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# Summability by Dirichlet convolutions

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Ingham (3) discusses the following summation method:

A series $\Sigma a_n$ will be said to be summable to $s$ if

$$\lim_{x\to\infty} \sum_{n\leqslant x} \frac{n}{x}\left[\frac{x}{n}\right] a_n = s,$$

where, as usual, $[x]$ indicates the greatest integer $\leqslant x$. (An equivalent method was introduced somewhat earlier by Wintner (8), but the notation $(I)$ for the above method and the attachment to Ingham's name seem to have become usual following [(1), Appendix IV].) The method $(I)$ is intimately connected with the prime number theorem and the fact that the Riemann zeta-function $\zeta(s)$ has no zeros on the line $\sigma = 1$. Ingham proved, among other results, that $(I)$ is not comparable with convergence but, nevertheless, for every $\delta > 0$, $(I) \Rightarrow (C, \delta)$ and for every $\delta$, $0 < \delta < 1$, $(C, -\delta) \Rightarrow (I)$, where the $(C, k)$ are Cesàro means of order $k$.

One generalization of the method $(I)$ was investigated by Pennington (6). In this paper we discuss a quite different general class of methods which includes $(I)$ and, in particular, the relationships between members of this class and the Cesàro methods, under suitable auxiliary conditions, when necessary.

Throughout $n$ and $d$ are positive integers, $\sum_{n\leqslant x}$ means a sum beginning at $n = 1$, $\sum_{d|n}$ means a sum over the positive divisors of $n$, and empty sums are to be interpreted as 0. $\Sigma a_n$ will always mean a series with first term $a_1$. The term regular will always mean 'sums every convergent series to its convergent sum'. All series are assumed real. The expression $(A) \Rightarrow (B)$ is to be read 'Every series summable by $(A)$ is summable by $(B)$ to the same sum'. All unexplained notation is as in (1).

Definition. *A series $\Sigma a_n$ will be said to be $(\mathscr{D}, h(n))$—summable to the sum $s$ if there exists a function on the positive integers $h(n)$ with $h(1) = 1$ such that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} n \sum_{d|n} h(n/d)\, a_d = s. \qquad (*)$$

*Remarks.* (i) It clearly (in sharp distinction to Riesz's arithmetic means) makes no difference whether $x\to\infty$ through all real values or only through the positive integers, and it will sometimes be convenient to specifically assume that $x$ takes integral values only.

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(ii) $(I)$-summability is $(\mathscr{D}, 1/n)$-summability.

(iii) We may write $(*)$ in the alternative form (interchanging the two summations)

$$\lim_{x\to\infty}\frac{1}{x}\sum_{d\leqslant x}da_d\sum_{m\leqslant x/d}mh(m)=\lim_{x\to\infty}\frac{1}{x}\sum_{d\leqslant x}dh(d)\sum_{m\leqslant x/d}ma_m=s \qquad (**)$$

which makes it clear that a series $\Sigma a_d$ is summable $(\mathscr{D}, h(n))$ if and only if the series $\Sigma h(d)$ is summable $(\mathscr{D}, a(n))$ where $a(n)=a_n$ (assuming $a(1)=1$); but we shall be interested in the nature of the set of series summed by a given fixed $h(n)$ and shall not exploit this reciprocal relationship.

(iv) The condition $h(1)=1$ is a normalization of $h(1)\neq 0$, which we require to ensure that $h(n)$ has an inverse under ordinary Dirichlet convolution which may be represented as a function on the integers. In fact we make the

DEFINITION. *Given $h(n)$ with $h(1)=1$, $h^*(n)$ denotes the (unique) function on the positive integers satisfying*

$$\sum_{d|n}h^*(d)\,h(n/d)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases}$$

It will also be convenient to make the

DEFINITION. *A summability method $(\mathscr{D}, h(n))$ will be called strong, if*

$$\int_1^x\frac{1}{t}\sum_{d\leqslant t}h^*(d)\,dt \qquad (1)$$

*converges absolutely to* 1 *as* $x\to\infty$.

