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Affine almost automorphic actions on compact nilmanifolds

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Abstract. We discuss conditions under which an affine automorphism of a compact nilmanifold is almost automorphic, and the structure of such automorphisms from dynamical as well as algebraic points of view.

1. *Introduction*

Let X be a compact second countable space and α a homeomorphism of X. A point $x \in X$ is said to be *almost automorphic* for the action of α if for any $y \in X$ and any sequence $\{k_i\}$ in $\mathbb{N}$ such that $\alpha^{k_i}(x) \to y$ we have $\alpha^{-k_i}(y) \to x$. The action of α on X is said to be *almost automorphic* if every $x \in X$ is almost automorphic for the α-action. It is generally recognized that such a behaviour is rare for a dynamical system. In this note we discuss almost automorphic points of affine automorphisms of compact nilmanifolds, and deduce certain necessary conditions on the structure of an affine automorphism of a compact nilmanifold, for it to be almost automorphic.

Let N be a simply connected nilpotent Lie group and Γ be a lattice in N (i.e. Γ is a discrete subgroup of N such that N/Γ is compact). Then N/Γ is a compact manifold, and the quotients arising in this way are called nilmanifolds. By an automorphism A of N/Γ we mean a self-map such that $A(g\Gamma) = \tilde{A}(g)\Gamma$ for all $g \in N$, where $\tilde{A}$ is a (Lie) automorphism of N leaving Γ invariant (namely such that $\tilde{A}(\Gamma) = \Gamma$). For $a \in N$, the map $T_a : N/\Gamma \to N/\Gamma$ given by $g\Gamma \mapsto ag\Gamma$, for all $g \in N$, is called the translation by a. A transformation of N/Γ of the form $T = T_a \circ A$, where $a \in N$ and A is an automorphism of N/Γ, is called an *affine automorphism* of N/Γ. An automorphism $\tilde{A}$ of N is said to

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be unipotent if the corresponding automorphism of the Lie algebra is a unipotent linear transformation. We call an automorphism of N/Γ unipotent if it is induced by a unipotent automorphism $\tilde{U}$ such that $\tilde{U}(\Gamma) = \Gamma$, and an affine automorphism of N/Γ of the form $T_a \circ U$, where $a \in N$ and U is a unipotent automorphism, is called a unipotent affine automorphism.

For almost automorphic points of unipotent affine automorphisms of compact nilmanifolds we prove the following (see Proposition 6.1, and the remark preceding it, for the motivation to restrict to unipotent affine transformations).

THEOREM 1.1. *Let N be a simply connected nilpotent Lie group and Γ be a lattice in N. Let $T = T_a \circ U$ be an affine automorphism of N/Γ, where U is a unipotent automorphism of N/Γ. Let $\tilde{U}$ be the automorphism of N corresponding to U. Let $x \in N$ be given. Then $x\Gamma$ is almost automorphic for the T-action if and only if there exists a closed connected abelian subgroup A of N such that $A\Gamma$ is closed, $x^{-1}a\tilde{U}(x) \in A$ and $\tilde{U}(g) = g$ for all $g \in A$.*

The 'if' part of the assertion is straightforward: let A be the closed connected abelian subgroup as in statement. Then the subset $C = xA\Gamma/\Gamma$ of N/Γ is T-invariant, since for any $g \in A$ we have $T(xg\Gamma) = a\tilde{U}(xg)\Gamma = a\tilde{U}(x)g\Gamma = x(x^{-1}a\tilde{U}(x))g\Gamma \in xA\Gamma$. We see also that if C is realized canonically as $B/(B \cap \Delta)$ where $B = xAx^{-1}$ and $\Delta = B \cap x\Gamma x^{-1}$, then the action of T on C corresponds to the translation by the element $a\tilde{U}(x)x^{-1} \in B$. As A is abelian, so is B and hence for the translations of $B/B \cap \Delta$ all points are almost automorphic.

The proof of the converse is based on the following theorem, which provides a more specific conclusion for a special class of systems; the general case is deduced by spotting such systems among the subsystems and their quotients (see Proposition 3.1).

THEOREM 1.2. *Let N be a simply connected nilpotent Lie group such that $[N, N]$ is one-dimensional, and let Γ be a lattice in N. Let $a \in N$ be such that the T_a-orbit of Γ is dense in N/Γ. Let $\mathbb{T} = [N, N]/([N, N] \cap \Gamma)$, realized as the circle group, and also (canonically) as the subset $[N, N]\Gamma/\Gamma$ of N/Γ. Let $\theta \in \mathbb{T}$ be given. Then there exist a sequence $\{k_i\}$ in $\mathbb{N}$, $\omega \in \mathbb{T}$ and ψ a root of unity in $\mathbb{T}$ such that $T_a^{k_i}(\Gamma) \to \omega$ and $T_a^{-k_i}(\omega) \to \theta\psi$; in particular, if θ is not a root of unity, then $\{T_a^{-k_i}(\omega)\}$ does not converge to Γ, and hence Γ is not almost automorphic.*

Remark 1.3. It turns out that Theorem 1.1 can also be deduced from certain results from the general theory of topological dynamical systems. This was realized by the authors while following up a comment by the referee on an earlier version of this paper; the authors are grateful to the referee in this respect. The argument however involves putting together various results (see Remark 4.3 for a sketch), and it seems to the authors that it would not be conveniently accessible to a broad readership, on account of the technical nature of the results involved. The present proof via Theorem 1.2, on the other hand, follows a direct route, and is more transparent on how the non-almost automorphic points arise.

The theorem implies the following towards unipotent affine automorphisms being almost automorphic.

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COROLLARY 1.4. *Let the notation be as in Theorem 1.1. Then T is almost automorphic if and only if there exists a closed connected abelian subgroup A of N such that $A\Gamma$ is closed, $x^{-1}a\tilde{U}(x) \in A$ for all $x \in N$ and $\tilde{U}(g) = g$ for all $g \in A$. A translation T_a of N/Γ is almost automorphic if and only if there exists a closed connected abelian normal subgroup A of N such that $a \in A$ and $A\Gamma$ is closed.*

The conclusion of the corollary means that, from the point of view of dynamics, almost automorphic affine automorphisms of compact nilmanifolds consist of a collection of translations of tori put together.

