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AN OPERATOR-SPLITTING GALERKIN/SUPG FINITE ELEMENT METHOD FOR POPULATION BALANCE EQUATIONS: STABILITY AND CONVERGENCE

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Abstract. We present a heterogeneous finite element method for the solution of a high-dimensional population balance equation, which depends both the physical and the internal property coordinates. The proposed scheme tackles the two main difficulties in the finite element solution of population balance equation: (i) spatial discretization with the standard finite elements, when the dimension of the equation is more than three, (ii) spurious oscillations in the solution induced by standard Galerkin approximation due to pure advection in the internal property coordinates. The key idea is to split the high-dimensional population balance equation into two low-dimensional equations, and discretize the low-dimensional equations separately. In the proposed splitting scheme, the shape of the physical domain can be arbitrary, and different discretizations can be applied to the low-dimensional equations. In particular, we discretize the physical and internal spaces with the standard Galerkin and Streamline Upwind Petrov Galerkin (SUPG) finite elements, respectively. The stability and error estimates of the Galerkin/SUPG finite element discretization of the population balance equation are derived. It is shown that a slightly more regularity, *i.e.* the mixed partial derivatives of the solution has to be bounded, is necessary for the optimal order of convergence. Numerical results are presented to support the analysis.

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1. Introduction

The numerical solution of a population balance equation (PBE) is highly demanded in many industrial applications such as crystallization, polymerization etc., see for example [3, 25, 26]. The PBE depends not only on time and space but also contains derivatives with respect to the properties of the particles (internal variables) such as the size, length, etc. Thus, the PBE is posed on a high-dimensional (more than three) domain. A population balance equation describing the particle size distribution u in a population balance system (PBS)

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of a crystallization process can be defined as:

$$\begin{aligned} \frac{\partial u}{\partial t} - \epsilon \Delta_x u + \mathbf{b} \cdot \nabla_x u + \mathbf{g} \cdot \nabla_\ell u &= f \qquad \text{in } (0,T] \times \Omega, \\ u(t,x,\ell) &= 0 \qquad \text{on } (0,T] \times \partial\Omega, \\ u(0,x,\ell) &= u_0(x,\ell) \text{ in } \Omega. \end{aligned} \tag{1.1}$$

Here, the computational domain Ω is the Cartesian product of the physical space (X-direction) domain $\Omega_X \subset \mathbb{R}^d$, $d = 2, 3$ and the internal property coordinate (L-direction) domain $\Omega_L \subset \mathbb{R}^s$, $s \geq 1$, *i.e.*, $\Omega := (\Omega_X \times \Omega_L) \subset \mathbb{R}^{d+s}$ with a polyhedral boundary $\partial\Omega$. Further, ∇_x and ∇_ℓ denote the gradient operators in Ω_X and Ω_L, respectively, whereas Δ_x denotes the Laplace operator in Ω_X. The fluid transport velocity $\mathbf{b}(t,x)$ and the growth rate $\mathbf{g}(t,x)$ are given d- and s-dimensional vector functions, respectively, $\epsilon > 0$ is a constant diffusion coefficient in Ω_X, u_0 is a given initial distribution of u, and T is a given final computational time. The source term f may be considered as a term arising from the aggregation and breakage. For simplicity, we assume $f \in C^0(0,T;L^2(\Omega))$ and use the homogeneous Dirichlet boundary condition. Further, to reduce the technicalities as much as possible, we assume that the fluid transport velocity $\mathbf{b}(t,x)$ is divergence free, that is,

$$\nabla_x \cdot \mathbf{b} = 0. \tag{1.2}$$

In a crystallization process the growth rate $\mathbf{g}(t,x)$ is often assumed to be independent of the particle size $\ell \in \Omega_L$. Therefore, we naturally have the property

$$\nabla_\ell \cdot \mathbf{g}(t,x) = 0. \tag{1.3}$$

One of the main challenges in the finite element solution of high-dimensional PBE is to discretize it spatially with the standard finite elements, especially when $d + s > 3$. Further, developing a fully-practical numerical scheme to solve a high-dimensional PBE will become more challenging, when the PBE is coupled with the flow and species concentration equations [8, 15]. Since the dimension of the PBE will be higher than the dimension of all other equations in a population balance system (PBS), a special care has to be taken in order to handle the coupling conditions [5, 13, 15, 18].

Several numerical methods, method of moments and its variants, discretization methods, finite difference, least square method, spectral and finite element methods, have been proposed and used to solve PBEs by several authors, see for example [16, 17, 22–26] and the references therein. However, most of these methods are restricted to one-dimensional (spatially) PBEs or applied to a system of one-dimensional (1D) PBEs which are obtained by splitting the high-dimensional PBE. For example, in [4] the PBE in $\mathbb{R}^2$ has been split into two 1D equations and a mixed Euler-Lagrange method has been applied. In the high-resolution finite volume computations [10, 19–21] of PBEs, the dimensional splitting has been applied to the PBE to obtain a system of 1D equations. Handling of the coupling conditions between the high-dimensional PBE and the 3D flow and species concentration equations will be more challenging when the PBE is split into a system of 1D equations. Further, a special technique is needed for the load balancing in parallel computations of the system of operator-split 1D equations [11].

In this paper, we present a heterogeneous finite element method for a high-dimensional population balance equation. In our approach, we split the PBE into two equations, that is, we split the $(d + s)$-dimensional PBE (1.1) into d- and s-dimensional equations. Then, we solve the d- and s-dimensional equations with the standard Galerkin and Streamline Upwind Petrov Galerkin (SUPG) finite element methods, respectively. The SUPG discretization suppress the spurious oscillations in the numerical solution due to the absence of the diffusion in the Ω_L. A rigorous numerical analysis of the operator-splitting (first order in time) Galerkin/SUPG finite element method for the population balance equation is presented in this paper. An advantage in our splitting approach is that the shape of the physical domain (Ω_X) can be arbitrary and it is not the case when the PBE is split into a system of 1D equations. Further, the coupling conditions between the PBE and the flow equations can be handled easily in our splitting [8]. Since the decomposition of the physical domain is sufficient

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for parallel computations, load balancing in the proposed splitting approach can be scheduled in an usual way [8]. Although the splitting is quite natural, to the best of the author's knowledge, the operator-splitting finite element method for high-dimensional PBEs has not been proposed before in the literature.

The paper is organized as follows. In the next section, we briefly discuss the standard discrete and algebraic form of the population balance equation. Then, we present the operator-splitting finite element method for the population balance equation in Section 3. After that, in Section 4, the stability and *a priori* error estimates of the operator-splitting Galerkin/SUPG finite element method for the population balance equation are presented. Finally, we present the numerical results in Section 5.

2. Finite element method for population balance equations

2.1. Preliminaries

Let $\Omega := \Omega_X \times \Omega_L \subset \mathbb{R}^{d+s}$ be a bounded domain. Assume that the particle size distribution function $u(t,x,\ell)$ in (1.1) is measurable, *i.e.*,

$$\int_0^T \int_{\Omega_X \times \Omega_L} |u(t,x,\ell)|\ d(x,\ell)\ \mathrm{d}t < \infty$$

then, due to Fubini's theorem we have

$$\begin{aligned}\int_0^T \int_{\Omega_X \times \Omega_L} u(t,x,\ell)\ d(x,\ell)\ \mathrm{d}t &= \int_0^T \int_{\Omega_X} \left(\int_{\Omega_L} u(t,x,\ell) \mathrm{d}\ell \right)\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^T \int_{\Omega_L} \left(\int_{\Omega_X} u(t,x,\ell) \mathrm{d}x \right)\ \mathrm{d}\ell\ \mathrm{d}t.\end{aligned}$$

Moreover, if $u(t,x,\ell) = g(t,x)h(t,\ell)$, then we have

$$\int_0^T \int_{\Omega_X \times \Omega_L} g(t,x)h(t,\ell)\ d(x,\ell)\ \mathrm{d}t = \int_0^T \int_{\Omega_X} g(t,x)\ \mathrm{d}x \left(\int_{\Omega_L} h(t,\ell)\ \mathrm{d}\ell \right)\ \mathrm{d}t.$$

2.2. Finite element spaces for the operator-splitting method

Let $(\cdot,\cdot)$ and $||\cdot||$ be the L^2-inner product and norm over Ω, respectively, that is,

$$(v,q) := \int_\Omega v(x,\ell)q(x,\ell), \qquad ||v||^2 = (v,v) \qquad \forall\ v,\ q \in L^2(\Omega).$$

Further, let $H^m(\Omega_X)$ and $H^m(\Omega_L)$ be the usual Sobolev spaces with the weak derivatives of order m. We now define

$$H^{m,m}(\Omega) := \left(H^m\left(\Omega_X; H^m(\Omega_L)\right)\right) \cap \left(H^m\left(\Omega_L; H^m(\Omega_X)\right)\right). \tag{2.1}$$

For each integer $m \geq 0$, the associated norm of a function $u \in H^m(\Omega_X; H^m(\Omega_L)$ can be defined as

$$\|u\|^2_{H^m(\Omega_X;H^m(\Omega_L)} := \sum_{|\beta|\leq m,\ |\alpha|\leq m} \|\partial_\ell^\beta \partial_x^\alpha u\|^2_{L^2(\Omega)}.$$

Hence, the associated norm and seminorm for a function $u \in H^{m,m}(\Omega)$ can be defined as

$$\|u\|^2_{H^{m,m}(\Omega)} := \sum_{|\beta|\leq m,\ |\alpha|\leq m} \|\partial_\ell^\beta \partial_x^\alpha u\|^2_{L^2(\Omega)}, \quad |u|^2_{H^{m,m}(\Omega)} := \sum_{|\beta|= m,\ |\alpha|= m} \|\partial_\ell^\beta \partial_x^\alpha u\|^2_{L^2(\Omega)}.$$

Note that the space $H^{m,m}(\Omega)$ is slightly more regular than the usual Sobolev space $H^m(\Omega)$, *i.e.*, the mixed partial derivatives of the functions from the space $H^{m,m}(\Omega)$ are bounded. This additional regularity is necessary in the analysis of operator-splitting finite element method.