For $(I)$-summability, $h^*(d)=\mu(d)/d$, where $\mu(n)$ is the Möbius function; it follows as a standard result of prime number theory that $(I)$ is a strong $(\mathscr{D}, h(n))$ method.

The interest of strong methods is that one can prove along the lines in (1):

THEOREM 1. *If the method $(\mathscr{D}, h(n))$ is strong then any series summable $(\mathscr{D}, h(n))$ is summable $(C, 1)$ to the same sum.*

*Proof.* Let $\mathscr{D}(x)=1/x\sum_{n\leqslant x}n\sum_{d|n}a_d h(n/d)$, and let $H^*(x)=\sum_{n\leqslant x}h^*(n)$. We then have

$$\begin{aligned}\int_1^x\frac{\mathscr{D}(t)H^*(x/t)}{t}dt&=\int_1^x\frac{H^*(x/t)}{t^2}\sum_{n\leqslant t}n\sum_{d|n}a_d h(n/d)\,dt\\&=\int_1^x\frac{H^*(x/t)}{t^2}\sum_{d\leqslant t}da_d\sum_{m\leqslant t/d}mh(m)\,dt\\&=\sum_{d\leqslant x}da_d\int_d^x\frac{H^*(x/t)}{t^2}\sum_{m\leqslant t/d}mh(m)\,dt\\&=\frac{1}{x}\sum_{d\leqslant x}da_d\int_1^{x/d}H^*(u)\sum_{m\leqslant x/ud}mh(m)\,du\\&=\frac{1}{x}\sum_{d\leqslant x}da_d\sum_{m\leqslant x/d}mh(m)\int_1^{x/md}H^*(u)\,du, \qquad (2)\end{aligned}$$

on letting $u=x/t$ and interchanging summation and integration.

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But by partial summation

$$\sum_{d \leqslant y} dh^*(d) = yH^*(y) - \int_1^y H^*(u)\,du$$

and hence (for $z \geqslant 1$)

$$\sum_{m \leqslant z} mh(m) \int_1^{z/m} H^*(u)\,du = z \sum_{m \leqslant z} h(m) \sum_{r \leqslant z/m} h^*(r) - \sum_{m \leqslant z} mh(m) \sum_{r \leqslant z/m} rh^*(r) = z - 1. \quad (3)$$

Setting $z = x/d$ and substituting (3) in (2) gives

$$\int_1^x \frac{\mathscr{D}(t)\,H^*(x/t)}{t}\,dt = \frac{1}{x} \sum_{d \leqslant x} (x-d)\,a_d. \quad (4)$$

Now, by hypothesis, $\mathscr{D}(x) = c + R(x)$ where $R(x) = o(1)$ as $x \to \infty$ and hence

$$\begin{aligned}
\int_1^x \frac{\mathscr{D}(t)\,H^*(x/t)}{t}\,dt &= \frac{1}{x}\int_1^x \frac{x}{u}\,\mathscr{D}\left(\frac{x}{u}\right) H^*(u)\,du \\
&= c\int_1^x \frac{H^*(u)}{u}\,du + \frac{1}{x}\left(\int_1^{\sqrt{x}} \frac{x}{u} R\left(\frac{x}{u}\right) H^*(u)\,du + \int_{\sqrt{x}}^x \frac{x}{u} R\left(\frac{x}{u}\right) H^*(u)\,du\right) \\
&= c + o(1) + \frac{1}{x}\int_1^{\sqrt{x}} o\left(\frac{x}{u}\right) |H^*(u)|\,du + \frac{1}{x}\int_{\sqrt{x}}^x O\left(\frac{x}{u}\right) |H^*(u)|\,du \\
&= c + o(1) + o\left(\int_1^{\sqrt{x}} \frac{|H^*(u)|}{u}\,du\right) + O\left(\int_{\sqrt{x}}^x \frac{|H^*(u)|}{u}\,du\right) = c + o(1), \quad (5)
\end{aligned}$$

since $(\mathscr{D}, h(n))$ was assumed to be a strong method.

