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THE TILED CIRCLE PROBLEM

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Abstract. How many square tiles are needed to tile a circular floor? Tiles are cut to fit the boundary. We give an algorithm for cutting, rotating and re-using the off-cut parts, so that a circular floor requires $\pi R^2 + O(\delta R) + O(R^{2/3})$ tiles, where R is the radius and δ is the width of the cutting tool. The algorithm applies to any oval-shaped floor whose boundary has a continuous non-zero radius of curvature. The proof of the error estimate requires methods of analytic number theory.

§1. *Introduction.* How many square tiles are needed to tile a circular floor? We draw the plan of the floor as a closed convex shape S in the x, y plane, scaled so that the square tiles are unit squares. Where an unit square intersects the boundary C of S, then the corresponding tile has to be cut to shape. The method that we give applies to any part of a floor with a curved wall, such as a semicircular bay in a rectangular building. In the "sacred geometry" associated with the painter and mathematician Piero della Francesca [**5**, **6**] a circle overlapping a square represents the meeting of heaven and earth. In this paper we take the shape S to be a complete oval, not necessarily a circle.

The shape S has area A, breadth B and height H when projected onto the x- and y-axes. Let N be the number of tiles used, and let A^*, B^* and H^* be the smallest integers with $A^* \geqslant A$, $B^* \geqslant B$ and $H^* \geqslant H$. There is an area inequality

$$N \geqslant A^*. \tag{1.1}$$

Squares that cross the boundary C correspond to tiles that must be cut. The curve C (anticlockwise) enters a square by one of its four sides, right, left, top or bottom. The curve enters at most B^* squares from the right, and similarly for the other three types of side. The curve enters at most $2B^* + 2H^*$ squares, each corresponding to a tile to be cut. If the off-cut parts of tiles are discarded, then

$$N \leqslant A + 2B^* + 2H^* \leqslant A + 2B + 2H + 4, \tag{1.2}$$

and the story ends here.

We suppose that the design on the tiles remains the same when rotated by 180°, so the off-cut parts of the tiles can be trimmed further and used elsewhere (Figure 1). The cutting tool has some finite width δ. The area inequality (1.1) still holds.

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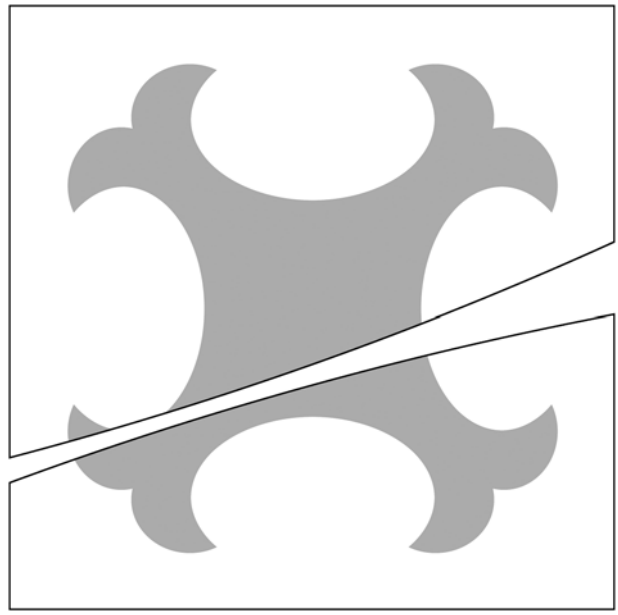


Figure 1: Cutting a tile to use in two places.

We obtain another lower bound from counting integer points as in the classic circle problem. Let M_1 integer points lie in the interior of S, and let M_2 lie on the boundary C. The corners of four tiles meet each interior integer point. At an integer point P on the boundary, the corners of three tiles meet in the general case, but two if either the x- or y-coordinate is maximum or minimum at P, and possibly only one if C has an acute-angled corner at P. Thus

$$N \geqslant M_1. \tag{1.3}$$

The bound corresponding to (1.2) is

$$M_1 \leqslant A + 2B + 2H + 1,$$

which cannot be improved when S is a rectangle.

We suppose that the boundary C of S is a curve with a tangent at every point, and that the arc length s along C is a differentiable function of the tangent angle ψ, and that the radius of curvature ρ of C lies in some range

$$c_1 R \leqslant \rho = \frac{ds}{d\psi} \leqslant c_2 R, \tag{1.4}$$

where $c_1 \leqslant 1 \leqslant c_2$. We call such a curve a C^2 curve. For $k \geqslant 3$ we call C a C^k curve if C is a C^2 curve, and s is a $k-1$-times differentiable function with bounds of the form

$$\left| \frac{d^i s}{d\psi^i} \right| \leqslant c_{i+1} R \quad \text{for } i = 2, \ldots, k-1. \tag{1.5}$$

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Finally we call C a C^∞ curve if C is a C^2 curve and (1.5) holds for all $i \geqslant 2$. The circle is the smoothest curve of all, with $c_1 = c_2 = 1$, $c_k = 0$ for $k \geqslant 3$.

For a C^2 curve we have

$$M_1 = A + O(R^{2/3}), \qquad M_2 = O(R^{2/3}), \tag{1.6}$$

a result of Voronoï [17] and Sierpiński [16] in special cases, and of Van der Corput [2] in general. The bounds (1.6) are best possible in order of magnitude by a construction of Jarník [12].

For C^k curves with $k \geqslant 3$, high analytic methods involving Fourier series become available. They give results of the form

$$M_1 = A + O(R^\kappa (\log R)^\lambda), \qquad M_2 = O(R^\kappa (\log R)^\lambda), \tag{1.7}$$

with $\kappa < 2/3$. Van der Corput [3, 4] and his followers developed an iterative method [7, 15] where the number of steps increases with n. Bombieri, Iwaniec and Mozzochi [1, 11] found a wonderful method that combines many ideas, starting from those of Voronoï, Sierpiński and Van der Corput. Its most recent form (Huxley [9, 10]) obtains (1.7) for a C^3 curve with $\kappa = 131/208$, $\lambda = 18627/8320$.

The finite width of the cutting tool sets another constraint on the number of tiles N. Let L be the length of C. The border set of points outside S, but within a distance δ of S, has area $\delta L + \pi\delta^2$. This border set can extend outside the squares cut by C, but only where there is an integer point close to C. A modification of the argument leading to (1.6) shows that

$$N \geqslant A + \frac{\delta L}{2} - O(\delta^2 R) - O(\delta R^{2/3}). \tag{1.8}$$

There is a factor $1/2$ in the second term because the border sets from the two halves of a cut tile overlap on the tile itself.

We describe a procedure for dividing the squares cut by the curve C into short blocks (of $O(R^{1/3})$ squares) with a rule that pairs some of the squares within each block. The parts of the shape S that fall within the two squares of a pair can be cut from a single tile, allowing for the width δ of the cutting tool.

THEOREM. *For a C^2 curve with R so large that*

$$c_1 R \geqslant 20, \tag{1.9}$$

the procedure can be implemented to tile the shape S with at most

$$A + \frac{\delta L}{2} + \frac{(20+\delta)c_2}{c_1^{1/3}} R^{2/3}(\log R + 3) + 96 + 6\delta \tag{1.10}$$

tiles. For R sufficiently large, a refined form of the procedure can tile S with at most

$$A + \frac{\delta L}{2} + O(R^{2/3}) \tag{1.11}$$

tiles.

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The circle with centre the origin, radius 100, is already a favourable case, with 20 integer points on the boundary circle C, and only 780 squares cut by C. There are 31 016 complete squares inside S, and $A^* = 31\,416$, so the excess number of tiles is $N - A^* = 380$. Our procedure with cutting width $\delta = 0.02$ and parameter $Q = 3$ in (2.4) gives $N - A^* = 260$. Trial and error matching cuts and off-cuts can get down to $N - A^* = 100$. Of course $R = 100$ is not large enough to distinguish contributions with orders of magnitude in the ratio $R^{1/3}$.

Any construction using Voronoï's Farey arcs method has curvature correction terms on each Farey arc whose total absolute value is at least $O(R^{2/3})$. When we estimate the area A by counting squares as M_1+M_2 using (1.7), these corrections have either sign, and they cancel to some extent over all the Farey arcs. In this problem the contribution to $N - A^*$ is always positive. There is probably no fortuitous configuration of a circle, or of any other smooth curve, in which the sum of these corrections is smaller than $O(R^{2/3})$. Matching cut parts and off-cuts in a set of randomly cut squares is expected to leave $O(R^{3/4})$ squares unpaired. By contrast, the expected error in estimating the area of a randomly placed C^2 curve by counting squares is $O(\sqrt{R})$ (Kendall [**14**]).

What is the complexity of the construction? There are $O(R)$ tiles to be cut anyway. The construction requires $O(Q^2)$ points $T(q, a)$, the centres and endpoints of the Farey arcs. To find the sequence of vectors (q, a) (the Farey sequence and its mediants), generating each vector from its predecessor, takes $O(Q^2 \log Q)$ operations. We need to know, by calculation or scale drawing, which squares they lie in. The points (n, y) or (x, n) where the curve enters each square must be known to sufficient accuracy to determine the integer part of the real number qy or ax, where (q, a) is the local integer vector in the direction of the curve. The integers a, q and Q are $O(R^{1/3})$.

