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NONSTANDARD MODELS IN RECURSION THEORY AND REVERSE MATHEMATICS

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Abstract. We give a survey of the study of nonstandard models in recursion theory and reverse mathematics. We discuss the key notions and techniques in effective computability in nonstandard models, and their applications to problems concerning combinatorial principles in subsystems of second order arithmetic. Particular attention is given to principles related to Ramsey's Theorem for Pairs.

§**1. Introduction.** The existence of a nonstandard model of Peano arithmetic (PA) was proved by Skolem [57] eighty years ago (see also [58] for an exposition twenty years later). In the ensuing years, the study of nonstandard models expanded in the main to higher order structures with applications to classical mathematics by way of nonstandard analysis, pioneered by Abraham Robinson [49]. However, the investigation of models of PA stayed very much within the theory itself (see Kaye [32]). The introduction in 1978 by Kirby and Paris [33] of a hierarchy of fragments of PA paved the way for applying nonstandard models to recursion theory. This began with Stephen G. Simpson's unpublished proof in 1984 of the Friedberg–Muchnik Theorem under the hypothesis of Σ_1 induction.

The study of recursion theory in nonstandard models is motivated by two general problems: First, capture the essence of computability from the perspective of definability. The equivalence between Σ_1 definability and recursive enumerability in $\mathbb{N}$ (the set of natural numbers) serves as an inspiration to investigate computability in general domains anchored on definability. Second, understand the proof-theoretic strength of theorems in recursion theory. In the spirit of reverse mathematics (see Simpson [56] for an introduction to the subject), the central question of reverse recursion theory asks for an analysis of the strength of mathematical induction required to prove theorems in recursion theory. Since induction is at the heart of any set construction in recursion theory, this question effectively summarizes the aims and objectives of the subject. These two considerations naturally lead to an

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axiomatic approach to computability theory, and with that the investigation of models—particularly nonstandard models—of fragments of PA.

The first explicit study of nonstandard models in recursion theory appeared in Slaman and Woodin [61] which gave a positive solution to Post's problem under the weaker assumption of Σ_1 bounding. This was followed by a systematic investigation (Groszek and Slaman [26]) of the basic properties of computability in models of fragments of PA. From the late 1980's, much effort was invested in characterizing the inductive strength of theorems proved using priority constructions (cf. Chong and Yang [14, 15], Mytilinaios [47], Mytilinaios, and Slaman [48]).

From its inception, the development of the subject was influenced by both reverse mathematics and higher recursion theory, especially α-recursion theory (see Chong [5] and Sacks [51] for an introduction to the subject). On the one hand, measuring the inductive strength of recursion-theoretic theorems was a program with foundational flavor grounded in the overall objective of reverse mathematics. On the other hand, a number of important notions and techniques it uses have their origins in α-recursion theory. For instance, the concepts of regularity and hyperregularity were first introduced there. The blocking method in priority constructions (Mytilinaios [47]) and the coding method in nonstandard models (Chong and Yang [14]) were inspired by Shore's earlier works [53, 54] (as will be discussed in Section 3). Of course, the model-theoretic approach that it adopts to study questions in recursion theory is reminiscent of the overall philosophy of α-recursion theory.

Before the turn of the century, the development of the subject was largely internal. The focus was on analyzing the proof-theoretic strength of a recursion-theoretic theorem, the characterization of various types of priority constructions (in terms of induction), and the structure of Turing degrees in models of systems weaker than PA. In recent years, the interest has widened to include applications of nonstandard models to reverse mathematics, especially problems concerning combinatorial principles. By definition, reverse mathematics and reverse recursion theory are proof-theoretic. This means that any question about the truth of a mathematical statement is not only relative to the standard model or an ω-model (i.e., a structure for second order arithmetic whose first order universe is $\mathbb{N}$). Indeed, there are problems which inherently concern nonstandard models. For example, the question whether over the base system RCA_0, Ramsey's Theorem for Pairs or the Chain Antichain Principle (CAC) implies Σ^0_2 induction. And there are questions that on surface do not concern nonstandard models but whose solutions are found by way of such models, for example, whether Ramsey's Theorem for Pairs is strictly stronger over RCA_0 than its stable counterpart (Chong, Slaman, and Yang [12]). The abundant supply of nonstandard models is a rich reservoir for the investigation of proof-theoretic questions such as these. In particular, the existence of a Σ_1 reflection model in which every definable real is coded is used as the starting point towards analyzing the complexity of Ramsey's Theorem for Pairs (see Section 4). Hence looking

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beyond the standard numbers offers a different perspective that has proved to be useful and fruitful.

In this paper, we give a survey of nonstandard models in recursion theory and their applications to reverse mathematics, focusing on combinatorial principles related to Ramsey's Theorem for Pairs. We discuss the main results and techniques used to prove them, with particular emphasis on the role of nonstandard models in the study. It is our aim to convey to the reader a sense of how different areas of recursion theory and model theory fit together in the investigation of interesting and challenging problems. Following the preliminaries in Section 2, we proceed to fragments of PA and recursion theory in Section 3. In the final section, we discuss subsystems of second order arithmetic and Ramsey type combinatorial principles. We avoid detailed proofs of any theorem, but provide sketches of the key ideas where appropriate.

§**2. Fragments of Peano Arithmetic and their models.** The language $\mathcal{L}$ of first order arithmetic consists of function symbols $S, +, \times,$ *and* exp, where S denotes the successor function and exp the exponential function $x \mapsto 2^x$, together with the constant symbol 0. Formulas are generated in the usual way, and we follow the convention of defining Σ^0_n and Π^0_n formulas (for $n \geq 0$) in terms of their complexity. For simplicity, we omit in this section the superscript 0 in Σ^0_n and Π^0_n. A relation is Δ_n over a theory in $\mathcal{L}$ if it is provably equivalent to a Σ_n relation and a Π_n relation in the theory. While our focus here is on notions defined in $\mathcal{L}$, many of the basic results about first order models to be discussed below extend to models of (fragments of) second order arithmetic and will be applied in Section 4.

2.1. The inductive hierarchy. Let PA denote the collection of axioms for Peano arithmetic and let P^- be PA minus the scheme of mathematical induction. $I\Sigma_n$ denotes the induction scheme for Σ_n formulas (allowing number parameters), and $B\Sigma_n$ denotes the bounding scheme

$$\forall x < a\, \exists y\, \varphi(x, y) \rightarrow \exists b\, \forall x < a\, \exists y < b\, \varphi(x, y),$$

for each Σ_n formula $\varphi(x, y)$ (note that "$<$" is definable from "$+$"). Intuitively, $B\Sigma_n$ says that Σ_n formulas are closed under bounded quantifiers.

Let Exp be the statement saying that the exponential function exp is total. In the following we assume that Exp holds. This assumption guarantees the existence of codes for "finite" sets of numbers in a model without Σ_1 induction.

Our starting point is the existence of a hierarchy of theories extracted from fragments of PA:

THEOREM 2.1 (Kirby and Paris, [33]). *Over* $P^- + I\Sigma_0 + \mathrm{Exp}$,

$$\cdots \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow B\Sigma_n \Rightarrow \cdots \Rightarrow I\Sigma_1 \Rightarrow B\Sigma_1,$$

and the implications do not reverse.