Substituting (5) in (4) proves the theorem.

*Remarks.* (i) If we only require that $\int_1^x \frac{1}{t} \sum_{n \leqslant t} h^*(n)\,dt$ converge absolutely as $x \to \infty$ to $A$ say, then the proof shows that for such a method $(\mathscr{D}, h(n))$, any series summable $(\mathscr{D}, h(n))$ to $s$ is $(C, 1)$ summable to $As$.

(ii) The same proof shows (under the condition of the previous remarks) that for such a method, $(\mathscr{D}, h(n))$-boundedness implies $(C, 1)$-boundedness.

It is well to note that all $(\mathscr{D}, h(n))$ methods have in common with $(I)$ the unpleasant feature of:

THEOREM 2. *No $(\mathscr{D}, h(n))$ summability method is regular.*

*Proof.* Suppose there were a function $h(n)$ such that the method $(\mathscr{D}, h(n))$ is regular. Then with $r$ a positive integer

$$\frac{1}{r} \sum_{n \leqslant r} n \sum_{d|n} h(n/d)\,a_d = \frac{1}{r} \sum_{d \leqslant r} da_d \sum_{m \leqslant r/d} mh(m)$$

and setting

$$C_{r,d} = \frac{d}{r} \sum_{m \leqslant r/d} mh(m) \quad (6)$$

it is well known that necessary and sufficient conditions for $(\mathscr{D}, h(n))$ to be regular are:

($\alpha$) $\lim_{r \to \infty} C_{r,d} = 1$ for each $d$.

($\beta$) $\sup_r \lim_{k \to \infty} \sum_{d=1}^{k} |C_{r,d} - C_{r,d+1}| < \infty$.

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Now if $k > r$, every term with $d > r$ in the sum $(\beta) = 0$, and so we need to evaluate sup over $r$ of

$$S_r = \sum_{d \leqslant r} \left| \frac{d}{r} \sum_{m \leqslant r/d} mh(m) - \frac{d+1}{r} \sum_{m \leqslant r/d+1} mh(m) \right|$$

$$= \sum_{\substack{d \leqslant r \\ [\frac{r}{d}] = [\frac{r}{d+1}]}} + \sum_{\substack{d \leqslant r \\ [\frac{r}{d}] \neq [\frac{r}{d+1}]}} \geqslant \sum_{d \leqslant r} \frac{1}{r} \left| \sum_{m \leqslant r/d} mh(m) \right|. \tag{7}$$

For each fixed $r$ and each integer $t$, $1 \leqslant t \leqslant r$, let $N(r,t)$ = the number of positive integers $d \leqslant r$ such that

$$\left[\frac{r}{d}\right] = \left[\frac{r}{d+1}\right] = t.$$

Then from (7) we get

$$S_r \geqslant \frac{1}{r} \sum_{t \leqslant r} N(r,t) \left| \sum_{m \leqslant t} mh(m) \right|. \tag{8}$$

and it remains to investigate the size of $N(r,t)$.

But if $\left[\frac{r}{d}\right] = \left[\frac{r}{d+1}\right] = t$, then $\frac{r}{d} - 1 < t \leqslant \frac{r}{d+1}$; i.e. $\frac{r}{t+1} < d \leqslant \frac{r}{t} - 1$; while, conversely, if $t$ is an integer and

$$\frac{r}{t+1} < d \leqslant \frac{r}{t} - 1, \quad \text{then} \quad \left[\frac{r}{d}\right] - 1 < t \leqslant \left[\frac{r}{d+1}\right],$$

and so

$$\left[\frac{r}{d}\right] \leqslant t \leqslant \left[\frac{r}{d+1}\right],$$

which, since $[x]$ is non-decreasing is impossible unless both equal signs hold.