Corollary 1.4 also shows that the condition of being almost automorphic imposes strong restrictions on the algebraic structure of the affine automorphism. We analyse this further and describe certain simple necessary conditions at the 'infinitesimal' (or the Lie algebra) level for the condition to hold.

COROLLARY 1.5. *Let the notation be as in Theorem 1.1. Furthermore, let $\mathfrak{N}$ be the Lie algebra of N, $\mathfrak{U}$ be the Lie automorphism of $\mathfrak{N}$ corresponding to $\tilde{U}$ and Ad_a be the adjoint transformation of $\mathfrak{N}$ corresponding to a. Let I and 0 denote the identity and zero transformations of $\mathfrak{N}$, respectively. Suppose that T is almost automorphic. Then the following hold:*

(i) $(\mathfrak{U} - I)(\mathrm{Ad}_a \circ \mathfrak{U} - I) = 0$*; in particular, if T is an automorphism (namely a is the identity element), then $\tilde{U}$ has nilrank at most one, namely $(\tilde{U} - I)^2 = 0$;*

(ii) *the image of $\mathrm{Ad}_a \circ \mathfrak{U} - I$ is an abelian Lie subalgebra.*

As we shall see, these conditions are not sufficient, even in the case of automorphisms; we shall describe some further necessary conditions for automorphisms of a class of compact nilmanifolds, along with an illustrative example.

2. *Proof of Theorem 1.2*

We recall that if N is a simply connected nilpotent Lie group, then $[N, N] \cap \Gamma$ is a lattice in $[N, N]$ and when the latter is one-dimensional $[N, N]/([N, N] \cap \Gamma)$ is the circle group.

Proof of Theorem 1.2. Let $\mathfrak{N}$ be the Lie algebra of N and $\exp : \mathfrak{N} \to N$ be the exponential map. Let Λ be the (additive) subgroup of $\mathfrak{N}$ generated by $\exp^{-1}(\Gamma)$. Then Λ is a lattice in $\mathfrak{N}$ and $\exp(\Lambda)$ is a lattice in N; (see [**4**, Theorem 2.12, addendum to the main statement]). Clearly $\exp(\Lambda)$ contains Γ as a subgroup of finite index. It therefore suffices to prove the proposition for the lattice $\exp(\Lambda)$ in place of Γ, and by modifying notation we shall assume $\Gamma = \exp(\Lambda)$.

We note that $[\mathfrak{N}, \mathfrak{N}] \cap \Lambda$ is an infinite cyclic subgroup, and choose a generator e_0. Then e_0 can be extended to a basis of Λ, as a free abelian group, say $\{e_0, e_1, \ldots, e_d\}$. Let W be the subspace spanned by $\{e_1, \ldots, e_d\}$. Then $\mathfrak{N}$ can be expressed as $W + \mathbb{R}$ with the Lie bracket operation given by $[(w_1, t_1), (w_2, t_2)] = (0, \alpha(w_1, w_2))$ for all $w_1, w_2 \in W$ and $t_1, t_2 \in \mathbb{R}$, where α is a bilinear form on W. Correspondingly N can be realized as $W \times \mathbb{R}$, with the multiplication given by $(w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \mu(w_1, w_2))$, for all $w_1, w_2 \in W$, $t_1, t_2 \in \mathbb{R}$, with μ a bilinear form on W, and the exponential map $\exp : \mathfrak{N} \to N$ is given by $\exp((w, t)) = (w, \frac{1}{2}\mu(w, w) + t)$ for all $w \in W$ and $t \in \mathbb{R}$.

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Let Δ be the integral lattice in W with respect to the basis $\{e_1, \ldots, e_d\}$. Then Δ is the image of Γ when W is viewed as $N/[N, N]$, and we note also that for any $\delta \in \Delta$, as a subset of $\mathfrak{N}$, we have $\exp(\delta) = (\delta, \frac{1}{2}\mu(\delta, \delta)) \in \Gamma$. Also $[N, N]$ is the subgroup $\{(0, t) \mid t \in \mathbb{R}\}$ and $\mathbb{T}$ as in the hypothesis is the quotient of $\mathbb{R}$ by $\frac{1}{2}\mathbb{Z}$.

Let

$$F = \left\{ u \in W \mid u = \sum_{i=1}^{d} \alpha_i e_i, \text{ with } |\alpha_i| \leq \frac{1}{2} \text{ for all } i \right\}.$$

Consider any $w \in W$. Then there exist $u \in F$ and $\delta \in \Delta$ such that $w = u + \delta$. Then in N, we have $(w, 0) = (u, -\mu(u, \delta) - \frac{1}{2}\mu(\delta, \delta)) \cdot (\delta, \frac{1}{2}\mu(\delta, \delta))$, and since $(\delta, \frac{1}{2}\mu(\delta, \delta)) \in \Gamma$ we get that $(w, 0)\Gamma = (u, -\mu(u, \delta) - \frac{1}{2}\mu(\delta, \delta))\Gamma$. For $w \in W$ such that u is in the interior of F the δ as above is unique and in this case we shall denote the element $(-\mu(u, \delta) - \frac{1}{2}\mu(\delta, \delta))$ by $\rho(w)$; hence, $(w, 0)\Gamma = (u, \rho(w))\Gamma$. We note also that given such a w, we have $(-w, 0) = (-u, -\mu(-u, -\delta) - \frac{1}{2}\mu(-\delta, -\delta)) \cdot (-\delta, \frac{1}{2}\mu(-\delta, -\delta))$, so $\rho(-w) = -\mu(-u, -\delta) - \frac{1}{2}\mu(-\delta, -\delta)) = \rho(w)$.