A model of $P^- + I\Sigma_0 + \mathrm{Exp}$ is a $B\Sigma_n$ *model* if it satisfies $B\Sigma_n$ but not $I\Sigma_n$. It is an $I\Sigma_n$ model if it satisfies $I\Sigma_n$ but not $B\Sigma_{n+1}$. Theorem 2.1 says that

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for each $n \geq 1$, there is a $B\Sigma_n$ model and an $I\Sigma_n$ model. Such models are constructed by taking appropriate Σ_n Skolem hulls inside a model.

It was also shown in [33] that $I\Sigma_n$, $I\Pi_n$, $L\Sigma_n$ (the least number principle for Σ_n sets) and $L\Pi_n$ are all equivalent. It turns out that $B\Sigma_n$ is equivalent to the Δ_n induction scheme $I\Delta_n$ as well as the least principle $L\Delta_n$ (Slaman [60]), thus presenting the Kirby–Paris hierarchy as a sequence of theories based entirely on inductive strength.

For the rest of the paper, $\mathcal{M}$ will always denote a model of a fragment of PA.

2.2. $\mathcal{M}$-finiteness and coding. Finiteness is a fundamental notion in recursion theory. It is known since the 1960's that in any implementation of the idea of computation to a general domain, a "correct" definition of finiteness is critical. This was hinted at in the writings of Gandy, Kreisel, Sacks, and Spector and explicitly expressed in Kreisel [34] where an illuminating discussion of this subject was presented.

If $\mathcal{M}$ is nonstandard, finiteness is defined in terms of codability:

DEFINITION 2.2. $K \subset M$ is *$\mathcal{M}$-finite* if it is coded by a number $c \in M$, i.e., for any i,

$$i \in K \leftrightarrow \text{ the } i\text{-th digit in the binary expansion of } c \text{ is 1.}$$

A set $A \subset M$ is *regular* if $A \upharpoonright x$ is $\mathcal{M}$-finite for every $x \in M$.

We will often not make a distinction between an $\mathcal{M}$-finite set and its canonical code in $\mathcal{M}$ (for example, in the definition of Turing reducibility involving quadruples in Section 2.3 below). Clearly every $\mathcal{M}$-finite set is bounded in M but the converse is false. For instance, in any nonstandard model of $P^- + I\Sigma_0 + \mathrm{Exp}$, ω is bounded but not $\mathcal{M}$-finite. The link between $\mathcal{M}$-finite sets and induction is summarized in the lemma below. The proof is straightforward and is omitted:

LEMMA 2.3 (H. Friedman (unpublished)). *Over $P^- + I\Sigma_0 + \mathrm{Exp}$, the following are equivalent*:

(1) $I\Sigma_n$.
(2) *Every bounded Σ_n set is $\mathcal{M}$-finite.*
(3) *Every partial Σ_n function maps a Σ_n definable bounded set onto an $\mathcal{M}$-finite set.*

DEFINITION 2.4. A nonempty bounded initial segment I of M is a *cut* if it is closed under the successor function S. If, in addition, I is Σ_n definable, then we say that I is a *Σ_n cut*.

For example, in the models constructed in [33] to separate $B\Sigma_n$ and $I\Sigma_n$, ω is a Σ_n cut.

LEMMA 2.5. *If $\mathcal{M}$ is a $B\Sigma_n$ model, then there is a Σ_n cut I with a Σ_n definable function $f : I \xrightarrow[\text{cofinal}]{\text{increasing}} M$.*

Lemma 2.3 says that in a model of $I\Sigma_n$, every bounded Σ_n definable set is $\mathcal{M}$-finite. A natural extension of this result would be that in a model of $B\Sigma_n$,

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every bounded Δ_n definable set is $\mathcal{M}$-finite. In fact something stronger is true.

DEFINITION 2.6.

(1) A set $X \subseteq A$ is *coded on* A if there is an $\mathcal{M}$-finite set $\hat{X}$ such that $\hat{X} \cap A = X$.
(2) $X \subseteq A$ is Δ_n *on* A if both $A \cap X$ and $A \cap \overline{X}$ are Σ_n definable.

LEMMA 2.7 (Chong and Mourad, [8]). *Let* $n \geq 1$, $\mathcal{M} \models P^- + B\Sigma_n + \mathrm{Exp}$ *and* $A \subseteq M$. *Then every bounded set that is* Δ_n *on* A *is coded on* A.

We refer to Lemma 2.7 as the *Coding Lemma*. In practice, the set A is often a Σ_n cut I or I^k, $k < \omega$. The Coding Lemma is a useful tool in recursion-theoretic constructions over $B\Sigma_n$ models. There are two ways in which it plays a role: On the one hand the complexity of a construction may be considerably reduced by appealing to a code (see Section 4 for an application of this idea to the study of Ramsey's Theorem for Pairs in second order $B\Sigma_2$ models), while on the other hand the presence of a code may imply the complete absence of certain sets in $B\Sigma_n$ models that have been constructed in the standard model (e.g. for $n = 2$ there is no maximal r.e. set and no incomplete high r.e. set; see Section 3).

2.3. Turing reducibility. A set is *recursively enumerable* (or computably enumerable) if it is Σ_1 definable over $\mathcal{M}$. A recursively enumerable (r.e.) set is *recursive* (computable) if its complement is also r.e. Since Exp holds in $\mathcal{M}$, one may apply Gödel's coding method to obtain a recursive enumeration of all r.e. sets.

Let Φ_e be the eth r.e. set of quadruples. If X and Y are subsets of M, then $X \leq_T Y$ ("X is *recursive in* Y") if there is an $e \in M$ such that $X = \Phi_e^Y$, i.e., for any $\mathcal{M}$-finite set $K \subset M$,

$$K \subset X \Leftrightarrow \exists P \subset Y \, \exists N \subset \overline{Y} \langle K, 1, P, N \rangle \in \Phi_e$$

and

$$K \subset \overline{X} \Leftrightarrow \exists P \subset Y \, \exists N \subset \overline{Y} \langle K, 0, P, N \rangle \in \Phi_e.$$

It is straightforward to verify that $\leq_T$ is a transitive relation. Turing degrees are defined in the usual way.

In the above definition, P is a "positive condition" of the oracle Y and N is a "negative condition". Notice that the reduction procedure is designed to answer questions about $\mathcal{M}$-finite sets K, rather than individual numbers x, with the help of an oracle. The reason is that in a weak system such as $P^- + I\Sigma_1$, pointwise Turing reduction is not transitive even for r.e. sets. Intuitively, if A is pointwise reducible to B and B is pointwise reducible to C, then computing an element in A may require an $\mathcal{M}$-finite amount of information X about B which is not computable using only $\mathcal{M}$-finite amount of information about C. This could occur even though C has the capability to make decisions on individual elements of X using $\mathcal{M}$-finite amount of information about itself. See [26] for the construction of such an example.

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2.4. Three important types of models. From the model-theoretic and proof-theoretic points of view, the study of recursion theory calls for investigation of the computational aspects of an arbitrary $\mathcal{M}$. While this sets out the general program, its success has been uneven. More progress is made for $B\Sigma_n$ models than for $I\Sigma_n$ models. A possible explanation is that one has a better understanding of the former. In contrast, our knowledge of $I\Sigma_n$ models is comparatively more limited.

