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Quillen's solution of Serre's Problem

by

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Abstract

In this note we recall the history of Serre's problem and its beautiful solution given by D. Quillen in 1976

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In the famous paper [6] which laid down the way to the development of modern algebraic geometry J.-P. Serre established correspondence between vector bundles over affine varieties and projective modules and pointed out that it was not known if there existed non trivial vector bundles over affine space $\mathbb{A}^n_k$ or equivalently if there existed non free projective modules over the polynomial ring $k[X_1,...,X_n]$. This problem became known as Serre's problem. It attracted attention of many mathematicians and for a long time was one of the stimuli for the development of algebraic K-theory.. The case $n = 1$ is trivial because $k[X]$ is a principal ideal domain and hence every finitely generated module without torsion is free. Thus the first non-trivial case of Serre's problem appears in dimension two. This case was treated by Seshadri [8] in 1958. Another important step was made by Serre himself [7], who proved that every finitely generated projective module P over $A = k[X_1,...,X_n]$ is stably free i.e. there exists a free module F such that $P \oplus F$ is free. When H. Bass proved his cancellation theorem [1] it showed that projective modules over $k[X_1,...,X_n]$ of rank $> n$ are free. This result was improved in 1974 by A. Suslin [9] who proved that modules of rank $> \frac{n+1}{2}$ are free. The same year P. Murthy and J. Towber [4] gave a complete solution of Serre's problem for polynomials in three variables. Around the same time L. Vasserstein and A. Suslin solved the problem for polynomials in three, four and five variables [13]. Finally the problem was solved in 1976 simultaneously and independently by D. Quillen [5]and A. Suslin [10].

The two methods used to prove that all vector bundles over $\mathbb{A}^n_k$ are trivial look rather different but the moving idea is the same. First one should find a sufficiently general claim valid for polynomials in one variable over arbitrary rings which will allow to deal with polynomials in n variables later. Here is the claim in question.

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Theorem 1 *(Quillen, Suslin) Let P be a finitely generated projective $A[X]$-module. Assume that P becomes extended from A after we invert all monic polynomials. Then P is extended from A.*

Once this theorem is proved everything becomes trivial. We proceed by induction on n Starting with a finitely generated projective module P over $k[X_1,...,X_n]$ we first note that P obviously becomes free after we invert all non-zero polynomials. As usual this implies that there exists a non-zero polynomial f such that P becomes free after we invert f. A classical lemma of Noether shows that making an appropriate change of variables we may assume that f is monic in X_n. Thus setting $A = k[X_1,...,X_{n-1}]$ and applying the above theorem we conclude that P is extended from A. Now it suffices to apply the inductive hypothesis.

Quillen's proof of the above theorem was amazingly simple and beautiful. It also introduced into algebra several new ideas and methods which were successfully used by other mathematicians later.

For one thing Quillen proved an absolutely unexpected fact that being extended is a local property with respect to A.

Theorem 2 *(Quillen) Let P be a finitely presented $A[X]$-module. The following conditions are equivalent*

1. *P is extended from A.*
2. *For any prime ideal $\mu \subset A$ the module P_μ is extended from A_μ*
3. *For any maximal ideal $\mu \subset A$ the module P_μ is extended from A_μ*

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. To prove that (3) implies (1) Quillen sets $I = \{a \in A : P_a \text{ is extended from } A_a\}$. Obviously it suffices to show that $1 \in I$. Quillen notes that condition (3) means that I is contained in no maximal ideal of A and next proves that the set I is an ideal. To prove this unexpected fact it suffices to show that if $a,b \in I$ and $x \in aA + bA$ then $x \in I$. Replacing A by A_x it suffices to establish that in case $a,b \in A$ are comaximal and we know that P_a is extended from A_a and P_b is extended from A_b then P itself is extended from A. Denote $P \otimes_{A[X]} A = P/XP$ by P_0. Thus P and $P_0[X]$ are isomorphic once we invert a or b. Moreover we may choose isomorphisms $\phi : P_a \xrightarrow{\sim} P_0[X]_a$ and $\psi : P_b \xrightarrow{\sim} P_0[X]_b$ so that both ϕ and ψ are congruent to identity modulo X. Straightforward verification shows that if $\phi_b = \psi_a$ then ϕ and ψ patch together to an isomorphism $P \xrightarrow{\sim} P_0[X]$. Finally Quillen takes a look on the automorphism $\theta = \psi_a \cdot \phi_b^{-1}$ of $P_0[X]_{ab}$. This θ serves as an obstruction to patching together of ϕ and ψ. Moreover changing ϕ and ψ we may replace θ by $\theta'_a \cdot \theta \cdot \theta''_b$ where θ' and θ'' are automorphisms of $P_0[X]_b$ and $P_0[X]_a$ respectively. Thus it suffices to establish

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that for any automorphism θ congruent to identity modulo X there exist θ' and θ'' such that $\theta = \theta'_a \cdot \theta''_b$. Now the idea is to get θ' and θ'' from θ substituting λX for X. Note that if we write θ in the form $\theta = \theta_0 + X\theta_1 + ... + X^n\theta_n$, where $\theta_i \in End_{A_{ab}}((P_0)_{ab})$, then $\theta_0 = Id$. Thus taking m sufficiently large and substituting $a^m X$ for X we may get rid of a in the denominators of θ_i. The same argument may be used to prove the following Lemma.