Hence we have

$$N(r,t) = \left[\frac{r}{t}\right] - 1 - \left[\frac{r}{t+1}\right] \geqslant \frac{r}{t} - \frac{r}{t+1} - 2 = \frac{r}{t(t+1)} - 2, \tag{9}$$

and also

$$0 \leqslant N(r,t) \leqslant \frac{r}{t} - \frac{r}{t+1} = \frac{r}{t(t+1)}. \tag{10}$$

Since $N(r,t)$ is integral-valued, (10) shows that if $t > \sqrt{r}$, $N(r,t) = 0$.

Hence using (9) and assuming $(\alpha)$ holds we get from (8)

$$S_r \geqslant \frac{1}{r} \sum_{t \leqslant \sqrt{r}} N(r,t) \left| \sum_{m \leqslant t} mh(m) \right| \geqslant \sum_{t \leqslant \sqrt{r}} \left( \frac{1}{t+1} - \frac{2t}{r} \right) |C_{t\,1}|$$

$$= (1 + o(1)) \log \sqrt{r} + O\left( \frac{1}{r} \sum_{t \leqslant \sqrt{r}} t \right) = \tfrac{1}{2} \log r + o(\log r) \tag{11}$$

contradicting $(\beta)$ which proves the theorem.

The relation between the $(\mathscr{D}, h(n))$ methods and $(C, -1)$ summability is elucidated by

THEOREM 3. *A necessary and sufficient condition on $h(n)$ that $(C, -1) \Rightarrow (\mathscr{D}, h(n))$ is*

$$\sum_{n \leqslant r} nh(n) = x + S(x),$$

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*where ($r$ is a positive integer)*

(i) $\lim_{r\to\infty}\frac{1}{r}\sum_{k\leqslant r}\frac{1}{k+1}S(r/k)=0,$

(ii) $\sup_r\frac{1}{r}\sum_{d\leqslant r}\left|S(r/d)-\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}S(r/k)\right|<\infty.$

*Proof.* As in the proof of the previous theorem, we have with $r$ a positive integer,

$$\frac{1}{r}\sum_{n\leqslant r}n\sum_{d|n}h(n/d)a_d=\sum_{d\leqslant r}C_{r,d}a_d,$$

where $C_{r,d}$ is defined by (6).

Let
$$E_{r,d}=\frac{d}{r}\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}\sum_{m\leqslant r/k}mh(m)\quad(1\leqslant d\leqslant r),\tag{12}$$

while for $d>r$, $E_{r,d}=0$.

Then

$$(d+1)E_{r,d}-dE_{r,d+1}=\frac{d(d+1)}{r}\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}\sum_{m\leqslant r/k}mh(m)$$
$$-\frac{d(d+1)}{r}\sum_{d+1\leqslant k\leqslant r}\frac{1}{k+1}\sum_{m\leqslant r/k}mh(m)=C_{r,d},$$

where an empty sum is interpreted as 0.

Hence by a theorem of Maddox (5) necessary and sufficient conditions that $(C,-1)\Rightarrow(\mathscr{D},h(n))$ are:

$$(\gamma)\quad\lim_{r\to\infty}E_{r,d}=1\quad\text{for each fixed } d,$$

$$(\delta)\quad\sup\sum_{d=1}^{\infty}|E_{r,d}-E_{r\,d+1}|<\infty.$$

But writing $\sum_{n\leqslant x}nh(n)=x+S(x)$, we get from (12) that

$$E_{r,d}=d\sum_{d\leqslant k\leqslant r}\frac{1}{k(k+1)}+\frac{d}{r}\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}S(r/k)=1-\frac{d}{r+1}+\frac{d}{r}\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}S(r/k)\tag{13}$$

and so $(\gamma)$ is equivalent to (i).

