Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2017.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

101.A (Martin Lukarevski)

Let P be a point inside the triangle ABC and let x, y, z denote the perpendicular distances from P to the sides BC, AC, AB respectively. Let R and r denote the circumradius and inradius of triangle ABC.

Show that $\sqrt{x} + \sqrt{y} + \sqrt{z} \leqslant (R + r)\sqrt{\dfrac{2}{R}}$.

101.B (Joseph Tonien)

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f\big(f(n) + 2017\big) + 2017 = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

101.C (Michael Fox)

An ellipse inscribed in triangle ABC touches BC at D, CA at E and AB at F.

(i) Prove that AD, BE and CF pass through a common point P.

(ii) Given that the areas PBC, PCA and PAB are in the ratios $\alpha : \beta : \gamma$, determine what fraction of area ABC is occupied by the ellipse.

101.D (Paul Stephenson)

A circle of unit radius is surrounded by a ring of $n \geqslant 3$ other circles (possibly of differing radii), each of which touches the unit circle and two others. The centres of adjacent circles in the ring are joined by straight lines to form a polygon with n sides – a *ring n-gon*.

Prove that a regular p-gon has the smallest perimeter of all ring n-gons with $n \leqslant p$.

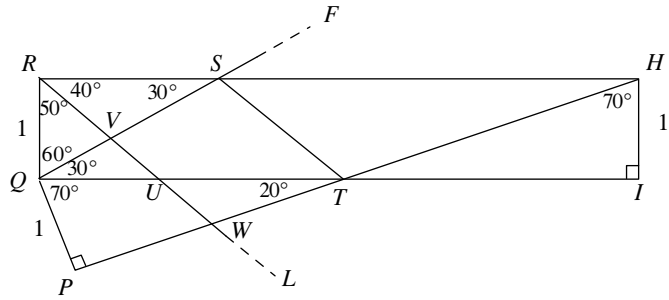
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Solutions and comments on **100.E, 100.F, 100.G, 100.H** (July 2016).

100.E (Chris Starr)

Let $ABC\ldots PQR$ be a regular 18-sided polygon. Lines QF and RH intersect at S; PH and QI intersect at T and RL and QI intersect at U. Prove that $RSTU$ is a parallelogram.

This attractive geometry problem was unusual in that almost every solution received used trigonometry (or something equivalent) at a key step of the proof. The relevant portion of the regular 18-gon is shown in the Figure: standard polygon angle properties quickly fill in the marked angles; V and W are the intersection points $RL \cap QF$, $RL \cap PH$ and we take the side-length to be 1.



FIGURE

It is clear that RS is parallel to UT; it remains to check either that RU and ST are parallel or that $RS = UT$.

For the former, Geoff Strickland showed that triangles QVU and QST are similar. The sine rule in triangle QVU gives $\dfrac{QV}{QU} = \dfrac{\sin 40°}{\sin 70°}$ and, from triangles QRS and QPT, $\dfrac{QS}{QT} = 2 \sin 20°$. Thus

$$\frac{QV}{QU} = \frac{\sin 40°}{\sin 70°} = \frac{2 \cos 20° \sin 20°}{\cos 20°} = \frac{QS}{QT}.$$

Neil Curwen gave an analogous argument using triangles HST and HRW.

For the latter, the following alternatives cover most of the other solutions submitted.

- $RS = \tan 60°$
  $UT = QT - QU = \operatorname{cosec} 20° - \tan 50°$

so we need to check that $\tan 50° + \tan 60° = \operatorname{cosec} 20°$. But

$$\tan 50° + \tan 60° = \frac{\sin 110°}{\cos 50° \cos 60°} = \frac{2 \cos 20°}{\sin 40°} = \operatorname{cosec} 20°.$$

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- $RS = \tan 60°$
  $UT = QI - TI - QU = \tan 80° - \tan 70° - \tan 50°$ (since $\angle QHI = 80°$)

So $RS = UT$ follows from the well-known fact that

$$\tan 50° + \tan 60° + \tan 70° = \tan 80° \left(= \tan 50° \tan 60° \tan 70°\right).$$

James Mundie also calculated the length of $RU = ST$: this results in a substantially messier expression than those above for $RS = UT$.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A.P. Harrison, G. Howlett, J. D. King, J. A. Mundie, Z. Retkes, V. Schindler, G. Strickland, E. Swylan, L. Wimmer and the proposer Chris Starr.

100.F (Osman Akar)

Let a, b, c, d be positive real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{2 + c(a-b)}{a+b+2c} + \frac{2 + d(b-c)}{b+c+2d} + \frac{2 + a(c-d)}{c+d+2a} + \frac{2 + b(d-a)}{d+a+2b} \geqslant 2.$$

Some respondents explored this problem using the method of Lagrange multipliers. Here, the difficulty lies in fully establishing that there is a global minimum when $a = b = c = d = 1$. However, there are also direct algebraic solutions such as the proposer's neat argument which follows.

