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Compressible flows with circular sector hodographs

II. Réthy flows

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Abstract. The general solution of the Chaplygin equation for 'simple wedge flows' is used for the particular case of Réthy flows. The drag coefficient is evaluated and comparisons made with the results obtained from the approximate hodograph equations of Tricomi and of Tomotika and Tamada. A distinction is made between subsonic and transonic régimes and the limitations on the lengths and pressure differences permissible in the various ranges are discussed.

1. *Introduction.* An important special case of the flows considered in Part I (McLaughlin and Pack (3)) is that of Réthy flows. These are defined by a circular sector hodograph with two singularities, one on a solid boundary and the other on the free boundary. Examples are the flow through an aperture, the flow from a funnel, jet flow against a wall or past a symmetrically placed wedge (finite or infinite), flow from a channel past a flat plate, etc.

A typical Réthy flow, the flow out of a semi-infinite nozzle past a symmetrically placed wedge, is sketched in Figure 1. The problem is defined by

(i) a stream with ratio of specific heats γ flowing from infinity upstream where there is uniform density ρ_1, speed q_1 and pressure p_1;

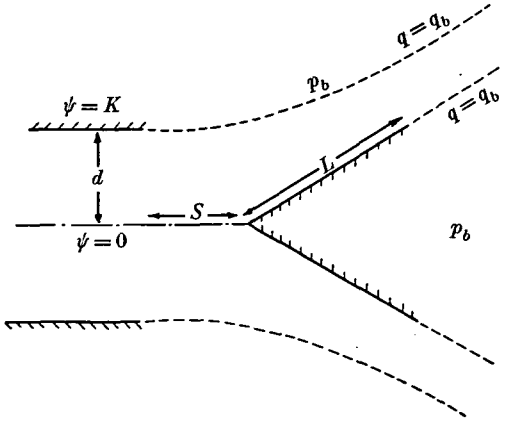


Fig. 1. Typical Réthy flow—the flow out of a nozzle past a symmetrically-placed wedge.

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(ii) a wedge of length L and angle $\alpha = 2\pi/\xi$ symmetrically placed at a distance S from an infinite nozzle of width $2d$;

(iii) a pressure p_b on the free boundaries and a final inclination $m = \pi/\mu$ of the jet at infinity downstream.

There are 10 specifying quantities, not all independent of course. From the formula for the physical coordinate z (Part I, section 10), L and S can be found in terms of the other constants. If γ is given, there are 7 degrees of freedom in the Réthy flow. From the constants, Bernoulli's equation and the adiabatic relationship we can determine immediately the pressure p_0 and density ρ_0 at a stagnation point, the speed q_b and density p_b on the free boundary, the values of τ at the significant points and the constant value of ψ on the free boundary, viz. $K(= dq_1\rho_1/\rho_0)$. The boundary conditions in the hodograph plane are shown in Figure 2.

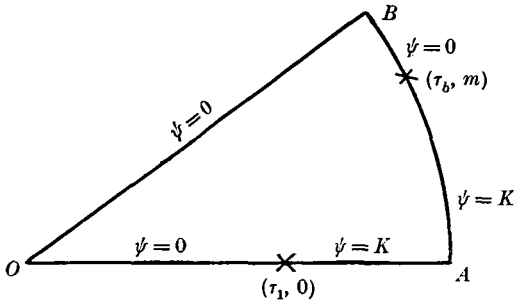


Fig. 2. Hodograph plane for typical Réthy flow.

In the general solution (Part I, 7·2) we have $f_0 = 0$, $f_1 = h_0 = K$, $h_1 = 0, g \equiv 0$. Thus

$$\psi = \psi^{(1)} + \psi^{(2)}$$

with
$$\psi^{(i)} = w^{(i)}(\theta) + (2\xi/\pi) \sum_{n=1}^{\infty} (1/n\xi)\, u^{(i)}(n\xi, \sigma) \sin n\xi\theta \quad (i = 1, 2), \tag{1·1}$$

where
$$w^{(1)}(\theta) = 0, \quad w^{(2)}(\theta) = w_q^{(2)}(\theta) = f_q(1 - \xi\theta/\pi),$$
$$u^{(1)}(n\xi, \sigma) = -K(1 - \cos mn\xi)\, L(n\xi, \sigma),$$
$$u_0^{(2)}(n\xi, \sigma) = K(1 - R'(n\xi, \sigma_1))\, L(n\xi, \sigma)$$

and
$$u_1^{(2)}(n\xi, \sigma) = K\{L(n\xi, \sigma) - L'(n\xi, \sigma_1)\, R(n\xi, \sigma)\}.$$