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Now, let $V := H_0^1(\Omega_X)$ and $Q := H_0^1(\Omega_L)$ be the usual Sobolev spaces, whose function values are zero on their respective boundaries. Let $\mathcal{T}_h$ and $\mathcal{S}_h$ be the triangulation of Ω_X and Ω_L respectively, and are assumed to be shape regular. Suppose $V_h \subset V$ and $Q_h \subset Q$ are conforming finite element (finite dimensional) spaces. We denote the diameter of the cells $K' \in \mathcal{T}_h$ and $K \in \mathcal{S}_h$ by $h_{K'}$ and h_K, respectively. Further, the global mesh size in each domain is defined as $h_x := \max\{h_{K'} \ : \ K' \in \mathcal{T}_h\}$ and $h_\ell := \max\{h_K \ : \ K \in \mathcal{S}_h\}$, respectively, and the global mesh size h in Ω is defined as $h := \max\{h_x, \ h_\ell\}$. Let $\phi_h := \phi_i(x)$, $i = 1, 2, \ldots, \mathcal{M}$, and $\psi_h := \psi_k(\ell)$, $k = 1, 2, \ldots, \mathcal{N}$, be the basis functions of V_h and Q_h, respectively, *i.e.*,

$$V_h = \text{span}\{\phi_i(x)\}, \qquad Q_h = \text{span}\{\psi_k(\ell)\}. \tag{2.2}$$

We then define the discrete finite element space W_h such that

$$W_h = V_h \otimes Q_h = \left\{ \xi_h : \ \xi_h = \sum_{j=1}^{\mathcal{M}} \sum_{l=1}^{\mathcal{N}} \xi_{j,l} \phi_j(x) \psi_l(\ell); \quad \xi_{j,l} \in \mathbb{R} \right\} \subset H_0^{1,1}(\Omega),$$

where

$$H_0^{1,1}(\Omega) = \left\{ \xi_h : \ \xi_h \in H^{1,1}(\Omega); \quad \xi(x,\ell) = 0, \ \forall \ \ (x,\ell) \in \partial\Omega \right\} \subset H_0^{1,1}(\Omega).$$

Also, the finite element functions are defined as follows

$$\begin{aligned} u_h &= \sum_{j=1}^{\mathcal{M}} \sum_{l=1}^{\mathcal{N}} u_{j,l} \phi_j \psi_l, & v_h &= \sum_{i=1}^{\mathcal{M}} \sum_{k=1}^{\mathcal{N}} v_{i,k} \phi_i \psi_k, \\ \nabla_x u_h &= \sum_{j=1}^{\mathcal{M}} \sum_{l=1}^{\mathcal{N}} u_{j,l} (\nabla_x \phi_j) \psi_l, & \nabla_x v_h &= \sum_{i=1}^{\mathcal{M}} \sum_{k=1}^{\mathcal{N}} v_{i,k} (\nabla_x \phi_i) \psi_k, \\ \nabla_\ell u_h &= \sum_{j=1}^{\mathcal{M}} \sum_{l=1}^{\mathcal{N}} u_{j,l} \phi_j (\nabla_\ell \psi_l), & \nabla_\ell v_h &= \sum_{i=1}^{\mathcal{M}} \sum_{k=1}^{\mathcal{N}} v_{i,k} \phi_i (\nabla_\ell \psi_k). \end{aligned} \tag{2.3}$$

Further, for all $K' \in \mathcal{T}_h$ and $K \in \mathcal{S}_h$, we define the mesh-dependent norm

$$\|v_h\|_{0,K',K}^2 := \int_K \int_{K'} v_h^2, \qquad \|v_h\|_{1,1,K',K}^2 := \int_K \int_{K'} \left(\partial_\ell \partial_x v_h\right)^2.$$

2.3. Galerkin/SUPG stabilized discretization

It is well known that the standard Galerkin discretization of convection diffusion equations is not stable for small diffusion coefficients (in comparison with convection), and induce spurious oscillations in the solution. In the considered problem (1.1), even if we assume a sufficiently large ϵ, the standard Galerkin discretization still induce spurious oscillations due to the absence of the diffusion in the L-direction. One possibility to circumvent the instability and suppress spurious oscillations is to use the SUPG method [6, 12]. SUPG is one of the most popular stabilization methods for finite element discretization, and it adds artificial diffusion along the streamlines of the solution, see for example [2, 14] and the references there in. Since we assumed that ϵ is sufficiently large, it is sufficient to stabilize the equation (1.1) only in the L-direction. Therefore, we use the standard Galerkin and the consistent SUPG stabilized discretizations in the X- and L-directions, respectively. Since we use different discretizations in different directions, we call it as a heterogeneous discretization method. Applying the Galerkin/SUPG discretization, the semi-discrete form of (1.1) reads:

For a given $u_h(0) = u_{h,0}$, find $u_h(t) \in W_h$ such that for all $t \in (0, T]$

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a_{LS}(u_h, v_h) + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\frac{\partial u_h}{\partial t}, \ \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K = (f, v_h) + F_S(f, v_h), \quad v_h \in W_h, \tag{2.4}$$

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where

$$a_{LS}(u_h, v_h) := \int_\Omega (\epsilon \nabla_x u_h \cdot \nabla_x v_h \ + \mathbf{b}_h \cdot \nabla_x u_h \ v_h + \mathbf{g}_h \cdot \nabla_\ell u_h \ v_h) + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\mathbf{g}_h \cdot \nabla_\ell u_h, \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K ,$$

$$F_S(f, v_h) := \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(f, \ \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K .$$

Here, $u_{h,0} \in W_h$ is a L^2-projection of the initial value u_0 onto W_h and $(\cdot,\cdot)_K$ denotes the L^2-inner product in the mesh cell $K \in \mathcal{S}_h$. Further, $\{\delta_K\}$ are the local stabilization parameters and in order to guarantee the coercivity of the bilinear form $a_{LS}(\cdot,\cdot)$ we assume

$$0 < \delta_K \leq \delta_0 h_K^2 \leq 1, \tag{2.5}$$

where $\delta_0 > 0$ is a constant.

Lemma 2.1 (coercivity of $a_{LS}(\cdot,\cdot)$)**.** *Let the discrete form of the assumptions (1.2) and (1.3) be satisfied and the assumption (2.5) be fulfilled. Then, the bilinear form associated with the Galerkin/SUPG discretization satisfies*

$$a_{LS}(u_h, u_h) \geq |||u_h|||^2. \tag{2.6}$$

Here, the mesh-dependent heterogeneous norm

$$|||u_h|||^2 := \sum_{K' \in \mathcal{T}_h} \sum_{K \in \mathcal{S}_h} \left(\epsilon \|\nabla_x u_h\|^2_{0,K',K} \ + \delta_K \|\mathbf{g}_h \cdot \nabla_\ell u_h\|^2_{0,K',K}\right). \tag{2.7}$$

Proof. Using the definition of the a_{LS}, and applying integration by parts we obtain

$$\begin{aligned}
a_{LS}(u_h, u_h) &= \int_{\Omega_L} \int_{\Omega_X} \left(\epsilon \nabla_x u_h \cdot \nabla_x u_h \ + \frac{1}{2} \mathbf{b}_h \cdot \nabla_x u_h^2\right) + \frac{1}{2} \int_{\Omega_X} \int_{\Omega_L} \mathbf{g}_h \cdot \nabla_\ell u_h^2 \\
&\quad + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \int_K \mathbf{g}_h \cdot \nabla_\ell u_h \ \mathbf{g}_h \cdot \nabla_\ell u_h, \\
&= \int_{\Omega_L} \int_{\Omega_X} \left(\epsilon \nabla_x u_h \cdot \nabla_x u_h \ - \frac{1}{2} \nabla_x \cdot \mathbf{b}_h u_h^2\right) - \frac{1}{2} \int_{\Omega_X} \int_{\Omega_L} \nabla_\ell \cdot \mathbf{g}_h u_h^2 \\
&\quad + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \int_K (\mathbf{g}_h \cdot \nabla_\ell u_h)^2, \\
&\geq \sum_{K' \in \mathcal{T}_h} \sum_{K \in \mathcal{S}_h} \left(\epsilon \|\nabla_x u_h\|^2_{0,K',K} \ + \delta_K \|\mathbf{g}_h \cdot \nabla_\ell u_h\|^2_{0,K',K}\right),
\end{aligned}$$

for all $u_h \in V_h$. □

2.4. Temporal discretization

Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a decomposition of the considered time interval $[0, T]$. Let us denote $\tau = \tau^n = t^n - t^{n-1}$, $1 \leq n \leq N$, be an uniform time step, and denote u_h^n be the approximation of $u(t^n, x, \ell)$ in W_h. Further, we denoted the one step finite difference operator

$$\bar{\partial} u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau}.$$

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After applying the backward Euler time discretization in (2.4), the heterogeneous discrete form of (1.1) can be written as:

For given f^n, $u_h^0 = u_{h,0}$, find $u_h^n \in W_h$ in the time interval (t^{n-1}, t^n) such that for all $v_h \in W_h$

$$\left(\bar{\partial} u_h^n,\ v_h\right) + a_{LS}(u_h^n, v_h) + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\bar{\partial} u_h^n,\ \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K = (f^n, v_h) + F_S(f^n, v_h). \tag{2.8}$$