Let (b, s), with $b \in W$ and $s \in \mathbb{R}$ be the pair corresponding to the element a as in the hypothesis, and let $T = T_a$. Then for any $w \in W$ and $t \in \mathbb{R}$ we have $T((w, t)\Gamma) = (b + w, s + t + \mu(b, w))\Gamma$. Recursively, we see that

$$T^k((w, t)\Gamma) = \left(kb + w, ks + t + \sum_{j=0}^{k-1} \mu(b, jb + w) \right)\Gamma$$

for all $k \geq 1$. Similarly, for all $w \in W$ and $t \in \mathbb{R}$ we have $T^{-1}((w, t)\Gamma) = (-b + w, -s + t - \mu(b, -b + w))\Gamma$ and, recursively,

$$T^{-k}((w, t)\Gamma) = \left(-kb + w, -ks + t - \sum_{j=1}^{k} \mu(b, -jb + w) \right)\Gamma$$

for all $k \geq 1$.

Let S be the translation $T_{(b,0)}$ of N/Γ and $\rho = \mu(b, b)([N, N] \cap \Gamma) \in \mathbb{T}$. Let $Y = (N/\Gamma) \times \mathbb{T}$ and $\tau : Y \to Y$ be the transformation given by $\tau(x, \alpha) = (Sx, \rho\alpha)$ for all $x \in N/\Gamma$ and $\alpha \in \mathbb{T}$.

Let θ be as in the hypothesis and $\varphi \in \mathbb{T}$ be such that $\varphi^2 = \theta$. We show that there exists a sequence $\{k_i\}$ in $\mathbb{N}$ such that $\tau^{k_i}(\Gamma) \to \varphi\psi$, with ψ a root of unity. This is evident if τ is minimal, with $\psi = 1$. Now suppose that τ is not minimal. The condition on a as in the hypothesis implies in particular that the coset $b + \Delta$ generates a dense subgroup of W/Δ. Hence, by Parry's theorem [**3**] S is minimal, and in particular there exists a sequence $\{k_i\}$ such that $S^{k_i}(\Gamma) \to \varphi$. Also, since τ is not minimal, the factor $T_b \times T_\rho$ on $(W/\Delta) \times (\mathbb{R}/\mathbb{Z})$ is also not minimal. This further implies that there exist $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and a linear form χ on W such that $\chi(\Delta) \subset \frac{1}{2}\mathbb{Z}$ and $\mu(b, b) = (\chi(b) + p)/q$. Since $S^{k_i}(\Gamma) \to \varphi \in \mathbb{T}$, $\{k_i b + \Delta\}$ converges to the identity element in W/Δ. Since $\mu(b, b) = (\chi(b) + p)/q$, with χ and p, q as above, we get that every limit point of $\{\rho^{k_i}\}$ in $\mathbb{T}$ is a qth root of unity. Thus, $\tau^{k_i}(\Gamma) \to \varphi\psi$, with ψ a root of unity, as sought to be shown.

Now let $\{k_i\}$ be a sequence as above. Recall that a is represented by (b, s), with $b \in W$ and $s \in \mathbb{R}$. Passing to a subsequence we may assume that $(0, k_i s)\Gamma \to \sigma \in \mathbb{T}$

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as $i \to \infty$. Then for all i we have $T^{k_i}(\Gamma) = (k_i b, k_i s + \lambda_i)\Gamma$, and $S^{k_i}(\Gamma) = (k_i b, \lambda_i)\Gamma$, where $\lambda_i = \sum_{j=0}^{k_i - 1} \mu(b, jb) = \frac{1}{2}(k_i - 1)k_i \mu(b, b)$. Let $u_i \in F$ and $\rho_i \in \mathbb{R}$ be such that $(k_i b, 0)\Gamma = (u_i, \rho_i)\Gamma$. Then for all i, $T^{k_i}(\Gamma) = (u_i, k_i s + \rho_i + \lambda_i)\Gamma$ and $S^{k_i}(\Gamma) = (u_i, \rho_i + \lambda_i)\Gamma$. Since $S^{k_i}(\Gamma) \to \varphi$ we get that $u_i \to 0$ and $(0, \rho_i + \lambda_i)\Gamma \to \varphi$, as $i \to \infty$. Since $(0, k_i s)\Gamma \to \sigma$ this shows that $T^{k_i}(\Gamma) \to \sigma\varphi$, as $i \to \infty$.

On the other hand, for all i, $T^{-k_i}(\Gamma) = (-k_i b, -k_i s + \lambda_i + k_i \mu(b, b))\Gamma$. Since $u_i \to 0$, for all large i we have $\rho_i = \rho(k_i b)$, and since $\rho(k_i b) = \rho(-k_i b)$ we get that $(-k_i b, 0)\Gamma = (-u_i, \rho_i)\Gamma$ for all large i. Hence, $T^{-k_i}(\Gamma) = (-u_i, -k_i s + \rho_i + \lambda_i + k_i \mu(b, b))\Gamma$, for all i. Using the convergences as above we see from this that as $i \to \infty$, $T^{-k_i}(\Gamma) \to \sigma^{-1}\varphi\psi \in \mathbb{T}$. Hence, $T^{-k_i}(\sigma\varphi) \to \varphi^2\psi = \theta\psi$. This proves the theorem, with $\omega = \sigma\varphi$. □

3. *Proof of Theorem 1.1 for translations*

In this section we prove Theorem 1.1 for all translations, in a slightly stronger form (see Theorem 3.2). We first note the following.

PROPOSITION 3.1. *Let N be a simply connected nilpotent Lie group and Γ be a lattice in N. Let $\Phi = \{a_t\}$ be a one-parameter subgroup in N. Then there exists a closed connected subgroup M of N such that the following conditions are satisfied:*

(i) *Φ is contained in M, $M\Gamma$ is closed, and $\overline{\Phi\Gamma/\Gamma} = M\Gamma/\Gamma$;*

(ii) *for all but countably many $t \in \mathbb{R}$, $\overline{\{a_{kt} \mid k \in \mathbb{N}\}\Gamma/\Gamma} = M\Gamma/\Gamma$;*

(iii) *either M is abelian or there exists a closed connected normal subgroup M' of M such that $(M \cap \Gamma)M'$ is closed, and the commutator subgroup $[M/M', M/M']$ is one-dimensional.*

Proof. (i) Let $H = \overline{\Phi[N, N]\Gamma}$. If $H = N$, then by a theorem of Parry [**3**] $\overline{\Phi\Gamma} = N$ and in this case we may choose $M = N$. If H is a proper subgroup, then H^0, the connected component of the identity in H, is a simply connected nilpotent subgroup of lower dimension, such that $H^0\Gamma = H\Gamma$, and the proof can be completed by an obvious inductive argument.