The physical coordinates (Part I, 10·1) are given by

$$z = z^{(1)} + z^{(2)},$$

where
$$z^{(i)} = \{\xi/\pi q(1-\tau)^{\beta}\}\,[d_i e^{i\theta} + (z_i^{(\psi)} + z_i^{(\varphi)})] + C_i,$$

with
$$z_i^{(\psi)} + z_i^{(\varphi)} = \sum_{n=1}^{\infty} \{u^{(i)}(n\xi, \sigma)\, \Omega(n\xi, \theta) + (1-\tau)^{\beta} v^{(i)}(n\xi, \sigma)\, \chi(n\xi, \theta)/n\xi\},$$
$$v^{(i)}(n\xi, \sigma) = (du^{(i)}/d\sigma)/(1-\tau_s)^{\beta},$$
$$\Omega(n\xi, \theta) = \exp\{(n\xi+1)\,\theta i\}/(n\xi+1) - \exp\{-(n\xi-1)\,\theta i\}/(n\xi-1),$$
$$\chi(n\xi, \theta) = \exp\{(n\xi+1)\,\theta i\}/(n\xi+1) + \exp\{-(n\xi-1)\,\theta i\}/(n\xi-1),$$
$$d_1 = 0, \quad d_{2,0} = 0, \quad d_{2,1} = -K;$$
$$C_1 = 0, \quad C_{2,0} = 0, \quad C_{2,1} = K \operatorname{cosec}(\pi/\xi) \exp(i\pi/\xi)/q_1(1-\tau_1)^{\beta}. \tag{1·2}$$

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The origin of these coordinates has been taken at the stagnation point; hence quantities such as the wedge length L and stand-off S can be calculated by substituting the appropriate values of τ and θ. To find the drag coefficient it is convenient to have a space coordinate measured along the face of the wedge. This coordinate l is given by

$$l = |z^{(1)}(\tau, \pi/\xi) + z_q^{(2)}(\tau, \pi/\xi)|, \quad \tau_q \leqslant \tau \leqslant \tau_{q+1} \quad (q = 0, 1).$$

The drag coefficient will be calculated exactly for Réthy flows, and for the special case of small wedge angle and sonic free-stream speed the results will be compared with the corresponding results obtained by using the approximate equations of Tricomi and of Tomotika and Tamada. The variable σ is more convenient for this work than τ is, and dashes below will refer to differentiations with respect to σ. We recall that the Wronskian $W(R, L) = 1$ and that the density at any speed is given by $\rho/\rho_0 = (1-\tau)^\beta$.

2. *General formula for the drag coefficient.* In series form we have

$$l = \begin{cases} (2K\rho_0/\rho_s\alpha q) \sum_{n=1}^{\infty} \{(-1)^n/(n^2\xi^2-1)\}\{\cos nm\xi - R'(n\xi, \sigma_1)\}\{L'(n\xi, \sigma) \\ \qquad\qquad - (\rho_s/\rho)L(n\xi, \sigma)\} \quad (\infty \geqslant \sigma \geqslant \sigma_1), \\ K\rho_0\{\operatorname{cosec}\alpha/\rho_1 q_1 - 1/\rho\alpha q\} \\ \qquad + (2K\rho_0/\rho_s\alpha q) \sum_{n=1}^{\infty} \{(-1)^n/(n^2\xi^2-1)\}[\cos nm\xi\{L'(n\xi, \sigma) - (\rho_s/\rho)\, L(n\xi, \sigma)\} \\ \qquad\qquad - L'(n\xi, \sigma_1)\{R'(n\xi, \sigma) - (\rho_s/\rho)\, R(n\xi, \sigma)\}] \quad (\sigma_1 \geqslant \sigma \geqslant \sigma_b). \end{cases} \tag{2·1}$$

These expressions can be written with integrals replacing the series, the integrals being taken over a contour C parallel to the imaginary axis and cutting the real axis between $z = 1$ and $z = 2$ (since the requisite integral is an analytic function of z except for simple poles at $z = \pm 1, n\xi$, and is exponentially small for large $|z|$, $\operatorname{Re} z > 0$. Thus

$$l = \begin{cases} (iK\rho_0/\rho_s q\pi) \int_C \dfrac{\{\cos mz - R'(z, \sigma_1)\}\{L'(z, \sigma) - (\rho_s/\rho)\, L(z, \sigma)\}\, dz}{(z^2-1)\sin\alpha z} \quad (\infty \geqslant \sigma \geqslant \sigma_1), \\ K\rho_0\{\operatorname{cosec}\alpha/\rho_1 q_1 - 1/\rho\alpha q\} \\ \qquad + (iK\rho_0/\rho_s q\pi) \int_C \dfrac{\cos mz\{L'(z, \sigma) - (\rho_s/\rho)\, L(z, \sigma)\} - L'(z, \sigma_1)\{R'(z, \sigma) - (\rho_s/\rho)\, R(z, \sigma)\}}{(z^2-1)\sin\alpha z}\, dz \\ \qquad\qquad (\sigma_1 \geqslant \sigma \geqslant \sigma_b) \end{cases} \tag{2·2}$$