Now, using the definition of finite element functions (2.3), the algebraic form of the discrete equation (2.8) can be written as

$$\left(M + M^S + \tau A + \tau A^S\right) \bar{U}^n = \tau F^n + \tau F^{S,n} + \left(M + M^S\right) \bar{U}^{n-1}. \tag{2.9}$$

Here, $\bar{U}^n = \mathrm{vec}(U^n)$ is the vectorization of the solution matrix $U^n = [u_{j,l}^n]_{\mathcal{M}\times\mathcal{N}}$. Further, the mass, stiffness and stabilization matrices are defined as follows:

$$M := M_x \otimes M_\ell, \quad M^S := M_x \otimes S_\ell \quad A := A_x \otimes M_\ell + M_x \otimes A_\ell, \quad A^S := M_x \otimes G_\ell, \tag{2.10}$$

where $\otimes$ denotes the Kronecker product of two matrices. Further,

$$\begin{aligned}
A_x &:= \int_{\Omega_X} \nabla_x \phi_j \cdot \nabla_x \phi_i + \int_{\Omega_X} \phi_j\ \mathbf{b}_h \cdot \nabla_x \phi_i, & M_x &:= \int_{\Omega_X} \phi_j \phi_i, \\
A_\ell &:= \int_{\Omega_L} \mathbf{g}_h \cdot \nabla_\ell \psi_l\ \psi_k, & M_\ell &:= \int_{\Omega_L} \psi_l \psi_k, \\
S_\ell &:= \sum_{K \in \mathcal{S}_h} \delta_K \left(\psi_l,\ \mathbf{g}_h \cdot \nabla_\ell \psi_k\right)_K, & G_\ell &:= \sum_{K \in \mathcal{S}_h} \delta_K \left(\mathbf{g}_h \cdot \nabla_\ell \psi_l,\ \mathbf{g}_h \cdot \nabla_\ell \psi_k\right)_K, \\
F^{S,n} &:= \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(f^n, \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K, & F^n &:= \int_{\Omega} f^n v_h.
\end{aligned} \tag{2.11}$$

Here, the entries in the mass matrix which are obtained by the Kronecker product of the matrices M_x and M_ℓ, are given by:

$$M = M_x \otimes M_\ell := \begin{bmatrix} [\Phi_{1,1}]_{k,l} & \cdots & [\Phi_{1,\mathcal{M}}]_{k,l} \\ \vdots & \vdots & \vdots \\ [\Phi_{\mathcal{M},1}]_{k,l} & \cdots & [\Phi_{\mathcal{M},\mathcal{M}}]_{k,l} \end{bmatrix}_{\mathcal{M}\mathcal{N}\times\mathcal{M}\mathcal{N}},$$

where the entries in the $\mathcal{N} \times \mathcal{N}$ block matrix $[\Phi_{i,j}]_{k,l}$, $1 \le k, l \le \mathcal{N}$ are evaluated by

$$[\Phi_{i,j}]_{k,l} := \int_{\Omega_X} \phi_i \phi_j \int_{\Omega_L} \psi_k \psi_\ell.$$

Note that solving an algebraic system of size $\mathcal{M}\mathcal{N} \times \mathcal{M}\mathcal{N}$ in each time step will be very expensive. Moreover, when the PBE is coupled with the flow equations in a population balance system, this large system has to be solved repeatedly several times in each time step to decouple the system of equations. This, requires enormous computing power.

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3. Operator-splitting finite element method

The operators Δ_x, ∇_x and ∇_ℓ in the population balance equation (1.1) are the decomposition of unmixed partial derivatives of the Cartesian coordinates x and ℓ, respectively. Thus, we can take advantage of the decomposition, and discretize equation (1.1) in space with d- and s-dimensional finite elements instead of $(d+s)$-dimensional finite elements. Using the Lie's operator-splitting method, see for *e.g.*, [9], in the time interval (t^{n-1}, t^n), the operator-split equations of (1.1) read:

Step 1 (L-direction).
For given $\hat{u}^{n-1} = u(t^{n-1}, x, \ell)$, find $\hat{u}^n$ in (t^{n-1}, t^n) such that for all $x \in \Omega_X$,

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} + \mathbf{g} \cdot \nabla_\ell \hat{u} &= f \qquad \text{in } \Omega_L, \\ \hat{u}(t, x, \ell) &= 0 \qquad \text{in } \partial\Omega_L, \end{aligned} \tag{3.1}$$

by considering x as a parameter. In this step the solution is updated in the L-direction. Then, this solution $\hat{u}^n$ is taken as the initial solution for the X-direction update.

Step 2 (X-direction).
For given $\tilde{u}^{n-1} = \hat{u}^n$, find $\tilde{u}$ in (t^{n-1}, t^n) such that for all for all $\ell \in \Omega_L$,

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} - \epsilon \Delta_x \tilde{u} + \mathbf{b} \cdot \nabla_x \tilde{u} &= 0 \qquad \text{in } \Omega_X, \\ \tilde{u}(t, x, \ell) &= 0 \qquad \text{in } \partial\Omega_X, \end{aligned} \tag{3.2}$$

by considering ℓ as a parameter. Here the solution u^n in the time step (t^{n-1}, t^n) is obtained by first updating L-direction operators (Eq. (3.1)) and then updating X-direction operators (Eq. (3.2)). Note that the L-direction equation (3.1) has to be solved only for inner point $x \in \Omega_X$ due to the Dirichlet boundary condition. However, if we consider non-Dirichlet boundary conditions, then equation (3.1) has to be solved also for all boundary points $x \in \partial\Omega_X$. Similar arguments hold for the X-direction equation (3.2).

Next, to derive the discrete forms of the operator-split equations (3.1) and (3.2), we define

$$\hat{u}_h^n(x_j, \ell) := \sum_{l=1}^{\mathcal{N}} \hat{u}_{j,l}^n \psi_l(\ell), \qquad \tilde{u}_h^n(x, \ell_l) := \sum_{j=1}^{\mathcal{M}} \tilde{u}_{j,l}^n \phi_j(x)$$

as the finite element functions for the equations (3.1) and (3.2), respectively. Here, $x_j \in \Omega_X$, $j = 1, \ldots, N_XP$, and $\ell_l \in \Omega_L$, $l = 1, \ldots, N_LP$ are the Cartesian coordinates which are necessary to evaluate the nodal functionals of the finite element spaces V_h and Q_h, respectively. As discussed before, the X-direction equation (3.2) is a standard convection-diffusion equation, and with sufficiently large ϵ in comparison to $|\mathbf{b}|$, it can be solved with the standard Galerkin method. However, the L-direction equation (3.1) is a pure advection equation and a stabilization method has to be used since the standard Galerkin method induce spurious oscillations in the solution. After applying the SUPG to (3.1) and the standard Galerkin to (3.2), the discrete form of the operator-split equations (3.1) and (3.2) in the time interval (t^{n-1}, t^n) with $u_h^0 = u_0$ read:

Step 1 (L-direction).
For a given f^n and $\hat{u}_h^{n-1} = u_h^{n-1}$, find $\hat{u}_h^n \in Q_h$ such that

$$\left(\bar{\partial}\hat{u}_h^n, \psi_h\right)_\ell + a_S(\hat{u}_h^n, \psi_h) + \sum_{K \in \mathcal{S}_h} \delta_K \left(\bar{\partial}\hat{u}_h^n,\ \mathbf{g}_h \cdot \nabla_\ell \psi_h\right)_K = (f^n, \psi_h)_\ell + F_S^L(f^n, \psi_h), \quad \forall\, \psi_h \in Q_h \quad \forall\, x \in \Omega_X, \tag{3.3}$$

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by considering x as a parameter. Here,

$$a_S(u_h,\psi_h) := \int_{\Omega_L} \mathbf{g}_h\cdot\nabla_\ell u_h\ \psi_h + \sum_{K\in\mathcal{S}_h}\delta_K\left(\mathbf{g}_h\cdot\nabla_\ell u_h,\ \mathbf{g}_h\cdot\nabla_\ell\psi_h\right)_K,$$
$$F_S^L(f,\psi_h) := \sum_{K\in\mathcal{S}_h}\delta_K\left(f,\ \mathbf{g}_h\cdot\nabla_\ell\psi_h\right)_K.$$

Step 2 (X-direction).
For the given $\tilde{u}_h^{n-1}=\hat{u}_h^n$, find $\tilde{u}_h^n\in V_h$, such that

$$\left(\bar{\partial}\tilde{u}_h^n,\phi_h\right)_x + a_X(\tilde{u}_h^n,\phi_h) = 0,\quad \forall\ \phi_h\in V_h,\quad \forall\ \ell\in\Omega_L, \tag{3.4}$$

by considering ℓ as a parameter. Here,

$$a_X(\tilde{u}_h,\phi_h) := \int_{\Omega_X}\epsilon\nabla_x\tilde{u}_h\cdot\nabla_x\phi_h\ +\ \int_{\Omega_X}\mathbf{b}_h\cdot\nabla_x\tilde{u}_h\ \phi_h.$$

Here, $(\cdot,\cdot)_\ell$ and $(\cdot,\cdot)_x$ in equations (3.3) and (3.4) denote the L^2-inner products in Ω_L and Ω_X, respectively. Finally, we obtain the global discrete solution

$$u_h^n(x,\ell) = \sum_{j=1}^{\mathcal{M}}\sum_{l=1}^{\mathcal{N}} u_{j,l}\phi_j(x)\psi_l(\ell)$$

by setting $u_{j,l}=\tilde{u}_{j,l}$. In the next section, we address the consistency error, the stability and the convergence of the operator-splitting Galerkin/SUPG finite element scheme for equations (3.3) and (3.4).

4. Analysis of the operator-splitting finite element method

To obtain the stability and *a priori* error estimates for the operator-split equations (3.3) and (3.4), we first derive the equivalent one-step operator-split discrete form of the operator-split equations (3.3) and (3.4).

