,

$$\text{tower}_{k-2}(q^2/6) < R_k(q) < \text{tower}_{k-1}(4q). \tag{4.5}$$

The last two bounds are due to Erdős, Hajnal and Rado (see, *e.g.*, [15]).

First let $k = 3$. If $q = c_1 \cdot \log\log n$, then by (4.3), First Player has a strong win in the $(\binom{n}{3}, \binom{q}{3})$ clique game, but the winning strategy is not known. On the other hand, by Theorem E under the same condition $q = c_1 \cdot \log\log n$ Maker has an *explicit* weak win strategy by playing copycat. Similarly, if $q = c_2 \cdot \log\log\log n$ then by (4.4) First Player has a strong win in the $(\binom{n}{4}, \binom{q}{4})$ clique game (the strategy is unknown), and under the same condition Maker has an *explicit* weak win strategy by playing copycat, and so on.

What happens if one replaces Theorem E with Theorem F? The Erdős–Selfridge-type weak win criterion (Theorem F) applies to the weak $(\binom{n}{k}, \binom{q}{k})$ clique game when

$$\binom{n}{q} > 2^{\binom{q}{k}-3}\binom{n}{k}\Delta_2.$$

For this particular family of winning sets the pair-degree Δ_2 satisfies the obvious inequality $\Delta_2 \leqslant \binom{n}{q-k-1}$. Indeed, two distinct k-sets cover at least $k+1$ points.

This leads to the inequality

$$\binom{n}{q} > 2^{\binom{q}{k}-3}\binom{n}{k}\binom{n}{q-k-1},$$

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which means $n \geqslant 2^{q^k/k!}$, or in terms of n, $q \leqslant (k! \log_2 n)^{1/k}$. Therefore, if

$$q = c_k (\log n)^{\frac{1}{k}}$$

then Maker has a weak win in the $(\binom{n}{k}, \binom{q}{k})$ clique game. This means that for k-graphs with $k \geqslant 4$ the weak Ramsey criterion (Theorem E) definitely *fails to give the true order of magnitude of the breaking point for a weak win* in the clique game. Indeed, by (4.5) the Ramsey theory threshold $R_k(q)$ is greater than the tower function $\text{tower}_{k-2}(q^2/6)$ of height $k-2$ (increasing with k). On the other hand, Maker can force a weak win around $n = 2^{q^k/k!}$, which has *constant height* independent of k. It is trivial to see that $\text{tower}_{k-2}(q^2/6)$ is asymptotically *much* larger than $n = 2^{q^k/k!}$ if $k \geqslant 4$; in fact they have completely different asymptotic behaviours.

By using more sophisticated arguments, I can find the *exact* game-theoretic breaking points for 'almost all' clique games ($k \geqslant 2$).

5. Beyond Ramsey theory

In Section 4 we learned that the clique game for k-graphs, where $k \geqslant 4$, is an example where the potential function technique (Theorem F) 'beats' Ramsey theory (Theorem E) in a dramatic fashion. Indeed, the weak game number is *much* smaller than the corresponding Ramsey number.

A different kind of example is a game where Ramsey theory (Theorem E) fails to give anything, *i.e.*, there is *no* Ramsey phenomenon whatsoever, but Theorem F still gives an interesting result. This kind of game is the unrestricted n-in-a-row game with *arbitrary slopes*: see subsection (2) below.

But first I recall the well-known unrestricted 5-in-a-row game, and in general, the unrestricted n-in-a-row game.

(1) Unrestricted means that it is played on an *infinite* chessboard (infinite in every direction). In the unrestricted 5-in-a-row game the players alternately occupy little squares of an infinite chessboard, First Player marks his squares by X, and Second Player marks his squares by O. That player wins who first gets 5 consecutive marks of his own in a row horizontally, or vertically, or diagonally (of slope 1 or -1). The unrestricted n-in-a-row differs in only one aspect: the winning size is n instead of 5.

Unrestricted n-in-a-row on the plane is a *semi-infinite* game (in fact, a semi-infinite strong positional game): the board is infinite but the winning sets are all finite. Since the board is infinite, we have to define the *length* of the game. We assume that the players take turns until either of them wins in a finite number of moves, or until they have taken their nth turns for every natural number n. In other words, the length of a play is at most ω, where ω denotes (as usual) the first countable (infinite) ordinal number. Semi-infinite strong (and weak) positional games are all *determined*, *i.e.*, Theorem A holds for semi-infinite games. Indeed, basically the same proof works: if player P has no winning strategy, then the opponent, player Q, can always make a next move so that player P still has no winning strategy, and this is exactly how player Q can force a draw. The point is that the winning sets are finite, so if a player wins, he wins in a finite number of moves – this is why player P cannot win and player Q forces a draw.

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The unrestricted 4-in-a-row game is an easy First Player's win. The unrestricted 5-in-a-row game is conjectured to be a First Player's win, too, but there is no proof. For 'reasonable board size', say 20 by 20, there is a First Player's winning strategy (computer-assisted proof; huge case study), but no one knows how to extend it to the whole plane ('extra set paradox'). In the other direction, the unrestricted 8-in-a-row is known to be a draw game (this is the best that we know). Here I include a simple pairing strategy proof of the slightly weaker fact that the unrestricted 9-in-a-row game is a draw: see p. 677 in [11]. Indeed, cover the infinite chessboard with disjoint copies of the following 8 by 8 matrix:

$$n = 8: \quad \begin{bmatrix} \backslash & \backslash & - & - & | & | & / & / \\ - & - & \backslash & \backslash & / & / & | & | \\ - & - & / & / & \backslash & \backslash & | & | \\ / & / & | & | & - & - & \backslash & \backslash \\ \backslash & \backslash & | & | & - & - & / & / \\ | & | & \backslash & \backslash & / & / & - & - \\ | & | & / & / & \backslash & \backslash & - & - \\ / & / & - & - & | & | & \backslash & \backslash \end{bmatrix}.$$

What this 8 by 8 matrix represents is a direction-marking of the $4 \cdot 8 = 32$ 'torus-lines' of the 8×8 torus. The direction marks $-$, $|$, $\backslash$, and $/$ mean (respectively) 'horizontal', 'vertical', 'diagonal of slope -1', and 'diagonal of slope 1'. Each one of the 32 torus-lines contains 2 marks of its own direction. The periodic extension of the 8 by 8 matrix over the whole plane gives a pairing strategy draw for the unrestricted 9-in-a-row game. Either player responds to the opponent's last move by taking the *nearest* similarly marked square in the direction indicated by the mark in the opponent's last move square.

(2) Unrestricted *n*-in-a-row with arbitrary slopes. Consider the following generalization of the unrestricted n-in-a-row: the players occupy lattice points instead of 'little squares', and n-in-a-row means consecutive lattice points on *any* straight line. The point is that *every* rational slope is allowed, not just the four Tic-Tac-Toe directions.

We challenge the reader to solve the following exercise.

Exercise. Show that there is no 'Ramsey phenomenon' here: it is possible to 2-colour the set of integer lattice points in the plane in such a way that every set of 100 consecutive lattice points on a straight line contains both colours.

Of course '100' just means a 'large constant'. The correct value is actually 4 (this elegant result is due to A. Dumitrescu and R. Radoicic), but it is much easier to prove 100.