In general, constructing sets with prescribed properties in an $\mathcal{M}$ with limited inductive strength is always a challenge. The difficulties may be resolved in certain models, or may be insurmountable in others, entailing the nonexistence of such a set. Here we discuss three types of models that figure prominently in this survey.

2.4.1. *Projection models.* As inferred from the name, $\mathcal{M}$ is a projection model if there is a definable bijection between M and a proper initial segment. The inspiration for the idea may be traced back to the works of Kleene, Spector, and Gandy, which collectively showed the existence of a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$-injection from ω_1^{CK} into ω. Jensen [29] made the idea explicit by introducing the notion of a Σ_n projectum and applied it to study the fine structure of Gödel's constructible hierarchy L.

LEMMA 2.8 (Groszek and Slaman [26]). *There is an $I\Sigma_1$ model $\mathcal{M}$ and a bijection $f : M \to \omega$ together with a recursive (in $\mathcal{M}$) approximation f' such that for all x,*

$$f(x) = \lim_s f'(x, s).$$

Such a model is obtained by the Kirby–Paris construction of an $I\Sigma_1$ model. The Σ_1 Skolem hull map provides the projection needed. Note that the function f is Δ_2 definable via the recursive approximation f'. The advantage of working with a projection model is that through the recursive approximation, it "transfers" a construction over $\mathcal{M}$ to one that is over its standard part, where the full suite of mathematical induction is available. For example, many well studied priority arguments may be implemented upon rearranging the requirements defined recursively over $\mathcal{M}$ into one that has order type ω. The tradeoff is that the ordering of priorities is now Δ_2 and not recursive, but one is still able to show by induction on ω that every requirement is satisfied. This idea first appeared in the context of metarecursion theory and is due to Kreisel and Sacks (cf. Sacks [50]).

2.4.2. *Saturated models.* The notion of saturation we introduce here has to do with the existence of codes. It is a notion weaker than that of ω_1-saturation in the literature (see [2]). Lemma 2.7 guarantees that in a $B\Sigma_n$ model, sets that are Δ_n on a cut are coded on it. This is a property internal to a $B\Sigma_n$ model. While such a property is sufficient in many recursion-theoretic applications (cf.Sections 3 and 4), stronger coding properties have proved to be useful for the study of problems in reverse mathematics.

We say that $\mathcal{M}$ is *arithmetically saturated* if $\omega \subsetneq M$ and every arithmetically defined (possibly with parameters) subset of ω is coded on ω. $\mathcal{M}$ is *saturated*

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if any arbitrary subset of ω is coded on ω. Clearly the universe of a saturated model is uncountable whereas an arithmetically saturated model can be countable.

LEMMA 2.9. *For every $n \geq 1$, there is a saturated $B\Sigma_n$ model and a countable arithmetically saturated $B\Sigma_n$ model, both with ω as a Σ_n cut.*

The existence of an ω_1-saturated model of PA implies that there is a saturated model of PA. In [61] a saturated $B\Sigma_1$ model is constructed. In general, given a saturated model of PA, one may apply the canonical construction of Kirby–Paris to obtain a saturated $B\Sigma_n$ model $\mathcal{M}$ with ω as a Σ_n cut. Using this, a countable arithmetically saturated $B\Sigma_n$ model is defined by generating an elementary substructure of $\mathcal{M}$ over $\omega \cup \{a\}$ where $a \in M$ is an upper bound of ω.

2.4.3. *Reflection models.* The third type of model $\mathcal{M}$ is one endowed with cofinally many initial segments that are Σ_n (or even full) elementary substructures of $\mathcal{M}$, with the additional property that each segment is also a model of PA. The ingredients for constructing such a model were in McAloon [45]. Lemma 2.10 borrows from these ingredients and blends them with arithmetical saturation. It provides a model that is used in Chong, Slaman, and Yang [12] to separate Stable Ramsey's Theorem for Pairs from Ramsey's Theorem for Pairs.

LEMMA 2.10. *There is a countable model $\mathcal{M}$ of $P^- + B\Sigma_2$ and a $\Sigma_2(\mathcal{M})$ increasing function g such that*

(1) *$\mathcal{M}$ is the countable union of an increasing sequence of cuts $\mathfrak{I}_i$ each of which is a Σ_1 elementary substructure of $\mathcal{M}$ and a model of* PA:
$$\mathfrak{I}_0 \prec_{\Sigma_1} \mathfrak{I}_1 \prec_{\Sigma_1} \cdots \prec_{\Sigma_1} \mathfrak{I}_i \prec_{\Sigma_1} \cdots \prec_{\Sigma_1} \mathcal{M}.$$
(2) *For each $i \in \omega$, $g(i) \in \mathfrak{I}_i$, and for $i > 0$, $g(i) \notin \mathfrak{I}_{i-1}$. Hence $\mathcal{M} \not\models I\Sigma_2$.*
(3) *$\mathcal{M}$ is arithmetically saturated.*

The reflection model is used in [12] to carry out iterations of Σ_1 definable constructions relative to an $\mathcal{M}$-finite set of parameters. Condition (1) allows one to argue that such a construction succeeds and ends within the cut in which the parameters are located. The situation may be phrased in the following abstract form involving the idea of a "self-generating function": Suppose $\mathcal{M}$ is an arbitrary $B\Sigma_2$ model and $b \in M$. Let f be a partial recursive function with unbounded damain and range. Let $A_0 = \{x \mid x \leq b\}$. For each z, let $A_{z+1} = \{x \mid \exists y \in A_z (x < f(y))\}$. By $B\Sigma_2$, each A_z, if defined, is an $\mathcal{M}$-finite initial segment of $\mathcal{M}$. There is no *a priori* bound on the size of each A_z. In general, the size of A_z may "go to infinity" as z ranges over a cut. However, condition (1) implies that in a reflection model there is no cut I such that $\sup_{z \in I} A_z$ is unbounded in $\mathcal{M}$. Indeed, for any a, A_z is always defined whenever $z \leq a$, and $\sup_{z \leq a} A_z$ is bounded in $\mathfrak{I}_i$ where $a \leq g(i)$.

§**3. Reverse Recursion Theory.** The thrust of reverse recursion theory is to study models of computation in weak systems of PA. There are two ways in which to view this: First, as the name entails, it may be considered to be a first order version of reverse mathematics. Theorems in classical recursion theory

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are classified according to the amount of mathematical induction needed to prove them over a base theory. The second is as generalized theories of computability, in which models of different axiom systems are investigated for their computational content. These two approaches merge at the point of analyzing the techniques developed to study individual problems.

In this section, we discuss theorems in the standard model that continue to hold in models of a weak system, and theorems that fail in what one might consider to be "reasonable" weak systems. We also discuss methods that have been introduced to study these problems. Concepts and ideas unique to nonstandard models will be highlighted.

It is not difficult to see that many theorems provable in PA are provable in $P^- + I\Sigma_n$ for some $n \geq 1$. The interesting problem quite often lies in identifying the least such n. These are broadly referred to respectively as sufficient and necessary conditions in the discussion below.