Lemma 4.1 (consistency of the operator-splitting method)**.** *The equivalent one-step discrete form of the operator-split equations* (3.3) *and* (3.4) *is*

$$\left(\bar{\partial}u_h^n,v_h\right) + a_{LS}(u_h^n,v_h) + a_{OS}(u_h^n,v_h) + \int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K\left(\bar{\partial}u_h^n,\ \mathbf{g}_h\cdot\nabla_\ell v_h\right)_K = (f^n,v_h) + F_S(f^n,v_h) \tag{4.1}$$

where the consistency error induced by the operator-splitting is

$$\begin{aligned} a_{OS}(u_h^n,\ v_h) = {}& \tau\int_\Omega \epsilon\mathbf{g}_h\cdot\nabla_\ell(\nabla_x u_h^n)\ \nabla_x v_h + \tau\int_\Omega \mathbf{g}_h\cdot\nabla_\ell(\mathbf{b}_h\cdot\nabla_x u_h^n)\ v_h \\ & +\tau\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K\left(\epsilon\mathbf{g}_h\cdot\nabla_\ell(\nabla_x u_h^n),\ \mathbf{g}_h\cdot\nabla_\ell(\nabla_x v_h)\right)_K \\ & +\tau\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K\left(\mathbf{g}_h\cdot\nabla_\ell(\mathbf{b}_h\cdot\nabla_x u_h^n),\ \mathbf{g}_h\cdot\nabla_\ell v_h\right)_K \\ & +\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K\left(-\epsilon\Delta_x u_h^n,\ \mathbf{g}_h\cdot\nabla_\ell v_h\right)_K \\ & +\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K\left(\mathbf{b}_h\cdot\nabla_x u_h^n,\ \mathbf{g}_h\cdot\nabla_\ell v_h\right)_K. \end{aligned} \tag{4.2}$$

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Proof. The algebraic form of the L-direction equation (3.3) can be written as

$$(M_\ell + S_\ell + \tau(A_\ell + G_\ell))\,\hat{U}^n = \tau(F_L^n + F_L^{S,n}) + (M_\ell + S_\ell)\,\hat{U}^{n-1}, \tag{4.3}$$

where

$$F_L^n := \int_{\Omega_L} f^n \psi_h, \qquad F_L^{S,n} = \sum_{K\in\mathcal{S}_h} \delta_K \left(f^n, \mathbf{g}_h \cdot \nabla\psi_h\right)_K .$$

Here, the matrices S_ℓ, G_ℓ and $F_L^{S,n}$ belong to the SUPG stabilization terms. Next, the algebraic form of the X-direction equation (3.4) can be written as

$$(M_x + \tau A_x)\,\tilde{U}^n = M_x \tilde{U}^{n-1}. \tag{4.4}$$

In the above equations (4.3) and (4.4),

$$\hat{U}^n = (U^n)^{\mathrm{T}}, \qquad \tilde{U}^n = U^n, \qquad F_\ell^n := \int_{\Omega_L} f^n \psi_h.$$

Now, multiply (4.3) by $M_x \otimes \mathbb{I}$, and (4.4) by $\mathbb{I} \otimes (M_\ell + S_\ell + \tau A_\ell + \tau G_\ell)$, we get

$$\begin{aligned}((M_x \otimes M_\ell) + (M_x \otimes S_\ell) + \tau(M_x \otimes A_\ell) + \tau(M_x \otimes G_\ell))\,\hat{U}^n = \tau(M_x \otimes F_L^n) + \tau(M_x \otimes F_L^{S,n}) \\ + ((M_x \otimes M_\ell) + (M_x \otimes S_\ell))\hat{U}^{n-1}\end{aligned}$$

and

$$\begin{aligned}&((M_x \otimes M_\ell) + (M_x \otimes S_\ell) + \tau\left\{(A_x \otimes M_\ell) + (A_x \otimes S_\ell) + (M_x \otimes A_\ell) + (M_x \otimes G_\ell)\right\} \\ &\quad + \tau^2\left\{(A_x \otimes A_\ell) + (A_x \otimes G_\ell)\right\})\,\tilde{U}^n = ((M_x \otimes M_\ell) + (M_x \otimes S_\ell) + \tau(M_x \otimes A_\ell) + \tau(M_x \otimes G_\ell))\,\tilde{U}^{n-1},\end{aligned}$$

respectively. Equating, the above equations, we get

$$\begin{aligned}&((M_x \otimes M_\ell) + (M_x \otimes S_\ell) + \tau\left\{(A_x \otimes M_\ell) + (A_x \otimes S_\ell) + (M_x \otimes A_\ell) + (M_x \otimes G_\ell)\right\} \\ &\quad + \tau^2\left\{(A_x \otimes A_\ell) + (A_x \otimes G_\ell)\right\})\,\bar{U}^n = \tau(M_x \otimes F_L^n) + \tau(M_x \otimes F_L^{S,n}) + ((M_x \otimes M_\ell) + (M_x \otimes S_\ell))\bar{U}^{n-1}.\end{aligned}$$

Using the definitions (2.10) in the above equation, we get

$$\begin{aligned}&\left(M + M^S + \tau A + \tau A^S + \tau^2(A_x \otimes A_\ell) + \tau^2(A_x \otimes G_\ell) + \tau(A_x \otimes S_\ell)\right)\bar{U}^n \\ &\qquad = \tau(M_x \otimes F_L^n) + \tau(M_x \otimes F_L^{S,n}) + \left(M + M^S\right)\bar{U}^{n-1}.\end{aligned} \tag{4.5}$$

For the $A_x \otimes A_\ell$ term, we have

$$\begin{aligned}(A_x \otimes A_\ell)\bar{U}^n &= \int_\Omega \epsilon u_{j,l}^n \nabla_x\phi_j \cdot \nabla_x\phi_i\ \mathbf{g}_h \cdot \nabla_\ell\psi_l\psi_k \ +\ u_{j,l}^n \mathbf{b}_h \cdot \nabla_x\phi_j\phi_i\ \mathbf{g}_h \cdot \nabla_\ell\psi_l\psi_k \\ &= \int_\Omega \epsilon\mathbf{g}_h \cdot \nabla_\ell(u_{j,l}^n \nabla_x\phi_j\psi_l)\ \nabla_x v_h \ +\ \mathbf{g}_h \cdot \nabla_\ell(\mathbf{b}_h \cdot (u_{j,l}^n \nabla_x\phi_j)\psi_l)\ v_h \\ &= \int_\Omega \epsilon\mathbf{g}_h \cdot \nabla_\ell(\nabla_x u_h^n)\ \nabla_x v_h \ +\ \mathbf{g}_h \cdot \nabla_\ell(\mathbf{b}_h \cdot \nabla_x u_h^n)\ v_h.\end{aligned}$$

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Similarly, for the cross term $A_x \otimes G_\ell$, we have

$$
\begin{aligned}
&(A_x \otimes G_\ell)\bar{U}^n \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \int_K \epsilon\delta_K u^n_{j,l} \nabla_x\phi_j \cdot \nabla_x\phi_i \ \mathbf{g}_h \cdot \nabla_\ell\psi_l \mathbf{g}_h \cdot \nabla_\ell\psi_k \ + \ \delta_K u^n_{j,l} \mathbf{b}_h \cdot \nabla_x\phi_j\phi_i \ \mathbf{g}_h \cdot \nabla_\ell\psi_l \mathbf{g}_h \cdot \nabla_\ell\psi_k \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \int_K \epsilon\delta_K u^n_{j,l} \mathbf{g}_h \cdot \nabla_\ell(\nabla_x\phi_j\psi_l) \ \mathbf{g}_h \cdot \nabla_\ell(\nabla_x\phi_i\psi_k) \ + \ \delta_K u^n_{j,l} \mathbf{g}_h \cdot \nabla_\ell(\mathbf{b}_h \cdot \nabla_x\phi_j\psi_l) \ \mathbf{g}_h \cdot \nabla_\ell(\phi_i\psi_k) \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \int_K \epsilon\delta_K \mathbf{g}_h \cdot \nabla_\ell(\nabla_x u^n_h) \ \mathbf{g}_h \cdot \nabla_\ell(\nabla_x v_h) \ + \ \delta_K \mathbf{g}_h \cdot \nabla_\ell(\mathbf{b}_h \cdot \nabla_x u^n_h) \ \mathbf{g}_h \cdot \nabla_\ell v_h.
\end{aligned}
$$

Also for the $(A_x \otimes S_\ell)$ term, we have

$$
\begin{aligned}
(A_x \otimes S_\ell)\bar{U}^n &= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \int_K \epsilon\delta_K u^n_{j,l} \nabla_x\phi_j \cdot \nabla_x\phi_i \ \psi_l \mathbf{g}_h \cdot \nabla_\ell\psi_k \ + \ \delta_K u^n_{j,l} \mathbf{b}_h \cdot \nabla_x\phi_j\phi_i \ \psi_l \mathbf{g}_h \cdot \nabla_\ell\psi_k \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \int_K -\phi_i\nabla_x \cdot \left(\epsilon u^n_{j,l} \nabla_x\phi_j \delta_K\psi_l \ \mathbf{g}_h \cdot \nabla_\ell\psi_k\right) \ + \ \delta_K u^n_{j,l} \mathbf{b}_h \cdot \nabla_x(\phi_j\psi_l) \ \mathbf{g}_h \cdot \nabla_\ell(\phi_i\psi_k) \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \int_K -\epsilon u^n_{j,l} \delta_K \Delta_x(\phi_j\psi_l) \ \mathbf{g}_h \cdot \nabla_\ell(\phi_i\psi_k) \ + \ \delta_K \mathbf{b}_h \cdot \nabla_x u^n_h \ \mathbf{g}_h \cdot \nabla_\ell v_h \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \int_K -\epsilon\delta_K \Delta_x u^n_h \ \mathbf{g}_h \cdot \nabla_\ell v_h \ + \ \delta_K \mathbf{b}_h \cdot \nabla_x u^n_h \ \mathbf{g}_h \cdot \nabla_\ell v_h.
\end{aligned}
$$