Even if the 'Ramsey phenomenon' fails here, I can still prove a 'game theorem': Maker can have n-in-a-row for *every* finite n. The proof is a straightforward application of Theorem F. Consider an $m \times m$ square lattice, in fact assume that the lower-left corner is the origin ($m = m(n)$ will be specified later). The set $(k + aj, l + j)$ $j = 0, 1, \ldots, n-1$ gives n *consecutive* lattice points on a line *inside* the $m \times m$ square if $0 \leqslant k \leqslant m/2$, $0 \leqslant l \leqslant m/2$, and $1 \leqslant a \leqslant \frac{m}{2n}$; we call it a (k, l, a)-set. Let $\mathscr{F}$ denote the family of all (k, l, a)-sets. $\mathscr{F}$ is an n-uniform hypergraph. Since $0 \leqslant k \leqslant m/2$, $0 \leqslant l \leqslant m/2$, and $1 \leqslant a \leqslant \frac{m}{2n}$, we have

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$|\mathscr{F}| \geqslant m^3/8n$. Since the 'board' $|V| = m^2$, and the pair-degree of $\mathscr{F}$ is at most n, Theorem F applies if

$$\frac{m^3}{8n} > 2^{n-3} \cdot n \cdot m^2. \tag{5.1}$$

We need a simple remark here: of course Maker cannot force Breaker to stay inside the $m \times m$ square, but if Breaker moves outside of the $m \times m$ square, that clearly just helps Maker to occupy an n-in-a-row faster.

By choosing $m = m(n) = n^2 \cdot 2^n + 1$, inequality (5.1) holds, and implies a Maker's weak win.

Theorem 1. *Playing 'unrestricted n-in-a-row with arbitrary slopes', Maker can occupy an n-in-a-row in less than* $n^4 \cdot 4^n$ *moves* ($n \geqslant 1$ *is arbitrary*). □

Another example of the same type is the *weak all-subset game.*

(3) Weak all-subset game. In the clique game the players take edges, that is, *pairs* of the vertex set. Here the players take *arbitrary subsets* of a 'ground set'. More precisely, there are two players, Maker and Breaker, who alternately take previously unselected subsets of the n-element set $[n] = \{1, 2, \ldots, n\}$. First Player (Maker) wins if he has all 2^{100} subsets of some 100-element subset of $[n]$; otherwise Second Player (Breaker) wins.

Can Maker win if n is sufficiently large? Well, one thing is very clear: Maker's opening move *has to be* the 'empty set'.

Observe that there is no 'Ramsey phenomenon' here. Indeed, colour the subsets depending on the parity of the size: 'even' means red and 'odd' means blue. This particular 2-colouring kills any chance of an 'all-subset Ramsey theorem'. But there is a 'game theorem': We show that Maker (First Player) has a winning strategy if n is sufficiently large.

How do we prove the 'game theorem'? Well, it seems a natural idea to use Theorem F. Unfortunately direct application does *not* work. Indeed, $|\mathscr{F}| = \binom{n}{100}$ is actually *less* than $|V| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{100}$, so Theorem F cannot give anything.

To get around this difficulty I use a technical trick: I prove the following statement.

Stronger Statement. *There is a* 101*-element subset S of* $[n]$ *such that Maker* (*First Player*) *can have all at-most-*100*-element subsets of S, assuming n is sufficiently large.*

In fact, I can prove it for arbitrary k (not just for 100), and then the Stronger Statement goes as follows: *There is a* $k+1$*-element subset S of* $[n]$ *such that Maker* (*First Player*) *can have all at-most-k-element subsets of S, assuming n is sufficiently large depending on k.*

The reason to switch to the Stronger Statement is that now $|\mathscr{F}| = \binom{n}{101}$ is roughly n-times greater than $|V| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{100}$. This makes it possible to *repeat the proof* of Theorem F. Note that direct application cannot work for the simple reason that the corresponding hypergraph is 'very degree-irregular', and it is 'bad' to work with the same pair-degree during the whole course of the play.

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Proof. The proof of the Stronger Statement goes as follows. As usual, in each move Maker makes the 'best choice': he chooses a subset of 'maximum value', using the usual power-of-two scoring system, and if there are two *comparable* subsets, say U and W with $U \subset W$, having the same 'absolute maximum', then Maker always chooses the *smaller set*, *i.e.*, U instead of W. (Note that if $U \subset W$, then the 'value' of U is *always* at least as large as the 'value' of W.) Maker's opening move is of course the 'empty set'. Let $U^{(i)}$ be an arbitrary move of Maker: $U^{(i)}$ is an i-element subset $(0 \leqslant i \leqslant k)$. Let W be Breaker's next move right after $U^{(i)}$. We know that W cannot be a subset of $U^{(i)}$ – indeed, otherwise Maker would prefer W instead of $U^{(i)}$ – so $|U^{(i)} \cup W| \geqslant i+1$. There are at most $\binom{n}{k+1-(i+1)}$ $(k+1)$-element subsets S containing *both* $U^{(i)}$ and W: this is the 'actual pair-degree'. Note that for $i = 0, 1, 2, \ldots, k$ Maker chooses an i-element set at most $\binom{n}{i}$ times, so by repeating the proof of Theorem F one obtains

$$C_{\text{last}} \geqslant C_{\text{start}} - \sum_{i=0}^{k} \binom{n}{i} \binom{n}{k+1-(i+1)} 2^{2^{k+1}-3}, \tag{5.2}$$

where C_j is the 'chance function' at the jth stage of the play. Clearly $C_{\text{start}} = C_0 = \binom{n}{k+1}$, so (5.2) is equivalent to

$$C_{\text{last}} \geqslant \binom{n}{k+1} - \sum_{i=0}^{k} \binom{n}{i} \binom{n}{k+1-(i+1)} 2^{2^{k+1}-3}. \tag{5.3}$$

A trivial calculation shows that

$$\binom{n}{k+1} > \sum_{i=0}^{k} \binom{n}{i} \binom{n}{k+1-(i+1)} 2^{2^{k+1}-3} \quad \text{if} \quad n > (k+1)!2^{2^{k+1}}. \tag{5.4}$$

By (5.3) and (5.4),

$$C_{\text{last}} > 0 \quad \text{if} \quad n > (k+1)!2^{2^{k+1}}. \tag{5.5}$$

If the 'chance function' is not zero at the end of a play, then Breaker could *not* block every 'winning set', so there must exist a 'winning set' completely occupied by Maker. In other words, there must exist a $(k+1)$-element subset S of $[n]$ such that Maker occupied all at-most-k-element subsets of S. This proves the Stronger Statement. □

The last step is trivial: throwing out an arbitrary integer from S we obtain a k-element subset S' of $[n]$ such that Maker occupied all 2^k subsets of S'. This solves the 'all-subset game'.

Note that inequality

$$n > (k+1)!2^{2^{k+1}}$$

holds (see (5.5)) if $k = (1 + o(1))\log_2\log_2 n$.

In the other direction, we show that the choice $k = (1 + o(1))\log_2\log_2 n$ is impossible. Indeed, the Erdős–Selfridge criterion applies if

$$\binom{n}{k} < 2^{2^k - 1}.$$

This holds for $k = (1 + o(1))\log_2\log_2 n$.

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Theorem 2. *The game-theoretic breaking point for the weak all-subset game is* $(1+o(1))\log_2\log_2 n$, *that is, the iterated binary logarithm of n.* □

Note that the power set, the goal of the all-subset game, is a 'very rapidly changing configuration': $2^{n+1} = 2 \times 2^n$, that is, changing the value of n by one doubles the size of the power set. This is why such a crude approach can still give a satisfying solution.

The Ramsey criteria (Theorems C and E) prove that the 'game world' is *at least* as large as the 'Ramsey world'. The 'unrestricted n-in-a-row with arbitrary slopes' and the all-subset game were the first examples to demonstrate that the 'game world' is actually bigger than the 'Ramsey world'. Another example is the following.

(4) Goal: 2-coloured AP. Let $N \geqslant W(n)$ where $W(n)$ is the van der Waerden threshold number for n-term arithmetic progressions (APs), and consider a play in the twice-as-long interval $[2N] = \{1, 2, \ldots, 2N\}$. If Second Player follows the 'copycat strategy': for First Player's x he replies by $2N + 1 - x$, then of course at the end of the play Second Player will occupy an n-term AP, and at the same time First Player will occupy an n-term AP. In other words, if Second Player is Mr. Red, and First Player is Mr. Blue, then Mr. Red can force the appearance of *both* a monochromatic *red* and a monochromatic *blue* n-term AP.