The drag D on the sloping surface of the wedge is easily found by calculating the loss of momentum across a square control surface at a large distance from the wedge

$$D = p_b L \sin\alpha + \rho_0 K\{q_1 - q_b \cos m + (p_1 - p_b)/\rho_1 q_1\}. \tag{2·3}$$

When the pressure is referred to that at boundary speed, the drag coefficient is defined as

$$C_D = (D - p_b L \sin\alpha)/\tfrac{1}{2}\rho_b q_b^2 L, \tag{2·4}$$

where L, from (2·2) with $\sigma = \sigma_b$, is

$$L = K\rho_0(\operatorname{cosec}\alpha/\rho_1 q_1 - 1/\rho_b q_b \alpha) + (iK\rho_0/\rho_b q_b \pi)\left[\frac{\rho_b}{\rho_s}\int_C \frac{L'(z, \sigma_b)\cos mz - L'(z, \sigma_1)}{(z^2-1)\sin\alpha z}\, dz + \int_C \frac{\cos mz}{(z^2-1)\sin\alpha z}\, dz\right] \tag{2·5}$$

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and use has been made of the fact that $L(n, \sigma_b) = -1$, $R(n, \sigma_b) = 0$ and $R'(n, \sigma_b) = 1$ for all n. The last integral may be moved to a contour consisting of the imaginary axis indented at the origin, after allowing for the pole at $z = 1$. Along this contour it receives a contribution from the indentation; the remaining contributions vanish, since the integrand is an odd function. Thus we obtain

$$L = (K\rho_0/\alpha\rho_b q_b)\left[(\alpha/\sin\alpha)(\rho_b q_b/\rho_1 q_1 - \cos m) - (\rho_b/\rho_s q_b)(\alpha/\pi i)\int_C \frac{L'(z,\sigma_b)\cos mz - L'(z,\sigma_1)}{(z^2-1)\sin\alpha z}\,dz\right]. \tag{2·6}$$

A special case is obtained by displacing the singularity on $\theta = 0$ to the corner of the hodograph diagram, i.e. by putting $\tau_1 = \tau_b$. This gives the flow of a jet past a symmetric wedge; the stream function for this flow was found in Part I, section 5.

For small values of α (large ξ) the contour integral in (2·6) can be evaluated approximately and L, and thus C_D, expressed as a series in ascending powers of α. The method consists of deforming the contour C (without traversing any singularities) to a new parallel contour C' on which $\mathrm{Re}\,(z) > \frac{1}{2}\xi$ and is therefore large. The expression $1/(z^2-1)$ may then be expanded in series and the functions $L'(z,\sigma)$ replaced by their asymptotic expansions for large $|z|$. The drag coefficient derived in this way will be compared with the corresponding ones found from the Tricomi and the Tomotika and Tamada approximations. The Tricomi approximation requires the speed to be near sonic; with this in mind, *only the special case of a sonic jet*, for which $\tau_b = \tau_s\,(\sigma_b = 0)$ *will be considered.*

3. *Asymptotic expansions.* By using the asymptotic formulae for $\psi_z(\tau)$ and $\psi'_z(\tau)$ given by Cherry (1), it can be deduced that, for large $|z|$ and $\sigma_b = 0$,

$$L'(z,\sigma) \sim \left(\frac{1-\tau_s}{1-\tau}\right)^{\beta} \frac{J_z(zt)\,[\{\tilde{q}_1^*(\tau)-1\}z + O(z^{-1})] + J'_z(zt)\,\{\tilde{q}_0^*(\tau)\,z^2 + O(1)\}}{J_z(z)\,\{z + O(z^{-1})\} + J'_z(z)\,[\{\tilde{q}_1(\tau_s) - \tilde{q}_0^*(\tau_s)\} + O(z^{-2})]}, \tag{3·1}$$

where t is a function of τ (and hence of σ) defined by the equation

$$\tanh^{-1}\sqrt{(1-t^2)} - \sqrt{(1-t^2)} = \tanh^{-1}\sqrt{\{(1-\tau/\tau_s)/(1-\tau)\}} - \tau_s^{-\frac{1}{2}}\tanh^{-1}\sqrt{\{(\tau_s-\tau)/(1-\tau)\}}$$

and is such that $0 < t \leqslant 1$ when $0 < \tau \leqslant \tau_s$, while the $\tilde{q}$'s are functions of τ which have been tabulated in Cherry's paper. Typical values, for $\gamma = 1{\cdot}4$ ($\tau_s = \frac{1}{6}$), are

τ	$\tilde{q}_0^*$	$\tilde{q}_1$	$\tilde{q}_1^*$
0	1·000	0·361	0·361
0·08	1·028	0·356	0·384
0·16	1·060	0·350	0·428
τ_s	1·063	0·349	0·433

In particular, $\tilde{q}_0^*(\tau)$ can be expressed analytically as

$$\tilde{q}_0^*(\tau) = (1-6\tau)^{\frac{1}{2}}/(1-\tau)^{\frac{1}{2}}\,(1-t^2)^{\frac{1}{2}}.$$

When $\tau_s = \frac{1}{6}$, this gives, as a limiting value, $\tilde{q}_0^*(\tau_s) = \{\frac{1}{2}(\gamma+1)\}^{\frac{1}{2}}$.