Next, we show that the source term $M_x \otimes F^n_L = F^n$. Each equation in the algebraic system (4.5) is obtained by applying summation to the ansatz indices j and k on both sides of the system. Therefore, the source term in the algebraic system (4.5) becomes

$$
M_x \otimes F^n_L = \int_{\Omega_X\times\Omega_L} \sum_{i=1}^{\mathcal{M}} \sum_{j=1}^{\mathcal{M}} \sum_{k=1}^{\mathcal{N}} f^n \phi_i\phi_j\psi_k.
$$

Thus, the right hand side vector, $rhs_{i,k}$, $i = 1,\ldots,\mathcal{M}$, $k = 1,\ldots,\mathcal{N}$, can be written as

$$
rhs_{i,k} = \int_{\Omega_X\times\Omega_L} \sum_{j=1}^{\mathcal{M}} f^n\phi_i\phi_j\psi_k = \int_{\Omega_X\times\Omega_L} f^n\phi_i\psi_k \sum_{j=1}^{\mathcal{M}} \phi_j = \int_{\Omega_X\times\Omega_L} f^n\phi_i\psi_k,
$$

which is the source term in the algebraic system (2.9). Thus, we have $M_x \otimes F^n_L = F^n$. Similar argument holds for $M_x \otimes F^{S,n}_L = F^{S,n}$. Hence, the statement of the lemma. □

Note that the discrete bilinear form (4.1) of the two-step operator-split equations is not same as the original discrete form (2.8). The difference is the consistency error due to the operator-splitting method.

Lemma 4.2. *Let the discrete form of the assumptions (1.2) and (1.3) be satisfied. Then, for all $v_h \in W_h$, we have*

$$
\begin{aligned}
&\int_\Omega \mathbf{g}_h \cdot \nabla_\ell(\mathbf{b}_h \cdot \nabla_x v_h) v_h = 0, \quad \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(\mathbf{g}_h \cdot \nabla_\ell(\mathbf{b}_h \cdot \nabla_x v^n_h), \ \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K = 0, \\
&\int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(\epsilon\Delta_x u^n_h, \ \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K = 0, \quad \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(\mathbf{b}_h \cdot \nabla_x v_h, \ \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K = 0.
\end{aligned}
$$

Proof. We use the definitions of the finite element spaces (2.2) to show this property. In particular, we repeatedly use the properties that the basis functions of V_h and Q_h are independent of $\ell \in \Omega_L$ and $x \in \Omega_X$, respectively.

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Also, in the integration by parts, we use the definition that the functions of W_h has zero value on the boundaries. For the first statement, we have

$$\begin{aligned}
\int_\Omega \mathbf{g}_h \cdot \nabla_\ell (\mathbf{b}_h \cdot \nabla_x v_h) v_h &= \int_\Omega \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h} \frac{\partial}{\partial \ell_q}\left(\mathbf{b}_{p,h}\frac{\partial v_h}{\partial x_p}\right) v_h = \int_\Omega \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h} \frac{\partial}{\partial \ell_q}\left(\mathbf{b}_{p,h}\frac{\partial(\phi_h\psi_h)}{\partial x_p}\right)\phi_h\psi_h \\
&= \int_\Omega \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h}\frac{\partial \psi_h}{\partial \ell_q}\psi_h\ \mathbf{b}_{p,h}\frac{\partial \phi_h}{\partial x_p}\phi_h = \frac{1}{4}\int_\Omega \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h}\frac{\partial \psi_h^2}{\partial \ell_q}\ \mathbf{b}_{p,h}\frac{\partial \phi_h^2}{\partial x_p} \\
&= \frac{1}{4}\int_\Omega \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h} \frac{\partial}{\partial \ell_q}\left(\mathbf{b}_{p,h}\frac{\partial \phi_h^2}{\partial x_p}\psi_h^2\right) = \frac{1}{4}\int_\Omega \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h} \frac{\partial}{\partial \ell_q}\left(\mathbf{b}_{p,h}\frac{\partial (\phi_h\psi_h)^2}{\partial x_p}\right) \\
&= \frac{1}{4}\int_\Omega \mathbf{g}_h \cdot \nabla_\ell \left(\mathbf{b}_h \cdot \nabla_x v_h^2\right) = -\frac{1}{4}\int_\Omega \nabla_\ell \cdot \mathbf{g}_h \left(\mathbf{b}_h \cdot \nabla_x v_h^2\right) = 0.
\end{aligned}$$

For the second statement, we have

$$\begin{aligned}
\int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\int_K \delta_K \mathbf{g}_h \cdot \nabla_\ell(\mathbf{b}_h \cdot \nabla_x v_h)\ \mathbf{g}_h \cdot \nabla_\ell v_h &= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h}\frac{\partial}{\partial \ell_q}\left(\mathbf{b}_{p,h}\frac{\partial \phi_h}{\partial x_p}\psi_h\right)\ \mathbf{g}_{q,h}\frac{\partial \psi_h}{\partial \ell_q}\phi_h \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{b}_{p,h}\frac{\partial \phi_h}{\partial x_p}\phi_h\ \mathbf{g}_{q,h}\frac{\partial \psi_h}{\partial \ell_q}\ \mathbf{g}_{q,h}\frac{\partial \psi_h}{\partial \ell_q} \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{b}_{p,h}\frac{\partial \phi_h^2}{\partial x_p}\ \mathbf{g}_{q,h}\frac{\partial \psi_h}{\partial \ell_q}\ \mathbf{g}_{q,h}\frac{\partial \psi_h}{\partial \ell_q} \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{b}_{p,h}\frac{\partial}{\partial x_p}\left(\phi_h^2\frac{\partial \psi_h}{\partial \ell_q}\frac{\partial \psi_h}{\partial \ell_q}\right)\ \mathbf{g}_{q,h}\ \mathbf{g}_{q,h} \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{b}_{p,h}\frac{\partial}{\partial x_p}\left(\frac{\partial v_h}{\partial \ell_q}\frac{\partial v_h}{\partial \ell_q}\right)\ \mathbf{g}_{q,h}\ \mathbf{g}_{q,h} \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K |\mathbf{g}_h|^2 \mathbf{b}_h \cdot \nabla_x(\nabla_\ell v_h \nabla_\ell v_h) \\
&= -\int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K |\mathbf{g}_h|^2 \nabla_x \cdot \mathbf{b}_h\ \nabla_\ell v_h \nabla_\ell v_h = 0,
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(\epsilon \Delta_x u_h^n,\ \mathbf{g}_h \cdot \nabla_\ell v_h\right)_K &= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \frac{\partial^2 \phi_h}{\partial x_p^2}\psi_h\ \mathbf{g}_{q,h}\frac{\partial \psi_h}{\partial \ell_q}\phi_h \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \frac{\partial^2 \phi_h}{\partial x_p^2}\ \phi_h\ \mathbf{g}_{q,h}\frac{\partial \psi_h^2}{\partial \ell_q} \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{g}_{q,h}\frac{\partial}{\partial \ell_q}\left(\phi_h\psi_h\frac{\partial^2 \phi_h}{\partial x_p^2}\ \psi_h\right) \\
&= \int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K \mathbf{g}_h \cdot \nabla_\ell\left(v_h \Delta_x v_h\right) \\
&= -\int_{\Omega_X} \sum_{K\in\mathcal{S}_h}\frac{1}{2}\int_K \delta_K \nabla_\ell \cdot \mathbf{g}_h\ v_h \Delta_x v_h = 0,
\end{aligned}$$

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$$\int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(\mathbf{b}_h\cdot\nabla_x v_h,\ \mathbf{g}_h\cdot\nabla_\ell v_h\right)_K = \int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\int_K \delta_K \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{b}_{p,h}\frac{\partial\phi_h}{\partial x_p}\psi_h\ \mathbf{g}_{q,h}\frac{\partial\psi_h}{\partial\ell_q}\phi_h$$
$$= \frac{1}{4}\int_\Omega \sum_{p=1}^{d}\sum_{q=1}^{s} \mathbf{b}_{p,h}\frac{\partial\phi_h^2}{\partial x_p}\ \mathbf{g}_{q,h}\frac{\partial\psi_h^2}{\partial\ell_q}$$
$$= \frac{1}{4}\int_\Omega \mathbf{g}_h\cdot\nabla_\ell\left(\mathbf{b}_h\cdot\nabla_x v_h^2\right) = -\frac{1}{4}\int_\Omega \nabla_\ell\cdot\mathbf{g}_h\left(\mathbf{b}_h\cdot\nabla_x v_h^2\right) = 0. \qquad \square$$

Lemma 4.3 (coercivity of $a_{OS}(\cdot,\cdot)$)**.** *Let the discrete form of the assumptions (1.2) and (1.3) be satisfied. Then, the bilinear form $a_{OS}(\cdot,\cdot)$ associated with the operator-splitting method satisfies*

$$a_{OS}(u_h^n, u_h^n) \geq \tau |||u_h^n|||_{OS}^2 \tag{4.6}$$

with

$$|||u_h^n|||_{OS}^2 = \sum_{K'\in\mathcal{T}_h}\sum_{K\in\mathcal{S}_h} \delta_K\left(\epsilon\|\mathbf{g}_h\cdot\nabla_\ell(\nabla_x u_h^n)\|_{1,1,K',K}^2\right).$$

Proof. Using the definitions of the finite element spaces V_h and Q_h, and the Lemma 4.2 in $a_{OS}(u_h^n, u_h^n)$, we get

$$a_{OS}(u_h^n, u_h^n) \geq \frac{\tau}{2}\int_\Omega \epsilon\mathbf{g}_h\cdot\nabla_\ell((\nabla_x u_h^n)^2) + \tau\int_{\Omega_X}\epsilon\sum_{K\in\mathcal{S}_h}\int_K \delta_K\mathbf{g}_h\cdot\nabla_\ell(\nabla_x u_h^n)\mathbf{g}_h\cdot\nabla_\ell(\nabla_x u_h^n)$$
$$= \tau\sum_{K'\in\mathcal{T}_h}\sum_{K\in\mathcal{S}_h}\delta_K\left(\epsilon\|\mathbf{g}_h\cdot\nabla_\ell(\nabla_x u_h^n)\|_{1,1,K',K}^2\right). \qquad \square$$

4.1. Stability of the operator-split finite element discretization

The stability of the operator-split discrete equation (4.1) is studied in this section. In the stability analysis, we use the following discrete form of the Grownwall's lemma.