(ii) This follows from the fact that by Parry's theorem both for actions of cyclic subgroups as well as one-parameter subgroups of N the action on N/Γ is minimal if and only if the action on the torus $N/[N, N]\Gamma$ is minimal, the corresponding assertion being true in the case of tori.

(iii) Suppose that M is non-abelian. If $[M, M]$ is one-dimensional, then we are through. To prove the assertion in general it suffices to show that there exists a closed connected normal subgroup M' of M, contained in $[M, M]$ such that M' has co-dimension one in $[M, M]$ and $(M \cap \Gamma)M'$ is closed. Suppose first that $[M, [M, M]]$ is the trivial subgroup. Then $[M, M]$ is central in M, thus a vector space, and $[M, M] \cap \Gamma$ is a lattice in $[M, M]$. Then the desired assertion holds if we choose M' to be any co-dimension-one vector subspace of $[M, M]$ intersecting $[M, M] \cap \Gamma$ in a lattice. If $[M, [M, M]]$ is not the trivial subgroup, then we consider $M/[M, [M, M]]$ in place of M (note that for this part N and M may be taken to be the same) and complete the argument by induction. □

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THEOREM 3.2. *Let N be a simply connected nilpotent Lie group and Γ be a lattice in N. Let $\Phi = \{a_t\}$ be a one-parameter subgroup of N and let $a = a_t$ for some $t \neq 0$. Then the following conditions are equivalent:*

(i) *Γ is almost automorphic for the action of T_a on N/Γ;*

(ii) *Γ is almost automorphic for the Φ-action on N/Γ;*

(iii) *there exists a closed connected abelian subgroup A of N such that Φ is contained in A and $A\Gamma$ is closed.*

Proof. The proof of equivalence of (i) and (ii) is routine and we shall omit it.

(ii) $\Longrightarrow$ (iii) Let M be the closed connected subgroup as in the conclusion of Proposition 3.1 (in the same notation). If M is abelian then statement (iii) holds for $A = M$. Suppose that M is not abelian and let M' be the subgroup as in Proposition 3.1. Let $N' = M/M'$ and $\Gamma' = (M \cap \Gamma)M'/M'$. Then Γ' is a lattice in N'. Let $t \in \mathbb{R}$ be such that statement (ii) in Proposition 3.1 holds, and let $a' = a_t M'$. Then the conditions of Theorem 1.2 are satisfied for N', Γ' and a' in place of N, Γ and a, respectively, and the theorem implies that Γ' is not almost automorphic for the action of $T_{a'}$. This implies in turn that $M \cap \Gamma$, as a point of $M/(M \cap \Gamma)$, is not almost automorphic for the action of T_a, and so Γ is not almost automorphic for the T_a action on N/Γ, which contradicts (i) (and hence (ii)). Therefore (ii) implies (iii).

(iii) $\Longrightarrow$ (i) When (iii) holds $A\Gamma/\Gamma$ is a closed invariant subset the restriction to which is equivalent to a translation on $A/A \cap \Gamma$ which is a torus, and in particular it is an isometry with respect to a metric. Hence, Γ is almost automorphic. This completes the proof. □

4. *Proofs of the main results*

In this section we shall deduce from the case of the translations the general case of the main theorem. In this respect we first recall the following simple facts, well known to experts, briefly indicating the proofs for the sake of completeness.

LEMMA 4.1. *Let N be a simply connected nilpotent Lie group. If M is a closed subgroup of N such that $M[N, N] = N$, then $M = N$.*

Proof. If N_i, $i = 1, \ldots, k$, is the central series of N, namely $N_1 = [N, N]$ and $N_{i+1} = [N, N_i]$ and N_k is trivial, then substituting the given condition $N = MN_1$ in $N_{i+1} = [N, N_i]$ we get successively that $N = MN_i$ for all i. Since N_k is trivial, $M = N$. □

LEMMA 4.2. *Let N, Γ and T be as in Theorem 1.1. Let $\mathfrak{o}$ be the orbit $\{T^k(\Gamma) \mid k \in \mathbb{Z}\}$ of Γ under T. Let $\mathcal{C}$ be the collection of all closed connected subgroups C such that C contains $\mathfrak{o}$ and $C\Gamma$ is closed. Then $\mathcal{C}$ has a unique minimal element.*

Proof. When $N = \mathbb{R}^d$, a vector space of dimension $d \geq 1$, we can realize Γ as $\mathbb{Z}^d$ and then the smallest rational subspace (namely defined by a system of linear equations with rational coefficients) containing $\mathfrak{o}$ has the desired property. Now consider the general case. Let $N/[N, N]$ be realized as $\mathbb{R}^d$ and the lattice $\Gamma[N, N]/[N, N]$ as $\mathbb{Z}^d$. Let $\eta : N \to \mathbb{R}^d$ be the quotient map. If $\eta(\mathfrak{o})$ is contained in a proper rational subspace, say W, then the questions reduces to the lower-dimensional subgroup $\eta^{-1}(W)$ and the proof may be

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completed by an obvious inductive argument. Suppose that $\eta(\mathfrak{o})$ is not contained in any proper rational subspace. Then for any $C \in \mathcal{C}$ we have $\eta(C) = \mathbb{R}^d$, and by Lemma 4.1 we get that $C = N$. This proves the lemma. □

The proof of the main theorem in the general case will be given by realizing the unipotent affine automorphisms as restrictions of translations of higher-dimensional compact nilmanifolds. This corresponds to considering the 'suspension' of the given affine automorphism, in the sense of dynamics, constructed as follows.