A computer program to direct the cutting tool has to calculate the curve as $y = f(x)$ or $x = g(y)$ anyway. This takes

$$O(DR(\log R)^E)$$

operations at least, where D is a large factor determined by the desired accuracy of the cut, and E is determined by the ease of calculation of $f(x)$ or $g(y)$. The extra work required to allocate off-cut tiles is $O(R)$, which is negligible by comparison.

Trial and error matching seems to require $O(\sqrt{R})$ cut squares to be compared and put in order in each of $O(\sqrt{R})$ arcs of the curve C, at least $O(R^{3/2})$ operations.

§2. *Step one: Farey arcs.* The curve C is drawn in the x, y plane. The tiles are inserted with their edges parallel to the axes, starting from four tiles with a common corner at the origin. We assume that for any integer vector (q, a), we can find the point $T(q, a)$ where (q, a) is a forward (anticlockwise) tangent vector. The tangent angle $\psi(q, a)$ at $T(q, a)$ satisfies

$$\tan \psi(q, a) = \frac{a}{q}. \tag{2.1}$$

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We divide the curve C into eight octants on which

$$(r-1)\frac{\pi}{4} \leqslant \psi \leqslant \frac{r\pi}{4}$$

for $r = 1, \ldots, 8$. The vector (q, a) is a forward tangent vector in the first octant if $q > 0$, $0 \leqslant a \leqslant q$. We describe the construction in the first octant. The construction in the other seven octants is obtained by considering the corresponding forward tangent vectors $(\pm q, \pm a)$ or $(\pm a, \pm q)$.

We fix a large integer $Q < R$. The Farey sequence $F(Q)$ in the interval $[0, 1]$ consists of the rational numbers a/q with $1 \leqslant q \leqslant Q$, $0 \leqslant a \leqslant q$ and highest common factor $(a, q) = 1$, numbered in ascending order $a_0/q_0 = 0/1$, $a_1/q_1 = 1/Q$, up to $a_F/q_F = 1/1$ (for some integer F). We write $T_i = T(q_i, a_i)$, $\psi_i = \psi(q_i, a_i)$, $U_i = T(q_i + q_{i+1}, a_i + a_{i+1})$, $\theta_i = \psi(q_i + q_{i+1}, a_i + a_{i+1})$. The basic properties of the Farey sequence are

$$a_{i+1}q_i - a_i q_{i+1} = 1, \qquad q_i + q_{i+1} \geqslant Q + 1. \tag{2.2}$$

Conversely, if a/q and b/r are two fractions in $[0, 1]$ with $bq - ar = 1$, they are consecutive fractions of $F(Q')$ for $\max(b, r) \leqslant Q' \leqslant b + r - 1$. From (2.1) and (2.2) we have

$$\begin{aligned}
\tan(\psi_{i+1} - \psi_i) &= \frac{1}{a_i a_{1+1} + q_i q_{i+1}}, \\
\tan(\theta_i - \psi_i) &= \frac{1}{a_i(a_i + a_{i+1}) + q_i(q_i + q_{i+1})}, \\
\tan(\psi_{i+1} - \theta_i) &= \frac{1}{a_{i+1}(a_i + a_{i+1}) + q_{i+1}(q_i + q_{i+1})}.
\end{aligned} \tag{2.3}$$

The first octant extends from $T_0 = T(1, 0)$ to $T_F = T(1, 1)$. The Farey arc $C(q_i, a_i)$ is the arc $U_{i-1}U_i$ of the curve C. Exceptionally, the point U_{-1} is interpreted as $T(Q + 1, -1)$, and the section $U_{-1}T_0$ of the Farey arc $C(1, 0)$ lies in the eighth octant. The point U_F is interpreted as $T(Q, Q + 1)$, and the section $T_F U_F$ of the Farey arc $C(1, 1)$ lies in the second octant. When the construction is repeated in the other seven octants, then the Farey arc $C(Q, -1)$ meets $C(1, 0)$ at U_0, and the Farey arc $C(1, 1)$ meets $C(Q - 1, Q)$ at U_F.

We divide the curve into octants so that x can be taken as the independent variable in the first, fourth, fifth and eighth octants with Farey arcs $C(\pm q, \pm a)$, and y on the remaining octants with Farey arcs $C(\pm a, \pm q)$. In the first octant let $L(q, a)$ be the length of the projection onto the x-axis of the part of $C(q, a)$ that lies in the first octant. We require $L(q, a) > q$ in order to use the periodicity of the lattice of integer points in the direction of the vector (q, a). This means that the parameter Q is at most $O(R^{1/3})$. A valid choice of Q in our procedure is

$$Q = [\tfrac{1}{2}(\sqrt{2}c_1 R)^{1/3} + \tfrac{1}{2}], \tag{2.4}$$

with $Q \geqslant 2$ by the condition (1.9); the square brackets mean "take the integer part". The reason for this precise choice of Q is explained in 4.

We prove two lemmas to bound the lengths $L(q, a)$.

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LEMMA 1. *For Farey arcs $C(q,a) = C(q_i,a_i)$ in the first octant, let $L^+(q,a)$ (for $i = 0,\dots,F-1$) be the length of the projection of T_iU_i onto the x-axis, and let $L^-(q,a)$ (for $i = 1,\dots,F$) be the length of the projection of $U_{i-1}T_i$ onto the x-axis. Then*

$$\frac{c_1R}{4q(Q+q)} \leqslant L^{\pm}(q,a) \leqslant \frac{c_2R}{q(Q+1)}. \tag{2.5}$$

Proof. We have

$$L^+(q_i,a_i) = \int_{\psi_i}^{\theta_i} \rho\cos\psi\,d\psi \leqslant c_2R\int_{\psi_i}^{\theta_i}\cos\psi\,d\psi \leqslant c_2R(\sin\theta_i - \sin\psi_i).$$

Now

$$\begin{aligned}\sin\theta_i - \sin\psi_i &= 2\sin\frac{\theta_i-\psi_i}{2}\cos\frac{\theta_i+\psi_i}{2} \leqslant 2\tan\frac{\theta_i-\psi_i}{2}\\ &\leqslant \tan(\theta_i-\psi_i) = \frac{1}{q_i(q_i+q_{i+1})} \leqslant \frac{1}{q(Q+1)},\end{aligned}$$

by (2.2) and (2.3). This gives the upper bound in (2.5).

For the lower bound we use similarly

$$L^+(q_i,a_i) \geqslant c_1R(\sin\theta_i - \sin\psi_i),$$

and, since the angles θ_i and ψ_i lie in the range $[0,\pi/4]$,

$$\begin{aligned}\sin\theta_i - \sin\psi_i &= 2\sin\frac{\theta_i-\psi_i}{2}\cos\frac{\theta_i+\psi_i}{2} \geqslant \sqrt{2}\sin\frac{\theta_i-\psi_i}{2}\\ &\geqslant \frac{1}{\sqrt{2}}\sin(\theta_i-\psi_i) = \frac{1}{\sqrt{2}}\tan(\theta_i-\psi_i)\cos(\theta_i-\psi_i)\\ &\geqslant \frac{1}{2}\tan(\theta_i-\psi_i) = \frac{1}{2(a_i(a_i+a_{i+1})+q_i(q_i+q_{i+1}))}\\ &\geqslant \frac{1}{4q(Q+q)}.\end{aligned}$$

For $L^-(q_i,a_i)$ we integrate from θ_{i-1} to ψ_i, and the estimates are the same. □

LEMMA 2. *The length $L(q,a)$ of the projection onto the x-axis of the part of the Farey arc $C(q,a)$ that lies in the first octant satisfies*

$$\sum_{F(Q)} L(q,a) \leqslant \frac{c_2R}{\sqrt{2}}, \tag{2.6}$$

$$\sum_{F(Q)} \frac{L(q,a)}{q} \leqslant \frac{c_2R(\log Q+1)}{Q}. \tag{2.7}$$

Proof. The first sum is simply

$$\int_0^{\pi/4}\rho\cos\psi\,d\psi \leqslant c_2R\int_0^{\pi/4}\cos\psi\,d\psi = c_2R\sin\frac{\pi}{4} = \frac{c_2R}{\sqrt{2}}.$$

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In the second sum we add $L^+(q,a)$ to $L^-(q,a)$ for $q \geqslant 2$, but we only count $L^+(1,0)$ and $L^-(1,1)$. Lemma 1 gives

$$\sum_{F(Q)} \frac{L(q,a)}{q} \leqslant \sum_{q=1}^{Q} \sum_{\substack{a \bmod q \\ (a,q)=1}} \frac{2c_2R}{qQ} \leqslant \frac{2c_2R}{Q} \sum_{q=1}^{Q} \frac{1}{q}$$
$$\leqslant \frac{2c_2R}{Q}\left(1 + \int_1^Q \frac{dx}{x}\right) = \frac{2c_2R(\log Q + 1)}{Q}. \qquad \square$$

§3. *Step two: blocks of squares.* The curve C has been divided into Farey arcs $C(q,a)$. In the first octant we can describe C by an equation $y = f(x)$ with x as the independent variable. We divide the squares cut by the first octant of the curve into blocks. A block $B_i(q,a)$ consists of all squares cut by the curve in a strip $n \leqslant x \leqslant n+q$, where n is an integer and $(n, f(n))$ is a point on the Farey arc $C(q,a)$. The first block $B_1(1,0)$ consists of the square containing the centre $T(1,0)$ of the Farey arc $C(1,0)$, either as an interior point, or on the left-hand edge, and possibly the square above. The blocks $B_2(1,0)$, $B_3(1,0)$, ... have width 1 and consist of a single square, or of two squares, one above the other. This pattern continues as far as the block containing U_0, either as an interior point, or on the right-hand edge. The next Farey arc is $C(Q,1)$, and we continue with blocks $B_i(Q,1)$ of width Q as far as the block containing U_1, either as an interior point, or on the right-hand edge. The next Farey arc is $C(Q-1,1)$, and we continue with blocks $B_i(Q-1,1)$ of width $Q-1$ as far as U_2, and so on. The last Farey arc in the first octant is $C(1,1)$, with $a_F/q_F = 1/1$, and we continue as far as the block containing $T_F = T(1,1)$.