Lemma 1 *There exists an integer m such that whenever $\lambda \equiv \lambda' \mod a^m$ the automorphism $\theta(\lambda X)\cdot\theta(\lambda' X)^{-1}$ has no a in the denominators and moreover extends to an automorphism of $P_0[X]_b$. In a similar way whenever $\lambda \equiv \lambda' \mod b^m$ the automorphism $\theta(\lambda X)\cdot\theta(\lambda' X)^{-1}$ has no b in the denominators and moreover extends to an automorphism of $P_0[X]_a$.*

Now it suffices to note that since a and b were assumed to be comaximal their powers a^m and b^m are also comaximal and hence there exists $\lambda \in A$ such that $\lambda \equiv 1 \mod b^m, \lambda \equiv 0 \mod a^m$ and hence $\theta = \theta(X)\cdot\theta(0)^{-1} = (\theta(X)\cdot\theta(\lambda X)^{-1})\cdot(\theta(\lambda X)\cdot\theta(0)^{-1})$. According to the choice of m the first factor in this formula extends to an automorphism θ' of $P_0[X]_b$ whereas the second factor extends to an automorphism θ'' of $P_0[X]_a$.

Theorem 1 follows easily from Theorem 2 because over local rings the corresponding result was established previously by Horrocks [3].

The ideology and the method used by Quillen in the proof of Theorem 2 are known now as Quillen's localization principle. This principle was used by Suslin [11] to solve the K_1-analogue of Serre's problem, i.e. to prove that every matrix of size ≥ 3 with determinant one over the polynomial ring is a product of elementary matrices. It was also used by Bass, Connell and Write [2] to show that every locally polynomial algebra is isomorphic to the symmetric algebra of a finitely generated projective module. Furthermore it was used by Tulenbaev [12] to solve the K_2 analogue of Serre's problem.

The general method used in the proof of Theorem 2, when one uses only principal open subsets and reduces the proof of the general claim to the situation when $Spec\, A$ is covered by two principal opens on which the problem already has positive solution is now known as Quillen's induction. This method was used to prove vanishing of certain cohomology groups and to study the Brauer group of affine schemes.

All in all Quillen's paper [5] is an outstanding piece of beautiful mathematics. It gives a solution of a well-known problem and develops new methods which turned out to be useful in many other situations.

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REFERENCES

1. H. Bass, *K-Theory and Stable Algebra*, Publ. Math. IHES **22** (1964), 5–60.
2. H. Bass, E.H.Connell, and D.L.Wright, *Locally Polynomial Algebras are Symmetric Algebras*, Invent. Math. **38** (1977), 279–299.
3. G. Horrocs, *Projective modules over an extension of a Local Ring*, Proc. London Math. Soc.**14** (1964), 714–718.
4. M.P.Murthy and J. Towber, *Algebraic Vector Bundles over* $\mathbb{A}^3$ *are trivial*, Invent. Math. **24** (1974),173–189.
5. D. Quillen, *Projective Modules over Polynomial Rings*, Invent. Math. **36** (1976), 166–172.
6. J.-P. Serre, *Faisceaux Algebriques Coherents*, Annals of Math. **61** (1955), 197–278.
7. J.-P. Serre, *Modules Projectifs et espace fibres a fibre vectoriel*, Seminaire P. Dubreil **11** (1958), 1–18.
8. C.S.Seshadri, *Triviality of Vector Bundles over the Affine Space* K^2, Proc. Nat. Acad. Sci. USA **44** (1958), 456–458.
9. A. Suslin, *On Projective Modules over Polynomial Rings*, Math USSR Sbornik, **22** (1974), 595–602.
10. A.Suslin, *Projective Modules over Polynomial Rings are Free*, Soviet Math. Dokl. **17** (1976), 1160–1164.
11. A.Suslin, *On the Structure of the Special Linear Group over Polynomial Rings*, Math USSR Izv. **11** (1977), 221–238.
12. M.S. Tulenbaev *Steinberg Group over Polynomial Rings*, Math USSR Sbornik **45** (1983), 139–154..
13. L. Vasserstein and A. Suslin,*Serre's Problem on Projective Modules over Polynomial Rings and Algebraic K-Theory*, Math. USSR Izv. **10** (1976), 937–1001.

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