Lemma 4.4 (discrete Grownwall's lemma)**.** *Assume that w^n, $n\geq 0$, satisfies*

$$w^n \leq \alpha^n + \sum_{k=0}^{n-1}\beta^k w^k, \quad \text{for } n\geq 0,$$

where α^n is a non-decreasing and $\beta^n\geq 0$. Then, we have

$$w^n \leq \alpha^n \exp\left\{\sum_{k=0}^{n-1}\beta^k\right\}.$$

Proof. The proof is rather technical, see for example [27]. $\square$

Lemma 4.5 (stability)**.** *For given $T>0$, let $\tau = T/N_T$, $N_T\geq 1$ and $\tau\leq 1/2$, be an uniform time step. Then, for $u_h^N\in W_h$ with the additional condition*

$$\delta\leq\frac{\tau}{2}, \qquad \text{where} \qquad \delta = \max\{\delta_K\}, \qquad \forall\ K\ \in\mathcal{S}_h, \tag{4.7}$$

we have the following stability estimate

$$\|u_h^N\|^2 + \tau\sum_{n=1}^{N}|||u_h^n|||^2 + 2\tau\sum_{n=1}^{N}\tau|||u_h^n|||_{OS}^2 \leq \left\{\|u_h^0\|^2 + 2\sum_{n=1}^{N}\tau(1+2\delta)\|f^n\|^2\right\}\exp(2T),$$

for $1\leq N\leq N_T$.

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Proof. Consider the operator-split discrete equation (4.1), and set $v_h = u_h^n$ to get

$$\left(u_h^n - u_h^{n-1},\ u_h^n\right) + \tau\ a_{LS}(u_h^n,\ u_h^n) + \tau\ a_{OS}(u_h^n,\ u_h^n) = \tau(f^n,\ u_h^n) + \tau \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(f^n,\ \mathbf{g}_h \cdot \nabla_\ell u_h^n\right)_K - \int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(u_h^n - u_h^{n-1},\ \mathbf{g}_h \cdot \nabla_\ell u_h^n\right)_K . \tag{4.8}$$

Now, applying the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ for the first term, and using the Lemmas 2.1 and 4.3 for the bilinear forms $a_{LS}(u_h^n,\ u_h^n)$ and $a_{OS}(u_h^n,\ u_h^n)$ in (4.8), we get

$$\|u_h^n\|^2 + \|u_h^n - u_h^{n-1}\|^2 + 2\tau|||u_h^n|||^2 + 2\tau^2|||u_h^n|||_{OS}^2 \ \le\ \|u_h^{n-1}\|^2 + 2\tau|(f^n, u_h^n)| + 2\tau\left|\int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(f^n,\ \mathbf{g}_h \cdot \nabla_\ell u_h^n\right)_K\right| + 2\left|\int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(u_h^n - u_h^{n-1},\ \mathbf{g}_h \cdot \nabla_\ell u_h^n\right)_K\right|.$$

Applying the Cauchy-Schwarz inequality and Young's inequality to the right hand side terms, we get

$$2\tau|(f^n, u_h^n)| \le \tau\|f^n\|^2 + \tau\|u_h^n\|^2,$$

$$2\tau\left|\int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(f^n,\ \mathbf{g}_h \cdot \nabla_\ell u_h^n\right)_K\right| \le 2\tau\delta\|f^n\|^2 + \frac{1}{2}\sum_{K\in\mathcal{S}_h}\sum_{K'\in\mathcal{T}_h} \tau\delta_K\|\mathbf{g}_h \cdot \nabla_\ell u_h^n\|_{0,K',K}^2,$$

$$2\left|\int_{\Omega_X} \sum_{K\in\mathcal{S}_h} \delta_K \left(u_h^n - u_h^{n-1},\ \mathbf{g}_h \cdot \nabla_\ell u_h^n\right)_K\right| \le 2\frac{\delta}{\tau}\|u_h^n - u_h^{n-1}\|^2 + \frac{1}{2}\sum_{K\in\mathcal{S}_h}\sum_{K'\in\mathcal{T}_h} \tau\delta_K\|\mathbf{g}_h \cdot \nabla_\ell u_h^n\|_{0,K',K}^2.$$

Using $\delta \le \tau/2$, we get

$$(1-\tau)\|u_h^n\|^2 + \tau|||u_h^n|||^2 + 2\tau^2|||u_h^n|||_{OS}^2 \ \le\ \|u_h^{n-1}\|^2 + \tau(1+2\delta)\|f^n\|^2$$

Divide the inequality by $(1-\tau)$, use $1 \le 1/(1-\tau) \le 1+2\tau \le 2$ for $\tau \le 1/2$, and sum over $n = 1, \ldots, N$, to get

$$\|u_h^N\|^2 + \sum_{n=1}^{N} \tau|||u_h^n|||^2 + 2\tau\sum_{n=1}^{N} \tau|||u_h^n|||_{OS}^2 \ \le\ \|u_h^0\|^2 + 2\sum_{n=1}^{N} \tau(1+2\delta)\|f^n\|^2 + 2\sum_{n=1}^{N-1} \tau\|u_h^n\|^2.$$

Finally, applying the discrete form of the Grownwall's lemma 4.4, we get the statement of the lemma. □

Remark 4.6. Note that we have used $\delta \le \tau/2$ in the above stability estimate which is a strong assumption. In [2], unconditional stability has been proved for smooth data. Also, for a special case of rough data this assumption has be relaxed to $\delta^2 \le \tau$ using the time dissipation introduced by the backward Euler method. However, a complete discussion of this topic is beyond the scope of this paper.

4.2. *A priori* error estimates

To derive the error estimate for the solution of the operator-split finite element discretization (4.1), let us introduce the following approximation properties, (*cf.* Thm. 4.8.12 and Cor. 4.8.15 in [1]).

(A1) Approximation property of V_h: there exist an interpolation operator $I_X \in \mathcal{L}(H_0^1(\Omega_X); V_h)$ such that for all $1 \le s \le r+1$

$$\|I_X u\|_{H^s(\Omega_X)} \le C\|u\|_{H^s(\Omega_X)}, \quad u \in H^s(\Omega_X) \cap H_0^1(\Omega_X),$$
$$\|u - I_X u\|_{L^2(\Omega_X)} + h_x\|u - I_X u\|_{H^1(\Omega_X)} \le Ch_x^s|u|_{H^s(\Omega_X)}, \quad u \in H^s(\Omega_X) \cap H_0^1(\Omega_X).$$

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(A2) Approximation property of Q_h: there exist an interpolation operator $I_L \in \mathcal{L}(H_0^1(\Omega_L); Q_h)$ such that for all $1 \le s \le r+1$

$$\begin{aligned} \|I_L u\|_{H^s(\Omega_L)} &\le C\|u\|_{H^s(\Omega_L)}, \quad u \in H^s(\Omega_L) \cap H_0^1(\Omega_L), \\ \|u - I_L u\|_{L^2(\Omega_L)} + h_\ell \|u - I_L u\|_{H^1(\Omega_L)} &\le C h_\ell^s |u|_{H^s(\Omega_L)} \quad u \in H^s(\Omega_L) \cap H_0^1(\Omega_L). \end{aligned}$$

Here, $\mathcal{L}(X;Y)$ denotes the set of linear and continuous mappings from X to Y. Now, we define a projection operator $I_h \in \mathcal{L}(H^{r+1,r+1}(\Omega) \cap H^{1,1}(\Omega); V_h \otimes Q_h)$ by

$$I_h : I_X I_L = I_L I_X.$$

Theorem 4.7. *Let u be the smooth enough solution of (1.1) and the approximation properties A1 and A2 be satisfied. Further, let the stabilization parameters fulfill (2.5) and (4.7) for all $K \in \mathcal{S}_h$. Then, the error $e_h^n := u(t^n) - u_h^n$, the estimate*

$$\|e_h^n\|^2 + \sum_{k=1}^{n} \tau |||e_h^n|||^2 + \tau \sum_{k=1}^{n} \tau |||e_h^n|||_{OS}^2 \le C_u \left(h^{2r} + \delta \epsilon^2 h^{2r-2} + \tau^2 \right), \qquad n = 1, \ldots, N,$$

holds true. Here, C_u is a constant depending on certain norms of the solution specified within the proof of the theorem.