Blocks $B_j(q_i,a_i)$ which extend past the endpoint U_i of their Farey arc $C(q_i,a_i)$ are called bad blocks. Other blocks are called good blocks. The part of the curve contained within a good block lies entirely within the Farey arc $C(q_i,a_i)$. The bad block containing U_i must not extend as far as T_{i+1}, the centre of the next Farey arc. This essential condition restricts the choice of Q; we show in § 4 that (2.4) gives a valid choice of Q.

In the eighth octant the blocks are laid out leftward of the square containing the centre $T(1,0)$ of the arc $C(1,0)$. This square has been assigned to a block in the first octant. The blocks in the fourth and fifth octants are laid out right and left from the square containing $T(-1,0)$. On the other octants y is the independent variable. For example in the second octant we start from the square containing $T(0,1)$, and lay out blocks $B_j(a,q)$ consisting of all squares cut by the curve in a range $n \geqslant y \geqslant n-q$, where n is an integer, and $(g(n), n)$ is a point on the Farey arc $C(a,q)$. The first and second octants meet at the point $T(1,1)$. The last block from the second octant may fit cleanly against the last block from the first octant, or it may overlap the blocks from the first octant by one or two squares. Our construction cannot use any off-cut tiles from the Farey arcs with $q = 1$. The upper bound calculations for the theorem allow for a block containing $T(1,1)$ in both octants. Thus overlapping blocks do not ruin the argument.

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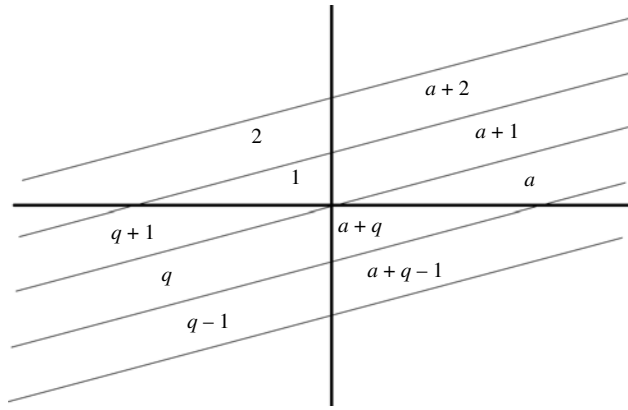


Figure 2: Band numbering at the corners of squares.

The lines $qx - ay = c$ with c an integer divide the plane into bands. There are q bands that enter a lattice square from the left, and a bands that enter it from below. We number the parts of the bands within each unit square, starting from 1 at the bottom right corner to $a+q$ at the top left corner (Figure 2). In an interval $n \leqslant x \leqslant n+q$ with n an integer, a band crosses $a+q$ squares, receiving a different number in each square.

We consider a block $B(q,a)$ of squares cut by the curve in an interval $n \leqslant x \leqslant n+q$. Squares cut by the curve represent tiles which have to be cut to fit the shape of the floor. We call the part of the square above the curve the cut part, and classify it as type $t = 1, \ldots, a+q$, where band t is the lowest-numbered band that meets the cut part.

We consider first the simple case when the curve does not cut a band boundary within the block $B(q,a)$. The curve stays in the same band, which cuts $a+q$ squares at a differently numbered position in each square. Each type from $t=1$ to $a+q$ occurs exactly once among the cut parts.

The width of the bands is $1/\sqrt{a^2+q^2}$. Let d be the integer with

$$d \geqslant \delta\sqrt{a^2+q^2} > d-1. \tag{3.1}$$

When the curve lies in band t, then the cutting tool removes parts of bands t, $t-1, \ldots, t-d+1$, and possibly part of band $t-d$ as well. For $t \geqslant d-2$, the off-cut part contains the whole of bands $1, \ldots, t-d+1$. We rotate the off-cut tile by 180°. The $t-d+1$ bands numbered $a+q+d+2-t$ to $a+q$ are complete. We can cut a shape of type

$$s = a+q+d+2-t \tag{3.2}$$

from the off-cut. If $s=t$, then we cannot use the off-cut tile within the same block $B(q,a)$, but we may be able to use it in an adjacent block. The case $s=t$ occurs when $a+q+d$ is even. We have $t < s$ when $t \leqslant [(a+q+d+1)/2]$.

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There are

$$\left[\frac{a+q+d+1}{2}\right]-(d+1)=\left[\frac{a+q-d-1}{2}\right]$$

values of t for which $1 \leqslant s \leqslant t-1$, and the off-cut part of the tile can be cut and used again in the same block. The tiles used to cut squares of types $1, \ldots, d+1$ cannot be used again, and if $a+q+d$ is even, the tile used to cut the shape t with $t=s$ cannot be used again in the same block $B(q,a)$. We can cover the $a+q$ cut squares on the block using parts of

$$\left[\frac{a+q+d+2}{2}\right] \tag{3.3}$$

tiles. This pairing is the basis of our procedure.

We compare the number of tiles used on the block $B(q,a)$ with the area to be covered, the total area of the cut parts of the squares. If we replace the curve by the mid-line of a band, of the form $qx-ay=c+1/2$ with c an integer, then the $a+q$ off-cut parts are congruent to the $a+q$ cut parts in a different order, so the area is $(a+q)/2$. If we know that the curve stays in the same band on the block $B(q,a)$, then we must subtract up to half the area of a band within a block, which forms a parallelogram with two sides length $1/q$, distance q apart. The total area of the cut parts of the squares in the block $B(q,a)$ is $\geqslant (a+q-1/2)$.

Let $E(q,a)$, called "the excess of the block $B(q,a)$", denote the number of tiles used minus the area of the cut squares on $B(q,a)$. In the theorem, $N-A^*$ is the sum of the excess over all the blocks of squares cut by the curve C. Let $\rho(t)$ be the periodic function $[t]-t+1/2$, with $\rho(0)=1/2$ and $\rho(1/2)=0$. Then

$$\begin{aligned} E(q,a) &\leqslant \left[\frac{a+q+d+2}{2}\right]-\frac{a+q-1}{2} \\ &= \frac{a+q+d+2}{2}+\rho\left(\frac{a+q+d+2}{2}\right)-\frac{1}{2}-\frac{a+q-1}{2} \\ &= \frac{d}{2}+1+\rho\left(\frac{a+q+d}{2}\right). \end{aligned} \tag{3.4}$$

When the curve C crosses one or more band boundaries on the block $B(q,a)$, then the pattern of cut types is disrupted. Let γ be the minimum of $qx-ay$ on the arc of C contained in the block $B(q,a)$. Then C lies above the reference line $qx-ay=\gamma$ on $B(q,a)$, and C meets the reference line at one point, which is either the centre $T(q,a)$ of the Farey arc, or the endpoint $x=n$ or $x=n+q$, whichever is closer to $T(q,a)$. We now call the type of a square cut by the curve, as defined above, the true type. The matching type of a square is the number of the band that the reference line passes through. If the reference line misses the square completely, then the matching type is not defined.

First we consider the case when the curve C cuts the same set of squares as the reference line, but there are c band crossings. In a square where the reference line passes through band t, the true type of the cut part may be $t, t+1, \ldots,$ or $t+c$. Since type $r+1$ is smaller than type r, the pairing of a square with matching type t with a square of matching type s given by (3.2) still works, and

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two cut parts of squares can be taken from a single tile. The number of tiles used is still given by (3.3). However the curve C lies up to c bands higher than the reference line. We subtract c from the lower bound estimate for the area of the cut squares. The estimate for the excess of tiles over area changes from (3.4) to

$$E(q,a) \leqslant c + \frac{d}{2} + 1 + \rho\left(\frac{a+q+d}{2}\right). \tag{3.5}$$

Crossing a band boundary may take the curve into a square which the reference line $qx - ay - \gamma$ does not enter. This changes both the number of tiles required and the sum of the areas of the cut parts. We consider a curved line across the block which moves continuously upwards from the reference line to the actual position of the curve C. Taking this moving line as a line for cutting along, we define the number of tiles needed, the area of the cut parts, and the excess in the same way as before. The excess varies continuously as the curve moves upwards, except where the curve passes the corner of a square and enters a different square, or misses a square which it had entered before. We consider the squares gained or lost one at a time.