Similarly, using (13), for $d\leqslant r$,

$$|E_{r,d}-E_{r,d+1}|=\left|\frac{1}{r+1}+\frac{d}{r}\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}S(r/k)-\frac{d+1}{r}\sum_{d+1\leqslant k\leqslant r}\frac{1}{k+1}S(r/k)\right|$$
$$=\left|\frac{1}{r+1}+\frac{1}{r}S(r/d)-\frac{1}{r}\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}S(r/k)\right|\tag{14}$$

while for $d>r$, $E_{r,d}=0$.

Hence $(\delta)$ is equivalent to (ii), which proves the theorem.

COROLLARY 1. *Sufficient conditions that $(C,-1)\Rightarrow(\mathscr{D},h(n))$ are* (i) *together with*

(iii) $\sum_{d\leqslant r}|S(r/d)|=O(r).$

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*Proof.* $$\sum_{d\leqslant r}\sum_{d\leqslant k\leqslant r}\frac{1}{k+1}|S(r/k)| = \sum_{k\leqslant r}\frac{k}{k+1}|S(r/k)| \leqslant \sum_{k\leqslant r}|S(r/k)|.$$

Hence (iii) implies (ii).

COROLLARY 2. (i) *and* (ii) *imply that* $S(x) = o(x)$.

*Proof.* $S(x) = o(x)$ is clearly a necessary condition that $(C,-1) \Rightarrow (\mathscr{D}, h(n))$ since the series $\Sigma a_n$, $a_1 = 1$, $a_n = 0$, $n \geqslant 2$ is $(C,-1)$ summable.

COROLLARY 3. $S(x) = O(x^\alpha)$, $0 < \alpha < 1$, *as* $x \to \infty$, *is a sufficient condition for* $(C,-1) \Rightarrow (\mathscr{D}, h(n))$.

*Proof.* If $S(x) = O(x^\alpha)$, $0 < \alpha < 1$, then (i) and (iii) clearly both hold.

*Remarks.* If we assume the stronger condition $S(x) = O(x^\alpha)$ instead of (i) and (ii), then using a theorem of Landau (4) one can prove along the lines of ((1), Appendix IV) that if $\Sigma a_n$ converges and $na_n$ is bounded below as $n \to \infty$, then $\Sigma a_n$ is $(\mathscr{D}, h(n))$ summable for all $h(n)$ satisfying this stronger condition, a result stronger than Corollary 3.

It is interesting to see what can be said about $(\mathscr{D}, h(n))$ methods when $\Sigma h(n)$ is itself subject to a summability condition. It turns out, not surprisingly, that if $\Sigma h(n)$ behaves too well, then the $(\mathscr{D}, h(n))$ method behaves violently. Precisely we have:

THEOREM 4. *If* $\Sigma h(n)$ *is* $(C,k)$*-summable for some* $k \geqslant 0$, *then the only* $(C,l)$*-summable series* $(l \geqslant 0)$, *which are also* $(\mathscr{D}, h(n))$*-summable, have* $(\mathscr{D}, h(n))$ *sum* 0.

*Proof.* Since $(C,k) \Rightarrow (C, k+\delta)$ for every $\delta > 0$, we can take $k$ and $l$ integral with no loss of generality.

Suppose $\Sigma h(n)$ is $(C,k)$-summable to $\beta$ and $\Sigma a_n$ is $(C,l)$-summable to $\alpha$, then (e.g. (2), Theorem 56), $\sum_{d|n} a_d h(n/d)$ is $(C, k+l+1)$ summable to $\alpha\beta$. Hence by the equivalence of Riesz's continuous arithmetic means and Cesaro means, as $x \to \infty$,

$$\sum_{n\leqslant x}(x-n)^{k+l+1}\sum_{d|n}a_d h(n/d) = (\alpha\beta + o(1))x^{k+l+1}. \tag{15}$$

Suppose now also that $\Sigma a_n$ is $(\mathscr{D}, h(n))$-summable to $\gamma$. Then

$$\sum_{n\leqslant x} n \sum_{d|n} a_d h(n/d) = \gamma x + o(x)$$

and so by partial summation, for $r > 0$, we have

$$\sum_{n\leqslant x} n^r \sum_{d|n} a_d h(n/d) = \frac{\gamma}{r}x^r + o(x^r) \tag{16}$$

as $x \to \infty$.