In order to obtain results that are uniformly valid for all values of source speed up to and including sonic speed, we make use of asymptotic expansions of the Bessel

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functions derived by Olver (4). These expansions are uniformly valid, for the real values of t with which we are concerned, in $|\arg z| < \frac{1}{2}\pi$.

$$J_z(zt) \sim \left(\frac{4\zeta}{1-t^2}\right)^{\frac{1}{4}}\left[\frac{\mathrm{Ai}\,(z^{\frac{2}{3}}\zeta)}{z^{\frac{1}{3}}}\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{z^{2s}}+\frac{\mathrm{Ai}'(z^{\frac{2}{3}}\zeta)}{z^{\frac{5}{3}}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{z^{2s}}\right],$$

$$J_z'(zt) \sim -\frac{2}{t}\left(\frac{1-t^2}{4\zeta}\right)^{\frac{1}{4}}\left[\frac{\mathrm{Ai}\,(z^{\frac{2}{3}}\zeta)}{z^{\frac{4}{3}}}\sum_{s=0}^{\infty}\frac{C_s(\zeta)}{z^{2s}}+\frac{\mathrm{Ai}'(z^{\frac{2}{3}}\zeta)}{z^{\frac{2}{3}}}\sum_{s=0}^{\infty}\frac{D_s(\zeta)}{z^{2s}}\right]. \tag{3·2}$$

Here ζ is defined by the relation

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \ln\left(\frac{1+\sqrt{(1-t^2)}}{t}\right)-\sqrt{(1-t^2)}$$

and in the usual notation $\mathrm{Ai}\,(z) = (z^{\frac{1}{2}}/\pi\sqrt{3})\,K_{\frac{1}{3}}(\frac{2}{3}z^{\frac{3}{2}})$,

$\mathrm{Ai}'(z) = -(z/\pi\sqrt{3})\,K_{\frac{2}{3}}(\frac{2}{3}z^{\frac{3}{2}})$.

The coefficients $A_s(\zeta)$, etc., are defined by the recurrence relations:

$$\left.\begin{aligned} A_0(\zeta) &= 1,\\ B_s(\zeta) &= \tfrac{1}{2}\zeta^{-\frac{1}{2}}\int_0^{\zeta}\zeta^{-\frac{1}{2}}[f(\zeta)\,A_s(\zeta)-A_s''(\zeta)]\,d\zeta,\\ A_{s+1}(\zeta) &= \tfrac{1}{2}B_s'(\zeta)+\tfrac{1}{2}\int f(\zeta)\,B_s(\zeta)\,d\zeta \quad \text{with} \quad A_{s+1}(-\infty) = 0,\\ \text{where} \qquad f(\zeta) &= \tfrac{5}{16}\zeta^2+\zeta t^2(t^2+4)/4(t^2-1)^3,\\ \text{and by} \qquad C_s(\zeta) &= \chi(\zeta)\,A_s(\zeta)+A_s'(\zeta)+\zeta B_s(\zeta),\\ D_s(\zeta) &= A_s(\zeta)+\chi(\zeta)\,B_{s-1}(\zeta)+B_{s-1}'(\zeta),\\ \text{where} \qquad \chi(\zeta) &= 1/4\zeta-\tfrac{1}{2}t^2\{\zeta/(1-t^2)^3\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{3·3}$$

In order to deal with the case $t = 1$ ($\zeta = 0, \sigma = 0, \tau = \tau_s$), we need the following limiting values:

$$\mathrm{Ai}\,(0) = 1/3^{\frac{2}{3}}\Gamma(\tfrac{2}{3}), \quad \mathrm{Ai}'(0) = -1/3^{\frac{1}{3}}\Gamma(\tfrac{1}{3}),$$

$$\lim_{\zeta\to 0}\{4\zeta/(1-t^2)\}^{\frac{1}{4}} = 2^{\frac{1}{3}}.$$

Furthermore, it is not difficult to show (with the aid of expressions given by Olver for $t(\zeta), f(\zeta)$ near $\zeta = 0$) that $\chi(\zeta), A_s(\zeta), B_s(\zeta), C_s(\zeta)$ and $D_s(\zeta)$ are all continuous (and finite) functions of ζ in the range $-\infty < \zeta \leqslant 0$, corresponding to $0 < t \leqslant 1$. Thus

$$\sum_{s=0}^{\infty}A_s(\zeta)/z^{2s} = A_0(\zeta)+o(z^0)$$

and the infinite sums in (3·2) are, to the first order, equivalent to $A_0(\zeta), B_0(\zeta), C_0(\zeta)$ and $D_0(\zeta)$ respectively. Near $\zeta = 0$ the analysis shows these to have the values

$$A_0(\zeta) = 1, \quad B_0(\zeta) = 2^{\frac{1}{3}}/70+O(\zeta), \quad C_0(\zeta) = \tfrac{1}{5}.2^{\frac{1}{3}}+O(\zeta), \quad D_0(\zeta) = 1. \tag{3·4}$$