Let N be a simply connected nilpotent Lie group, Γ a lattice in N, U a unipotent automorphism of N/Γ and $\tilde{U}$ the corresponding automorphism of N. Then there exists a one-parameter subgroup $\Phi = \{\tilde{U}_t\}$ consisting of unipotent automorphisms of N, with $\tilde{U} = \tilde{U}_1$. Let $M = \Phi \ltimes N$, the semidirect product of Φ and N with respect to the action of Φ by automorphisms. Then M is a simply connected nilpotent Lie group. Since $\tilde{U}(\Gamma) = \Gamma$, it follows that $\Delta := \{\tilde{U}^n \mid n \in \mathbb{Z}\} \ltimes \Gamma$ is a subgroup of M. Furthermore Δ is a lattice in M. We realize N/Γ canonically as a subset of M/Δ. We note also that the U-action on N/Γ is the restriction of the translation of M/Δ by the element $\tilde{U}$; for $g \in N$, $(\tilde{U} \cdot g)\Delta = \tilde{U}(g)\tilde{U}\Delta = \tilde{U}(g)\Delta$ and the latter is the same as $U(g)\Gamma$ under the realization of N/Γ as a subset of M/Δ.

Proof of Theorem 1.1. We follow the notation as before. We first note that it suffices to prove the theorem when x is the identity element; the general case can then be deduced from it by considering the lattice $x\Gamma x^{-1}$ in place of Γ; with the change of the base point the T-action corresponds to $T_{a_x} \circ U_x$, where $a_x = a\tilde{U}(x)x^{-1}$, and U_x is the automorphism of $N/x\Gamma x^{-1}$ induced by the automorphism $\tilde{U}_x$ given by $\tilde{U}_x(g) = x\tilde{U}(x^{-1}gx)x^{-1}$, and if A is the subgroup arrived at in the special case, for the lattice $x\Gamma x^{-1}$, then the assertion as in the theorem holds with xAx^{-1} as the desired subgroup in place of A.

Next let C be a minimal closed subgroup as in Lemma 4.2, such that $\mathfrak{o} := \{T^k(\Gamma) \mid k \in \mathbb{Z}\}$ is contained in C and $C\Gamma$ is closed. It is easy to see that C is invariant under the T-action. Hence, replacing N by C we may assume that N contains no proper closed subgroup C such that $\mathfrak{o}$ is contained in $C\Gamma/\Gamma$.

We realize N/Γ, as seen above, as a subset of M/Δ such that the U-action on N/Γ is the restriction of a translation of M/Δ, by an element, specifically $\tilde{U}$ viewed as an element of M. Then T is also the restriction of a translation, by the element $a\tilde{U} \in M$, and we note that Γ being almost automorphic for the T-action implies that Δ is almost automorphic for the translation by $a\tilde{U}$. Hence, there exists a closed connected abelian subgroup B of M, containing $a\tilde{U}$, such that $B\Delta$ is closed. Let $A = B \cap N$. Since M is a simply connected nilpotent Lie group we get that A is a connected closed subgroup. Moreover, since $B \cap \Delta$ and $N \cap \Delta$ are lattices in B and N, respectively, it follows that $A \cap \Gamma = A \cap \Delta$ is a lattice in A and, hence, $A\Gamma$ is closed. Clearly $\mathfrak{o}$ is contained in $A\Gamma/\Gamma$ and, hence, by our assumption $A = N$. Thus, in particular, $a \in A$. Also, since $a\tilde{U} \in B$ and $a \in A \subset B$, we get also that $\tilde{U} \in B$. Now for any $g \in A$, $\tilde{U}(g) = \tilde{U}g\tilde{U}^{-1} = g$, as B is abelian. This proves the theorem. □

Remark 4.3. As mentioned in the introduction, Theorem 1.1 can also be deduced from certain results from the general theory of topological dynamical systems. We give here a

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sketch of the argument. It can be seen easily that for any almost automorphic point the orbit closure is a minimal subset. It follows from Theorems 1.1 and 1.2 of [**5**] that if *all* of the points in the orbit closure are almost automorphic, then the action on the minimal subset is equicontinuous. The latter condition implies that there is a compact abelian group C acting transitively on the minimal set and the automorphism corresponds to the action of one of the elements of C (generating a dense subgroup of C). While *a priori* C may have no relation with N, from the existence of such a group one can deduce, in the case of the translations considered above (with U as the identity automorphism), that C is, in fact, of the form B/Δ where B is a closed abelian subgroup of N and Δ is a lattice in B, and from this one can conclude that there exists a connected, closed abelian subgroup A with a closed orbit containing the minimal set as above. For general unipotent affine automorphisms the proof can be completed by considering suspensions as in the above proof. This would complete the proof provided all points in the minimal set are almost automorphic.

It turns out that for *distal* automorphisms, if a point is almost automorphic, then all points in its orbit closure are almost automorphic (cf. [**2**] for instance). Since unipotent affine automorphisms are distal, the condition as above holds and hence the argument as above yields a proof of the theorem.

Proof of Corollary 1.4. Suppose that T is almost automorphic. Let $x \in N$ be arbitrary. Then $x\Gamma$ is almost automorphic for the T-action. We view N/Γ as $N/x\Gamma x^{-1}$, choosing $x\Gamma$ as the base point. Then the T-action corresponds to $T_{a_x} \circ U_x$, where $a_x = a\tilde{U}(x)x^{-1}$, and U_x is the automorphism of $N/x\Gamma x^{-1}$ induced by the automorphism $\tilde{U}_x$ given by $\tilde{U}_x(g) = x\tilde{U}(x)^{-1}\tilde{U}(g)\tilde{U}(x)x^{-1}$. Applying Theorem 1.1 we get that there exists a closed connected subgroup A_x such that $A_x x\Gamma$ is closed, $a_x \in A_x$ and $\tilde{U}_x(g) = g$ for all $g \in A_x$.