How about if Mr. Red's goal is an *arbitrary* 2-*coloured* n-term AP? Of course Ramsey theory cannot help here. But again the potential function technique works very well!

The precise definition of the '2-coloured goal game' goes as follows. Fix an arbitrary Red–Blue sequence of length 100: (say) $R, B, R, R, R, B, B, R, B, \ldots$. We call it the 'goal sequence'. Mr. Red and Mr. Blue alternately take new integers from the interval $[N] = \{1, 2, \ldots, N\}$, and colour them with their own colours. Mr. Red wins if at the end of the play there is an arithmetic progression of length 100 which is coloured exactly like the given Red–Blue 'goal sequence'; otherwise Mr. Blue wins. I call this the *goal sequence game*. Is it true that Mr. Red has a winning strategy in the goal sequence game if N is sufficiently large?

The answer is 'yes'. As in the case of the all-subset game, one cannot directly apply Theorem F. Instead one has to repeat the whole proof. Of course 100 can be replaced by an arbitrary finite number.

Theorem 3. *Consider an arbitrary Red–Blue 'goal sequence' S of length n. Assume that Mr. Red and Mr. Blue play the goal sequence game on the interval* $[N] = \{1, 2, \ldots, N\}$, *where the 'goal sequence' is S. If* $N \geqslant n^3 \cdot 2^{n-2}$ *then at the end of the play Mr. Red can force the appearance of an arithmetic progression of length n which is coloured exactly the same way as the given Red–Blue 'goal sequence' S.*

Proof. For simplicity assume that Mr. Red is the first player. Assume that we are in the middle of a play: so far Mr. Red has coloured integers $x_1, x_2, \ldots, x_i$ ('red points') and Mr. Blue has coloured integers $y_1, y_2, \ldots, y_i$ ('blue points'). This defines a partial 2-colouring of $[N]$.

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Let $\mathscr{F} = \mathscr{F}(N,n)$ be the family of all n-term APs in $[N]$; clearly $|\mathscr{F}| > N^2/4(n-1)$. The partial 2-colouring of $[N]$ defines a partial 2-colouring of every n-term AP $A \in \mathscr{F}$. We introduce the following natural adaptation of the power-of-two scoring system: if the partial 2-colouring of an n-term AP $A \in \mathscr{F}$ *contradicts* the given 'goal sequence' S, then A has *zero value*, and we call it a 'dead set'. The rest of the elements of $\mathscr{F}$ are called 'survivors'; the partial 2-colouring of a 'survivor' $A \in \mathscr{F}$ has to be consistent with the given 'goal sequence' S. The *value* of a 'survivor' $A \in \mathscr{F}$ is 2^j if j points of A were coloured in the partial 2-colouring.

We define the 'chance function' in the standard way:

$$C_i = \sum_{A \in \mathscr{F}} \text{value}_i(A).$$

As usual, we study how the consecutive moves x_{i+1} and y_{i+1} affect the 'chance function'.

For an arbitrary n-term AP $A \in \mathscr{F}$, 2-colour the elements of A copying the given 'goal sequence' S. Then let $\text{red}_S(A)$ denote the set of red elements of A, and let $\text{blue}_S(A)$ denote the set of blue elements of A.

We have

$$\begin{aligned} C_{i+1} = C_i &+ \sum_{\substack{A \in \mathscr{F}: \\ x_{i+1} \in \text{red}_S(A)}} \text{value}_i(A) - \sum_{\substack{A \in \mathscr{F}: \\ x_{i+1} \in \text{blue}_S(A)}} \text{value}_i(A) \\ &+ \sum_{\substack{A \in \mathscr{F}: \\ y_{i+1} \in \text{blue}_S(A)}} \text{value}_i(A) - \sum_{\substack{A \in \mathscr{F}: \\ y_{i+1} \in \text{red}_S(A)}} \text{value}_i(A) \\ &- \sum_{\substack{A \in \mathscr{F}: \\ \{x_{i+1}, y_{i+1}\} \in \text{red}_S(A)}} \text{value}_i(A) - \sum_{\substack{A \in \mathscr{F}: \\ \{x_{i+1}, y_{i+1}\} \in \text{blue}_S(A)}} \text{value}_i(A) \\ &+ \sum_{\substack{A \in \mathscr{F}: \\ x_{i+1} \in \text{red}_S(A) \\ y_{i+1} \in \text{blue}_S(A)}} \text{value}_i(A) + \sum_{\substack{A \in \mathscr{F}: \\ x_{i+1} \in \text{blue}_S(A) \\ y_{i+1} \in \text{red}_S(A)}} \text{value}_i(A). \end{aligned}$$

How should Mr. Red choose his next move x_{i+1}? Well, Mr. Red's optimal move is to compute the numerical value of the function

$$f(u) = \sum_{\substack{A \in \mathscr{F}: \\ u \in \text{red}_S(A)}} \text{value}_i(A) - \sum_{\substack{A \in \mathscr{F}: \\ u \in \text{blue}_S(A)}} \text{value}_i(A)$$

for every unoccupied integer $u \in [N]$, and to choose that $u = x_{i+1}$ for which the *maximum* is attained. This choice implies

$$C_{i+1} \geqslant C_i - \binom{n}{2} 2^{n-2}.$$

Indeed, there are at most $\binom{n}{2}$ n-term arithmetic progressions containing both x_{i+1} and y_{i+1}. Therefore,

$$C_{\text{last}} \geqslant C_{\text{start}} - \frac{N}{2} \binom{n}{2} 2^{n-2}.$$

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Since $C_{\text{start}} = C_0 = |\mathscr{F}| \geqslant N^2/4(n-1)$, it suffices to guarantee the inequality

$$\frac{N^2}{4(n-1)} > \frac{N}{2}\binom{n}{2}2^{n-2},$$

which is equivalent to $N > n(n-1)^2 \cdot 2^{n-2}$. Under this condition $C_{\text{last}} > 0$, that is, at the end of the play there must exist an n-term AP which is 2-coloured by Mr. Red and Mr. Blue exactly the same way as the 'goal sequence' S. Theorem 3 follows. □

We leave the 'clique version' of Theorem 3 to the reader in the form of an exercise.

Exercise. Fix an arbitrary 2-coloured clique $K_{100}(\text{red};\text{blue})$ of 100 vertices ('2-coloured goal graph'). Consider the following game: Mr. Red and Mr. Blue alternately take edges of a complete graph K_n on n vertices, and colour them with their own colours. Mr. Red wins if at the end of the play there is an isomorphic copy of $K_{100}(\text{red};\text{blue})$; otherwise Mr. Blue wins. Show that Mr. Red has a winning strategy if n is sufficiently large.

6. Bending the rules: (2 : 2) and biased versions

(1 : 1) versus (2 : 2): copycat breakdown. Every game that we have been studying so far was played in the usual (1 : 1) way: each player occupied one new point per move. A minor change is to consider the (2 : 2) play instead of the (1 : 1) play: the players still alternate, but either player occupies *two* new points per move. The game remains *fair*, but there is a surprising difference: copycat pairing does *not* work any longer. I illustrate this 'breakdown' on two examples. I begin with the ordinary (1 : 1) versions.

(1) Consider the following arithmetic progression game. The board is $\mathbb{Z}$, the set of all integers. The two players alternate, and each takes one new integer per move. First Player wins if at some point, at an even stage of the play (*i.e.*, right after Second Player's move), his longest arithmetic progression (AP) is longer than Second Player's longest AP by at least *one*. If this never happens Second Player wins.

A simple copycat pairing yields that Second Player wins. Indeed, Second Player chooses a 'half-integer' (say) $1/2$ as the 'centre', and reflects his opponent's moves: if First Player's last move was m, then Second Player replies by $1-m$. □

The second example is a clique version of game (1).