When the curve moves past an integer point which is not on the boundary $x = n$ or $x = n + q$ of the block $B(q, a)$, then in Figure 2, instead of passing through band $a + q$ of the lower right square, it passes through band 1 of the upper left square. The lower right square had matching type $a + q, a + q - 1, \ldots,$ or $a + q - c$, and it was paired with another cut square if the matching type was at least $d + 2$. The upper left square has type 1, and it cannot be paired with another cut square. The number of tiles required increases by 1 except possibly when $a + q \leqslant c + d + 1$.

At the moment of passing the integer point, the cut part of the lower right square in Figure 2 has area 0, whilst the cut part of the upper left square has area 1. The total area increases by 1. The excess is unchanged unless $a + q \leqslant c + d + 1$, when it may decrease by 1.

There are two exceptional cases. If the new square entered in band 1 lies in $n + q - 1 \leqslant x \leqslant n + q$, then no square is lost, one new square is entered. The new square requires an extra tile, and the total area of the cut parts of squares increases by 1. So the excess is unchanged.

The second exceptional case is when the integer point passed as the curve moves upwards lies on the left-hand boundary of the block, the line $x = n$. In Figure 2 the curve in the upper right square now passes from band $a+1$ to band a. The curve does not enter the lower right square. The lower right square had been paired with another cut square if its matching type was at least $d + 2$, so the number of tiles required remains the same except possibly when $a+q \leqslant c+d+1$, when the number of tiles required might decrease by 1. The areas of the cut parts in the upper and lower right-hand squares change continuously, with the area in the lower right square decreasing to 0. In this case also the excess is unchanged unless $a + q \leqslant c + d + 1$, when it may decrease by 1.

Hence the estimate (3.5) for the excess on a block in which the curve C crosses c band boundaries remains valid even when crossing the band boundaries takes the curve into different squares from those entered by the reference line.

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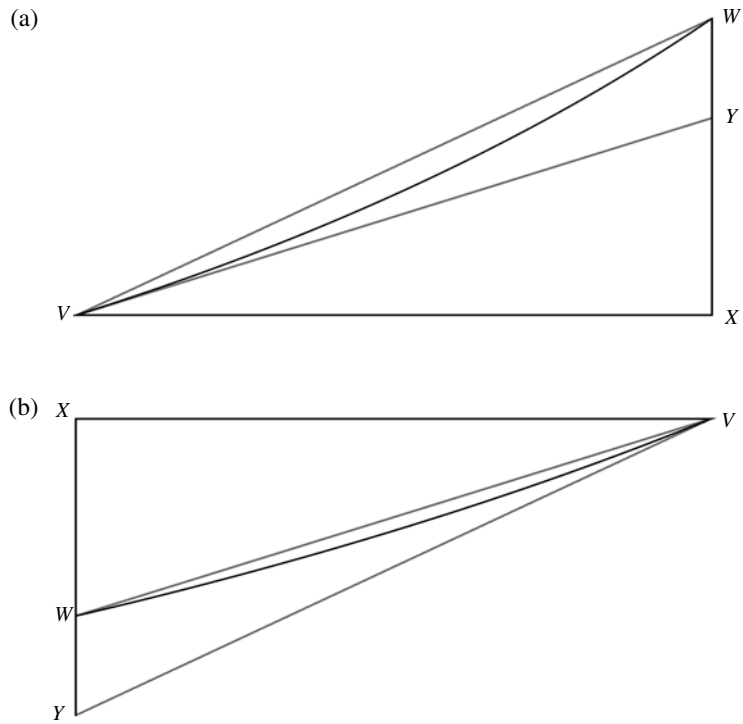


Figure 3: The curve C and the reference line.

§4. *Step three. Band crossings.* The difficulty in the proof of the theorem is to bound the number of band boundaries that the curve C crosses within a block $B(q,a)$ defined by $n \leqslant x \leqslant n+q$ for some integer n. There is a reference line $qx - ay = \gamma$ that meets the curve C at one point V within the block. The point V is either an endpoint with $x = n$ or $x = n+q$, or the centre $T = T(q,a)$ of the Farey arc $C(q,a)$. We compare the arc VW of C with its projections VY onto the reference line and VX onto the line through V parallel to the x-axis (Figures 3a and 3b). The line XW is either the boundary $x = n+q$ of the block (case a), or the boundary $x = n$ (case b).

Let $C(q,a)$ be the ith Farey arc $C(q_i, a_i)$. The reference line VY makes an angle $\alpha = \psi_i$ with the horizontal line VX, and some angle β with the chord VW. Then

$$XY = XV \tan\alpha \leqslant q \tan\alpha.$$

In Figure 3(a) we have on good blocks

$$\begin{aligned} XW &= XV \tan(\alpha+\beta), \\ YW &= XV(\tan(\alpha+\beta) - \tan\alpha) \leqslant XV(\tan\theta_i - \tan\alpha) \\ &\leqslant q\left(\frac{a+a_{i+1}}{q+q_{i+1}} - \frac{a}{q}\right) = \frac{1}{q+q_{i+1}} \leqslant \frac{1}{Q+1} < \frac{1}{q}. \end{aligned} \tag{4.1}$$

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In Figure 3(b) we have on all blocks

$$\begin{aligned} XW &= XV\tan(\alpha-\beta),\\ YW &= XV(\tan\alpha-\tan(\alpha-\beta)) \leqslant XV(\tan\alpha-\tan\theta_{i-1})\\ &\leqslant q\left(\frac{a}{q}-\frac{a+a_{i-1}}{q+q_{i-1}}\right)=\frac{1}{q+q_{i-1}}\leqslant\frac{1}{Q+1}<\frac{1}{q}. \end{aligned} \tag{4.2}$$

The band boundaries meet the line VX at points $1/q$ apart. The arc VW crosses at most one band boundary, and the true type of a square is at most one greater than its matching type.

On the bad block the point U_i lies between V and W in Figure 3(a), and W lies between U_i and T_{i+1}. Thus $\beta\leqslant\psi_{i+1}-\alpha$, which gives only

$$YW\leqslant q(\tan\psi_{i+1}-\tan\alpha)=q\left(\frac{a_{i+1}}{q_{i+1}}-\frac{a}{q}\right)=\frac{1}{q_{i+1}}. \tag{4.3}$$

We need a sharper inequality than (4.3). Let $\psi=\xi$ at V and $\psi=\eta$ at W. Since $VX=q$ in Figure 3(a), we have

$$q=\int_\xi^\eta \rho\cos\psi\,d\psi\geqslant c_1R\int_\xi^\eta\cos\psi\,d\psi=c_1R\cos^3\zeta\int_\xi^\eta\sec^2\psi\,d\psi$$

for some ζ in $\xi<\zeta<\eta$. Hence

$$q\geqslant\frac{c_1R}{2\sqrt{2}}(\tan\eta-\tan\xi).$$

A sufficient condition for W to lie between U_i and T_{i+1} is

$$\tan\eta-\tan\xi\leqslant\tan\psi_{i+1}-\tan\theta_i\leqslant\frac{1}{q_{i+1}(q+q_{i+1})}. \tag{4.4}$$

We show that (2.4) implies (4.4) by way of

$$\left(Q-\frac{1}{2}\right)^3\leqslant\frac{c_1R}{4\sqrt{2}}.$$

We have

$$qq_{i+1}(q+q_{i+1})\leqslant Q(Q-1)(2Q-1)\leqslant 2\left(Q-\frac{1}{2}\right)^3\leqslant\frac{c_1R}{2\sqrt{2}},$$

so that

$$\tan\eta-\tan\xi\leqslant\frac{2\sqrt{2}\,q}{c_1R}\leqslant\frac{1}{q_{i+1}(q+q_{i+1})}=\tan\psi_{i+1}-\tan\theta_i, \tag{4.5}$$

as required in (4.4).

On the bad block we now have

$$\beta<(\theta_i-\alpha)+(\eta-\xi),$$

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so by (4.5)

$$\begin{aligned}\tan(\alpha+\beta)-\tan\theta_i &= \tan(\alpha+\beta-\theta_i)(1+\tan(\alpha+\beta)\,\tan\theta_i)\\ &< 2\tan(\alpha+\beta-\theta_i) < 2\tan(\eta-\xi)\\ &= \frac{2(\tan\eta-\tan\xi)}{1+\tan\eta\,\tan\xi} < 2(\tan\eta-\tan\xi) < \frac{4\sqrt{2}\,q}{c_1R}\\ &\leqslant \frac{q}{(Q-1/2)^3} \leqslant \frac{64q}{27Q^3}.\end{aligned}$$

In place of (4.1), we have

$$\begin{aligned}YW &= q(\tan(\alpha+\beta)-\tan\alpha) = q(\tan\theta_i-\tan\alpha+\tan(\alpha+\beta)-\tan\theta_i)\\ &\leqslant \frac{1}{Q+1}+\frac{64q^2}{27Q^3} < \frac{4}{q}.\end{aligned}$$

We see that the arc VW crosses at most four band boundaries on a bad block.