REMARK 3.1. It is tempting to raise the following philosophical issue. If one adopts the formalist position, then the provability of a theorem (for example, the existence of an incomplete r.e. set, see below) in one weak system but not another may only be statement of a mathematical fact. On the other hand, there is no *a priori* justification for choosing one system over another based on the preference for a theorem to hold. Hence should the study of computability gravitate towards a particular model, and is the standard model the most natural mathematical structure in which to study arithmetical sets?

3.1. Analysis of proofs in the standard model. Mathematical induction is arguably the central notion of recursion theory and this is seen, for example, in the construction of an r.e. set and in the verification that every requirement is satisfied. (Subsection 3.1.1). In some cases, induction may be applied implicitly, such as the assumption that a bounded Σ_n set is finite (in the sense of the model. See Subsection 3.1.3). We illustrate these points with examples where use of induction beyond Σ_0 is crucial.

3.1.1. *Priority argument.* In general, for $n = 1, 2$ or 3, $I\Sigma_n$ is sufficient for the implementation of $\mathbf{0^{(n)}}$-priority arguments. We exhibit three representative theorems whose proofs are anchored on an analysis of the complexity of the original arguments. For (3) of Theorem 3.2, recall that a degree $\boldsymbol{a}$ is *branching* if $\boldsymbol{a}$ is the infinmum of two other degrees.

THEOREM 3.2. *Over* P^-,

(1) ($\mathbf{0'}$*-priority*) $I\Sigma_1$ *proves the Friedberg–Muchnik Theorem*;
(2) ($\mathbf{0''}$*-priority*) $I\Sigma_2$ *proves the existence of a minimal pair of r.e. degrees.*
(3) ($\mathbf{0'''}$*-priority*) $I\Sigma_3$ *proves the density of branching degrees.*

The Friedberg–Muchnik Theorem offers an immediate view of where $I\Sigma_1$ comes into the picture. In verifying that every requirement is satisfied in a construction for this theorem, the key step is to show that each requirement is injured at most finitely often. Simpson noted that $I\Sigma_1$ guaranteed this: We make one observation: The set $S = \{e :$ the eth requirement is injured at most 2^e times$\}$ contains 0, is Π_1 and closed

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under successor. However, there is no Π_1 cut in a model of $P^- + I\Sigma_1$ (by an application of $L\Sigma_1$). Hence S is the entire model and so every requirement is satisfied.

Consider now the problem of obtaining a minimal pair via a tree construction. In this setting each requirement has one of two possible outcomes: either a Π_2 or a Σ_2 outcome. The source of the challenge lies in arguing that a true path exists. This is the leftmost path on the priority tree whose initial segments are visited infinitely often during the construction. Its existence is shown by appealing to $I\Sigma_2$: Note that for any e,

$$\{(\delta, k) : k \leq e \wedge \delta \text{ is a string of length } k \text{ on the priority tree} \wedge \delta \text{ is visited infinitely often}\}$$

is bounded and Π_2, and is therefore $\mathcal{M}$-finite under the assumption that $\mathcal{M}$ satisfies Σ_2 induction. It follows that for each e, there is a leftmost string of length e on the priority tree visited infinitely often. $I\Sigma_2$ enables one to extend this to a true path and the path is shown to be $\varnothing''$-recursive. By the same analysis, the existence of a high incomplete r.e. set and the Sacks' Jump Inversion Theorem follows from $I\Sigma_2$ as well.

Slaman [59] proved the density of branching r.e. degrees. The proof is an instance of a $\mathbf{0}'''$-priority argument, where the truth of a Σ_3 fact is approached by a process of approximation relative to a $\varnothing''$-oracle. More precisely, since Σ_3 statements may be viewed as Σ_1 *relativized to* $\varnothing''$, a $\mathbf{0}'''$-priority argument is essentially a finite injury argument relative to the true path Λ of a $\mathbf{0}''$-priority tree. As for the minimal pair construction observed above, the true path Λ is in general recursive in $\varnothing''$, and is shown to exist in any model of $I\Sigma_2$. Now each true Σ_3 outcome, which is $\Sigma_1(\Lambda)$, corresponds to a Friedberg–Muchnik type injury relativized to Λ. Hence to prove the density of branching degrees, one only needs $I\Sigma_1(\Lambda)$, which is no worse than $I\Sigma_1$ relative to $\varnothing''$. Thus, $P^- + I\Sigma_3$ proves Theorem 3.2 (3).

3.1.2. *Dynamic blocking.* Even for finite injury (i.e., $\mathbf{0}'$-priority), there are constructions that do not conform to the Friedberg–Muchnik condition of an effective bound on the number of times a requirement is injured. Such constructions do not automatically generalize to all models of $I\Sigma_1$. An instructive example is the Sacks Splitting Theorem, whose characteristic feature in the original proof is the absence of an *a priori* Friedberg–Muchnik type recursive bound. The main strategy in Sacks' proof of splitting a nonrecursive r.e. set A is to preserve every computation, and hence length of agreement, due to the requirement of highest priority. The idea is that if the requirement fails to be satisfied, then the length of agreement and computations being preserved will be unbounded and that forces A to be recursive. If one mimics the construction in an arbitrary model $\mathcal{M}$ of $I\Sigma_1$, then there is no assurance that each requirement is satisfied. Indeed it is possible that the set of requirements that are satisfied constitute a (proper) Σ_2 cut in $\mathcal{M}$.

The proof of the Sacks Splitting Theorem for models of $I\Sigma_1$, due to Mytilinaios [47], exploits a technique called *blocking* in α-recursion theory introduced by Shore [53]. Roughly speaking, blocking groups the set of

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requirements into a "short sequence" of blocks of requirements such that requirements in the same block are accorded the same priority. Such a short sequence enables a modified priority construction to be implemented in a Σ_1 admissible ordinal that would otherwise require Σ_2 admissibility. The method is adapted to the arithmetic setting, so that a model of $I\Sigma_1$ now performs the work of models of $I\Sigma_2$.

Here is a brief description of blocking with details glossed over. Suppose we wish to split A into the disjoint union of A_0 and A_1. First fix $e_0 > 0$ and consider the requirements R_e, $e \leq e_0$ and of the type $\Phi_e^{A_0} \neq A_1$, as one block B_0 all given the same (highest) priority. In splitting the nonrecursive r.e. set A, every computation and length of agreement due to a requirement in B_0 is preserved, as we do not allow a requirement in B_0 to injure another requirement in the block. The length of agreement for B_0 is the sum of the lengths of agreement for requirements in the block. Collectively, if the size of preserved computations in B_0 "goes to infinity" over time, one argues that A is recursive and arrives at a contradiction. Hence there is a least bound e_1 (which exists by $I\Sigma_1$) on the length of agreement for B_0. Now the requirements in $[e_0+1, e_1]$ of the type $\Phi_e^{A_1} \neq A_0$ form the next block B_1 and a similar argument shows that there is a bound e_2 on the length of agreement for B_1. The construction continues in this way by induction.