(2) The board is an infinite complete graph K_∞. The two players alternately take new edges, one edge per move. First Player wins if at some stage of the play his largest clique is larger than Second Player's largest clique by at least *two* (vertices). If this never happens Second Player wins.

Again Second Player wins by a copycat pairing. Indeed, for simplicity assume that the vertex set is $\{1,2,3,\dots\} \cup \{-1,-2,-3,\dots\}$, then Second Player makes the following natural pairing: $i \leftrightarrow (-i)$ $(i = 1,2,3,\dots)$. Every edge $\{i,j\}$ has its 'mirror image' $\{-i,-j\}$; the 'mirror image' edge is a different edge, except for the 'fixpoint edges' $\{i,-i\}$. Second

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Player's strategy is to take the 'mirror image' of First Player's last move; if First Player's last move was a 'fixpoint edge' then Second Player replies arbitrarily.

Let K_q denote First Player's largest clique at an arbitrary even stage (*i.e.*, right after Second Player's move). It is easy to see that K_q cannot contain two 'fixpoint edges' (we recommend the reader to draw a picture).

Removing an endpoint of a possible 'fixpoint edge' from K_q leads to a 'fixpoint-edge-free' subclique K_{q-1}. The 'mirror image' of K_{q-1} must belong to Second Player, so his largest clique has at least $q-1$ vertices. □

How about the (2 : 2) version of games (1) and (2)? Copycat pairing clearly breaks down, which leads to the following open problem.

Open Problem 8.

(a) *Consider the* (2 : 2) *version of the infinite arithmetic progression game* (1). *Can Second Player guarantee that, during the* whole *course of the play, his longest AP and First Player's longest AP differ in length by at most* 100*?*

(b) *Consider the* (2 : 2) *version of the infinite clique game* (2). *Can Second Player guarantee that, during the* whole *course of the play, his largest clique and First Player's largest clique differ in vertex size by at most* 100*?*

Of course, 100 simply means 'large constant'; any absolute constant would do. The point in Open Problem 8 is to control the opponent's 'lead' during the *whole course* of the play. How about if we want 'balance' at the *end* only? To understand the problem, I study the following finite (2 : 2) version of clique game (2).

(3) The board is a finite complete graph K_n ($n \geqslant 3$). The two players alternately take new edges, two edges per move. First Player wins if at the end of the play his largest clique is larger than Second Player's largest clique by at least *three* (vertices); otherwise Second Player wins.

By using the *strategy stealing* argument I show that Second Player has a winning strategy in the (2 : 2)-type game (3). This is an indirect proof: *assume* that First Player has a winning strategy Str, and I will derive a contradiction.

I begin with a simple observation. If one slightly changes the rules so that Second Player may 'pass' when he wishes to do so (*i.e.*, to skip a move, or just to take one edge), then of course strategy Str still guarantees for First Player a lead of $\geqslant 3$ at the end of the play.

Second Player can 'steal' and use Str as follows. Let $\{1, 2, 3, \ldots, n\}$ be the vertex set, and without loss of generality we can assume that First Player's opening move by Str is either

Case 1: $\{1, 2\}$ and $\{3, 4\}$; or
Case 2: $\{1, 2\}$ and $\{1, 3\}$.

In Case 1 Second Player deletes 1 and 3, and adds $1'$ and $3'$ to the vertex set. Second Player plays on the *virtual board* K_n', where the new vertex set is $\{1', 2, 3', 4, 5, \ldots, n\}$, by using strategy Str with the slight change that '1' means the new vertex $1'$ and '3' means $3'$. Second Player's opening move in K_n' is of course $\{1', 2\}$ and $\{3', 4\}$. Suppose that

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First Player follows his strategy Str, and Second Player keeps replying to the opponent's moves by Str (as we explained above). At the end of the play, let K_s denote Second Player's largest clique, and let K_f denote First Player's largest clique in K'_n. By hypothesis $s \geqslant f + 3$. This holds for the virtual game on K'_n. In the *real* game on K_n vertices 1 and 3 are added back and $1'$ and $3'$ are removed. It follows that at the end of this play in the real game, First Player's largest clique has at most $f + 2$ vertices, and Second Player's largest clique has at least $s - 2$ vertices. Since First Player uses Str, his clique in the real game is bigger than Second Player's largest clique by at least 3, implying $f + 2 \geqslant (s - 2) + 3$. This means $f + 1 \geqslant s$, which contradicts the inequality $s \geqslant f + 3$ obtained from the virtual game.

Note that Case 2 is even simpler, since Second Player has to remove only one vertex, namely 1, instead of two, and add $1'$ to the vertex set in the virtual game.

In both cases we obtain a contradiction. This proves that First Player cannot have a winning strategy in the (2 : 2) game (3), so it is Second Player who has a winning strategy. □

As usual with the strategy stealing argument, we don't know any explicit Second Player's winning strategy.

What happens for the finite (2 : 2) version of game (1)? I have to admit, I don't know. The proof for game (3) fails to work for APs, since removing an integer from an AP breaks it into two shorter pieces (the worst case is to have two 'halves').

After the (2 : 2) game, which is still fair, let's move to a real biased game, and consider (say) the (2 : 1)-type game where Second Player is the 'underdog', *i.e.*, Second Player takes only one per move. In general, if player P takes p points per move, and the opponent, player Q, takes q points per move from a given hypergraph, and (say) $p < q$, then we call player P the *underdog* and player Q the *topdog*.

(4) For simplicity assume that the players play a (2 : 1) game on an n-uniform hypergraph $\mathscr{F}$. Moreover, assume that $\mathscr{F}$ has at least 3^n pairwise disjoint winning sets. Then First Player, who takes 2 points per move, always wins the strong game: First Player can always occupy a whole winning set *first*.

Proof. The proof is rather simple. Again I use strategy stealing. There are two cases: either

Case 1: playing the (2 : 1) strong game on $\mathscr{F}$, Second Player has a drawing strategy; or
Case 2: playing the (2 : 1) strong game on $\mathscr{F}$, First Player has a winning strategy.

Of course in Case 2 we are done. In Case 1 let Str denote Second Player's drawing strategy. First Player is going to use Str; to explain how, consider a play (First Player marks X and Second Player marks O)

X X O X X O X X O X X O X X O X X O X X O X X O X X O X X O X X O

and decompose it into blocks as follows (the first block has 7 marks, the rest have 6 each):

X X O X X O X–X O X X O X–X O X X O X–X O X X O X–X O X X O X– ⋯

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First Player's first 4 Xs are arbitrary ('free'), his 5th X is a reply to the 2 Os of Second Player by using Str. The next 3 Xs of First Player are arbitrary ('free'), his 9th X is a reply to the last 2 Os of Second Player by using Str. The next 3 Xs of First Player are arbitrary ('free'), his 13th X is a reply to the last 2 Os of Second Player by using Str, and so on. This means the play is divided into blocks of length 6; the first block is exceptional: it has 7 marks. Each block has one X which is determined by Str (replying to the last two Os), 3 'free' Xs which are arbitrary, and 2 Os. By using the 'reply' Xs, First Player can prevent Second Player from occupying a winning set first.

On the other hand, by using his 3 'free' Xs against the opponent 2 Os (per block), First Player can definitely occupy a whole winning set. Indeed, $(3:2)$ is biased in favour of First Player, and there are at least 3^n pairwise disjoint n-element winning sets. Actually, First Player will occupy a whole n-element winning set in n phases. In phase 1, by using his free Xs, First Player marks 3^n disjoint winning sets, one X per set. Second Player can block at most $2/3$ of them (by putting an O), so at the end of phase 1 there are $\geqslant 3^{n-1}$ n-sets which are O-free and have one X each. Similarly, at the end of phase 2 there are $\geqslant 3^{n-2}$ n-sets which are O-free and have 2 Xs each. Repeating this argument, at the end of phase n there exists $\geqslant 3^{n-n} = 1$ n-set which is O-free and has n Xs, *i.e.*, this winning set is completely occupied by First Player. This shows that

(i) Second Player cannot occupy a winning set first (since First Player uses Str); and
(ii) First Player can occupy a whole winning set.