The behaviour of the Airy functions in (3·2) depends upon the order of magnitude of $z^{\frac{2}{3}}\zeta$. The expansions for the Bessel functions are to be substituted in the formula for $L'(z, \sigma_1)$ and then integrated along a contour on which $|z| = O(\alpha^{-1})$. The value $\zeta = \zeta_1$ corresponding to $\sigma = \sigma_1$ depends upon τ_1/τ_s. Let $1-\tau_1/\tau_s = \alpha^{\epsilon}$, and consider all ϵ in

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the range $0 < \epsilon \leqslant \infty$, corresponding to $0 < \tau_1 \leqslant \tau_s$. Then, from the definitions of ζ and t we find that

$$\zeta_1 = 2^{-\frac{2}{3}}(1-\tau_s)^{-\frac{1}{3}}[\alpha^\epsilon + \{(4\beta-1)/10\beta\}\alpha^{2\epsilon} + O(\alpha^{3\epsilon})], \tag{3·5}$$

where $\beta = 1/(\gamma-1)$ as usual, and

$$t_1 = 1 - 2^{-1}(1-\tau_s)^{-\frac{1}{3}}\alpha^\epsilon + O(\alpha^{2\epsilon}), \tag{3·6}$$

whence

$$\{4\zeta_1/(1-t_1^2)\}^{\frac{1}{4}} = 2^{\frac{1}{3}}\{1 + O(\alpha^\epsilon)\}. \tag{3·7}$$

The argument of the Airy functions is therefore

$$z^{\frac{2}{3}}\zeta = O(\alpha^{\epsilon-\frac{2}{3}})$$

and is thus large or small according as $\epsilon <$ or $> \frac{2}{3}$. The value $\epsilon = \frac{2}{3}$ divides the régimes of subsonic flow ($\epsilon < \frac{2}{3}$) and transonic flow ($\epsilon > \frac{2}{3}$), and affects radically the form of the drag coefficient, as will be seen. For $\epsilon < \frac{2}{3}$, from the inequalities

$$\begin{aligned} \mathrm{Ai}\,(z) &< A(1+|z|^{\frac{1}{4}})^{-1}\,|\exp(-\tfrac{2}{3}z^{\frac{3}{2}})|, \\ \mathrm{Ai}'(z) &< A(1+|z|^{\frac{1}{4}})\,|\exp(-\tfrac{2}{3}z^{\frac{3}{2}})|, \end{aligned}$$

given by Olver (A is a finite constant in these inequalities), it can be deduced that $J_z(zt_1)$ and $J_z'(zt_1)$ are O [exp $\{-\frac{2}{3}\alpha^{-(1-\frac{3}{2}\epsilon)}\}$] and are thus exponentially small when α is small. The contribution from the term $L'(z, \sigma_1)$ may therefore be ignored in subsonic (as distinct from transonic) flow. When $\epsilon \geqslant \frac{2}{3}$, the Airy function and its derivative are both $O(1)$. For the particular case $\epsilon = \infty$, corresponding to $\tau_1 = \tau_s$, the expansions (3·2) reduce to

$$\left.\begin{aligned} J_z(z) &\sim (\tfrac{2}{9})^{\frac{1}{3}}/\Gamma(\tfrac{2}{3})\,z^{\frac{1}{3}} - (\tfrac{4}{3})^{\frac{1}{3}}/70\Gamma(\tfrac{1}{3})\,z^{\frac{5}{3}} + o(z^{-\frac{5}{3}}), \\ J_z'(z) &\sim (\tfrac{4}{3})^{\frac{1}{3}}/\Gamma(\tfrac{1}{3})\,z^{\frac{2}{3}} - (\tfrac{2}{9})^{\frac{1}{3}}/5\Gamma(\tfrac{2}{3})\,z^{\frac{4}{3}} + o(z^{-\frac{4}{3}}). \end{aligned}\right\} \tag{3·8}$$

These expansions are those given by Watson (5), with the second extended by one term. After substitution in (3·1) we obtain

$$L'(z, 0) \sim A^* z^{\frac{2}{3}} + B_1^* - 1 + o(1), \tag{3·9}$$

where

$$A^* = \{3(\gamma+1)\}^{\frac{1}{3}}\,\Gamma(\tfrac{2}{3})/\Gamma(\tfrac{1}{3}) \quad \text{and} \quad B_1^* = \tilde{q}_1^*(\tau_s) - \tfrac{1}{5}\tilde{q}_0^*(\tau_s).$$

This expansion is required for the computation of L and C_D for all ϵ.

