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Some link-polynomial relations

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The polynomial invariants of classical links that relate to the Jones polynomial have various inter-connections. Here a new one is given: the *square* of the Jones polynomial is obtained from the F-polynomial by means of a certain substitution of variables. This explains and corrects some similarities between known evaluations of these polynomials. The proof uses the 'Dubrovnik' version of the F-polynomial; this is first proved to be equivalent to the F-polynomial. Finally a correlation between the F and P polynomials is included as a provocative curiosity.

In what follows, L, with various suffices, will be an oriented link in S^3 with $c(L)$ components, and D will be an oriented (generic) planar diagram for L. Recall that two such diagrams are said to be *regularly isotopic* if they differ by a sequence of Reidemeister moves of the second and third types. In the usual way the orientation defines a sign, $+1$ or -1, at each crossing of D, and the *writhe*, $w(D)$, is the sum of the signs at all the crossings; this is an integer-valued regular isotopy invariant. Suppose that L is changed to L_1 by just changing the orientation of one component γ of L; this changes D to D_1. Then $w(D)-w(D_1)$ is four times the linking number of γ with $L-\gamma$. Thus $w(D) \equiv w(D_1)$ modulo 4, and the regular isotopy invariant $i^{w(D)}$, where $i^2 = -1$, is independent of the choice of orientation on the link. Most of the diagram invariants that will be used apply only to unoriented diagrams but if such an invariant is applied to D then D is assumed first to be disoriented.

Recall that the semi-oriented two-variable Laurent polynomial invariant of Kauffman [**4**] is a function

$$F\colon \{\text{Oriented links in } S^3\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}].$$

It is defined uniquely by $F(L) = a^{-w(D)}\Lambda(D)$, where

$$\Lambda\colon \{\text{Unoriented diagrams of links}\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

satisfies

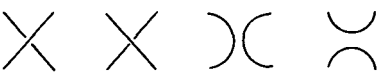
(i) $\Lambda\left(\bigcirc\right) = 1$,

(ii) Λ is invariant under regular isotopy,

(iii) $\Lambda\left(\boxed{D}\right) = a\Lambda(D)$,

(iv) $\Lambda(D_+)+\Lambda(D_-) = z(\Lambda(D_0)+\Lambda(D_\infty))$.

Here, D_+, D_-, D_0 and D_∞ are identical diagrams except near a point where they are



D_+ D_- D_0 D_∞

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The first theorem explains the relationship between the F-polynomial and another similar polynomial here called the F^*-polynomial (called the Dubrovnik polynomial in [4]). This result has already been quoted by Kauffman [4] and by Turaev [12].

THEOREM 1. *Let L be an oriented link with planar diagram D. Let $F^*(L) \in \mathbb{Z}[\alpha^{\pm 1}, \omega^{\pm 1}]$ be defined by*

$$F^*(L) = (-1)^{c(L)-1}[F(L)]_{(a,z)=(i\alpha,-i\omega)}.$$

Then $F^*(L) = \alpha^{-w(D)}\Lambda^*(D)$ where

$$\Lambda^*: \{\text{Unoriented diagrams of links}\} \to \mathbb{Z}[\alpha^{\pm 1}, \omega^{\pm 1}]$$

satisfies (and is uniquely determined by)

(i)* $\Lambda^*\left(\bigcirc\right) = 1$,

(ii)* Λ^* *is invariant under regular isotopy,*

(iii)* $\Lambda^*\left(\boxed{D}\right) = \alpha\Lambda^*(D)$,

(iv)* $\Lambda^*(D_+) - \Lambda^*(D_-) = \omega(\Lambda^*(D_0) - \Lambda^*(D_\infty))$.

Here, D_+, D_-, D_0 and D_∞ are identical diagrams except near a point where they are related as before.