(i) and (ii) together give that First Player can occupy a whole winning set *first*, which contradicts the hypothesis of Case 1. This contradiction proves that Case 1 is impossible, *i.e.*, First Player can force a strong win. □

Again I don't know any explicit First Player's strong-win strategy in the $(2:1)$-type game (4).

This means that, playing the $(2:1)$ game, the underdog *always* loses the strong game, unless the hypergraph is very small. In other words, 'doing it first' is a hopeless task for the underdog. The relevant question is 'Can the underdog do it or not?', the 'reasonable' goal is a weak win, and the relevant game is the weak game.

Weak win for the underdog. If there is a 'density theorem', then of course the underdog can achieve a weak win by simply showing up. For example, in the $(2:1)$ version of the (N, n) van der Waerden game, at the end of a play the underdog has $N/3$ integers from the interval $1, 2, \ldots, N$. By the deep Szemerédi–Gowers theorem, *every* $N/3$-element subset of the interval $1, 2, \ldots, N$ contains an n-term AP if N is larger than a five-times iterated exponential function of n.

Unfortunately we don't know any *effective* density version of the Hales–Jewett theorem, and the density version of the Ramsey theorem is simply *not true*. What can we do then? Can large *chromatic number* help us out here? Is it true that, if the chromatic number of the hypergraph is sufficiently large, then playing the $(2:1)$ game on the hypergraph, the underdog can achieve a weak win? Unfortunately, the answer is 'no', even if the underdog

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is the first player, *i.e.*, the (1 : 2) game. The counter-example is due to F. Galvin: for every $r \geqslant 3$ there are uniform hypergraphs with chromatic number $\geqslant r$ such that, playing the (1 : 2) game on the hypergraph, the second player can prevent the first from occupying a whole winning set. Galvin's construction is very complicated.

Since large chromatic number does not necessarily guarantee the underdog's weak win, Ramsey theory becomes 'useless' for biased games. In other words, there is no underdog analogue of Theorem E. But what actually *does* work here is the potential function technique. One can easily formulate and prove the biased version of Theorem F (weak win criterion), and also the biased version of Theorem D (Erdős–Selfridge theorem). These two generalizations (see Theorems G–H [6] below) are the basic tools to handle biased games.

Theorem G (biased weak win). *If*

$$\sum_{A\in\mathscr{F}} \left(\frac{p+q}{p}\right)^{-|A|} > p^2\cdot q^2\cdot (p+q)^{-3}\cdot \Delta_2(\mathscr{F})\cdot |V(\mathscr{F})|,$$

where $\Delta_2(\mathscr{F})$ is the pair-degree of hypergraph $\mathscr{F}$, and $V(\mathscr{F})$ is the board, then First Player can completely occupy a winning set $A \in \mathscr{F}$ in the biased $(p : q)$ play on $\mathscr{F}$ (First Player takes p new points and Second Player takes q new points per move).

Applying the biased criterion to a fair game. I give a surprising application of Theorem G: I apply it to a *fair* tournament game! We recall that a 'tournament' means a 'directed complete graph' such that every edge of a complete graph is directed by one of the two possible orientations; it represents a tennis tournament where any two players played with each other, and an arrow points from the winner to the loser.

Fix an arbitrary tournament T_k on k vertices. The two players are Red and Blue, who alternately take new edges of a complete graph K_n, and for each new edge choose one of the two possible orientations ('arrow'). Either player colours his arrow with his own colour. At the end of a play, the players create a 2-coloured tournament on n vertices. Red wins if there is a red copy of T_k; otherwise Blue wins. Is it true that, if n is sufficiently large compared to k, then Red always has a winning strategy?

The answer is 'yes', and I prove it by a simple application of the biased weak win criterion Theorem G. This is rather surprising, since the tournament game itself is *fair*. The idea is to associate with the fair tournament game a biased (1 : 3) hypergraph game! To understand the application, we recommend the reader to draw a picture. The board $V = V(\mathscr{F})$ of the biased (1 : 3) hypergraph game is the set of $2\binom{n}{2}$ arrows of $K_n(\uparrow\downarrow)$, where $K_n(\uparrow\downarrow)$ means that every edge of the complete graph K_n shows up with both orientations. The winning sets $A \in \mathscr{F}$ are the arrow-sets of all possible copies of T_k in $K_n(\uparrow\downarrow)$. So $\mathscr{F}$ is a $\binom{k}{2}$-uniform hypergraph, and trivially $|\mathscr{F}| \geqslant \binom{n}{k}$. If the mth move of Red and Blue are, respectively, $i_1 \to j_1$ and $i_2 \to j_2$, then these two moves (arrows) automatically exclude the extra arrows $j_1 \to i_1$ and $j_2 \to i_2$ from $K_n(\uparrow\downarrow)$ for the rest of the play. (There may be some coincidence among i_1, j_1, i_2, j_2, but it does not make any difference.) This means 1 arrow

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for Maker, and 3 arrows for Breaker in the hypergraph game on $\mathscr{F}$, which explains how the biased (1 : 3) play enters the story.

All that is left is to apply Theorem G to the (1 : 3) hypergraph game on $\mathscr{F}$. If

$$\sum_{A\in\mathscr{F}} 4^{-\binom{k}{2}} > \frac{9}{64}\cdot\Delta_2(\mathscr{F})\cdot|V(\mathscr{F})|,$$

where $\Delta_2(\mathscr{F})$ is the pair-degree and $V(\mathscr{F})$ is the board, then Red can force a win in the tournament game. We have the trivial equality $|V(\mathscr{F})| = 2\binom{n}{2}$, and the less trivial inequality $|\Delta_2(\mathscr{F})| \leqslant \binom{n-3}{k-3}\cdot k!$. Indeed, a tournament T_k *cannot* contain parallel arrows (*i.e.*, both orientations of an edge), so a pair of red and blue arrows contained by a copy of T_k must span at least 3 vertices, and there are at most $\binom{n-3}{k-3}\cdot k!$ ways to extend an unparallel arrow-pair to a copy of T_k.

Combining these facts, in view of Theorem G, it suffices to check that

$$\binom{n}{k} > \frac{9}{64}\cdot 4^{\binom{k}{2}}\cdot\binom{n-3}{k-3}\cdot k!\cdot 2\binom{n}{2}.$$

This inequality trivially follows if $n \geqslant c_0\cdot k^{k+3}\cdot 4^{\binom{k}{2}}$. The threshold $n_0(k) = c_0\cdot k^{k+3}\cdot 4^{\binom{k}{2}}$ works uniformly for all goal tournaments T_k on k vertices. □

We conclude this section with the biased version of the Erdős–Selfridge theorem (see [6]).

Theorem H (biased Erdős–Selfridge). *If*

$$\sum_{A\in\mathscr{F}}(1+q)^{-|A|/p} < 1, \quad \text{or} \quad < \frac{1}{1+q},$$

then First Player (*or Second Player*) *can block every winning set* $A\in\mathscr{F}$ *in the biased* $(q : p)$ (*or* $(p : q)$) *play on* $\mathscr{F}$.