4. *Drag coefficient for subsonic flow.* For subsonic flow ($\epsilon < \frac{2}{3}$) we ignore $L'(z, \sigma_1)$ as mentioned above and expand the integrand of the contour integral of (2·6) as indicated in section 2. We obtain a sum of integrals of the form

$$I_r = (\alpha/\pi i)\int_{C'} \frac{\cos mz}{\sin \alpha z} z^{-r}\,dz,$$

where C' is the contour, parallel to C, already defined (Re $z > \frac{1}{2}\xi$).

By the theory of residues we may write this integral as

$$I_r = -2(\alpha/\pi)^r \sum_{n=1}^{\infty} \cos n\pi \cos n\pi\nu/n^r = -\alpha^r f(r, \nu), \quad \text{where} \quad \nu = m/\alpha,$$

$$f(r, \nu) = \{\zeta(1-r, \overline{1+\nu}/2) + \zeta(1-r, \overline{1-\nu}/2\}/2^{1-r}\Gamma(r)\cos\tfrac{1}{2}r\pi$$

and $\zeta(s, c)$ is the generalized Riemann Zeta Function.

After some simplification the following results are obtained:

$$L = P\alpha^{-1} + Qf(\tfrac{4}{3}, \nu)\,\alpha^{\frac{1}{3}}, \tag{4·1}$$

$$C_D = A\alpha + + Bf(\tfrac{4}{3}, \nu)\,\alpha^{\frac{7}{3}} + \{C + D_1 f(2, \nu)\}\alpha^3 + o(\alpha^3), \tag{4·2}$$

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where
$$f(2,\nu) = \tfrac{1}{2}\nu^2 - \tfrac{1}{6},$$
$$P = -K\rho_0(1+b_1)/\rho_s q_s \quad (>0),$$
$$Q = (K\rho_0/\rho_s q_s)\, 6^{\frac{1}{3}}\Gamma(\tfrac{2}{3})\, \tilde{q}_0^*(\tau_s)/\Gamma(\tfrac{1}{3}),$$
$$A = 2(1+a_1)/(1+b_1),$$
$$B = -AQ/P,$$
$$C = -(1+2b_1+a_1b_1)/3(1+b_1)^2,$$
$$D = -2[1+b_1-(1+a_1)\,B_1^*)]/(1+b_1)^2,$$
$$a_1 = (p_s - p_1 - \rho_1 q_1^2)/\rho_1 q_1 q_s,$$
$$b = -\rho_s q_s/\rho_1 q_1.$$

These results have been compared with those to be obtained from the Tomotika and Tamada and the Tricomi approximations. The details will not be given here; the appropriate forms for the independent solutions of the hodograph equations have been discussed by Mackie (2). The drag coefficients may be written in one formula, namely

$$C_D = A\alpha + Bf(\tfrac{4}{3},\nu)\,\alpha^{\frac{7}{3}} + \{C + D_i f(2,\nu)\}\alpha^3 + o(\alpha^3) \quad (i = 1,2,3),$$

where D_1 is the Chaplygin value given above,

$$D_2 = -2\left[(1+b_1) - (1+a_1)\left\{1 - \frac{1}{5}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{3}}\right\}\right]\Big/(1+b_1)^2 \quad \text{(Tomotika and Tomada)},$$

$$D_3 = -2[(1+b_1)-(1+a_1)]/(1+b_1)^2 \quad \text{(Tricomi)}.$$

It is interesting to note that in all cases the first two terms are in complete agreement; also, that the error involved in retaining the third term is of the same order for the Tricomi case as for the (better) approximation of Tomotika and Tamada. A similar result was obtained by Mackie (2) for Helmholtz flow. This suggests that, at least for drag computation, the error introduced by the poor fit near the stagnation point of the equation of state implied by the Tricomi equation is not as serious as might have been supposed.

5. *Drag coefficient for a wedge in a sonic jet.* For a wedge in a sonic jet,

$$\sigma_1 = 0, \quad a_1 = b_1 = -1.$$

The result of these substitutions is

$$C_D = \tilde{A}^*\alpha^{\frac{5}{3}} - \tfrac{1}{2}B_1^*(\tilde{A}^*)^2\alpha^{\frac{7}{3}} + o(\alpha^{\frac{7}{3}}),$$

where
$$1/\tilde{A}^* = A^*\nu^{-2}\{f(\tfrac{4}{3},\nu) + 2\pi^{-\frac{4}{3}}(1-2^{-\frac{1}{3}})\,\zeta(\tfrac{4}{3})\}$$

and use has been made of the fact that

$$\frac{\alpha}{2\pi i}\int_{C'}\frac{z^{-r}\,dz}{\sin\alpha z} = \left(\frac{\alpha}{\pi}\right)^r(1-2^{1-r})\,\zeta(r) \quad (\mathrm{Re}\, r > 1).$$