Proof. It is assumed here that F is well defined and so trivially F^* is well defined. Let Λ^* be defined by $\Lambda^*(D) = \alpha^{w(D)}F^*(L)$. Then

$$\Lambda^*(D) = (-1)^{c(L)-1}i^{-w(D)}[\Lambda(D)]_{(a,z)=(i\alpha,-i\omega)}.$$

It follows immediately that Λ^* satisfies (i)*, (ii)* and (iii)*. As mentioned above, $i^{-w(D)}$ can be found from any convenient orientation of D. If $i^{-w(D)}$ takes the value ϵ when D is D_0 then the values taken when D is D_+, D_- and D_∞ respectively are

(a) $-i\epsilon$, $i\epsilon$ and ϵ when $c(L_\pm) < c(L_0)$,

(b) $i\epsilon$, $-i\epsilon$ and ϵ when $c(L_\pm) < c(L_\infty)$,

(c) $-i\epsilon$, $i\epsilon$ and $-\epsilon$ when $c(L_\pm) > c(L_0)$.

Hence, in each of these cases, if $(-1)^{c(L)-1}i^{-w(D)}$ takes the value δ on D_0 then it takes values $i\delta$, $-i\delta$ and $-\delta$ on D_+, D_- and D_∞. Thus

$$\Lambda^*(D_+) - \Lambda^*(D_-) - \omega(\Lambda^*(D_0) - \Lambda^*(D_\infty)) = \delta[i\Lambda(D_+) + i\Lambda(D_-) - \omega(\Lambda(D_0) + \Lambda(D_\infty))]_{(a,z)=(i\alpha,-i\omega)}.$$

By (iv) the latter expression is zero.

The usual induction argument on the number of crossings, combined with a procedure of repeatedly changing crossings to achieve a no-crossing diagram, shows that Λ^* can be calculated from (i)* to (iv)*. ∎

It should be observed that if (a,z) is changed to $(-a,-z)$, $\Lambda(D)$ changes to $(-1)^n\Lambda(D)$, where n is the number of crossings in D, and $F(L)$ is unchanged. If $F^{**}(L)$ is defined to be $(-1)^{c(L)-1}[F^*(L)]_{(a,\omega)=(ia,-iz)}$ then $F^{**}(L) = F(L)$. Theorem 1 shows that the oriented link invariant F^*, when created by means of formulae (i)* to (iv)*, is nothing more than the invariant F in thin disguise; it contains no more information. However, often F^* seems to occur the more naturally. An example of this is the following result announced in [7].

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PROPOSITION 2. *For any oriented link* L, $[F(L)]_{a=i} = (-1)^{c(L)-1}$.

Proof. It is clear that $\Lambda^*(D) = 1$ for all D is the solution of (i)* to (iv)* when $\alpha = 1$. Thus $[F^*(L)]_{\alpha=1} = 1$, and the result follows from Theorem 1. ∎

It is known [6], [4], that the original Jones polynomial $V(L) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ is obtained from the $F(L)$ by the substitution $(a, x) = (t^{-\frac{3}{4}}, -(t^{-\frac{1}{4}} + t^{\frac{1}{4}}))$. The next result shows that, surprisingly, $(V(L))^2$ is also obtainable from $F(L)$ by a substitution.

THEOREM 3. *For any oriented link* L,

$$[F(L)]_{(a,z)=(q^3, q^{-1}+q)} = (-1)^{c(L)-1}[(V(L))^2]_{t=-q^{-2}}.$$

Proof. Recall that, if D is any diagram for L, then $V(L)$ can be defined by $V(L) = [(-A^3)^{-w(D)}\langle D\rangle]_{A=t^{-\frac{1}{4}}}$. This formula uses the bracket polynomial [5], a regular isotopy invariant defined by

$$\langle\ \rangle : \{\text{Unoriented link diagrams}\} \to \mathbb{Z}[A^{\pm 1}] \quad \text{such that}$$

(*a*) $\langle U^c\rangle = (-(A^{-2} + A^2))^{c-1}$,

(*b*) $\left\langle \times \right\rangle = A \left\langle \smile\!\!\frown \right\rangle + A^{-1} \left\langle \bigcirc\bigcirc \right\rangle$.