Let $\mathcal{A}$ be the set of all closed connected abelian subgroups A such that $A\Gamma$ is closed. Then $\mathcal{A}$ is countable; this follows from the fact that the Lie subalgebra corresponding to $A \in \mathcal{A}$ is determined by linear equations with rational coefficients. We have $x^{-1}A_x x \in \mathcal{A}$ for all $x \in N$ and hence it follows that there exists $A \in \mathcal{A}$ such that the set E consisting of all x such that $x^{-1}A_x x = A$ (which is closed and, hence, a Borel subset), has positive Haar measure. We have $a_x \in A_x$ and, hence, $x^{-1}a\tilde{U}(x) \in x^{-1}A_x x = A$ for all $x \in E$. Since the set of x such that $x^{-1}a\tilde{U}(x) \in A$ is a closed submanifold of N and E has positive Haar measure, this implies that $x^{-1}a\tilde{U}(x) \in A$ for all $x \in N$.

Now consider any $x \in E$. We have $\tilde{U}_x(g) = x\tilde{U}(x)^{-1}\tilde{U}(g)\tilde{U}(x)x^{-1} = g$ for all $g \in A_x$. Thus, $\tilde{U}(x^{-1}gx) = x^{-1}gx$ for all $g \in A_x$, and since $A = x^{-1}A_x x$ this means that $\tilde{U}(g) = g$ for all $g \in A$. This shows that A has the desired properties and thus proves the 'only if' part of the corollary.

Now suppose that there exists a subgroup A as in the statement. Let $x \in N$ be arbitrary. Then $xA\Gamma/\Gamma$ is a closed subset of N/Γ. For any $g \in A$ we have $T(xg\Gamma) = a\tilde{U}(xg)\Gamma = a\tilde{U}(x)\tilde{U}(g)\Gamma = a\tilde{U}(x)g\Gamma$, since $\tilde{U}(g) = g$, and $a\tilde{U}(x) \in xA$ by hypothesis, so we get that $T(xg\Gamma) \in xA\Gamma/\Gamma$. Thus, $xA\Gamma/\Gamma$ is T-invariant. Furthermore, the above relation also shows that under the canonical correspondence of $xA\Gamma/\Gamma$ with $A\Gamma/\Gamma$ the restriction of T to the former corresponds to the translation action by the element $x^{-1}a\tilde{U}(x)$ on

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the latter. Now $A\Gamma/\Gamma$ is a torus and the translation is an isometry with respect to a metric on it, so the map is almost automorphic. Since this holds for every x it follows that T is almost automorphic. □

Proof of Corollary 1.5. By Corollary 1.4, for every $x \in N$, $x^{-1}a\tilde{U}(x)$ and a are fixed by $\tilde{U}$, and hence so is $x^{-1}a\tilde{U}(x)a^{-1}$. Given $\xi \in \mathfrak{N}$, applying this to $\exp t\xi$, $t \in \mathbb{R}$, in place of x, and differentiating with respect to t we get that $\mathrm{Ad}_a \circ \mathfrak{U}(\xi) - \xi$ is fixed by $\mathfrak{U}$. Thus $\mathfrak{U}(\mathrm{Ad}_a \circ \mathfrak{U}(\xi) - \xi) = \mathrm{Ad}_a \circ \mathfrak{U}(\xi) - \xi$ for all $\xi \in \mathfrak{N}$, that is, $(\mathfrak{U} - I)(\mathrm{Ad}_a \circ \mathfrak{U} - I) = 0$. This proves (i).

Also, by Corollary 1.4, the elements $x^{-1}a\tilde{U}(x)a^{-1}$ commute with each other. Let $\mathfrak{M} = (\mathrm{Ad}_a \circ \mathfrak{U} - I)(\mathfrak{N})$. Then $\mathfrak{M}$ is readily seen to be a Lie subalgebra of $\mathfrak{N}$. Given $\xi \in \mathfrak{M}$ it is tangent to the curve $x_t^{-1}a\tilde{U}(x_t)a^{-1}$ where $x_t = \exp t\xi$ for all $t \in \mathbb{R}$. Since for ξ, $\eta \in \mathfrak{M}$ the elements in the corresponding curves as above commute with each other, it follows that ξ and η commute in $\mathfrak{N}$. Hence, $\mathfrak{M}$ is commutative. This proves the Corollary. □

5. *Some further consequences*

In this section we discuss some auxiliary results around the theme of the main results.

For $N = \mathbb{R}^d$, $d \geq 1$ we have the following characterization of almost automorphic affine automorphisms, which may be compared with Corollary 1.5.

PROPOSITION 5.1. *Let $N = \mathbb{R}^d$, $d \geq 1$ and $\Gamma = \mathbb{Z}^d$. Let $T = T_a \circ U$ be an affine automorphism of N/Γ, where $a \in N$ and U is an automorphism of N/Γ induced by a unipotent automorphism $\tilde{U}$ of N. Then T is almost automorphic if and only if $(\tilde{U} - I)^2 = 0$ (here I and 0 are the identity and zero matrices, respectively) and $\tilde{U}(a) = a$.*

Proof. Suppose that T is almost automorphic. Then by Corollaries 1.5 and 1.4 $(\tilde{U} - I)^2 = 0$ and $\tilde{U}(a) = a$. Conversely suppose $(\tilde{U} - I)^2 = 0$ and $\tilde{U}(a) = a$. Then for every $g \in N$, the T-orbit of g is contained in gC, where $C = \{x \in \mathbb{T}^d \mid U(x) = x\}$, and restricted to gC the T-action is equivalent to a translation. Hence, T is almost automorphic. □

The following example shows that the converse of Corollary 1.5 is however not true in general, even for automorphisms; for the affine case we already have another necessary condition, that a is fixed by $\tilde{U}$, not incorporated in Corollary 1.5, which focuses on the conditions at the Lie algebra level.