Proof. The error analysis of (4.1) starts by decomposition of the error into two parts in which the first measures the interpolation error and the other measures the difference of the interpolation and the discrete solution.

$$e_h^n := u(t^n) - u_h^n = (u(t^n) - I_h u(t^n)) + (I_h u(t^n) - u_h^n) =: \eta^n + \xi^n.$$

The interpolation error η^n can be estimated using the approximation properties of the finite element spaces. For the error $\xi^n \in W_h$, apply $\xi^n = u(t^n) - u_h^n - \eta^n$ and $v_h = \xi^n$ in (4.1) to obtain

$$\begin{aligned} &\left(\bar{\partial}\xi^n,\ \xi^n\right) + a_{LS}(\xi^n,\ \xi^n) + a_{OS}(\xi^n,\ \xi^n) + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\bar{\partial}\xi^n,\ \mathbf{g}_h \cdot \nabla_\ell \xi^n\right)_K \\ &\quad = \left(\bar{\partial} u(t^n), \xi^n\right) + a_{LS}(u(t^n),\ \xi^n) + a_{OS}(u(t^n),\ \xi^n) + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\bar{\partial} u(t^n),\ \mathbf{g}_h \cdot \nabla_\ell \xi^n\right)_K \\ &\qquad - (f^n,\ \xi^n) - F_S(f^n, \xi^n) \\ &\qquad - \left(\bar{\partial}\eta^n,\ \xi^n\right) - a_{LS}(\eta^n,\ \xi^n) - a_{OS}(\eta^n,\ \xi^n) - \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\bar{\partial}\eta^n,\ \mathbf{g}_h \cdot \nabla_\ell \xi^n\right)_K \\ &\quad = \left(\bar{\partial} u(t^n) - \frac{\partial u(t^n)}{\partial t}, \xi^n\right) + a_{OS}(u(t^n),\ \xi^n) + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\bar{\partial} u(t^n) - \frac{\partial u(t^n)}{\partial t},\ \mathbf{g}_h \cdot \nabla_\ell \xi^n\right)_K \\ &\qquad + \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\epsilon \Delta_x u(t^n),\ \mathbf{g}_h \cdot \nabla_\ell \xi^n\right)_K - \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\mathbf{b}_h \cdot \nabla_x u(t^n),\ \mathbf{g}_h \cdot \nabla_\ell \xi^n\right)_K \\ &\qquad - \left(\bar{\partial}\eta^n,\ \xi^n\right) - a_{LS}(\eta^n,\ \xi^n) - a_{OS}(\eta^n,\ \xi^n) - \int_{\Omega_X} \sum_{K \in \mathcal{S}_h} \delta_K \left(\bar{\partial}\eta^n,\ \mathbf{g}_h \cdot \nabla_\ell \xi^n\right)_K, \end{aligned} \tag{4.9}$$

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where (2.4) is used in the second step. Multiplying (4.9) with τ, we get

$$\begin{aligned}\left(\xi^n-\xi^{n-1},\xi^n\right)+\tau a_{LS}(\xi^n,\xi^n)+\tau a_{OS}(\xi^n,\xi^n)=\tau(E_1^n,\xi^n)+\tau\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K(E_2^n,\ \mathbf{g}_h\cdot\nabla_\ell\xi^n)_K\\ -\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K\left(\xi^n-\xi^{n-1},\ \mathbf{g}_h\cdot\nabla_\ell\xi^n\right)_K\\ +\tau\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K(E_3^n,\ \mathbf{g}_h\cdot\nabla_\ell\nabla_x\xi^n)_K+\tau(E_4^n,\nabla_x\xi^n),\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}E_1^n&:=\bar\partial u(t^n)-\frac{\partial u(t^n)}{\partial t}+\tau\mathbf{g}_h\cdot\nabla_\ell(\mathbf{b}_h\cdot\nabla_x u(t^n))-\bar\partial\eta^n-\mathbf{g}_h\cdot\nabla_\ell\eta^n-\mathbf{b}_h\cdot\nabla_x\eta^n-\tau\mathbf{g}_h\cdot\nabla_\ell(\mathbf{b}_h\cdot\nabla_x\eta^n),\\ E_2^n&:=E_1^n+\epsilon\Delta_x\eta^n,\\ E_3^n&:=\tau\epsilon\mathbf{g}_h\cdot\nabla_\ell(\nabla_x u(t^n))-\tau\epsilon\mathbf{g}_h\cdot\nabla_\ell(\nabla_x\eta^n),\\ E_4^n&:=E_3^n-\epsilon\nabla_x\eta^n.\end{aligned}$$

The error equation (4.10) is similar to the operator-split discrete form (4.8) used in the stability estimate, except the E_3 and E_4 terms. Therefore, to use the stability estimate (Lem. 4.5) in (4.10), we first derive the estimates for E_3 and E_4 terms. Using the Cauchy-Schwarz's inequality and Young's inequality, we get

$$|(E_4^n,\nabla_x\xi^n)|\le\frac{1}{2\epsilon}\|E_4^n\|^2+\frac{\epsilon}{2}\|\nabla_x\xi^n\|^2\tag{4.11}$$

and

$$\begin{aligned}\left|\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K(E_3^n,\ \mathbf{g}_h\cdot\nabla_\ell\nabla_x\xi^n)_K\right|&=\left|\int_{\Omega_X}\sum_{K\in\mathcal{S}_h}\delta_K\left(\frac{1}{(\tau\epsilon)^{1/2}}E_3^n,\ (\tau\epsilon)^{1/2}\mathbf{g}_h\cdot\nabla_\ell\nabla_x\xi^n\right)_K\right|\\ &\le\frac{1}{4\epsilon}\|E_3^n\|^2+\frac{\tau\epsilon}{2}\sum_{K'\in\mathcal{T}_h}\sum_{K\in\mathcal{S}_h}\delta_K\|\mathbf{g}_h\cdot\nabla_\ell\nabla_x\xi^n\|^2,\end{aligned}\tag{4.12}$$

where the assumption $\delta\le\tau/2$ is applied. Using the estimates (4.11) and (4.12), and applying the Lemma 4.5 to the error equation (4.10), we get

$$\|\xi^N\|^2+\tau\sum_{n=1}^N|||\xi^n|||^2+\tau\sum_{n=1}^N\tau|||\xi^n|||_{OS}^2\le\exp(2T)\left\{\|\xi^0\|^2+2\sum_{n=1}^N\tau\left(\|E_1^n\|^2+2\delta\|E_2^n\|^2+\frac{1}{2\epsilon}\|E_3^n\|^2+\frac{1}{\epsilon}\|E_4^n\|^2\right)\right\}.\tag{4.13}$$

Now, using the Cauchy-Schwarz's inequality and applying the Taylor's theorem with remainder for the E_2 term, we get

$$\begin{aligned}\|E_1^n\|^2&\le C\left(\tau\int_{t^{n-1}}^{t^n}\left\|\frac{\partial^2u(s)}{\partial s^2}\right\|^2\,\mathrm{d}s+\frac{1}{\tau}\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\eta(s)}{\partial s}\right\|^2\,\mathrm{d}s\right)+\tau^2|\mathbf{b}|^2|\mathbf{g}|^2\|\nabla_\ell\nabla_x u(t^n)\|^2\\ &\quad+\tau^2|\mathbf{b}|^2|\mathbf{g}|^2\|\nabla_\ell\nabla_x\eta^n\|^2+|\mathbf{g}|^2\|\nabla_\ell\eta^n\|^2+|\mathbf{b}|^2\|\nabla_x\eta^n\|^2,\\ \delta\|E_2^n\|^2&\le\frac{\tau}{2}\|E_1^n\|^2+\delta\epsilon^2\|\Delta_x\eta^n\|,\\ \frac{1}{\epsilon}\|E_3^n\|^2&\le\tau^2|\mathbf{g}|^2\|\nabla_\ell\nabla_x u(t^n)\|^2+\tau^2|\mathbf{g}|^2\|\nabla_\ell\nabla_x\eta^n\|^2,\\ \frac{1}{\epsilon}\|E_4^n\|^2&\le\frac{1}{\epsilon}\|E_3^n\|^2+\|\nabla_x\eta^n\|^2,\end{aligned}$$

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where $|\mathbf{b}|^2 = \|\mathbf{b}\|^2_{L^\infty(0,T;\Omega_X)}$ and $|\mathbf{g}|^2 = \|\mathbf{g}\|^2_{L^\infty(0,T;\Omega_X)}$. Collecting all these bounds, we get

$$\begin{aligned}\|\xi^N\|^2 + \sum_{n=1}^{N}\tau|||\xi^n|||^2 + \tau\sum_{n=1}^{N}\tau|||\xi^n|||^2_{OS} \le C\exp(2T)\Bigg(&\|\xi^0\|^2 + \tau^2\left\|\frac{\partial^2 u}{\partial s^2}\right\|^2_{L^2(0,T;L^2(\Omega))}\\ &+ \tau^2|\mathbf{b}|^2|\mathbf{g}|^2\|\nabla_\ell\nabla_x u(t^n)\|^2_{l^2(0,T;L^2(\Omega))}\\ &+ \tau^2|\mathbf{g}|^2\|\nabla_\ell\nabla_x u(t^n)\|^2_{l^2(0,T;L^2(\Omega))} + \int_0^T\left\|\frac{\partial\eta(s)}{\partial s}\right\|^2 \mathrm{d}s + \sum_{n=1}^{N}\tau|\mathbf{b}|^2\|\nabla_x\eta^n\|^2\\ &+ \sum_{n=1}^{N}\tau|\mathbf{g}|^2\|\nabla_\ell\eta^n\|^2 + \sum_{n=1}^{N}\tau\|\nabla_x\eta^n\|^2 + \tau^2\sum_{n=1}^{N}\tau|\mathbf{b}|^2|\mathbf{g}|^2\|\nabla_\ell\nabla_x\eta^n\|^2\\ &+ \tau^2\sum_{n=1}^{N}\tau|\mathbf{g}|^2\|\nabla_\ell\nabla_x\eta^n\|^2 + \sum_{n=1}^{N}\tau\delta\epsilon^2\|\Delta_x\eta^n\|^2\Bigg).\end{aligned} \tag{4.14}$$