In an actual implementation, the number e_1 is obtained in the limit and therefore not recursively computable. Hence the size of the block B_1 is dynamic and is a function of the construction itself (it is in fact $\Sigma_2(\mathcal{M})$ definable). As a consequence, $\{d : B_d \text{ is defined}\}$ is a Σ_2 cut in the model $\mathcal{M}$ of $I\Sigma_1$. It is worth noting that the dynamic aspect of blocking, where the size of each block is determined by the construction, is necessary and unique to nonstandard models (dynamic blocking has also been applied to study $B\Sigma_2$ models). It ensures that each requirement will belong to some B_d, even if the map $d \mapsto B_d$ is not total. Such a dynamic approach to forming blocks is not present in the case of admissible ordinals, where the Σ_2 cofinality function for the ordinal essentially predetermines the size of each block.

3.1.3. *Implicit inductive strength.* Priority constructions are not the only venue where Σ_n induction is invoked. Two "priority-free" theorems about r.e. sets due to Friedberg [24] use $I\Sigma_2$ in the proof and, as it turns out, fail in its absence.

THEOREM 3.3. *Over* P^-, $I\Sigma_2$ *proves*:

(1) *There exists a maximal r.e. set.*
(2) *There is a Friedberg numbering, i.e., an effective enumeration of all r.e. sets without repetition.*

Suppose $\{W_e\}_{e\in M}$ is a standard enumeration of all r.e. sets. Construction of a maximal set typically applies the e-state method: Given n, the *e-state of n at stage s* is $\sum_{e'\leq e, n\in W_{e',s}} 2^{e-e'}$. In the standard model, one proves the existence of a maximal e-state i_e which is the eventual e-state of infinitely many n's. Now i_e is the maximal element of the following Π_2 definable

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bounded set:

$$S_e = \{i < 2^e : \forall m\, \exists n > m\, \exists s\, (i \leq e\text{-state of } n \text{ at stage } s)\}.$$

$I\Sigma_2$ guarantees that S_e has a maximal element and hence maximal sets exist in models of $P^- + I\Sigma_2$.

An analogous situation applies in the construction of a Friedberg numbering, which as shown in Kummer [35], can be priority free. The proof exploits the fact that given e and any effective list of r.e. sets $\{A_e\}$,

$$\{i : i \leq e \ \wedge A_i = A_e\},$$

has a least element. This is an instance of $L\Sigma_2$ which as noted earlier is equivalent to Σ_2 induction.

3.2. Recursion theory of $B\Sigma_n$ models. The discussion in Section 3.1 raises the natural question of *necessity*: whether $\mathbf{0^{(n)}}$-priority constructions can be implemented without $I\Sigma_n$. In fact, one could even ask whether the theorems whose proofs historically relied on the priority method could be established in its absence and with weaker induction hypothesis. More specifically, can the conclusions in Theorem 3.2 and Theorem 3.3 be proved, without the use of the priority method and in systems weaker than those stated? It turns out that in many cases the answers are negative. We discuss examples of theorems which are provably equivalent to $I\Sigma_n$ over the base theory $B\Sigma_n$ for some n. These results say that the failure of a priority construction in a model entails the failure of the theorem itself, so that the use of $I\Sigma_n$ in the construction was crucial.

We take as our base theory $P^- + B\Sigma_n$ and examine recursion-theoretic questions in $B\Sigma_n$ models, with particular focus on $n = 1, 2$, and 3.

3.2.1. *Cuts and Codes.* In Section 2 it was observed that every $B\Sigma_n$ model has a Σ_n cut. A Σ_n cut is a canonical example of a nonregular Σ_n set. More than just a curiosity, Σ_n cuts carry with them useful degree-theoretic and coding information that has proved to be important to the construction of sets. Here is a summary of the key properties of a Σ_n cut:

THEOREM 3.4 ([9, 42, 46, 61]). *Suppose I is a Σ_n cut in a $B\Sigma_n$ model. Then*

(1) *All Σ_n cuts have the same Turing degree*;
(2) *I is neither recursive nor Σ_n-complete, hence a solution of Post's problem for $n = 1$*;
(3) *I and $\varnothing^{(n-1)}$ form a minimal pair. If $n = 1$, then I is of minimal Σ_n degree, and therefore a counterexample to Sacks' Splitting Theorem*;
(4) *Every bounded set recursive in $\varnothing^{(n)}$ is recursive in I.*

The proof of Theorem 3.4 makes extensive use of special features of a Σ_n cut including its coding property. To give the reader a taste of how one may exploit these, we describe the proof of Theorem 3.4 (2) and (3) when $n = 1$. This is one place in the paper where we delve into technical details.

We first give a short proof that a Σ_1 cut I is a nonrecursive incomplete r.e. set, and then show that it is in fact a set of minimal r.e. degree.

Clearly I is r.e. and nonrecursive. To construct an r.e. set A such that $A \not\leq_T I$, we note two points:

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(i) Since I is downward closed, a computation using I as an oracle may be simplified: A positive condition is contained in I if and only if its maximal element is in I, and a negative condition is contained in $\overline{I}$ if and only if its minimal element is in $\overline{I}$.

(ii) Let $f : I \to M$ be a Σ_1 cofinal function. This function enables one to "compress" space and time for the purpose of construction. In particular a construction is completed in I-many stages if at stage $i \in I$ one executes $f(i)$-many steps of computation.

Suppose we wish to construct an r.e. set A so that for each e, the eth requirement R_e: $\Phi_e^I \neq A$ is satisfied. Fix a to be an upper bound of I. For R_e, we assign the space $[\langle e, 0\rangle, \langle e, a\rangle]$ in which to perform the required diagonalization. At stage i, we enumerate triples $(\langle e, x\rangle, p, n) < f(i)$ such that $0 \leq x \leq a$, $p \in I$, n is the largest number that appears to be in $\overline{I}$, and $(\langle e, x\rangle, 0, \{p\}, \{n\}) \in \Phi_{e,f(i)}$. For the largest n where such a triple is enumerated, choose a corresponding $\langle e, x\rangle$ that is not yet in A and enumerate it into A. Since $[\langle e, 0\rangle, \langle e, a\rangle]$ has size $a+1$, there is an $\langle e, x\rangle$ that never enters A. If $\Phi_e^I = A$, then $\Phi_e^I(\langle e, x\rangle) = 0$ and this fact is enumerated at some stage i with correct positive and negative conditions. But then by the construction, at stage i some $\langle e, y\rangle$ with a negative condition at least as large as that for $\langle e, x\rangle$ is enumerated into A. Thus, $\Phi_e^I(\langle e, y\rangle) = 0 \neq A(\langle e, y\rangle)$.

Imbedded in a Σ_n cut is an abundance of codes which may be uncovered using the Coding Lemma (2.7). Theorem 3.4 (3) is proved applying this idea. We discuss how this is done for $n = 1$, following the notations used above:

Assume that $\Phi^I = A$. If A is not regular, we claim that A computes I. Let $i \in I$ be fixed so that $A \upharpoonright f(i)$ is not regular. Then $G = \{(j, x) : x < f(i) \wedge x \in A_{f(j)} \setminus A_{f(j-1)}\}$ is Δ_1 on $I \times [0, f(i)]$ and hence coded on $I \times [0, f(i)]$ by a $\hat{G}$ according to Lemma 2.7. Then $j \notin I$ if and only if there is a $j' \leq j$ and $x \notin A$ such that $(j', x) \in \hat{G}$. Hence $I \leq_T A$.