Expanding (15) by the binomial theorem and using (16) we have (since $k$ and $l$ are integers)

$$\begin{aligned} x^{-k-l-1}\sum_{n\leqslant x}(x-n)^{k+l+1}\sum_{d|n}a_d h(n/d) &= \sum_{n\leqslant x}\sum_{r=0}^{k+l+1}(-1)^r (n/x)^r \binom{k+l+1}{r}\sum_{d|n}a_d h(n/d) \\ &= \sum_{n\leqslant x}\sum_{d|n}a_d h(n/d) + \sum_{r=1}^{k+l+1}(-x)^{-r}\binom{k+l+1}{r}\frac{1}{r}x^r(\gamma + o(1)) \\ &= \sum_{n\leqslant x}\sum_{d|n}a_d h(n/d) + \gamma K + o(1), \end{aligned} \tag{17}$$

where $K$ is a constant depending only on $k$ and $l$.

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Hence by (15) $\sum_{n\leqslant x}\sum_{d|n} a_d h(n/d)$ converges, and so $\sum_{n\leqslant x} n \sum_{d|n} a_d h(n/d) = o(x)$, which proves the theorem.

COROLLARY 1. *If $\Sigma h(n)$ is summable $(C,k)$ for some $k \geqslant 0$, then the method $(\mathscr{D}, h(n))$ is not strong.*

*Proof.* Suppose it were, and let $\Sigma a_n$ be any series which is $(\mathscr{D}, h(n))$-summable. Then since $(\mathscr{D}, h(n))$ is presumed a strong method, $\Sigma a_n$ is $(C,1)$-summable, and so by the theorem its $(\mathscr{D}, h(n))$-sum must be 0.

Hence to prove the corollary it is enough to prove that given any $(\mathscr{D}, h(n))$ method, there exists a $(\mathscr{D}, h(n))$-summable series whose $(\mathscr{D}, h(n))$-sum is not zero. But this is almost obvious, e.g. let

$$a_n = \frac{1}{n^2}\sum_{d|n} d^2 h^*(d)\,\sigma(n/d),$$

where as usual $\sigma(n) = \sum_{d|n} d$. Then

$$\frac{1}{x}\sum_{n\leqslant x} n \sum_{d|n} a_d h(n/d) = \frac{1}{x}\sum_{n\leqslant x}\frac{\sigma(n)}{n} = \pi^2/6 + o(1)$$

as is easily seen, either by considering Dirichlet series generating functions or by direct computation.

COROLLARY 2. *Let $h(n)$ be any function on the positive integers, $h(1) = 1$, and $h^*(n)$ its ordinary Dirichlet inverse. Then if $\int_1^x 1/t \sum_{n\leqslant t} h^*(n)\,dt$ converges absolutely to some non-zero-value as $x\to\infty$, then $\Sigma h(n)$ is not $(C,k)$-summable for any $k$.*

*Proof.* This is simply a reformulation of Corollary 1 in terms of $h^*(n)$, observing Remark (i) following Theorem 1.

We turn our attention briefly to Tauberian results. With the obvious modifications necessitated by replacing $I$-summability by $(\mathscr{D}, h(n))$-summability, the various variants of ((7), Theorem 1) go through exactly as there, being based on the Wiener–Pitt theory applied to formula (4) above. Thus, for example, we have

THEOREM 5. *Suppose $(\mathscr{D}, h(n))$ is a strong method, and further that $H^*(x) = \sum_{n\leqslant x} h^*(n)$ satisfies the condition, $\int_0^\infty H^*(u)\,u^{-1+iy}\,du \neq 0$ for any real $y$; then if $\Sigma a_n$ is $(C,1)$-summable, and $\sum_{d|n} a_d h(n/d) = O(1/n)$, then $\Sigma a_n$ is $(\mathscr{D}, h(n))$-summable.*

At the expense of adding a further condition on $H^*(x)$ (to enable an application of Littlewood's Tauberian Theorem, as in (7)), one can replace $(C,1)$-summability in the above theorem by Abel-summability.