Here in (*a*), U^c denotes any c-component diagram with no crossing at all, and (*b*) refers to three diagrams the same except near a point. It follows that

$$\left\langle \boxed{D} \infty \right\rangle = -A^3 \langle D \rangle.$$

Suppose that formula (*b*) is interpreted as refering to a selected crossing of a diagram. Squaring the formula gives

$$\left\langle \times \right\rangle^2 = A^2 \left\langle \smile\!\!\frown \right\rangle^2 + 2 \left\langle \smile\!\!\frown \right\rangle \left\langle \bigcirc\bigcirc \right\rangle + A^{-2} \left\langle \bigcirc\bigcirc \right\rangle^2.$$

Apply this to the diagram with the selected crossing switched to obtain

$$\left\langle \times \right\rangle^2 = A^2 \left\langle \bigcirc\bigcirc \right\rangle^2 + 2 \left\langle \bigcirc\bigcirc \right\rangle \left\langle \smile\!\!\frown \right\rangle + A^{-2} \left\langle \smile\!\!\frown \right\rangle^2.$$

Subtracting this from the previous formula gives

$$\left\langle \times \right\rangle^2 - \left\langle \times \right\rangle^2 = (A^2 - A^{-2}) \left(\left\langle \bigcirc\bigcirc \right\rangle^2 - \left\langle \smile\!\!\frown \right\rangle^2 \right).$$

Together with the fact that $\left\langle \boxed{D} \infty \right\rangle^2 = A^6 \langle D \rangle^2$, this means that $\langle D\rangle^2$ satisfies the same characterizing formulae, (i)* to (iv)*, as does $\Lambda^*(D)$ when $\alpha = A^6$ and $\omega = (A^2 - A^{-2})$. Thus

$$A^{-6w(D)}\langle D\rangle^2 = [F^*(L)]_{(\alpha,\omega)=(A^6, (A^2-A^{-2}))}.$$

Use of Theorem 1 then produces the required result. ∎

Remark 4. In [3] it was noted that $[V(L)]_{t=-1} = \det(L)$, where $\det(L)$ was the normalized Alexander polynomial when $t = -1$. In [1] it was shown that $[F(L)]_{(a,z)=(1,2)} = (\det(L))^2$ where $\det(L)$ was the determinant of the Goeritz matrix.

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These evaluations are displayed side by side in [7] and perhaps that is misleading; $[V(L)]_{t=-1}$ can be a complex number when $c(L)$ is even. Theorem 3 shows that the relation between these two 'determinants' is

$$[F(L)]_{(a,z)=(1,2)} = (-1)^{c(L)-1}[(V(L))^2]_{t=-1}.$$

Other confirmation of Theorem 3 follows from the result

$$[V(L)]_{t=e^{i\pi/3}} = \pm i^{c(L)-1}(i\sqrt{3})^{d(D(L))}$$

from [9] and [11], and from the following from [1]

$$[F(L)]_{(a,z)=(1,-1)} = (-3)^{d(D(L))}.$$

Here $d(D(L))$ is the dimension of the first homology, with $\mathbb{Z}_3$ coefficients, of the double cover of S^3 branched over L.

It follows from Turaev's work [12] on the Yang–Baxter equation approach to the defining of $F(L)$ that the evaluations of $F(L)$ at $(a,z) = (\pm(i)^{n+1}q^n, (q^{-1}+q))$ can in a sense be regarded as basic. The evaluations shown in Table 1 are now known. The penultimate line of this table is due to Yamada[13]. There $\lambda(L)$ denotes the sum of the linking numbers of all pairs of components of L and $X^{p(2)}$ is the untwisted 2-parallel of the sublink X with parallel pairs oriented in opposite directions.

Table 1

(a,z)	$F(L)_{(a,z)}$
$(q^3, q^{-1}+q)$	$(-1)^{c(L)-1}[(V(L))^2]_{t=-q^{-2}}$
$(q, q^{-1}+q)$	zero when L is a split link
$(i, q^{-1}+q)$	$(-1)^{c(L)-1}$
$(-q, q^{-1}+q)$	$\frac{1}{2}(-1)^{c(L)-1}\sum_{X\subset L} q^{4\,\text{linking number}(X, L-X)}$, see [10]
$(-iq^2, q^{-1}+q)$	$[t^{2\lambda(L)}(t^{-\frac{1}{2}}+t^{\frac{1}{2}})(t^{-1}+1+t)^{-1}\sum_{X\subset L}(-1)^{c(X)}V(X^{p(2)})]_{t=-tq^{-1}}$
$(-q^3, q^{-1}+q)$	$[V(L)]_{t=-q^{-4}}$