On the other hand, if A is regular then we claim that A either computes I or is recursive. Regularity implies that $A \upharpoonright f(i)$ is $\mathcal{M}$-finite for every $i \in I$. By $B\Sigma_1$, $A \upharpoonright f(i) = A_{f(k)} \upharpoonright f(i)$ for some k. Let $(i, j, k) \in G$ if and only if $A_{f(j)} \upharpoonright f(i) = A_{f(k)} \upharpoonright f(i)$. Since G is Δ_1 on I^3, by the Coding Lemma it is coded, say by $\hat{G}$. Suppose $A_{f(j)} \upharpoonright f(i) = A \upharpoonright f(i)$. Then for all $k > j$ in I, $(i, j, k) \in G \subset \hat{G}$. Since I has no maximal element in M, by overspill $k_{i,j} = \max\{k > j : (i, j, k) \in \hat{G}\} \in \overline{I}$. The set $J = \{k_{i,j} : i < j \in I \wedge A_{f(j)} \upharpoonright f(i) = A \upharpoonright f(i)\}$ is r.e. in A. There are two possibilities to consider: The first is that J is cofinal *downwards* in $\overline{I}$ and that implies that $\overline{I}$ is r.e. in A as well and so $I \leq_T A$. The other possibility is that there is an $a \in \overline{I}$ which is a lower bound of J. Then for any $i < j \in I$,

$$A_{f(j)} \upharpoonright f(i) = A \upharpoonright f(i) \leftrightarrow \exists k > a \, \forall k' \in [j, k] \, (i, j, k') \in \hat{G},$$

which implies that A is recursive.

Note that in the above argument, essential use was made of the recursive enumerability of A to enable the coding of G. This is not always possible if A has a higher complexity such as Δ_2. However, if $\mathcal{M}$ is saturated then the

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existence of codes paints a vastly different picture for the structure of Turing degrees, even for those below $\mathbf{0}'$:

THEOREM 3.5 ([8, 42]). *Suppose $\mathcal{M}$ is a saturated $B\Sigma_1$ model and I is a Σ_1 cut in $\mathcal{M}$. Then all degrees below $\mathbf{0}'$ are r.e. Furthermore, I is an r.e. set with a minimal Turing degree.*

This leads to the first two questions that we pose in this paper:

QUESTION 1. *If $\mathcal{M}$ is a $B\Sigma_1$ model, is the Σ_1 cut I necessarily of minimal degree*?

QUESTION 2. *Is there a $B\Sigma_1$ model in which a non-r.e. degree exists below $\mathbf{0}'$*?

Together with Theorem 3.4 (3), Theorem 3.6 implies, among other things, that the existence of a low r.e. set is not provable in $P^- + B\Sigma_1$:

THEOREM 3.6 ([15, 17]). *Let $n \geq 1$ and let $\mathcal{M}$ be a $B\Sigma_n$ model with Σ_n cut I. Then the jump of I is not recursive in $\varnothing^{(n)}$.*

Returning to the origin of the priority method, we remark that the Friedberg–Muchnik Theorem continues to hold even when $\mathbf{0}'$-priority constructions break down. The proof makes use of nonregular sets in the framework of "union of cuts" where diagonalization takes place to produce r.e. sets of incomparable degrees:

THEOREM 3.7 ([8]). *$P^- + B\Sigma_1$ proves the Friedberg–Muchnik Theorem.*

By Theorem 3.4 (3), any pair of incomparable r.e. sets has to consist of nonregular sets whose Turing degrees lie strictly above that of I. Another example where the original priority construction (Cooper [19]) completely fails but the theorem remains true concerns the existence of a proper d-r.e. degree (a degree that contains a set that is the difference of two r.e. sets, but contains no r.e. set) stated below, whose proof exploits the coding power of a Σ_1 cut:

THEOREM 3.8 (Li [42]). *$P^- + B\Sigma_1$ proves there is a proper d-r.e. degree.*

REMARK 3.9. There is an obvious nonuniformity in the proofs of Theorem 3.7 and Theorem 3.8: If $\mathcal{M}$ is a model of $P^- + I\Sigma_1$, then the $\mathbf{0}'$-priority method is applicable. On the other hand, if $I\Sigma_1$ fails, then one works with nonregular sets using completely different, nonpriority constructions to establish the results. In fact, in the case of Theorem 3.8, one has the rather unexpected situation that every d-r.e. degree below $\mathbf{0}'$ is r.e., and hence the proper d-r.e. degree constructed in Li [42] does not lie below $\mathbf{0}'$.

QUESTION 3.

(1) *Is there a uniform proof of the Friedberg Muchnik Theorem in $P^- + B\Sigma_1$*?
(2) *Is there a uniform proof of the existence of a proper d-r.e. degree in $P^- + B\Sigma_1$*?

We end our discussion of $B\Sigma_1$ models with the observation that Theorem 3.6 implies that the existence of a low r.e. set is equivalent over $P^- + B\Sigma_1$ to Σ_1 induction. This is somewhat surprising since in the classical proof, the construction of a low r.e. set uses a finite injury method that is not significantly different from that of an incomparable pair of r.e. sets for the Friedberg–Muchnik Theorem or a proper d-r.e. set, and yet Theorems 3.7

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and 3.8 imply that the low set theorem and the two latter theorems have different proof-theoretic strengths.

We now shift our attention to the subsystem $P^- + B\Sigma_2$ and consider constructions of r.e. sets that typically involve the use of $I\Sigma_2$. These include questions such as the existence of a maximal r.e. set, the existence of a Friedberg numbering (see Theorem 3.3), the existence of an incomplete high r.e. set, the Sacks Density Theorem, and the Sacks Jump Inversion Theorem. It turns out that except for the Density Theorem, all of these are equivalent to $I\Sigma_2$ over the base theory $P^- + B\Sigma_2$. As will be seen, an underlying thread that links these negative results is the coding power of a Σ_2 cut. The discussion begins with another application of coding concerning maximal r.e. sets. Coding will be discussed in conjunction with the notion of hyperregularity in the Subsection 3.2.2.

THEOREM 3.10. *Let $\mathcal{M}$ be a $B\Sigma_2$ model and A be an r.e. set whose complement is not $\mathcal{M}$-finite. Then there exists an r.e. set W such that both $\overline{A} \cap W$ and $\overline{A} \setminus W$ are unbounded. Hence A is not maximal.*

PROOF. (Sketch) Since the complement of A is a Π_1 set, we have a Δ_2 increasing enumeration h of the elements of $\overline{A}$. We consider two cases, depending on whether the domain of h is a cut or all of M, which informally says whether the order type of $\overline{A}$ is short or long.