In the limit as $\nu \to 0$ we recover the flow of an infinite stream at sonic speed past a finite wedge with sonic break-away (Helmholtz flow). The drag for this problem was given by Mackie (2). The appropriate expansion is

$$C_D = \tilde{A}_0^*\alpha^{\frac{5}{3}} - \tfrac{1}{2}B_1^*(\tilde{A}_0^*)^2\alpha^{\frac{7}{3}} + o(\alpha^{\frac{7}{3}}),$$

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where $$1/\bar{A}_0^* = (2^{\frac{5}{3}}-1)(\gamma+1)^{\frac{1}{3}}\zeta(\tfrac{5}{3})\,\Gamma(\tfrac{5}{6})/3^{\frac{2}{3}}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{3}),$$

and we have already commented in the previous paragraph on the similarity between the results for the three hodograph equations in this case.

6. *Wedge in transonic flow.* For $\infty > \epsilon > \frac{2}{3}$, the analysis of (3·2) shows that only the first terms of each series need be retained. In the calculation of L (2·5) it must be remembered that q_1 and therefore ρ_1, K and σ_1, are to be expressed as powers of α. The expansion for $L'(z, \sigma_1)$ is

$$L'(z,\sigma_1) \sim \mathfrak{F}(\tau_1)\,\mathrm{Ai}(z^{\frac{2}{3}}\zeta_1)\{1+o(z^0)\} - \mathfrak{G}(\tau_1)\,\mathrm{Ai}'(z^{\frac{2}{3}}\zeta_1)\{z^{\frac{2}{3}}+o(z^{\frac{2}{3}})\}$$

with $$\mathfrak{F}(\tau_1) = 3^{\frac{2}{3}}\Gamma(\tfrac{2}{3})\{(1-\tau_s)/(1-\tau_1)\}^{\beta}\{\tilde{q}_1^*(\tau_1) - 1 - 2^{\frac{1}{3}}t_1^{-1}C_0(\zeta_1)\,\tilde{q}_0^*(\tau_1)\}$$

and $$\mathfrak{G}(\tau_1) = 2^{\frac{1}{3}}3^{\frac{2}{3}}\Gamma(\tfrac{2}{3})\{(1-\tau_s)/(1-\tau_1)\}^{\beta}t_1^{-1}\tilde{q}_0^*(\tau_1).$$

The functions $\tilde{q}_0^*(\tau_1)$, etc., have also to be expanded in powers of α. After a considerable amount of algebra, involving substitutions in the integrals in the equation for L and evaluation of residues, the series for L in ascending powers of α is obtained as

$$\begin{aligned} L/d = {} & \tfrac{1}{8}(\gamma+1)\,\alpha^{2\epsilon-1} + O(\alpha^{3\epsilon-1}) + A^*\{f(\tfrac{4}{3},\nu) - \Sigma_1(\zeta_1,\alpha)\}\,\alpha^{\frac{1}{3}} \\ & + O(\alpha^{\frac{1}{3}+2\epsilon}) + \{\tfrac{1}{6} + B_1^* f(2,\nu) + (B_1^*-1)\,\Sigma_2(\zeta_1,\alpha)\}\,\alpha \\ & + O(\alpha^{1+2\epsilon}), \end{aligned} \tag{6·1}$$

where $$\Sigma_1(\zeta_1,\alpha) = 2.3^{\frac{1}{3}}\Gamma(\tfrac{1}{3})\pi^{-\frac{4}{3}}\sum_{n=1}^{\infty}\cos n\pi\,\mathrm{Ai}'\{(n\pi/\alpha)^{\frac{2}{3}}\zeta_1\}\,n^{-\frac{4}{3}}$$

and $$\Sigma_2(\zeta_1,\alpha) = 2.3^{\frac{2}{3}}\Gamma(\tfrac{2}{3})\pi^{-2}\sum_{n=1}^{\infty}\cos n\pi\,\mathrm{Ai}\{(n\pi/\alpha)^{\frac{2}{3}}\zeta_1\}\,n^{-2}.$$

The Airy functions are of order in α less than or equal to 1. The order of importance of the terms in (6·1) depends on the value of ϵ. The drag coefficient may now be obtained by use of (2·4).

7. *Wedge length and stand-off.* The wedge length and the stand-off of the jet from the nozzle are both derived quantities as a result of the method of solution. In a given physical situation, on the other hand, these quantities are likely to be prescribed. This implies that the upstream conditions may not then be freely chosen.