Example 5.2. Let $\mathfrak{N}$ be the four-dimensional Lie algebra with a set of basis vectors $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ and the Lie bracket satisfying the conditions $[\xi_1, \xi_2] = \xi_3 = -[\xi_2, \xi_1]$, $[\xi_2, \xi_3] = \xi_4 = -[\xi_3, \xi_2]$ and $[\xi_i, \xi_j] = 0$ for (i, j) different from $(1, 2)$, $(2, 1)$, $(2, 3)$ and $(3, 2)$. It can be seen that $\mathfrak{N}$ is a nilpotent Lie algebra. Let $\mathfrak{U} : \mathfrak{N} \to \mathfrak{N}$ be the linear map such that $\mathfrak{U}(\xi_1) = \xi_1 + \xi_2$ and $\mathfrak{U}(\xi_i) = \xi_i$ for $i = 2, 3$ and 4. It can be verified that $\mathfrak{U}$ is a Lie algebra automorphism and $(\mathfrak{U} - I)^2 = 0$. Also the image of $\mathfrak{U} - I$ is the (one-dimensional) span of ξ_2, which is an abelian Lie subalgebra. Let N be the simply connected Lie group corresponding to $\mathfrak{N}$ and let $\tilde{U}$ be the Lie automorphism of N corresponding to $\mathfrak{U}$. Let Λ be the subgroup of $\mathfrak{N}$ generated by the basis $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ and $\Gamma = \exp(\Lambda)$, where exp denotes the exponential map of $\mathfrak{N}$ onto N. Since the basis has rational structure constants it follows that Γ is a lattice in N (cf. [**4**, Theorem 2.12]).

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Also, since Λ is $\mathfrak{U}$-invariant, it follows that Γ is $\tilde{U}$-invariant. Let U be the corresponding automorphism of N/Γ. Let M be the connected subgroup of N corresponding to the Lie subalgebra spanned by $\{\xi_2, \xi_3, \xi_4\}$; then M is also the set of fixed points of $\tilde{U}$. For all $t \in \mathbb{R}$ let $x_t = \exp t\xi_1$, and consider the subset $E = \{x_t^{-1}\tilde{U}(x_t) \mid t \in \mathbb{R}\}$ of M. We see that E generates M; this may be proved by observing that in $M/[M, M]$, which may be viewed canonically as $\mathbb{R}^2$, the image of E is an affine line. In particular E is not contained in an abelian subgroup. Hence, by Corollary 1.4 we get that U is not almost automorphic, even though conditions (i) and (ii) as in the conclusion of Corollary 1.5 hold for U. □

The phenomenon involved in the above example is seen in a more general form in the following proposition.

PROPOSITION 5.3. *Let $\mathfrak{N}$ be a nilpotent Lie algebra generated, as a Lie algebra, by two elements ξ and η, and $\mathfrak{U}$ be a unipotent Lie automorphism of $\mathfrak{N}$ such that $\mathfrak{U}(\xi) = \xi + \eta$. Let N be a simply connected nilpotent Lie group with $\mathfrak{N}$ as its Lie algebra. Let $\tilde{U}$ be the Lie automorphism of N corresponding to $\mathfrak{U}$. Let Γ be a lattice in N invariant under $\tilde{U}$ and let U be the automorphism of N/Γ induced by $\tilde{U}$. If U is almost automorphic, then $N = \{\exp t\xi\} \ltimes M$, a semidirect product, where M is an abelian normal connected Lie subgroup such that $M \cap \Gamma$ is a lattice in M and the action of $\tilde{U}$ on M is trivial.*

Proof. Since $\mathfrak{N}$ is generated by ξ and η it follows that $\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ is a two-dimensional vector space spanned by the images of ξ and η. Since $\mathfrak{U}(\xi) = \xi + \eta$ this implies that $\langle\eta\rangle + [\mathfrak{N}, \mathfrak{N}]$ is a $\mathfrak{U}$-invariant Lie ideal in $\mathfrak{N}$. Let $\mathfrak{M} = \langle\eta\rangle + [\mathfrak{N}, \mathfrak{N}]$ and M be the simply connected subgroup of N corresponding to $\mathfrak{M}$. Then M is a $\tilde{U}$-invariant subgroup and N is the semidirect product of $\{\exp t\xi\}$ and M. Also, since $[N, N] \cap \Gamma$ is a lattice in $[N, N]$ and $M/[N, N]$ is the set of fixed points of the factor of $\tilde{U}$ on $N/[N, N]$ it follows that $M \cap \Gamma$ is a lattice in M. We conclude the proof by showing that M is abelian and the $\tilde{U}$ action on M is trivial.

For all $t \in \mathbb{R}$ let $x_t = \exp t\xi$. Then as seen before, for every $t \in \mathbb{R}$, $x_t M\Gamma/\Gamma$ is U-invariant and when it is viewed as $M/(M \cap \Gamma)$ the U-action on $x_t M\Gamma/\Gamma$, $t \in \mathbb{R}$ corresponds to the affine automorphism induced by $T_{a_t} \circ \tilde{U}$, where $a_t = {x_t}^{-1}\tilde{U}(x_t) = \exp(-t\xi)\exp(t(\xi + \eta))$. Let $L = M/[M, M]$ and $\Delta = [M, M](M \cap \Gamma)$. Let $\mathfrak{L}$ be the Lie algebra of L and $\tau : \mathfrak{L} \to \mathfrak{L}$ be the map defined by $\tau(\mu + [\mathfrak{M}, \mathfrak{M}]) = [\xi, \mu] + [\mathfrak{M}, \mathfrak{M}]$ for all $\mu \in \mathfrak{M}$. Since $\mathfrak{N}$ is generated by ξ and η it follows that $\mathfrak{L}$ is spanned by $\{\eta, \tau(\eta), \ldots, \tau^{n-1}(\eta)\}$, where n is the least positive integer such that $\tau^n(\eta) = 0$, and furthermore the set forms a basis of $\mathfrak{L}$.

Now $N/[M, M]$ may be viewed as the semidirect product of the one-parameter subgroup $\Phi := \{x_t\}$ and $\mathfrak{L}$, identifying L with $\mathfrak{L}$ canonically. In turn, the group may be realized as a group of $(n+1) \times (n+1)$ matrices, with x_t represented by $\left(\begin{smallmatrix} \exp t\nu & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\nu = (\nu_{ij})$ is the $n \times n$ matrix given by $\nu_{ij} = 1$ if $j = i + 1$ and 0 otherwise, and the elements of $\mathfrak{L}$ are represented by $\left(\begin{smallmatrix} I_n & v \\ 0 & 1 \end{smallmatrix}\right)$ with I_n the $n \times n$ identity matrix and v a column vector ${}^t(v_1, \ldots, v_n)$; specifically the entries $v_1, \ldots, v_n$ are given by the coordinates of the element with respect to the basis $\{\tau^{n-1}(\eta), \ldots, \eta\}$ (in that order). From this the image of a_t in L can be computed explicitly to be $\sum_{k=0}^{n-1} c_k t^{k+1}\tau^k(\eta)$, where $c_k = (-1)^k/k!$ for all k.