7. An illustration of the BigGame SmallGame Decomposition

A major technical difficulty in studying the n^d game comes from the highly irregular nature of the n^d hypergraph. For example, if n is odd, the maximum degree (attained in the centre) is $(3^d-1)/2$, which is much larger than the average degree:

$$\text{AverDeg}(n^d\text{—hypergraph}) = \frac{n\cdot\text{family size}}{\text{board size}} = \frac{n\big((n+2)^d - n^d\big)/2}{n^d} \approx \frac{n}{2}\big(e^{\frac{2d}{n}}-1\big).$$

Therefore, the average degree of the n^d-hypergraph is about (very roughly) the nth root of the maximum degree. (Basically the same holds for n even; then the maximum degree is the smaller but still exponential $2^d - 1$.)

On the other hand, if we switch to the n^d *torus*, and consider the family of all 'torus-lines' (instead of all 'geometric lines' in the n^d hypercube), then we obtain a *degree-regular* hypergraph. This is the family of winning sets of the n^d *torus game*. The 8^2 torus game already came up in Section 6: we proved that this particular torus game was a pairing

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strategy draw, and the given explicit pairing implied that the 'unrestricted 9-in-a-row' was a draw game.

The n^d torus is an abelian group, implying that the n^d torus-hypergraph is translation-invariant (any two points 'look the same'). The n^d torus-hypergraph is degree-regular: every point has degree $(3^d - 1)/2$ (which is by the way the same as the degree of the centre in the n^d hypergraph when n is odd). So the total number of winning sets (torus-lines) is $(3^d - 1)n^{d-1}/2$. Of course, the board size remains n^d. We owe the reader a formal definition of the concept of 'torus-line'. A torus-line L is formally defined by a point $P \in L$ and a vector $\mathbf{v} = (a_1, \ldots, a_d)$ where each coordinate a_i is either 0, or $+1$, or -1 $(1 \leqslant i \leqslant d)$. The n points of line L are $P + k\mathbf{v} \pmod{n}$ where $k = 0, 1, \ldots, n-1$.

A peculiarity of this new 'line-concept' is that two different torus-lines may have more than one point in common! We recommend the reader to study the 4^2 torus game.

We show that this 'pair-intersection' cannot happen when n is *odd*, and if n is *even* then it can happen only under very special circumstances.

Statement. Any two different torus-lines have at most *one* common point if n is *odd*, and at most *two* common points if n is *even*. In the second case the distance between the two common points along either torus-line is always $n/2$.

Proof. The proof of this elementary fact goes as follows. Let L_1 and L_2 be two different torus-lines with (at least) two common points P and Q. Then there exist $k, l, \mathbf{v}, \mathbf{w}$ with $1 \leqslant k, l \leqslant n-1$, $\mathbf{v} = (a_1, \ldots, a_d)$, $\mathbf{w} = (b_1, \ldots, b_d)$ (where a_i and b_i are either 0, or $+1$, or -1 $(1 \leqslant i \leqslant d)$), $\mathbf{v} \neq \pm\mathbf{w}$, such that $Q \equiv P + k\mathbf{v} \equiv P + l\mathbf{w} \pmod{n}$. It follows that $k\mathbf{v} \equiv l\mathbf{w} \pmod{n}$, or equivalently, $ka_i \equiv lb_i \pmod{n}$ for every $i = 1, 2, \ldots, d$. Since a_i and b_i are either 0, or $+1$, or -1 $(1 \leqslant i \leqslant d)$, the only solution is $k = l = n/2$ (no solution if n is *odd*). □

What can we say about the two-dimensional n^2 torus game? Well, we know everything. From Section 6 we know that the 8^2 torus game is a pairing strategy draw, and this is a sharp result (since the point/line ratio of the 7^2 torus is $7^2/4 \cdot 7 = 7/4$: less than 2).

The Erdős–Selfridge theorem applies if $4n + 4 < 2^n$, which gives that the n^2 torus game is a blocking draw for every $n \geqslant 5$. The 4^2 torus game is also a draw (mid-size 'case study'), but I don't know any elegant proof. On the other hand, the 3^2 torus game is an easy First Player's win.

Next consider the three-dimensional n^3 torus game. The Erdős–Selfridge theorem applies if $13n^2 + 13 < 2^n$, which gives that the n^3 torus game is a blocking draw for every $n \geqslant 11$. I am convinced that the 10^3 torus game is also a draw, but I don't know how to prove it.

How about the four-dimensional n^4 torus game? The Erdős–Selfridge theorem applies if $40n^3 + 40 < 2^n$, which gives that the n^4 torus game is a blocking draw for every $n \geqslant 18$. This $n = 18$ can be improved to $n = 15$ by using a new method called 'BigGame SmallGame Decomposition'. This method was developed for proving the best known draw-criterion for the n^d game (multi-dimensional Tic-Tac-Toe). That application is rather complicated, and gives 'good' results in relatively large dimensions only (for example, by 'beating' any pairing strategy for $d \geqslant 33$; see the Hales–Jewett conjecture in Section 3).

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The following proof is perhaps the best illustration of this method on a (relatively) simple low-dimensional example.

Theorem 4. *The* 15^4 *torus game is a blocking draw.*

Proof. Since $n = 15$ is odd, the corresponding hypergraph is almost disjoint. This fact will be used repeatedly during the whole proof.

The basic idea is a *decomposition* of the 15^4 torus game into *two non-interacting games* with disjoint boards; I call them the *big game* and the *small game*. This is in Breaker's mind only; Maker does not know anything about the decomposition. Whenever Maker picks a point from the *big board* (board of the big game) then Breaker responds in the big board; whenever Maker picks a point from the *small board* (board of the small game) then Breaker responds in the small board (that is, Breaker follows the 'same board rule').

Non-interacting games means that playing the big game Breaker has to make his moves without any knowledge of the action in the small game, and similarly, playing the small game Breaker has to make his moves without any knowledge of the action in the big game, even if, of course, Breaker *does* know everything ('game of perfect information'). In other words, we *assume* that Breaker is 'schizophrenic': he has two 'personalities'; one for the big game and one for the small game, and the two personalities know nothing about each other ('Iron Curtain principle'). This assumption, at first sight weird, is crucial to the proof.

The *small game* deals with the winning sets (torus-lines) that are *dangerous*, where Maker is close to winning in the sense that all but 2 points are occupied by Maker. The *small board* is a kind of 'emergency room': in the small game, Breaker's goal is to block the 2-element unoccupied parts of the 'dangerous' winning sets by using a trivial pairing strategy. The small game is Breaker's last chance to block the 'dangerous' sets before it is too late.

Breaker's goal in the *big game* is to prevent too complex winning-set-configurations ('forbidden configurations') from graduating into the *small game*. For example, a pairing strategy works in the small game *only* if the 2-element unoccupied parts of the dangerous winning sets remain *pairwise disjoint*. The big game has to enforce, among many other things, this 'disjointness property'. This is how the big game guarantees that Breaker's pairing strategy in the small game will indeed work. This is why we must fully understand the small game first. The big game plays a auxiliary role only: to keep the small game under control.

In the *big game* Breaker uses the Erdős–Selfridge power-of-two scoring system, and *not* the Erdős–Selfridge theorem itself; see Lemma 5 below. The key numerical fact is that the number of *big sets* depends primarily on the maximum degree $D = (3^4 - 1)/2 = 40$ (rather than on the much larger total number $40 \cdot 15^3$ of winning sets). This is why the number of big sets is not too large. This is how Breaker can force – by using the power-of-two scoring system – the small game to remain 'trivial'; so simple that even a pairing strategy can block every 'dangerous' winning set.