A series for the stand-off S is obtained at once from the expression for the physical coordinates (Part I, section 10) as $\mathrm{Re}\,z(\tau_s, 0)$. This may be written in integral form as

$$\begin{aligned} S/d = {} & \mathrm{cosec}\,\alpha[\cos\alpha - (\rho_1 q_1/\rho_s q_s)\cos(\alpha-m)] \\ & - (\rho_1 q_1/\rho_s q_s)\frac{1}{\pi i}\left\{\int_C \frac{L'(z,\sigma_1)\cos\alpha z\,dz}{(z^2-1)\sin\alpha z} - \int_C \frac{L'(z,0)\cos\{(\alpha-m)z\}\,dz}{(z^2-1)\sin\alpha z}\right\}, \end{aligned}$$

which gives after some algebra the ascending series

$$\begin{aligned} S/d = {} & \tfrac{1}{8}(\gamma+1)\,\alpha^{2\epsilon-1} + O(\alpha^{3\epsilon-1}) + A^*\{\bar{f}(\tfrac{4}{3},\nu) - \bar{\Sigma}_1(\zeta_1,\alpha)\}\,\alpha^{\frac{1}{3}} + O(\alpha^{2\epsilon+\frac{1}{3}}) \\ & + \{-\tfrac{1}{3} + B^*\bar{f}(2,\nu) + (B^*-1)\,\bar{\Sigma}_2(\zeta_1,\alpha)\}\,\alpha + O(\alpha^{2\epsilon+1}), \end{aligned}$$

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where $$\bar{f}(r,\nu) = \{\zeta(1-r,\tfrac{1}{2}\nu)+\zeta(1-r,1-\tfrac{1}{2}\nu)\}/2^{1-r}\Gamma(r)\cos\tfrac{1}{2}\pi r,$$

$$\bar{\Sigma}_1(\zeta_1,\alpha) = 2.3^{\frac{1}{3}}\Gamma(\tfrac{1}{3})\sum_{n=1}^{\infty}\mathrm{Ai}'\{(n\pi/\alpha)^{\frac{2}{3}}\zeta_1\}(n\pi)^{-\frac{4}{3}}$$

and $$\bar{\Sigma}_2(\zeta_1,\alpha) = 2.3^{\frac{2}{3}}\Gamma(\tfrac{2}{3})\sum_{n=1}^{\infty}\mathrm{Ai}\{(n\pi/\alpha)^{\frac{2}{3}}\zeta_1\}(n\pi)^{-2}.$$

We observe at once that L and S are always of the same order of magnitude. This may at first sight seem paradoxical, since L and S are at our disposal. The solution of the apparent paradox is that L measures the distance along the wedge at which sonic speed occurs, and we have already seen in the theorem on sonic jets (Part I) that this speed is achieved in a finite distance;† longer wedges end in uniform jets with $\nu = 1$.

For $1-\tau_1/\tau_s = \alpha^{\epsilon}$ and small α we have the following results:

(i) $0 < \epsilon < \frac{1}{2}$, i.e. $0 < \tau_1 < (1-\alpha^{\frac{1}{2}})\tau_s$;

$$L/d = S/d = \tfrac{1}{8}(\gamma+1)\alpha^{-(1-2\epsilon)} + O(\alpha^{-(1-3\epsilon)}),$$

implying that both L and S are large compared with d.

(ii) $\epsilon = \frac{1}{2}$ requires L and S to be of the same order of magnitude as d.

(iii) $\frac{1}{2} < \epsilon < \frac{2}{3}$ requires $L/d = S/d = \frac{1}{8}(\gamma+1)\alpha^{2\epsilon-1} + O(\alpha^{\frac{1}{3}})$.

Now L and S are approximately equal but both are small compared with d.

(iv) $\epsilon > \frac{2}{3}$, i.e. $\tau_1 > (1-\alpha^{\frac{2}{3}})\tau_s$;

L/d and S/d have the terms in $\alpha^{\frac{1}{3}}$ as the leading quantities. It is only in this range that the asymptotic direction $\theta = m$ enters the leading term (through the quantity ν), and that L and S are not equal to this order of approximation. The variation of

ν	0·2	0·4	0·6	0·8
$L/d\alpha^{\frac{1}{3}}$	0·029	0·118	0·284	0·577

L with ν can be seen from the Table above (computed for $\gamma = 1{\cdot}4$). It can be shown analytically that $L/d = O(\nu^2)$ for small ν, hence $L/d = O(\alpha^{\frac{1}{3}}\nu^2)$. For a short wedge of length $L = O(\alpha^{\frac{1}{3}+\lambda})$ it follows that $\nu = O(\alpha^{\frac{1}{2}\lambda})$. If S is chosen independently of L, then q_1 (and τ_1) is determined.

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For fuller details of the analysis in this paper and in Part I the reader is referred to Mrs McLaughlin's doctoral thesis (1968) submitted to the University of Strathclyde.

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† Similarly, S turns out to be finite for the problem of section 5. On account of the sonic jet theorem the changes take place within this distance upstream although the jet itself extends to infinity in that direction.

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