Let $S = \sum_{\text{cyc}} \frac{2 + c(a-b)}{a+b+2c}$ denote the cyclic sum on the left-hand side of the inequality. Then

$$\begin{aligned} 2S &= \sum_{\text{cyc}} \left[\frac{4 + 2c(a-b)}{a+b+2c} + b - a \right] \\ &= \sum_{\text{cyc}} \frac{2b^2 + c^2 + d^2}{a+b+2c}, \text{ since } a^2 + b^2 + c^2 + d^2 = 4 \\ &\geqslant \frac{\left(\sum_{\text{cyc}} \sqrt{2b^2 + c^2 + d^2}\right)^2}{4(a+b+c+d)}, \qquad (*), \end{aligned}$$

by the Cauchy-Schwarz inequality, $\mathbf{x} \cdot \mathbf{y} \leqslant |\mathbf{x}|\,|\mathbf{y}|$, applied to the vectors whose four components are the cyclic permutations of $\frac{\sqrt{2b^2 + c^2 + d^2}}{\sqrt{a+b+2c}}$ (for $\mathbf{x}$) and $\sqrt{a+b+2c}$ (for $\mathbf{y}$). Now

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$$\left(\sum_{\text{cyc}}\sqrt{2b^2+c^2+d^2}\right)^2 = 4\left(a^2+b^2+c^2+d^2\right)+2\sum\sqrt{2a^2+b^2+c^2}\sqrt{2b^2+c^2+d^2}$$

$$\geqslant 4\sum a^2+2\sum\left(b^2+c^2+ab+ad\right)$$

$$= 4\sum a^2+4\sum a^2+2\left(\sum a\right)^2$$

$$= 32+2\left(\sum a\right)^2,$$

using the Cauchy-Schwarz inequality applied to vectors such as $\mathbf{x} = (b, c, a, a)$ and $\mathbf{y} = (b, c, b, d)$.

Substituting into $(*)$, we then deduce the required inequality

$$2S \geqslant \frac{32+2\left(\sum a\right)^2}{4\sum a} = \frac{8}{\sum a}+\frac{\sum a}{2} \geqslant 4 \text{ (by the AM-GM inequality).}$$

Moreover, there is equality throughout if, and only if, $a = b = c = d = 1$.

Correct solutions were received from: S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, P. Kitchenside, J. A. Mundie and the proposer Osman Akar.

100.G (Stan Dolan)

Find, up to similarity, all triangles such that each angle (in degrees) and the square of each side are whole numbers.

Answer: Up to similarity, there are four such triangles with side-lengths proportional to $\{1, 1, 1\}, \{1, 1, \sqrt{2}\}, \{1, 1, \sqrt{3}\}$ and $\{1, \sqrt{3}, 2\}$.

All solvers made use of cases of the following well-known Lemma, often providing their own proof: other references are [1, 2, 3].

Lemma: If r is rational, then the only rational values possible for $\cos(180r)^\circ$ are $0, \pm\frac{1}{2}, \pm 1$.

With the standard triangle notation, if triangle ABC satisfies the hypotheses of the problem, then

$$\cos 2A = 2\cos^2 A - 1 = \frac{\left(b^2+c^2-a^2\right)^2}{2b^2c^2} - 1$$

is rational. By the Lemma, $2A \in \{60°, 90°, 120°, 180°, 240°, 270°, 300°\}$ so that $A \in \{30°, 45°, 60°, 90°, 120°, 135°, 150°\}$ with the same possibilities for B and C. With these constraints, the only triples of angles which sum to 180° are $\{60°, 60°, 60°\}, \{45°, 45°, 90°\}, \{30°, 30°, 120°\}$,

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$\{30°, 60°, 90°\}$ and these correspond to the side-lengths listed in the answer above.

As several solvers noted, the Lemma generates the same four triangles if the angle hypothesis of **100.G** is weakened to 'A, B, C are each a rational number of degrees'.

References

1. H. G. Forder, Three short proofs, *Math. Gaz.* **53** (May 1969) pp. 177-178.
2. J. D. King, Regular polygons with integer coordinates, *Math. Gaz.* **94** (November 2010) pp. 495-498.
3. I. Niven, *Irrational numbers*, Math. Assoc. of America (1967) pp. 36-41.

Correct solutions were received from: M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, J. D. King, J. A. Mundie, Z. Retkes, A. C. Robin, E. Swylan, L. Wimmer and the proposer Stan Dolan,

100.H (M. N. Deshpande)

(a) For every integer $n \geqslant 2$, consider the isosceles triangle ABC in which $AB = AC = n^2 + n + 1$ and $BC = 2n^2 + 2n - 1$. Let D be on BC with $\angle ADB = 60°$. Show that triangle ABD has integer side-lengths.

(b) Find another example of a family of isosceles triangles in which AB and BC are quadratic functions of n which have the same property as (a).