Case 1. The domain of h is a Σ_2-cut I. Let a be an upper bound of I. Recursively enumerate $\mathcal{M}$ into a-many pairwise disjoint sets $\{S_e : e < a\}$ so that each S_e contains at most one element in $\overline{A}$. This can be done since the order type of $\overline{A}$ is "less than" a (see Chong and Yang [16] for details). Consider the set:

$$G = \{(i, e) : i \in I \wedge e < a \wedge h(i) \in S_e\}.$$

Observe that G is Δ_2 on $I \times [0, a)$. By the Coding Lemma (2.7), there is an $\mathcal{M}$-finite set $\hat{G}$ such that $\hat{G} \cap I \times [0, a) = G$. Let $W = \bigcup\{S_e : \exists k(2k, e) \in \hat{G}\}$. W is r.e. since $\hat{G}$ is a parameter. Clearly, $W \cap \overline{A} \supseteq \{h(2k) : 2k \in I\}$ and $\overline{A} \setminus W \supseteq \{h(2k+1) : 2k+1 \in I\}$ are both unbounded.

Case 2. The domain of h is M. The idea is to define an r.e. set $A^* \supset A$ with complement of order type I, and then apply Case 1.

⊣

REMARK 3.11. The technique to show that maximal r.e. sets do not exist in some models of computation goes back to Lerman and Simpson [39] who proved that there is no maximal $\aleph_\omega^L$-r.e. set. In their proof, the existence of codes for definable subsets of ω was used to recursively split any unbounded Π_1 set.

3.2.2. *Hyperregularity in the absence of Σ_2 Induction.* The notion of hyperregularity also has its origin in α-recurion theory. It was adapted by Mytilinaios and Slaman in [48] to study infinite injury priority constructions in $B\Sigma_2$ models, especially in relation to the existence of nonelow r.e. sets.

DEFINITION 3.12. Let $\mathcal{M}$ be a model of $P^- + I\Sigma_0 + \exp$. A set $A \subseteq M$ is *hyperregular* if every partial A-recursive function maps a bounded set onto a bounded set.

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In a model of $P^- + I\Sigma_1$, every recursive set is hyperregular. This is false for a $B\Sigma_1$ model. A hyperregular set may be characterized in several ways, but they all point to the limited computational power of the set:

LEMMA 3.13 ([48]). *Let $\mathcal{M}$ be a model of $P^- + I\Sigma_0$+Exp. Let $A \subseteq M$ be regular. Then the following are equivalent.*

(1) *A is hyperregular.*
(2) *$I\Sigma_1$ relative to A holds.*
(3) *For every e, W_e^A is regular.*

Shore [54] first made the connection between a hyperregular α-r.e. set and its jump. He showed, for example, that every incomplete $\aleph_\omega^L$-r.e. set is low. In the arithmetic setting, this was first seen through the lens of a saturated model:

THEOREM 3.14 (Mytilinaios and Slaman [48]). *Let $\mathcal{M}$ be an arithmetically saturated $B\Sigma_2$ model with ω as Σ_2 cut. If A is r.e. and hyperregular in $\mathcal{M}$, then A is low.*

The key idea of the proof is as follows. Let f be an increasing Σ_2 cofinal function with domain ω. Given $i \in \omega$, the set $A' \cap [0, f(i)]$ is $\Sigma_1(A)$ and its enumeration is completed by stage $f(j)$ for some $j \in \omega$ according to the hyperregularity of A. Let $j(i)$ be the least such j. Then by arithmetical saturation, $G = \{(i, j(i) : i \in \omega\}$ is coded by an $\mathcal{M}$-finite set $\hat{G}$. Then $\varnothing'$ may use $\hat{G}$ to compute A', so that A is low.

At the other end of the scale are the nonhyperregular r.e. sets. Far from being low, they all occupy the highest r.e. degree (Chong and Yang [14], Groszek, Mytilinaios, and Slaman [25]):

THEOREM 3.15. *Let $\mathcal{M}$ be a $B\Sigma_2$ model. Then every nonhyperregular r.e. set in $\mathcal{M}$ is complete.*

The proof is yet another application of the Coding Lemma (2.7). Suppose A is nonhyperregular and r.e. Let h be a partial A-recursive function from a bounded set with unbounded range. By redefining h if necessary, we may assume that the domain of h is a cut I (which is $\Sigma_1(A)$). For each $i \in I$, the effective enumeration of $\varnothing' \upharpoonright h(i)$ is completed by stage $h(j)$ for some $j \in I$. Then $G = \{(i, j) : \varnothing' \upharpoonright h(i) = \varnothing'_{h(j)} \upharpoonright h(i)\}$ is Δ_2 on $I \times I$ and hence coded by an $\mathcal{M}$-finite set $\hat{G}$. This gives A an algorithm to compute $\varnothing'$: For each $i \in I$, let $j(i)$ be the least j such that $(i, j) \in \hat{G}$. Then $\varnothing' \upharpoonright h(i) = \varnothing'_{h(j(i))} \upharpoonright h(i)$.

Theorems 3.14 and 3.15 together say that in an arithmetically saturated $B\Sigma_2$ model, the following dichotomy holds for an r.e. set A: Either A is nonhyperregular and hence complete, or A is hyperregular, hence $I\Sigma_1^A$ holds and A is low. In fact, the proof of Theorem 3.14 can be easily modified to show that hyperregular Δ_2 sets are low. In an arithmetically saturated $B\Sigma_2$ model, this means that every incomplete Δ_2 set is low. The abundance of low sets is exploited to separate Ramsey's Theoren for Pairs from its stable counterpart. We will elaborate on how this is done in Section 4.

The conclusion of Theorem 3.14 relies heavily on the use of codes for definable subsets of a Σ_2 cut. Such codes may not be available for arbitrary $B\Sigma_2$ models. In particular, not every hyperregular r.e. set is automatically low.

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By a finer analysis of hyperregularity, Chong and Yang [14] characterized the possible jump of an r.e. set in an arbitrary $B\Sigma_2$ model, in terms of a "Three-point Theorem":

THEOREM 3.16. *Let $\mathcal{M}$ be a $B\Sigma_2$ model. The jump of an r.e. set is Turing equivalent to either $\varnothing'$, $\varnothing''$ or $I \oplus \varnothing'$, where I is a Σ_2 cut in $\mathcal{M}$. Moreover, no incomplete r.e. set is high.*

The above theorem together with Theorem 3.7 relativized to the degrees above the degree of $I \oplus \varnothing'$ gives us the strength of Sacks' Jump Inversion Theorem:

COROLLARY 3.17. *Over the base theory $P^- + B\Sigma_2$, the Sacks Jump Inversion Theorem is equivalent to Σ_2 induction.*

By Theorem 3.15 every incomplete r.e. set is hyperregular. This is a key ingredient in the proof of the following theorem due to Groszek, Mytilinaios, and Slaman [25], where they exploit the fact that given r.e. sets $A <_T B$ in a $B\Sigma_2$ model, A is hyperregular and therefore $\Sigma_1(A)$ induction holds. This allows a $\mathbf{0''}$-priority construction to succeed:

THEOREM 3.18. *$P^- + B\Sigma_2$ proves the Sacks' Density Theorem.*

REMARK 3.19. One can show that the Sacks Density Theorem holds in the projection model. Since such a model satisfies $I\Sigma_1$ but not $B\Sigma_2$, we conclude that this theorem is not equivalent to $B\Sigma_2$ over the base system $P^- + I\Sigma_1$. Indeed, every theorem discussed in this paper proved using a $\mathbf{0''}$-priority construction holds in the projection model. It is still open whether the Sacks' Density Theorem holds in every $I\Sigma_1$ model.