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We can find t such that $1, c_0t, c_1t^2, \ldots, c_{n-1}t^n$ are linearly independent over $\mathbb{Q}$; this is because all c_k being non-zero, the set of t for which the elements are linearly dependent is countable. For t satisfying this condition, as discussed above, the action $T_{a_t} \circ U$ on L/Δ is minimal, and in turn the $T_{a_t} \circ U$ action on $M/(M \cap \Gamma)$ is minimal. Since it is almost automorphic, by Corollary 1.4 we get that M is abelian and the $\tilde{U}$ action on M is trivial. □

6. *Miscellanea*

In this section we discuss the issue in a broader context. First, we prove the following result which extends the scope of applicability of the results of the earlier sections.

We shall say that a homeomorphism φ of a locally compact second countable space X *admits a convergent trajectory* if there exists $x \in X$ such that $\varphi(x) \neq x$ and either $\{\varphi^k(x)\}_{k=0}^{\infty}$ or $\{\varphi^{-k}(x)\}_{k=0}^{\infty}$ converges in X, as $k \to \infty$; we note that in either case the limit is a fixed point of φ. Clearly, if φ admits a convergent trajectory, or more generally if a factor of φ admits a convergent trajectory, then φ is not almost automorphic. This means in particular that if T as in the hypothesis of the next proposition is also assumed to be almost automorphic, then only the second alternative as in the conclusion can hold.

PROPOSITION 6.1. *Let N be a simply connected nilpotent Lie group and Γ be a lattice in N. Let $T = T_a \circ A$ be an affine automorphism of N/Γ, where $a \in N$ and A is an automorphism of N/Γ. Let $\tilde{A}$ be the automorphism corresponding to A. Then one of the following holds:*

(i) *there exists a proper closed connected normal $\tilde{A}$-invariant subgroup M of N such that $M\Gamma$ is closed, and the factor $\bar{T}$ of T on $N/M\Gamma$ admits a convergent trajectory;*

(ii) *there exists $r \in \mathbb{N}$ such that $\tilde{A}^r$ is unipotent.*

Proof. First consider the case when $N = \mathbb{R}^d$, for some $d \geq 1$. Let V be the largest $\tilde{A}$-invariant subspace of $\mathbb{R}^d$ on which 1 is the only eigenvalue, and $W = \mathbb{R}^d/V$. Since $\tilde{A}$ leaves Γ invariant, it follows that the image $\Delta = (\Gamma + V)/V$ of Γ in W is a lattice in W. It suffices to prove the proposition for the factor of T on W and hence without loss of generality we may assume that V is trivial and $W = \mathbb{R}^d$; that is, 1 is not an eigenvalue of $\tilde{A}$. The last condition implies, via simple linear algebra, that $T_a \circ \tilde{A}$ has a fixed point $b \in \mathbb{R}^d$. Then $T_a \circ \tilde{A} = T_b \circ \tilde{A} \circ T_b^{-1}$. If $\tilde{A}$ has an eigenvalue of absolute value different from 1, then $\tilde{A}$ admits a convergent trajectory and hence so does $T_a \circ \tilde{A}$, and so statement (i) holds in this case. Now suppose that all of the eigenvalues of $\tilde{A}$ are of absolute value 1. Since $\tilde{A}$ leaves Γ invariant this further implies that all of the eigenvalues are roots of unity (one proves this by considering the characteristic polynomials of $\tilde{A}^k$, $k \in \mathbb{N}$, and showing that under the given condition their roots, which are eigenvalues of $\tilde{A}^k$, would belong to a fixed finite set). Hence, there exists $r \in \mathbb{N}$ such that $\tilde{A}^r$ is unipotent. This proves the proposition when N is abelian.

Now consider the general case. Let $\bar{A}$ denote the factor of $\tilde{A}$ on $N/[N, N]$. Suppose that alternative (i) does not hold. From the special case we then get that there exists $r \in \mathbb{N}$ such that $\bar{A}^r$ is unipotent. Therefore, in proving the proposition we may without loss of generality assume that $\bar{A}$ is unipotent. By an argument well known to Lie group theorists this implies that $\tilde{A}$ is also unipotent. Let us recall the argument here, briefly, for the reader's convenience: let $\mathfrak{N}$ be the Lie algebra of N and $\mathfrak{M}$ be the largest $d\tilde{A}$-invariant subspace

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on which $d\tilde{A}$ is unipotent. It can be seen that $\mathfrak{M}$ is a Lie subalgebra of $\mathfrak{N}$, and as $\bar{A}$ is unipotent we get $\mathfrak{M} + [\mathfrak{N}, \mathfrak{N}] = \mathfrak{N}$. A Lie algebra version of Lemma 4.1 now shows that $\mathfrak{M} = \mathfrak{N}$, and hence that $\tilde{A}$ is unipotent. This completes the proof of the proposition. □

We conclude with the following remark.

Remark 6.2. We recall that a homeomorphism φ of a locally compact second countable space X is said to be *distal* if for $x, y \in X$, $x \neq y$, there does not exist any sequence $\{k_i\}$ in $\mathbb{Z}$ such that $\varphi^{k_i}(x)$ and $\varphi^{k_i}(y)$ converge to the same point as $i \to \infty$. Proposition 6.1 also shows that if an affine automorphism $T = T_a \circ A$ is assumed to be distal, then A^r is unipotent for some $r \in \mathbb{N}$; then T^r is a unipotent affine automorphism. Related general results for distal group actions by group automorphisms, with analogous conclusions about the action of a subgroup of finite index, may be found in [**1**]. The preceding conclusion can indeed be deduced, albeit not directly, from the results of [**1**]. On the other hand, Proposition 6.1 provides a simple and direct argument in the case at hand.

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