(a) There are several ways of verifying that triangle ABD has integer side-lengths. For example, if M is the midpoint of BC, then

$$AM = \sqrt{(n^2 + n + 1)^2 - \tfrac{1}{4}(2n^2 + 2n - 1)^2}$$

$$= \sqrt{\tfrac{3}{4}(4n^2 + 4n + 1)} = \tfrac{\sqrt{3}}{2}(2n + 1).$$

It follows that $AD = \dfrac{AM}{\sin 60°} = 2n + 1$ and

$$BD = \tfrac{1}{2}BC + AD\cos 60° = \tfrac{1}{2}(2n^2 + 2n - 1) + \tfrac{1}{2}(2n + 1) = n^2 + 2n$$

are both integers.

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(b) Any triangle ABD with integer side-lengths and $\angle D = 60°$ can be used to construct an isosceles triangle ABC by extending side BD as shown in Figure 1.

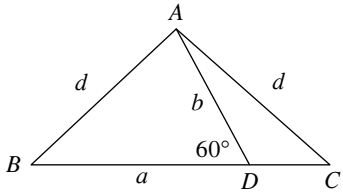


FIGURE 1

Since

$$BC = 2(BD - AD\cos 60°) = 2a - b,$$

triangle ABC has integer side-lengths and, together with D, provides a solution.

Moreover, triangle ABD has a natural 'dual' shown in Figure 2 where triangle ABE also has integer side-lengths and $\angle E = 60°$.

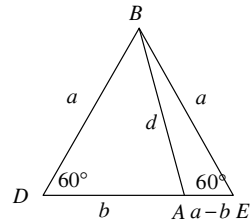


FIGURE 2

As Geoff Strickland and Stan Dolan noted, the dual of $d = n^2 + n + 1$, $b = 2n + 1$, $a = n^2 + 2n$ in (a) gives the new family $d = n^2 + n + 1$, $b = n^2 - 1$, $a = n^2 + 2n$ with $AB = n^2 + n + 1$ and $BC = n^2 + 4n + 1$.

Respondents also made good use of formulae parametrising 60° triangles with integer side-lengths: proofs of these often feature in the *Gazette* [1, 2, 3, 4]. One such parametrisation from [1] is:

$$d = n^2 + mn + m^2, \qquad b = m^2 + 2mn, \qquad a = n^2 + 2mn$$

with $1 \leqslant m < n$. (Some solvers used the equivalent alternatives given in [3, 4].)

Here, $AB = n^2 + mn + m^2$, $BC = 2n^2 + 2mn - m^2$ and the dual family has $d = n^2 + mn + m^2$, $b = n^2 - m^2$, $a = n^2 + 2mn$ with $AB = n^2 + mn + m^2, BC = n^2 + 4mn + m^2$.

A wide range of choices of m, n (other than $m = 1$ above) were offered, including:

- $m = 2 : AB = n^2 + 2n + 4, BC = 2n^2 + 4n - 4$
- $m = 3 : AB = n^2 + 3n + 9, BC = 2n^2 + 6n - 9$
- $m \to n - 1, n \to n + 1 : AB = 3n^2 + 1, BC = 6n^2 - 2$ (from the dual formula)
- $m \to 2m, n \to n - m : AB = n^2 + 3m^2, BC = 2n^2 - 6m^2$.

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2. K. Selkirk, Integer-sided triangles with an angle of 120°, *Math. Gaz.* **67** (December 1983) pp. 251-255.
3. B. Burn, Triangles with a 60° angle and sides of integer length, *Math. Gaz.* **87** (March 2003) pp. 148-153.
4. E. Read, On integer-sided triangles containing angles of 120° or 60°, *Math. Gaz.* **90** (July 2006) pp. 299-305.

Correct solutions were received from: M. V. Channakeshava, N. Curwen, P. De, S. Dolan, M. G. Elliott, F. Farrokhpay, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, P. F. Johnson, J. D. King, J. A. Mundie, Z. Retkes, C. Starr, G. Strickland, B. Sury, E. Swylan, L. Wimmer and the proposer M. N. Deshpande.

Postscript

A published solution in Problem Corner is never the final word on a problem: in this section, we report on any fresh insights that readers have discovered.

John Mahony comments that the circumscribing rectangle to an ellipse featured in **100.B** has a practical application in electrical engineering discussed in his article on the polarisation ellipse in the March 2010 *Gazette*, pp. 94-102. In particular, the calculation on pp. 98-100 provides another solution to **100.B**.

I also apologise for two typos that occurred in my write-up of **100.C** where the final two displayed equations should read

$$\frac{bc}{r_a} = \frac{4R(s-a)}{a} = 2R\left(\frac{b}{\boldsymbol{a}} + \frac{c}{a} - 1\right)$$

and

$$\sum \frac{bc}{r_a} = 2R\left(\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{\boldsymbol{b}} - 3\right) \geqslant 6R.$$

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