The number of blocks $B_j(q,a)$ of length q derived from the Farey arc $C(q,a)$ is

$$J(q,a) < \frac{L(q,a)}{q}+1.$$

At most one of these blocks is a bad block. A block $B_j(q,a)$ contributes towards the excess $N-A^*$ at most

$$\begin{aligned}E_j(q,a) &\leqslant c+\frac{d}{2}+1+\rho\left(\frac{a+q+d}{2}\right)\\ &\leqslant c+1+\frac{1}{2}\left(\delta\sqrt{a^2+q^2}+1\right)+\frac{1}{2} \leqslant c+2+\frac{\delta\sqrt{a^2+q^2}}{2},\end{aligned}$$

where $c \leqslant 1$ for good blocks and $c \leqslant 4$ for the bad block. The total excess from the blocks on the Farey arc $C(q,a)$ is

$$\leqslant \left(3+\frac{\delta\sqrt{a^2+q^2}}{2}\right)J(q,a)+3 \leqslant \frac{\delta\sqrt{a^2+q^2}\,J(q,a)}{2}+\frac{3L(q,a)}{q}+6. \quad (4.6)$$

Let W_1 be the point where the curve C enters the first block $B_1(q,a)$, and let W_2 be the point where C leaves the last block $B_J(q,a)$. Let $G(q,a)$ be the length of the arc W_1W_2 of C.

LEMMA 3. *For a Farey arc $C(q,a)$ in the first octant*

$$J(q,a)\sqrt{a^2+q^2} \leqslant G(q,a)+\frac{c_2R}{2q^2(Q+1)^2}. \quad (4.7)$$

Proof. Let $\psi=\alpha-\beta_1$ at W_1, and $\psi=\alpha+\beta_2$ at W_2. Then

$$\begin{aligned}qJ(q,a) &= \int_{\alpha-\beta_1}^{\alpha+\beta_2}\rho\cos\psi\,d\psi\\ &\leqslant \int_{\alpha-\beta_1}^{\alpha+\beta_2}\rho\cos\alpha\,d\psi+\int_{\alpha-\beta_1}^{\alpha}\rho(\cos\psi-\cos\alpha)\,d\psi\\ &= G(q,a)\cos\alpha+\int_{\alpha-\beta_1}^{\alpha}2\rho\sin\frac{\alpha-\psi}{2}\,\sin\frac{\alpha+\psi}{2}\,d\psi.\end{aligned}$$

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We have

$$\begin{aligned}\int_{\alpha-\beta_1}^{\alpha} 2\rho \sin\frac{\alpha-\psi}{2}\,\sin\frac{\alpha+\psi}{2}\,d\psi &\leqslant 2c_2R\sin\alpha\int_{\alpha-\beta_1}^{\alpha}\sin\frac{\alpha-\psi}{2}\,d\psi\\ &\leqslant c_2R\sin\alpha\int_{\alpha-\beta_1}^{\alpha}\frac{\sin(\alpha-\psi)}{\cos^3(\alpha-\psi)}\,d\psi\\ &= -\frac{c_2R\sin\alpha}{2}[\tan^2(\alpha-\psi)]_{\alpha-\beta_1}^{\alpha}\\ &= \frac{c_2R}{2}\sin\alpha\tan^2\beta_1.\end{aligned}$$

From the calculations of (4.1) we have

$$\tan\beta_1 = \frac{\tan\alpha-\tan(\alpha-\beta_1)}{1+\tan\alpha\,\tan(\alpha-\beta_1)} < \tan\alpha-\tan(\alpha-\beta_1) \leqslant \frac{1}{q(Q+1)}.$$

The result of the lemma follows on dividing by $\cos\alpha$ and noting that $\tan\alpha < 1$. □

Let G_1 be the length of the first octant of the curve C. The block of the Farey arc containing $C(1,0)$ containing the centre $T(1,0)$ extends into the eighth octant, and the block of $C(1,1)$ containing the centre $T(1,1)$ extends into the second octant. The second term on the right of (4.7) arises from the part of $C(q,a)$ to the left of $T(q,a)$, so we do not include it for the Farey arc $C(1,0)$. Hence

$$\sum_{F(Q)} J(q,a)\sqrt{a^2+q^2} \leqslant G_1+1+\sqrt{2}+\sum_{q=1}^{Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{c_2R}{2q^2(Q+1)^2}.$$

As in the proof of Lemma 2 we have

$$\sum_{q=1}^{Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{1}{q^2} \leqslant \log Q+1,$$

so we have

$$\sum_{F(Q)}\frac{\delta J(q,a)\sqrt{a^2+q^2}}{2} \leqslant \frac{\delta G_1}{2}+\frac{(1+\sqrt{2})\delta}{2}+\frac{\delta c_2R(\log Q+1)}{12Q}, \tag{4.8}$$

where we have used $Q \geqslant 2$.

Since $Q \geqslant 2$, (2.4) gives

$$\frac{3Q}{4} \leqslant Q-\frac{1}{2} \leqslant \frac{1}{2}(\sqrt{2}c_1R)^{1/3} < Q+\frac{1}{2} \leqslant \frac{5Q}{4}.$$

Hence

$$\begin{aligned}6\sum_{F(Q)}1 &\leqslant 6\left(2+\sum_{1}^{Q}(q-1)\right) \leqslant 12+3Q(Q-1)\\ &\leqslant 12+3\left(Q-\frac{1}{2}\right)^2 \leqslant \frac{3}{4}(\sqrt{2}c_1R)^{2/3}+12.\end{aligned} \tag{4.9}$$

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By Lemma 2

$$3\sum_{F(Q)}\frac{L(q,a)}{q}\leqslant\frac{3c_2R}{Q}(\log Q+1)$$
$$\leqslant 3c_2R.\frac{5}{2}\frac{1}{(\sqrt{2}c_1R)^{1/3}}\left(1+\log\frac{2}{3}+\frac{1}{6}\log 2+\frac{1}{3}\log c_1R\right)$$
$$\leqslant\frac{15c_2}{2c_1^{1/3}}R^{2/3}\left(\frac{1}{3}\log R+1+\frac{7}{6}\log 2-\log 3\right)$$
$$\leqslant\frac{5c_2}{2c_1^{1/3}}R^{2/3}\left(\log R+\frac{5}{2}\right),\tag{4.10}$$

since $\log 2<7/10$ and $\log 3>1$.

Summing (4.6) over Farey arcs, we see from (4.8), (4.10) and (4.9) that the total excess in the first octant is

$$\leqslant\frac{\delta G_1}{2}+\left(1+\frac{\delta}{36}\right)\frac{5c_2R^{2/3}}{2c_1^{1/3}}\left(\log R+\frac{5}{2}\right)$$
$$+\frac{3}{4}(\sqrt{2}\,c_1R)^{2/3}+12+\frac{(1+\sqrt{2})\delta}{2}$$
$$\leqslant\frac{\delta G_1}{2}+\left(1+\frac{\delta}{20}\right)\frac{5c_2R^{2/3}}{2c_1^{1/3}}(\log R+3)+12+\frac{3\delta}{4},$$

where we have used $\sqrt{2}<3/2$ and $2^{1/3}<4/3$. Hence the total excess over all eight octants is

$$\leqslant\frac{\delta L}{2}+(20+\delta)\frac{c_2R^{2/3}}{c_1^{1/3}}(\log R+3)+96+6\delta.\qquad\square$$

§5. *Step four. The refined procedure.* On Farey arcs $C(q,a)$ with q small we have short blocks and broad bands. Band crossings are very rare. For $q\leqslant Q/10$ we divide each band into k narrow bands with

$$2\leqslant k\leqslant\frac{Q}{q}.\tag{5.1}$$

In the next section we see that k of size about $(Q/q)^{4/11}$ is a good choice. Each unit square in a block $B(q,a)$ is crossed by $k(a+q)<Q$ narrow bands. At the end of this section we estimate X, the number of times that the curve C crosses a narrow band boundary within a block of the Farey arc $C(q,a)$. The condition (5.1) ensures that for most blocks, the curve C stays within the same narrow band throughout the block. We number the narrow bands within each broad band from 1 to k (from bottom right to top left in the first octant). The narrow type t of a block is defined to be the number of the lowest narrow band that the curve occupies within the block.

We take the integer D with

$$D\geqslant\delta k\sqrt{a^2+q^2}\geqslant D-1.\tag{5.2}$$

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We note that (3.1) gives

$$dk \geqslant \delta k\sqrt{a^2+q^2} \geqslant dk-k,$$

so that d and D are related by

$$D = dk - \ell, \quad 0 \leqslant \ell \leqslant k-1. \tag{5.3}$$

Narrow band t within broad band b of a square is the Tth narrow band counting from the bottom right of the square, with $T = (b-1)k + t$. When the curve C occupies the Tth narrow band, then the cutting tool removes part of the Tth band, and may cut as far as the $(T-D)$th narrow band in the square. If $T \geqslant D+2$, the off-cut contains the first $T-D-1$ narrow bands complete. After we rotate by 180°, the off-cut contains complete bands from the T'th to the last, where

$$T' = k(a+q-1) - (T-D-1) = k(a+q-1-b) + D + 2 - t. \tag{5.4}$$

We write

$$T' = (b'-1)k + s, \tag{5.5}$$

with $1 \leqslant s \leqslant k$. We can pair the off-cut with a cut part of type b' in a block of narrow type s. We write

$$s = D + 2 - t - rk \equiv D + 2 - t \mod k, \tag{5.6}$$

with r some integer. This is the analogue for narrow bands of the pairing (3.2).