In the other direction, it was shown in (7) that the series $\Sigma a_n$ where

$$a_n = \sum_{d|n}\mu(d)\,(-1)^{n/d} \quad \text{and} \quad \mu(n)$$

is the Möbius function, is $(I)$-bounded and $(C,1)$-summable, but not $(I)$-summable. In fact this series is $(C,\delta)$-summable for every $\delta > 0$.

For general strong $(\mathscr{D}, h(n))$ methods we have the following weaker.

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THEOREM 6. *If* $(\mathscr{D}, h(n))$ *is a strong method then the series* $\Sigma a_n$ *where*

$$a_n = \sum_{d|n} (-1)^{n/d} h^*(d) \tag{18}$$

*is* $(\mathscr{D}, h(n))$*-bounded and* $(C, 1+\delta)$*-summable for every* $\delta > 0$*, but not* $(\mathscr{D}, h(n))$*-summable.*

*Proof.* That the series $\Sigma a_n$, where $a_n$ is defined by (18), is $(\mathscr{D}, h(n))$-bounded but not $(\mathscr{D}, h(n))$-summable is almost trivial, since

$$\begin{aligned}\frac{1}{x}\sum_{n\leqslant x} n \sum_{d|n} a_d h(n/d) &= \frac{1}{x}\sum_{n\leqslant x} n \sum_{d|n} h(n/d) \sum_{k|d} (-1)^{d/k} h^*(k) \\ &= \frac{1}{x}\sum_{n\leqslant x} n \sum_{e|n} (-1)^e \sum_{k|n/e} h\left(\frac{n}{ke}\right) h^*(k) = \frac{1}{x}\sum_{n\leqslant x} n(-1)^n \\ &= \frac{1}{x}\left[\frac{x+1}{2}\right](-1)^{[x]}. \end{aligned} \tag{19}$$

Since $(\mathscr{D}, h(n))$ is by hypothesis a strong method, by Theorem 1, Remark (ii), (19) also shows that $\Sigma a_n$ is $(C, 1)$-bounded.

Now $\Sigma(-1)^n$ is $(C, 1)$-summable, and $\Sigma h^*(n)$ is clearly $(\mathscr{D}, h(n))$ and hence $(C, 1)$-summable, by Theorem 1. Hence [e.g. (2), Theorem 56] $\Sigma \sum_{d|n} (-1)^d h^*(n/d)$ is $(C, 3)$-summable, and so by the convexity theorem for Cesàro means $(C, 1+\delta)$-summable for every $\delta > 0$.

*Remarks.* It is not known whether $(C, 1+\delta)$ can be replaced by $(C, 1)$ in the above theorem, either for the specified series or for some other series, for general strong $(\mathscr{D}, h(n))$ methods.

On the other hand if we further subject $h^*(n)$ to the conditions (satisfied, for example, by $\mu(n)/n = h^*(n)$ for $(I)$-summability)

$$\sum_{n\leqslant x} h^*(n) = o(1) \quad \text{and} \quad \sum_{n\leqslant x} n\left|h^*(n)\right| = O(x),$$

then the method used in (7) in conjunction with Axer's theorem (see e.g. (4)) can be used to show that the series $\Sigma a_n$ where $a_n$ is defined by (18) is $(C, 1)$-summable.

It might be finally remarked that even for $(I)$-summability, questions of the type considered in Theorem 6 are not completely settled. Thus it is apparently not known whether $(I)$-boundedness and convergence together imply $(I)$-summability.

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