The board of the big game is going to shrink during a play (more precisely: the *unoccupied part* of the big game is going to shrink). The reason for the 'shrinking' is that

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the 2-element 'emergency sets' are constantly removed from the big board, and added to the small board. Consequently, the small board – which, by definition, is exactly the *complement* of the big board – keeps growing during a play. At the beginning of a play the big board is equal to the whole board $V = n^d$-torus (*i.e.*, the small game is 'not born yet'). Let $V_{\mathrm{BIG}}(i)$ and $V_{\mathrm{small}}(i) = V \setminus V_{\mathrm{BIG}}(i)$ denote the big board and the small board after Maker's ith move and before Breaker's ith move. Then

$$V = V_{\mathrm{BIG}}(0) \supseteq V_{\mathrm{BIG}}(1) \supseteq V_{\mathrm{BIG}}(2) \supseteq V_{\mathrm{BIG}}(3) \supseteq \cdots,$$
$$\emptyset = V_{\mathrm{small}}(0) \subseteq V_{\mathrm{small}}(1) \subseteq V_{\mathrm{small}}(2) \subseteq V_{\mathrm{small}}(3) \subseteq \cdots.$$

The big game is played on the family of *big sets*, and the small game is played on the family of *emergency sets*. What are the big sets, and what are the emergency sets? Well, we already explained, at least in a heuristic way, what the emergency sets are. After the long 'heuristic introduction' it is time now to introduce the precise definitions.

Let $x_1, x_2, \ldots, x_i, \ldots$ and $y_1, y_2, \ldots, y_i, \ldots$ denote, respectively, the points of Maker and Breaker in a particular play. At the beginning, when the board of the small game is empty (*i.e.*, the small game is 'not born yet'), Breaker chooses his points $y_1, y_2, y_3, \ldots$ according to an Erdős–Selfridge type scoring system applied to the family $\mathscr{B}$ of big sets ($\mathscr{B}$ will be defined later). The hypergraph of the game is the family of all 'torus-lines of the 15^4 torus'; we denote it by $\mathscr{F}$. $\mathscr{F}$ is a 15-uniform hypergraph, and $|\mathscr{F}| = 40 \cdot 15^3$. In the course of a play in the big game a 15-element winning set $A \in \mathscr{F}$ (torus-line) is called 'dead' when it contains a mark of Breaker. Note that dead winning sets (*i.e.*, elements of $\mathscr{F}$) no longer represent any danger (they are marked by Breaker, so Maker cannot completely occupy them). At any time in the big game, the elements of $\mathscr{F}$ which are not yet dead are called 'survivors'. A survivor $A \in \mathscr{F}$ becomes 'dangerous' when Maker occupies its 13th point (all 13 points have to be in the big board); then the unoccupied 2-element part of this dangerous $A \in \mathscr{F}$ becomes an 'emergency set'. Whenever an emergency set arises (in the big game), then it is *removed* from the board of the big game, and at the same time it is added to the board of the small game. This is why the big game is shrinking. The board of the small game is precisely the union of *all* 2-*element emergency sets*, and consequently, the board of the big game is precisely the complement of the union of all emergency sets.

If the (growing) family of 2-element emergency sets remains 'disjoint' (*i.e.*, the 2-element emergency sets never overlap during the whole course of a play), then Breaker can easily block them in the small game (on the small board) by using the following trivial pairing strategy: when Maker takes a member of a 2-element emergency set, then Breaker takes the other one. The big game is designed exactly to enforce, among other properties, the 'disjointness of the emergency sets'. Therefore, Breaker must prevent the appearance of any 'forbidden configuration of type 1' (to understand the definition we *strongly recommend* the reader to draw a picture).

Forbidden configuration of type 1. At some stage of the play there exist two *dangerous* sets $A_1 \in \mathscr{F}$ and $A_2 \in \mathscr{F}$ such that their 2-element emergency parts $E_1 (\subset A_1)$ and $E_2 (\subset A_2)$ have a common point. (Since $\mathscr{F}$ is almost disjoint, they cannot have more than one point in common.)

If there exists a forbidden configuration of type 1, then at some stage of the play Maker occupied $13 + 13 = 26$ points of a 'pair-union' $A_1 \cup A_2$ (where $A_1, A_2 \in \mathscr{F}$) in the

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big board (during the big game), and Breaker could not yet put a single mark in $A_1 \cup A_2$ in the big game (perhaps Breaker could do it in the small game, but that does not count).

Note that the total number of 'intersecting pairs' $\{A_1, A_2\}$ with $|A_1 \cap A_2| = 1$ ($A_1, A_2 \in \mathscr{F}$) is exactly

$$15^4 \binom{40}{2}. \tag{7.1}$$

Indeed, the torus has 15^4 points, and each point has the same degree 40.

Another 'potentially bad configuration', that Breaker is advised to prevent, is any 'forbidden configuration of type 2' (again we *strongly recommend* the reader to draw a picture).

Forbidden configuration of type 2. There exists a *survivor* $A_0 \in \mathscr{F}$
(a) which never graduates into a *dangerous set*; and at some stage of the play,
(b) A_0's intersection with the big board is completely occupied by Maker, or possibly empty; and at the same time,
(c) A_0's intersection with the small board is completely covered by pairwise disjoint 2-element emergency sets.

The 'danger' of (a)–(c) is obvious: since A_0 never graduates into a *dangerous set*, A_0's intersection with the small board remains 'invisible' for Breaker in the whole course of the small game, so there is a real chance that A_0 will be completely occupied by Maker.

In (c) let k denote the size of the intersection of A_0 with the small board; (a) implies that the possible values of k are $3, 4, \dots, 15$. Accordingly we can talk about a forbidden configuration of type $(2, k)$ where $3 \leqslant k \leqslant 15$.

Let A_0 be a forbidden configuration of type $(2, k)$ ($3 \leqslant k \leqslant 15$); then there are k disjoint 2-element emergency sets $E_1, E_2, \dots, E_k$ which cover the k-element intersection of A_0 with the small board (see (c)). Let $A_1, A_2, \dots, A_k$ denote the *supersets* of $E_1, E_2, \dots, E_k$: $E_i \subset A_i$ where every $A_i \in \mathscr{F}$ is a dangerous set ('almost disjointness' implies that for every E_i there is a unique A_i).

Then, at the particular stage of the play described by (b)–(c), Maker occupied at least $(15-2)+(15-3)+\cdots+(15-k)+(15-k-1)+(15-k) = (30+25k-k^2)/2$ points (almost disjointness) of a union set $\bigcup_{i=0}^{k} A_i$ in the big board (in the big game), and Breaker could not yet put a single mark in $\bigcup_{i=0}^{k} A_i$ in the big game.

Note that the total number of configurations $A_0, \{A_1, A_2, \dots, A_k\}$ satisfying

(α) $A_0, A_1, \dots, A_k \in \mathscr{F}$, and
(β) $|A_0 \cap A_i| = 1$ for $1 \leqslant i \leqslant k$, and
(γ) $A_0 \cap A_i$ with $1 \leqslant i \leqslant k$ are k distinct points,

is at most

$$(40 \cdot 15^3) \cdot \binom{15}{k} \cdot (40-1)^k. \tag{7.2}$$

Indeed, first choose A_0, then choose the k distinct points $A_0 \cap A_i$ with $1 \leqslant i \leqslant k$, and finally choose the k sets $A_i \in \mathscr{F}$ ($i = 1, \dots, k$) (use almost disjointness).

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Note that after an arbitrary move of Maker in the big game, the big board may shrink (because of the possible appearance of new dangerous sets; the 2-element emergency parts are removed from the big board), but the big board does *not* change after a move of Breaker.

In order to prevent all forbidden configurations (*i.e.*, type 1 and type $(2,k)$ where $k = 3, 4, \ldots, 15$), Breaker needs the following 'shrinking' variant of the Erdős–Selfridge theorem.