Comparing the expressions (5.4) and (5.5) for T' and substituting (5.6), we have

$$b' = a + q + r + 2 - b, \tag{5.7}$$

in place of (3.2). The integer r is determined by the narrow band type t of the first block, which determines the narrow band type s of the block where we use the off-cuts. Substituting (5.3) into (5.6), we have

$$s + t = D + 2 - rk = (d-r)k + 2 - \ell. \tag{5.8}$$

If $r = d$, then $s + t = 2 - \ell$, which is possible when

$$t = 1, \quad \ell = 0. \tag{5.9}$$

If $r = d-1$, then $s + t = k + 2 - \ell$, which is possible when

$$2 - \ell \leqslant t \leqslant k + 1 - \ell. \tag{5.10}$$

If $r = d-2$, then $s + t = 2k + 2 - \ell$, which is possible when

$$t \geqslant k + 2 - \ell, \quad \ell \geqslant 2. \tag{5.11}$$

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We can now explain the refined pairing. There are $J = J(q, a)$ blocks on the Farey arc $C(q, a)$. We number the blocks $B_j(q, a)$ in order from left to right. For each $t = 1, \ldots, k$, we list the values of j for which $B_j(q, a)$ has narrow type t. The average number of blocks on each list is J/k. Let the actual size of the tth list be $J/k + E_t$. Suppose that narrow type t pairs with narrow type s in (5.6). In general $s \neq t$, and we pair the first block on list t with the first block on list s, then the next block on list t with the next block on list s, and so on as far as possible. If the remainders E_s and E_t have the same sign, then there are $|E_s - E_t|$ blocks unpaired on the longer list. If E_s and E_t have opposite signs, then there are $|E_s| + |E_t|$ blocks unpaired on the longer list. In the exceptional case $s = t$, we pair the first block on the list with the second block, and the $(2r - 1)$th block with the $2r$th block, leaving at most one block unpaired at the end of the list. The total number of unpaired blocks is U, with

$$U \leqslant \sum_{t=1}^{k} |E_t| + 1. \tag{5.12}$$

Suppose that block $B_i = B_i(q, a)$ of type t is paired with block $B_j = B_j(q, a)$ of type s. If there are no broad band crossings in either block (the usual case), then both blocks consist of $a + q$ squares, one of each type. The rule (5.7) pairs the block of type b in B_i with the block of type b' in B_j. The rule fails when $b \leqslant r + 1$ or when $b' \leqslant r + 1$. In each block, $r + 1$ squares remain unpaired, and the other $a + q - r - 1$ squares are paired with a square in the other block. We use

$$a + q + r + 1 \tag{5.13}$$

tiles to form the cut parts of the squares on the two blocks.

There may be exceptional cases with $s = t$ in (5.6); this happens for at most two values of t. If the number of blocks of type t is odd, then the last block is paired with itself. If $a + q + r$ is even in (5.7), then there is a square with $b = b'$, which cannot be paired with itself. This block requires an extra tile, increasing the number of tiles in (5.13) by one.

In an unpaired block we use the original procedure, pairing the square of type b with the square of type b' according to the rule (3.2) with b' for s and b for t. The number of tiles required is given by (3.3).

The integer r in (5.13) changes with t. If $\ell = 0$ in (5.3), then $r = d$ for $t = 1$ by (5.9), $r = d - 1$ for $t \geqslant 2$ by (5.10). If $\ell = 1$, then $r = d - 1$ for all t by (5.10). If $\ell \geqslant 2$, then $r = d - 1$ for $t \leqslant k + 1 - \ell$ by (5.10), $r = d - 2$ for $t \geqslant k + 2 - \ell$ by (5.11). In all three cases, the average value of r over $t = 1, \ldots, k$ is

$$d - 1 - \frac{\ell - 1}{k} = \frac{D + 1}{k} - 1. \tag{5.14}$$

All blocks, whether paired or unpaired, require

$$\frac{a + q}{2} + \frac{d}{2} + O(1) = \frac{a + q}{2} + \frac{D + 1}{2k} + O(1) \tag{5.15}$$

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tiles. Let

$$V = \sum_{t=1}^{k} |E_t|. \tag{5.16}$$

Comparing (5.14) and (5.15), we see that the total number of tiles used for the J blocks on the Farey arc $C(q, a)$ is

$$\begin{aligned} &\left(\frac{a+q}{2} + \frac{D+1}{2}\right)J + O\left(\sum_{1}^{k} |E_t|\right) + O(U) \\ &= \left(\frac{a+q}{2} + \frac{D+1}{2}\right)J + O(V+1) \end{aligned} \tag{5.17}$$

where the first error term on the left of (5.17) comes from the irregularities in the number of paired blocks of each type, and the second error term comes from the unpaired blocks.

In the absence of narrow band crossings, the total area of the cut parts within a block is known to within $1/2k$, half the area of a narrow band crossing a block, as in §3. Again as in §3, if there are c narrow band crossings within a block which do not change the set of squares entered by the curve C from those entered by the reference line, then the error in the total area of the cut parts of squares within a block increases by $O(c/k)$. Finally we saw in §3 that when a band crossing takes the curve C into a new square not cut by the reference line, or out of an old square cut by the reference line, then the number of tiles needed and the total area of the cut parts both change by the same integer 0 or ± 1. Hence the Farey arc $C(q, a)$ contributes to the excess $N - A^*$ an amount

$$\begin{aligned} &\frac{D+1}{2}J + O\left(\frac{X}{k}\right) + O(V+1) \\ &= \frac{\delta J\sqrt{a^2+q^2}}{2} + O\left(\frac{J}{k}\right) + O\left(\frac{X}{k}\right) + O(V+1), \end{aligned} \tag{5.18}$$

where X is the total number of narrow band crossings. Lemma 2 gives

$$J(q, a)\sqrt{a^2+q^2} \leqslant G(q, a) + O\left(\frac{J}{qQ}\right) \leqslant G(q, a) + O\left(\frac{J}{k}\right).$$

We bound the contribution (5.18) to the excess from the Farey arc $C(q, a)$ as

$$\leqslant G(q, a) + O\left(\frac{J}{k}\right) + O\left(\frac{X}{k}\right) + O(V+1). \tag{5.19}$$

We change to Cartesian coordinates to estimate X, the number of band crossings. In the first octant the equation of the curve C is $y = f(x)$ with

$$\frac{dy}{dx} = \tan\psi, \qquad \frac{dx}{d\psi} = \rho\cos\psi,$$

$$\frac{d^2y}{dx^2} = \frac{1}{\rho\cos\psi}, \qquad \frac{d}{d\psi}\frac{dy}{dx} = \frac{1}{\rho\cos\psi}.\sec^2\psi = \frac{\sec^3\psi}{\rho}.$$

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Let $T(q,a)$, the centre of the Farey arc $C(q,a)$, be (x_0, y_0). The Second Mean Value Theorem gives

$$y - y_0 = \frac{a}{q}(x - x_0) + E(x), \qquad E(x) = \frac{1}{2}(x - x_0)^2 f''(\xi) \tag{5.20}$$

for some ξ between x and x_0, with

$$|E(x)| \leqslant \frac{1}{2}(x - x_0)^2 . \frac{2\sqrt{2}}{c_1 R} = O\left(\frac{(x - x_0)^2}{R}\right). \tag{5.21}$$

Let the endpoints W_1 and W_2 be (n_1, y_1) and (n_2, y_2). Then

$$n_1 < x_0 < n_2 = n_1 + Jq.$$

The number of narrow band crossings between T and W_2 is

$$[kqy_2] - [kqy_0] = kqE(n_2) + O(1).$$

Similarly the number of narrow band crossings between W_1 and T is $kqE(n_1) + O(1)$, and we have

$$X = kqE(n_2) + kqE(n_1) + O(1) = O\left(\frac{kqJ^2q^2}{R}\right) = O\left(\frac{kqJ}{Q}\right) = O(J)$$

by (5.1). The term $O(X/k)$ in (5.18) is absorbed by the previous error term $O(J/k)$.

We choose k with

$$k \geqslant \frac{1}{2}\left(\frac{Q}{q}\right)^{4/11}.$$

The terms $O(J/k)$ sum over all Farey arcs in the first octant to

$$\begin{aligned} O\left(\sum_{q=1}^{Q}\sum_{a=0}^{q}\frac{Q^2}{q^2}.\left(\frac{q}{Q}\right)^{4/11}\right) &= O\left(Q^{18/11}\sum_{q=1}^{Q}\frac{1}{q^{7/11}}\right) \\ &= O\left(Q^{18/11}\left(1 + \int_1^Q \frac{dx}{x^{7/11}}\right)\right) \\ &= O(Q^{18/11}\, Q^{4/11}) = O(Q^2). \end{aligned}$$

The same estimate holds on the other seven octants.