The Λ-polynomial is defined and calculated recursively using (iv), the formula for the Λ-polynomial of four related unoriented diagrams. It is a little troublesome to give four prototype *oriented* diagrams to define the F-polynomial (see [6]). However $P(L)$, the oriented polynomial, which was the first two-variable generalization of the Jones polynomial [2], [8], is completely defined by means of three prototype oriented diagrams. Recall that P: {*Oriented* links} $\to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ is an invariant of ambient isotopy, $P(\text{unknot}) = 1$, and

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0;$$

here L_+, L_-, L_0 are links identical except near a point where respectively they have a positive crossing, a negative crossing and no crossing (see [8]). The final result of this note is a remark to the effect that there are four prototype oriented diagrams and a simple linear formula relating the F-polynomial of these diagrams, and that the P-polynomial satisfies essentially the *same* formula. It is known that these two polynomials carry different information as they distinguish different pairs of knots (see [7] for example) so this formula is insufficient to define the polynomials.

THEOREM 5. *Let L_3, L_2, L_1 and L_0 be four oriented links that are exactly the same except near a point where they are as shown.*

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L_3 L_2 L_1 L_0

Then

$$a^4 F(L_3) - (a^2 + a^3 z) F(L_2) + (a^2 + az) F(L_1) - F(L_0) = 0,$$

$$l^4 P(L_3) - (l^2 - l^3 m) P(L_2) + (l^2 - lm) P(L_1) - P(L_0) = 0.$$

Proof. Let D_3, D_2, D_1 and D_0 be the unoriented diagrams corresponding to L_3, L_2, L_1 and L_0, and, as before, let D_∞ be the other unoriented diagram with no crossing near the point under consideration. Then (ii), (iii) and (iv) imply that

$$\Lambda(D_3) + \Lambda(D_1) = z(\Lambda(D_2) + a^{-2}\Lambda(D_\infty)),$$

$$\Lambda(D_2) + \Lambda(D_0) = z(\Lambda(D_1) + a^{-1}\Lambda(D_\infty))$$

and, eliminating $\Lambda(D_\infty)$, this gives

$$a\Lambda(D_3) - (1 + az)\Lambda(D_2) + (a + z)\Lambda(D_1) - \Lambda(D_0) = 0.$$

The result follows from the fact that $\Lambda(D) = a^{w(D)} F(L)$. The recursive formula for the P-polynomial gives

$$lP(L_3) + mP(L_2) + l^{-1} P(L_1) = 0,$$

$$lP(L_2) + mP(L_1) + l^{-1} P(L_0) = 0.$$

Subtracting l times the second of these equations from l^3 times the first gives the required formula. ∎

REFERENCES

[1] R. D. Brandt, W. B. R. Lickorish and K. C. Millett. A polynomial invariant for unoriented knots and links. *Invent. Math.* **84** (1986), 563–573.

[2] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu. A new polynomial invariant of knots and links. *Bull. Amer. Math. Soc.* **12** (1985), 239–246.

[3] V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc.* **12** (1985), 103–111.

[4] L. H. Kauffman. An invariant of regular isotopy. (To appear.)

[5] L. H. Kauffman. State models and the Jones polynomial. *Topology* **26** (1987), 395–407.

[6] W. B. R. Lickorish. A relationship between link polynomials. *Math. Proc. Cambridge Philos. Soc.* **100** (1986), 109–112.

[7] W. B. R. Lickorish. The panorama of polynomials for knots, links and skeins. Proceedings of conference on Artin's Braid Group, Santa Cruz (1986). (To appear.)

[8] W. B. R. Lickorish and K. C. Millett. A polynomial invariant of oriented links. *Topology* **26** (1987), 107–141.

[9] W. B. R. Lickorish and K. C. Millett. Some evaluations of link polynomials. *Comment. Math. Helv.* **61** (1986), 349–359.

[10] W. B. R. Lickorish and K. C. Millett. An evaluation of the F-polynomial of a link. (To appear.)

[11] A. S. Lipson. An evaluation of a link polynomial. *Math. Proc. Cambridge Philos. Soc.* **100** (1986), 361–364.

[12] V. G. Turaev. The Yang–Baxter equations and invariants of links. LOMI preprint E-3-87.

[13] S. Yamada. An operator on regular isotopy invariants of link diagrams. (To appear.)

https://doi.org/10.1017/S0305004100001390 Published online by Cambridge University Press