Lemma 5. *Let $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_l$ be a sequence of finite hypergraphs. Let $m_1, m_2, \ldots, m_l$ be positive integers. Consider the following 'shrinking' game. Let V be the union set of the l hypergraphs; we call V the 'initial board'. The two players, called White and Black, alternate. A 'move' of White is to take a previously unoccupied point of the board and at the same time White may remove an arbitrary unoccupied part from the board (if he wants any). A 'move' of Black is standard: he takes a previously unoccupied point of the board. After White's move the board may shrink, and the players are always forced to take the next point from the 'available board' (which is a 'decreasing' subset of the initial board). White wins, if at some stage of the play, there exist $i \in \{1, \ldots, l\}$ and $B \in \mathscr{B}_i$ such that, White has m_i points from B and Black has none. Otherwise Black wins.*

Assume that

$$\sum_{i=1}^{l} \big(|\mathscr{B}_i| + \mathrm{MaxDeg}(\mathscr{B}_i)\big) 2^{-m_i} < 1.$$

Then Black has a winning strategy, no matter whether Black is the first or second player.

First I explain how to apply Lemma 5 (and prove it later). To prevent the appearance of any forbidden configuration of type 1, one has to control all possible candidates (of course no one knows in advance which winning set will eventually become *dangerous*), so let

$$\mathscr{B}_1 = \left\{ A_1 \cup A_2 : \{A_1, A_2\} \in \binom{\mathscr{F}}{2},\ |A_1 \cap A_2| = 1 \right\} \text{ and let } m_1 = 2(15-2) = 26.$$

To avoid the appearance of any forbidden configuration of type $(2,3)$, define hypergraph $\mathscr{B}_2$ as follows:

$$\left\{ A_0 \cup \cdots \cup A_3 : \{A_0, \ldots, A_3\} \in \binom{\mathscr{F}}{4}, A_0 \cap A_i, 1 \leqslant i \leqslant 3, \text{ are distinct points} \right\},$$

and let $m_2 = (15-2) + (15-3) + (15-4) + (15-3) = 48$. In general, to avoid the appearance of any forbidden configuration of type $(2, j+1)$, define hypergraph $\mathscr{B}_j$ as follows:

$$\left\{ A_0 \cup \cdots \cup A_{j+1} : \{A_0, \ldots, A_{j+1}\} \in \binom{\mathscr{F}}{j+2}, A_0 \cap A_i, 1 \leqslant i \leqslant j+1, \text{ are distinct points} \right\}, \tag{7.3}$$

and let

$$m_j = (15-2) + (15-3) + \cdots + (15-j-1) + (15-j-2) + (15-j-1) = \frac{54 + 23j - j^2}{2}. \tag{7.4}$$

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For technical reasons we use definition (7.3)–(7.4) for $j = 2, 3, \dots, 6$ only. This takes care of forbidden configurations of type $(2, k)$ with $3 \leqslant k \leqslant 7$. To prevent the appearance of any forbidden configuration of type $(2, k)$ where $8 \leqslant k \leqslant 15$, we use a *single* extra hypergraph $\mathscr{B}_7$. Let hypergraph $\mathscr{B}_7$ be defined as

$$\left\{ A_0 \cup A_1 \cup \cdots \cup A_8 : \{A_0, A_1, \dots, A_8\} \in \binom{\mathscr{F}}{9},\ A_0 \cap A_i, 1 \leqslant i \leqslant 8, \text{ are distinct points} \right\}, \tag{7.5}$$

and let

$$m_7 = (15-2)+(15-3)+(15-4)+(15-5)+(15-6)+(15-7)+(15-8)+(15-9) = 76. \tag{7.6}$$

In the definition of m_7 we didn't include the number of marks of Maker in A_0: this is how we can deal with all types $(2, k)$ where $8 \leqslant k \leqslant 15$ *at once*. Indeed, any forbidden configuration of type $(2, k)$ with $8 \leqslant k \leqslant 15$ contains a *subconfiguration* which is 'covered' by $\mathscr{B}_7$ and m_7 (we strongly recommend the reader to draw a picture).

Next we estimate the sum

$$\sum_{i=1}^{7} \big(|\mathscr{B}_i| + \mathrm{MaxDeg}(\mathscr{B}_i)\big) 2^{-m_i}.$$

By (7.1)–(7.2) we have

$$|\mathscr{B}_1| = 15^4 \binom{40}{2}, \text{ and } |\mathscr{B}_j| = (40 \cdot 15^3) \cdot \binom{15}{j+1} \cdot (40-1)^{j+1} \text{ for } 2 \leqslant j \leqslant 7.$$

Since the 15^4-torus is a group, the hypergraphs $\mathscr{B}_i$ $(1 \leqslant i \leqslant 7)$ are all degree-regular; every point has the same degree, namely the average degree (this observation simplifies the calculations). In $\mathscr{B}_1$ every set has the same size $2 \cdot 15 - 1 = 29$, and for each $j = 2, \dots, 7$ every $B \in \mathscr{B}_j$ has size $\geqslant (15-2)+(15-3)+\cdots+(15-j-1)+(15-j-2)+(15-j-1) = (54+23j-j^2)/2$. It follows that

$$|\mathscr{B}_1| + \mathrm{MaxDeg}(\mathscr{B}_1) = 15^4 \cdot \binom{40}{2}\left(1 + \frac{29}{15^4}\right),$$

and for $j = 2, \dots, 7$,

$$|\mathscr{B}_j| + \mathrm{MaxDeg}(\mathscr{B}_j) \leqslant (40 \cdot 15^3) \cdot \binom{15}{j+1} \cdot (40-1)^{j+1}\left(1 + \frac{54+23j-j^2}{2 \cdot 15^4}\right).$$

Therefore, easy calculations give

$$\begin{aligned}\sum_{i=1}^{7} \big(|\mathscr{B}_i| + \mathrm{MaxDeg}(\mathscr{B}_i)\big) 2^{-m_i} &\leqslant 15^4 \cdot \binom{40}{2}\left(1 + \frac{29}{15^4}\right) 2^{-26} \\ &+ \sum_{j=2}^{6} (40 \cdot 15^3) \cdot \binom{15}{j+1} \cdot (40-1)^{j+1}\left(1 + \frac{54+23j-j^2}{2 \cdot 15^4}\right) 2^{-(54+23j-j^2)/2} \\ &+ (40 \cdot 15^3) \cdot \binom{15}{8} \cdot (40-1)^8 \left(1 + \frac{54+23 \cdot 8 - 8^2}{2 \cdot 15^4}\right) 2^{-76} < \frac{9}{10} < 1.\end{aligned}$$

Now at last we are ready to define the *big sets*, *i.e.*, the forbidden configurations.

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Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_7$ be the family of big sets. Applying Lemma 5 (Maker is 'White' and Breaker is 'Black') we obtain that, in the big game, played on the family of big sets, Breaker can prevent the appearance of any forbidden configuration of type 1 or type $(2,k)$ with $3 \leqslant k \leqslant 15$. We claim that, combining the 'Lemma 5 strategy' with the trivial pairing strategy in the family of emergency sets (small game), Breaker can block hypergraph $\mathscr{F}$, that is, Breaker can put his mark in every torus-line of the 15^4 torus game. Indeed, an arbitrary $A \in \mathscr{F}$ (torus-line)

(1) either eventually becomes *dangerous*; then its 2-element emergency set will be blocked by Breaker in the small board (in the small game) by the trivial pairing strategy ('disjointness of the emergence sets' is enforced by preventing forbidden configurations of type 1),

(2) or A never becomes *dangerous*; then it will be blocked by Breaker in the big board: indeed, otherwise there is a forbidden configuration of type $(2,k)$ with some $k \in \{3,\ldots,15\}$.

It remains to prove Lemma 5.

Proof of Lemma 5. The proof is a standard application of the power-of-two scoring system. If $B \in \mathscr{B}_i$ is marked by Black then it scores 0. If $B \in \mathscr{B}_i$ is unmarked by Black and has w points of White, then it scores 2^{w-m_i}. The 'target value' (*i.e.*, White's win) is $\geqslant 2^0 = 1$; the 'initial value' is less than 1 (by hypothesis). Black can guarantee the usual 'decreasing property', which implies that no play is possible if the 'target value' is larger than the 'initial value'. Lemma 5 follows. □

This completes the proof of Theorem 4. □

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