The main term in (5.18) has been replaced by $\delta G(q,a)/2$, which sums over all Farey arcs in all eight octants to $\delta L/2$. We shall complete the proof of the theorem by estimating $V + 1$ in (5.18) as $O(J/k)$.

§6. *Step five: counting narrow band types.* The blocks on a Farey arc $C(q,a)$ have narrow types t in $1 \leqslant t \leqslant k$. There are J blocks, and the number of blocks of type t is $J/k + E_t$. We want to show

$$\sum_{t=1}^{k} |E_t| = O\left(\frac{J}{k}\right). \tag{6.1}$$

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Band crossings are always upwards away from the centre $T = T(q, a)$ of the Farey arc $C(q, a)$. The point at which the curve C cuts the line $x = n$ that forms the boundary between two blocks determines the narrow type of the block for which the line $x = n$ is the inner boundary. The $J - 1$ points at which C cuts the internal boundaries between blocks determine the narrow band types of all the blocks except the central block containing T. These values of n form an arithmetic progression $n_j = n_0 + jq$ for $j = 1, \dots, J - 1$.

The narrow bands are numbered $s = 1, \dots, k$ corresponding to the intercept y on the left-hand boundary $x = n$ of the block by

$$\frac{s-1}{k} < t \leqslant \frac{s}{k}, \quad t \equiv qy \bmod 1.$$

The counter for t to be in the narrow band s is

$$\rho\left(t - \frac{s-1}{k}\right) - \rho\left(t - \frac{s}{k}\right) + \frac{1}{k}.$$

Since t is congruent to qy modulo one, we have

$$E_s = \sum_{j=1}^{J-1} \left(\rho\left(qf(n_j) - \frac{s-1}{k}\right) - \rho\left(qf(n_j) - \frac{s}{k}\right)\right) \tag{6.2}$$

for all but one value of s. The exceptional s is the type of the block containing T, the centre of the Farey arc, which is not counted in (6.2), so we add one to the right of (6.2)

We approximate $\rho(t)$ by Fourier series with period 1, using the notation $e(t) = \exp 2\pi i t$ for complex exponentials. Weyl's criterion for uniform distribution says that the distribution modulo 1 of a sequence t_j (here $t_j = qf(n_j)$) is governed by the size of the exponential sums

$$S(h) = \sum e(ht_j) = \sum_{j=1}^{J-1} e(hqf(n_j)), \tag{6.3}$$

for $h = 0, \pm 1, \pm 2, \dots$. To work to an accuracy of $O(1/k)$ in t_j, we need the sums $S(h)$ up to $h = O(k)$ (the "uncertainty principle"). But we also need the errors E_t to be smaller by a factor $O(1/k)$. This requires sums $S(h)$ up to $H = O(k^2)$.

We use the Féjèr kernel

$$L(H, x) = \left|\sum_{h=-H}^{H} e(hx)\right|^2 = \sum_{h=-2H}^{2H-1} (2H - 1 - |h|)e(hx). \tag{6.4}$$

For $|x| \leqslant 1/6H$, we have

$$\operatorname{Re} \sum_{-H}^{H} e(hx) \geqslant \sum_{-H}^{H} \cos 2\pi hx \geqslant \frac{2H+1}{2} > H.$$

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Let $\|x\|$ denote the distance of x from the nearest integer. By periodicity we have

$$L(H,x) > H^2 \quad \text{for } \|x\| \leqslant \frac{1}{6H}. \tag{6.5}$$

For x not an integer we sum a geometric progression to obtain

$$\begin{aligned} L(H,x) &= \left| \frac{e((H+1/2)x) - e(-(H+1/2)x)}{e(x/2) - e(-x/2)} \right|^2 \\ &= \frac{\sin^2(2H+1)\pi x}{\sin^2 \pi x} \leqslant \min\left((2H+1)^2, \frac{1}{\pi^2 \|x\|^2} \right). \end{aligned} \tag{6.6}$$

Let $L(H,x)^2$ have the expansion

$$L(H,x)^2 = \sum_{h=-4H}^{4H} b(h) e(hx), \tag{6.7}$$

where $0 \leqslant b(h) \leqslant b(0)$. Integrating $L(H,x)^2$ from x to $1/2$ and estimating carefully gives our next lemma, which is a restatement of [**9**, Lemma 5.3.1] in the notation of (6.4) and (6.7).

LEMMA 4 (The row-of-teeth function smoothly truncated). *We have*

$$\rho(t) = \sum_{h \neq 0} \frac{b(h)}{b(0)} \frac{e(ht)}{2\pi i h} + O\left(\frac{1}{H} + \min\left(1, \frac{1}{H^3 \|t\|^3} \right) \right). \tag{6.8}$$

Using the notation (6.3), we have

$$\sum_{j=i}^{J-1} \rho\left(qf(n_j) - \frac{s}{k} \right) = \sum_{h \neq 0} \frac{1}{2\pi i h} \frac{b(h)}{b(0)} e\left(\frac{-hs}{k} \right) S(h) + O\left(\frac{J}{H} \right) + O(\Delta_s), \tag{6.9}$$

where

$$\Delta_s = \sum_{j=1}^{J-1} \min\left(1, \frac{1}{H^3 \|qf(n_j) - s/k\|^3} \right). \tag{6.10}$$

We take $H = k^2$. We subtract from (6.9) the corresponding expression involving $s-1$ to expand (6.2) as

$$\begin{aligned} E_s &= \sum_{h \neq 0} \frac{1}{2\pi i h} \frac{b(h)}{b(0)} \left(e\left(\frac{h}{k} \right) - 1 \right) e\left(\frac{-hs}{k} \right) S(h) \\ &\quad + O\left(\frac{J}{H} \right) + O(\Delta_s + \Delta_{s-1}) \\ &= \sum_{h \neq 0} \frac{b(h)}{b(0)} \frac{\sin \pi h/k}{\pi h} e\left(\frac{-h}{k} \left(s - \frac{1}{2} \right) \right) S(h) + O\left(\frac{J}{k^2} \right) + O(\Delta_s + \Delta_{s-1}). \end{aligned}$$

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Adding $O(1)$ for the exceptional block containing the centre $T(q,a)$ of the Farey arc, we have

$$\sum_{s=1}^{k}|E_s| = \sum_{s=1}^{k}\left|\sum_{h\neq 0}\frac{b(h)}{b(0)}\,\frac{\sin \pi h/k}{\pi h}\,e\left(\frac{-h}{k}\left(s-\frac{1}{2}\right)\right)S(h)\right| + O\left(\frac{J}{k}\right) + O\left(\sum_{s=1}^{k}\Delta_s\right) + O(1). \tag{6.11}$$

The term $O(1)$ can be absorbed by $O(J/k)$.

It is easier to bound the sum of the squares of the main sums in (6.11) than the sum of their moduli. The same phenomenon was seen in the study [**8**] of almost-consecutive prime numbers. We have

$$\begin{aligned}
&\sum_{s=1}^{k}\left|\sum_{h\neq 0}\frac{b(h)}{b(0)}\,\frac{\sin \pi h/k}{\pi h}\,e\left(\frac{-h}{k}\left(s-\frac{1}{2}\right)\right)S(h)\right|^2\\
&\quad=\sum_{g\neq 0}\sum_{h\neq 0}\frac{b(g)b(h)}{b(0)^2}\,\frac{\sin \pi g/k\,\sin \pi h/k}{\pi^2 gh}\,\overline{S(g)}S(h)\\
&\qquad\times\sum_{s=1}^{k}e\left(\frac{(g-h)}{k}\left(s-\frac{1}{2}\right)\right)\\
&\quad=k\sum_{g\neq 0}\sum_{\substack{h\neq 0\\ g\equiv h \bmod k}}\frac{b(g)b(h)}{b(0)^2}\,\frac{\sin \pi g/k\,\sin \pi h/k}{\pi^2 gh}\,\overline{S(g)}S(h)\\
&\quad=k\sum_{c \bmod k}\left|\sum_{\substack{h\equiv c \bmod k\\ h\neq 0}}\frac{b(h)}{b(0)}\,\frac{\sin \pi h/k}{\pi h}\,S(h)\right|^2.
\end{aligned} \tag{6.12}$$

The case $c\equiv 0$ contributes 0 to the sum in (6.12).

We must now use the structure of the sum $S(h)$. The expansion of $y=f(x)$ in (5.20) gives

$$hqf(n_j) = hqy_0 + ah(n_j - x_0) + hqE(n_j),$$

so we have in (6.3)

$$S(h) = e(hqy_0 - ahx_0)\sum_{j=1}^{J-1}e(hqE(n_j)). \tag{6.13}$$

This idea for treating Farey arc exponential sums was used by Karatsuba [**13**], and it forms an important step in the Bombieri–Iwaniec method [**1**, **11**]. Here we approximate the sum on the right of (6.13) by an integral.