Finally, we estimate the interpolation error to get

$$\begin{aligned}\|\eta\| = \|u - I_X I_L u\| &\le \|u - I_X u\| + \|I_X u - I_X I_L u\|\\ &\le \left(\int_{\Omega_L}\|u - I_X u\|^2_{L^2(\Omega_X)}\right)^{\frac{1}{2}} + \left(\int_{\Omega_X}\|I_X u - I_L I_X u\|^2_{L^2(\Omega_L)}\right)^{\frac{1}{2}}\\ &\le Ch^r\left(\int_{\Omega_L}|u|^2_{H^r(\Omega_X)}\right)^{\frac{1}{2}} + Ch^r\left(\int_{\Omega_X}\|I_X u\|^2_{H^s(\Omega_L)}\right)^{\frac{1}{2}}\\ &\le Ch^r\left(\|u\|^2_{L^2(\Omega_L;H^r(\Omega_X))} + |u|^2_{L^2(\Omega_X;H^s(\Omega_L))}\right).\end{aligned} \tag{4.15}$$

Similarly, for the derivatives of η we get the following bounds

$$\begin{aligned}\left\|\frac{\partial\eta}{\partial s}\right\| &\le Ch^r\left(\left\|\frac{\partial u}{\partial t}\right\|_{L^2(\Omega_L;H^r(\Omega_X))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(\Omega_X;H^r(\Omega_L))}\right),\\ \|\nabla_x\eta\| &\le \|\nabla_x u - \nabla_x I_X u\| + \|\nabla_x I_X u - I_L\nabla_x I_X u\|\\ &\le Ch^r\left(\|u\|_{L^2(\Omega_L;H^{r+1}(\Omega_X))} + \|u\|_{H^1(\Omega_X;H^r(\Omega_L))}\right),\\ \|\nabla_\ell\,\eta\| &\le \|\nabla_\ell\, u - \nabla_\ell\, I_L u\| + \|\nabla_\ell\, I_L u - I_X\nabla_\ell\, I_L u\|\\ &\le Ch^r\left(\|u\|_{L^2(\Omega_X;H^{r+1}(\Omega_L))} + \|u\|_{H^1(\Omega_L;H^r(\Omega_X))}\right),\\ \|\nabla_x\nabla_\ell\,\eta\| &\le \|\nabla_x\nabla_\ell\, u - \nabla_x I_X\nabla_\ell\, u\| + \|\nabla_\ell\,\nabla_x I_X u - \nabla_\ell\, I_L\nabla_x I_X u\|\\ &\le Ch^r\left(\|u\|_{H^1(\Omega_L;H^{r+1}(\Omega_X))} + \|u\|_{H^1(\Omega_X;H^{r+1}(\Omega_L))}\right),\\ \|\Delta_x\eta\| &\le \|\Delta_x u - \Delta_x I_X u\| + \|\Delta_x I_X u - I_L\Delta_x I_X u\|\\ &\le Ch^{r-1}\left(\|u\|_{L^2(\Omega_L;H^{r+1}(\Omega_X))} + \|u\|_{H^2(\Omega_X;H^r(\Omega_L))}\right).\end{aligned}$$

Substituting the above estimates, and using $\|\xi^0\| \le \|\eta^0\|$, we get the statement of the theorem. □

5. Numerical results

From the computational point of view, the communication of the finite element solution $\hat{u}_h^n$ from L-direction step to the X-direction step and *vice versa* is one of the challenging task in splitting schemes. Two variants of operator-splitting finite element algorithms based on the nodal points of the finite elements and the quadrature points have been presented in [7]. Here, we use the more efficient nodal point based operator-splitting algorithm to support the analysis of the proposed Galerkin/SUPG operator-splitting finite element scheme in the previous

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section. As a test example, we consider the equation (1.1) in $\Omega_X = (0,1)^2$ and $\Omega_L = (0,1)$. Further, we choose $\epsilon = 1$, $\mathbf{b} = (0,0)$, $\mathbf{g} = 1$ and $T = 1$. The source term f is chosen such that

$$u(t,x,\ell) = \mathrm{e}^{-0.1t}\sin(\pi x_1)\cos(\pi x_2)\cos(\pi \ell)$$

is the solution of (1.1) with the above data. For this configuration, the operator-split population balance equations (3.1) and (3.2) will reduce to a pure advection and a time dependent diffusion equations, respectively. At level 1, the initial grid of Ω_X contains four quadrilaterals, whereas Ω_L contains two intervals. The higher grid levels of Ω_X and Ω_L are obtained by successively refining their respective initial grids uniformly. In this numerical study, we used $Q_1 \otimes P_1$, that is, bilinear and linear finite elements on quadrilaterals and intervals, respectively, and $Q_2 \otimes P_2$, that is biquadratic and quadratic finite elements on quadrilaterals and intervals, respectively. Further, the backward Euler scheme with $\tau = h^2$ for $Q_1 \otimes P_1$ and $\tau = h^3$ for $Q_2 \otimes P_2$ is used in all computations. To calculate the error in space and time we use

$$\ell^\infty(0,T;L^2(\Omega)) := \sup_{n=1,\ldots,N} \|u(t^n) - u_h(t^n)\|_{L^2(\Omega)}$$

$$\ell^2(0,T;L^2(\Omega)) := \left(\sum_{n=1}^{N} \delta t \|u(t^n) - u_h(t^n)\|^2_{L^2(\Omega)}\right)^{1/2},$$

where the L^2 error is calculated by applying the quadrature rules in X- and L-directions, *i.e.*

$$||u||^2_{L^2(\Omega)} = \int_\Omega u^2 \,\mathrm{d}x\,\mathrm{d}\ell = \int_{\Omega_X}\left(\int_{\Omega_L} u^2(x,\ell)\mathrm{d}\ell\right)\mathrm{d}x \approx \sum_{K_i^X}\sum_{m=1}^{NX} w_m^x \int_{\Omega_L} u^2(x_m,\ell)\mathrm{d}\ell$$

$$\approx \sum_{K_i^X}\sum_{m=1}^{NX} w_m^x \sum_{K_j^L}\sum_{l=1}^{NL} w_l^\ell u^2(x_m,\ell_l).$$

Here, K_i^X and K_j^L are cells in Ω_X and Ω_L, respectively. Further, NX, w_m^x and NL, w_l^ℓ are the number of quadrature points, quadrature weights in each K_i^X and K_j^L, respectively. The computational results obtained using the Galerkin/Galerkin and Galerkin/SUPG discretizations are presented in Figure 1. For the $Q_1 \otimes P_1$ finite element pair, the numerical errors obtained in both the Galerkin/Galerkin and Galerkin/SUPG discretizations are similar. For the $Q_2 \otimes P_2$ finite element pair, the numerical error obtained with the Galerkin/SUPG discretization is slightly less than the numerical error obtained with Galerkin/Galerkin discretization. Nevertheless, in all cases the optimal order of convergence is obtained. These computational results show that we can use tailored discretization methods in the operator-splitting finite element scheme. Further, it show that the consistency error induced by the splitting in the backward Euler heterogeneous finite element scheme does not affect the optimal order of convergence.

6. Summary

We have presented a novel operator-splitting Galerkin/SUPG finite element method for high-dimensional population balance equations, which depend on both physical and internal property coordinates. The proposed scheme alleviates the "curse of dimensionality" associated with the solution of a population balance equation in a population balance system. In our scheme, we split the population balance equation into two low-dimensional equations, where the first equation in the physical space domain and the second equation in the internal space domain. This splitting facilitates to use different discretizations in physical and internal spaces. Thus, to suppress the spurious oscillations in the numerical solution due to the absence of the diffusion in the internal direction, the standard Galerkin and the Streamline Upwind Petrov Galerkin (SUPG) finite element discretizations are used for the physical and internal spaces, respectively. Further, we were able to estimate the operator-splitting

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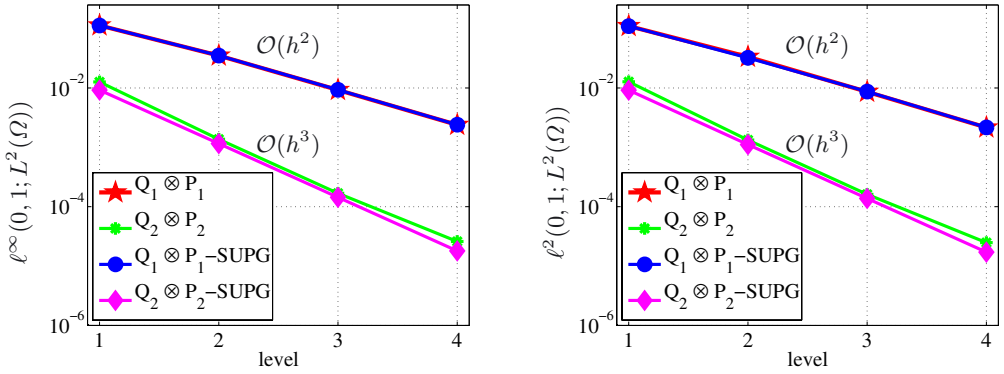


FIGURE 1. Numerical errors in $\ell^\infty(0,1;L^2(\Omega))$ and $\ell^2(0,1;L^2(\Omega))$ norms for the 2D + 1D test example.

error and prove the stability of the Galerkin/SUPG finite element method for the population balance equation. In the error estimate, it is shown that a slightly more regularity, *i.e.*, mixed partial derivative of the solution should be bounded, is required to obtain the optimal order of convergence. Further, the numerical results were presented to support the analysis and the optimal order of convergence was obtained for the first and second order spatial approximations.

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