LEMMA 5. *Let $F(x)$ be twice continuously differentiable for $n-q/2\leqslant x\leqslant n+q/2$. Then*

$$\left|\int_{-q/2}^{q/2}F(n+u)\,du - qF(n)\right| \leqslant \frac{q^3}{24}\max_{-q/2\leqslant u\leqslant q/2}|F''(n+u)|. \tag{6.14}$$

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Proof. By the Mean Value Theorem we have

$$F(n+u) = F(n) + uF'(n) + \frac{u^2}{2}F''(n+v(u))\,du,$$

for some $v(u)$ between 0 and u. Hence

$$\int_{-q/2}^{q/2} F(n+u)\,du = qF(n) + \int_{-q/2}^{q/2} \frac{u^2}{2}F''(n+v(u))\,du,$$

which gives the estimate (6.14) of the lemma. □

We apply Lemma 5 with

$$\begin{aligned} F(x) &= e(hqE(x)),\\ F'(x) &= 2\pi hqE'(x)e(hqE(x)),\\ F''(x) &= (2\pi ihqE''(x) - 4\pi^2h^2q^2E'(x)^2)e(hqE(x)). \end{aligned}$$

Differentiating the definition (5.20) of $E(x)$ gives

$$E'(x) = f'(x) - \frac{a}{q}, \qquad E''(x) = f''(x).$$

Thus

$$E'(x) = \int_{x_0}^{x} f''(t)\,dt = (x-x_0)f''(w) \tag{6.15}$$

for some w between x_0 and x_1. Since $F''(x) = O(1/R)$, we have

$$\begin{aligned} F''(x) &= O\left(\frac{hq}{R} + \frac{h^2q^2}{R^2}(x-x_0)^2\right) = O\left(\frac{hq}{Q^3}\right) + O\left(\frac{h^2q^2}{R^2}\cdot\frac{R^2}{q^2Q^2}\right)\\ &= O\left(\frac{h^2}{Q^2}\right). \end{aligned}$$

We write $n_{j+1/2}$ for $n_j + q/2$. Then Lemma 5 gives

$$e(hqE(n_j)) = \frac{1}{q}\int_{n_{j-1/2}}^{n_{j+1/2}} e(hqE(x))\,dx + O\left(\frac{h^2q^2}{Q^2}\right),$$

so that in (6.13)

$$S(h) = \frac{1}{q}e(hqy_0 - ahx_0)\int_{n_{1/2}}^{n_{J-1/2}} e(hqE(x))\,dx + O(h^2), \tag{6.16}$$

a stationary phase integral. We choose x_1 and x_2 with

$$x_2 - x_0 = x_0 - x_1 = \sqrt{\frac{R}{|h|q}}.$$

Then

$$\int_{\max(x_1,n_{1/2})}^{\min(x_2,n_{J-1/2})} e(hqE(x))\,dx = O\left(\sqrt{\frac{R}{|h|q}}\right).$$

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When $x > n_{1/2}$, then we integrate by parts using

$$\begin{aligned}\int_{n_{1/2}}^{x_1} e(hqE(x))\,dx &= \left[\frac{e(hqE(x))}{2\pi i hqE'(x)}\right]_{n_{1/2}}^{x_1} + \frac{1}{2\pi i hq}\int_{n_{1/2}}^{x_1}\frac{E''(x)}{E'(x)^2}e(hqE(x))\,dx\\ &= O\left(\frac{1}{|h|q(x_0-x_1)\min|f''(x)|}\right)\\ &\quad + O\left(\frac{1}{|h|q}\int_{n_{1/2}}^{x_1}\frac{\max|f''(x)|}{(x_0-x)^2\min|f''(x)|^2}\,dx\right)\\ &= O\left(\frac{R}{|h|q(x_0-x_1)}\right) = O\left(\sqrt{\frac{R}{|h|q}}\right),\end{aligned}$$

where we have used (6.15). We obtain the same estimate for the integral from x_2 to $N_{J-1/2}$ when $x_2 < n_{J-1/2}$.

We now have in (6.16)

$$S(h) = O\left(\sqrt{\frac{R}{|h|q^3}}\right) + O(h^2) = O\left(\sqrt{\frac{Q^3}{|h|q^3}} + h^2\right). \tag{6.17}$$

In the sum in (6.12) we can suppose that $1 \leqslant |c| \leqslant k/2$. Then

$$\begin{aligned}&\sum_{h\equiv c \bmod k}\frac{b(h)}{b(0)}\frac{\sin\pi hk}{\pi h}S(h)\\ &\quad= O\left(\frac{1}{k}\sqrt{\frac{Q^3}{|c|q^3}} + k^2\right) + \sum_{r=1}^{k+1}\frac{1}{rk}\left(\sqrt{\frac{Q^3}{rkq^3}} + r^2k^2\right)\\ &\quad= O\left(\frac{1}{k\sqrt{|c|}}\sqrt{\frac{Q^3}{q^3}}\right) + O(k^3).\end{aligned}$$

Hence (6.12) gives

$$\begin{aligned}&\sum_{s=1}^{k}\left|\sum_{h\neq 0}\frac{b(h)}{b(0)}\frac{\sin\pi h/k}{\pi h}e\left(-\frac{h}{k}\left(s-\frac{1}{2}\right)\right)S(h)\right|^2\\ &\quad= O\left(k\sum_{1\leqslant c\leqslant k/2}\left(\frac{1}{ck^2}\frac{Q^3}{q^3} + k^6\right)\right) = O\left(\frac{Q^3\log k}{kq^3} + k^8\right),\end{aligned}$$

and in (6.11) Cauchy's inequality gives

$$\begin{aligned}\sum_{s=1}^{k}\left|\sum_{h\neq 0}\frac{b(h)}{b(0)}\frac{\sin\pi h/k}{\pi h}e\left(-\frac{h}{k}\left(s-\frac{1}{2}\right)\right)S(h)\right| &= O\left(\sqrt{\frac{Q^3\log k}{q^3} + k^9}\right)\\ &= O\left(\frac{J}{k}\right),\end{aligned} \tag{6.18}$$

provided that

$$k = O\left(\left(\frac{Q}{q}\right)^{4/11}\right) = O(J^{2/11}). \tag{6.19}$$

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The second error term in (6.11) requires more work. We have

$$\sum_{s=1}^{k} \Delta_s = \sum_{j=1}^{J-1} \sum_{s=1}^{k} \min\left(1, \frac{1}{k^6 \|qf(n_j) - s/k\|^3}\right).$$

For fixed j, let s' be the value of s (or one of the two values of s) for which the second term in the minimum is greatest. Then for $1 \leqslant r \leqslant k/2$, we have

$$\left\| qf(n_j) - \frac{s \pm r}{k} \right\| \geqslant \frac{r - 1/2}{k}.$$

Thus

$$\begin{aligned}\sum_{s=1}^{k} \min\left(1, \frac{1}{k^6 \|qf(n_j) - s/k\|^3}\right) &\leqslant \min\left(1, \frac{1}{k^6 \|qf(n_j) - s'/k\|^3}\right) \\ &\quad + 2 \sum_{1 \leqslant r \leqslant k/2} \frac{k^3}{k^6} \frac{1}{(r - 1/2)^3} \\ &\leqslant \min\left(1, \frac{1}{k^3 \|kqf(n_j)\|^3}\right) + O\left(\frac{1}{k^3}\right).\end{aligned}$$

We have

$$\sum_{s=1}^{k} \Delta_s = \Delta + O\left(\frac{J}{k^3}\right), \tag{6.20}$$

with

$$\Delta = \sum_{j=1}^{J-1} \min\left(1, \frac{1}{k^3 \|kqf(n_j)\|^3}\right) = \frac{1}{k^3} \sum_{j=1}^{J-1} K(kqf(n_j), \tag{6.21}$$

where $K(x)$ is a periodic "row-of-glasses" function,

$$K(x) = \min\left(k^3, \frac{1}{\|x\|^3}\right). \tag{6.22}$$

LEMMA 6 (Bounding the row-of-glasses by Féjèr's kernel). *We have*

$$K(x) \leqslant 216\left(1 + \sum_{r=0}^{R} 2^r L(2^r, x)\right), \tag{6.23}$$

where R is the smallest integer with $2^R \geqslant k/3$.

Proof. We use the following bounds for $K(x)$:

$$K(x) \leqslant \begin{cases} 2^3 3^3 & \text{for } \|x\| \geqslant 1/6, \\ 2^{3(r+1)} 3^3 & \text{for } 1/6.2^r \leqslant \|x\| \leqslant 1/3.2^r, r \geqslant 0, \\ k^3 & \text{for } \|x\| \leqslant 1/3.2^R \leqslant 1/k. \end{cases}$$

For each of these ranges we use (6.5) with a different value of H.

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We expand each Féjèr kernel $L(2^r, x)$ in (6.23) using (6.4). We have $hk \leqslant 2k^2$, and coefficients of size $O(k^2)$. In (6.21)

$$\Delta = O\left(\frac{1}{k}\left(J + \sum_{h=1}^{2k} |S(hk)|\right)\right) = O\left(\frac{1}{k}\left(J + \sqrt{\frac{Q^3}{q^3}} + k^5\right)\right) = O\left(\frac{J}{k}\right),$$

by the bound (6.17) and the assumption (6.19). This completes the proof of (6.1). □

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