3.3. Promptness. The next three results all pertain to the notion of *promptness* which is a property about dynamic enumeration of r.e. sets introduced by Maass [44]. One of its most striking applications was the characterization due to Ambos-Spies, Jockusch, Shore, and Soare [1] of prompt r.e. sets as precisely those that are not half of a minimal pair, an equivalence which continues to hold in any $B\Sigma_2$ model.

DEFINITION 3.20. Assume $I\Sigma_1$. An r.e. set A is *prompt* if there is a total recursive function h such that for all infinite r.e. set W,

$$\exists s\, \exists x\, (x \text{ is enumerated into } W \text{ at stage } s \wedge A_s \upharpoonright x \neq A_{h(s)} \upharpoonright x).$$

A rather unexpected phenomenon about promptness is its omnipresence in the nonrecursive r.e. sets of any $B\Sigma_2$ model (Chong, Qian, Slaman, and Yang [6]):

THEOREM 3.21. *Let $\mathcal{M}$ be a $B\Sigma_2$ model. Then any nonrecursive r.e. set is prompt. Hence there is no minimal pair of r.e. sets in $\mathcal{M}$.*

The main idea of proving Theorem 3.21 is as follows. Let $\mathcal{M}$ be a $B\Sigma_2$ model with Σ_2 cut I. Let a be an upper bound of I. Given a nonrecursive r.e. set A, construct a-many (partial) recursive functions $\{g_n : n < a\}$. The intention is to make each g_n witness the promptness of A. For a given r.e. set W, when an element x enters W say at stage s, we delay the definition of $g_n(s)$ until we see a change in $A \upharpoonright x$. If W offers us infinitely many chances

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and A is not recursive, then we will see such a change in A and g_n will witness the promptness of A for W. Of course W may be finite, in which case g_n is partial. The lack of Σ_2 induction is exploited to make this strategy succeed.

In addition to not being half of a minimal pair, prompt r.e. sets are also low cuppable in the standard model: An r.e. degree $\boldsymbol{a}$ is *cuppable* if there is an r.e. degree $\boldsymbol{b}$ such that $\boldsymbol{a} \vee \boldsymbol{b} = \mathbf{0}'$. It is low cuppable if $\boldsymbol{b}$ is low. In [1] it was proved that any prompt r.e. set is low cuppable. The argument may be adapted to any model of $P^- + B\Sigma_2$. Thus, by Theorem 3.21,

COROLLARY 3.22. *Over the base theory* $P^- + B\Sigma_2$, *the following are equivalent to* $I\Sigma_2$:

(1) *The existence of a minimal pair of r.e. degrees*;
(2) *The existence of a noncuppable r.e. degree.*

3.4. Σ_3 **induction.** The existence of a minimal pair says that $\mathbf{0}$ is a branching degree, namely the infimum of two incomparable degrees. Given an r.e. set A, by relativising the proof of Theorem 3.21 to A, we conclude that $\deg(A)$ is branching implies $I\Sigma_2(A)$. In particular when A is high, $I\Sigma_2(A)$ is equivalent to $I\Sigma_3$. Consequently the density of branching degrees is provably equivalent to $I\Sigma_3$: The sufficiency of $I\Sigma_3$ (Theorem 3.2 (3)) follows from an adaptation of the original $\mathbf{0}'''$-priority argument in [59]. The other direction can be shown as follows: If $\mathcal{M}$ is a $B\Sigma_3$ model in which the branching degrees are dense, apply $I\Sigma_2$ to fix two high degrees $\boldsymbol{h}_1 < \boldsymbol{h}_2$. Then there is a high branching degree $\boldsymbol{h}$ in $\mathcal{M}$ such that $\boldsymbol{h}_1 < \boldsymbol{h} < \boldsymbol{h}_2$. By the remark above, $\mathcal{M} \vDash I\Sigma_3$. Thus

THEOREM 3.23. *Over the base theory* $P^- + B\Sigma_3$, *the density of branching degrees is equivalent to* $I\Sigma_3$.

As far as we know, the density of branching degrees is the only known "natural" example of a theorem about r.e. degrees equivalent to $I\Sigma_n$ for $n > 2$. By "natural" we mean a theorem that is not part of a family of results obtained by relativizing a theorem provably equivalent to $I\Sigma_2$ over $B\Sigma_2$, such as the existence of high$_n$ or low$_{n+1}$ r.e. degrees for $n > 1$. One suspects that there exist other "textbook examples" concerning r.e. degrees that demonstrate a proof-theoretic strength beyond Σ_2 induction.

QUESTION 4. *Is Lachlan's nonsplitting theorem* [36] *or nonbounding theorem* [37] *equivalent to* $I\Sigma_3$ *over the base theory* $P^- + B\Sigma_3$? *The same question applies to the existence of a Slaman triple* [55].

A problem related to the above discussion concerns base theory: Taking $P^- + I\Sigma_k$ as the base theory, for each $n > k$, is there a theorem about r.e. degrees provably equivalent to $I\Sigma_n$? A partial answer is provided in Chong, Shore, and Yang [10]:

THEOREM 3.24. *Over the base theory* $P^- + I\Sigma_4$, *for each* $n > 4$ *there is a sentence* φ_n *about r.e. degrees equivalent to* $I\Sigma_n$.

3.4.1. *Minimal degrees.* Sacks' construction of a minimal degree below $\mathbf{0}'$ may be carried out in models of $P^- + I\Sigma_2$. However, the same construction

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encounters insurmountable hurdles in $B\Sigma_2$ models. The situation is very similar to that in α recursion theory for Σ_2 inadmissible cardinals such as $\aleph_\omega^L$. Rather unexpectedly, every Σ_2 cut in a saturated $B\Sigma_2$ model is a set of minimal degree. The same is true of a saturated $B\Sigma_1$ model. In this case a Σ_1 cut, which is r.e., is a set of minimal degree. Hence over the base theory $P^- + B\Sigma_2$, the existence of a minimal degree does not imply $I\Sigma_2$. Nevertheless, since every set in a $B\Sigma_2$ model that is recursive in $\varnothing'$ is regular, the following question makes sense and seems to be the right question to ask:

QUESTION 5. *Over the base theory $P^- + B\Sigma_2$, does the existence of a minimal degree below $\mathbf{0}'$ imply $I\Sigma_2$?*

3.5. Summary of results. We compile a list of what is known about sets and Turing degrees, in the order of increasing inductive strength.

- $P^- + I\Sigma_0 + B\Sigma_1$+Exp implies the Friedberg–Muchnik Theorem.
- Over the base theory $P^- + I\Sigma_0 + B\Sigma_1$+Exp, the following are equivalent [17, 42, 46, 47]:
  (1) $I\Sigma_1$;
  (2) The existence of a low nonrecursive r.e. degree;
  (3) The existence of a proper d-r.e. degree below $\mathbf{0}'$;
  (4) The Sacks Splitting Theorem;
  (5) The Low Basis Theorem.
- $P^- + B\Sigma_2$ implies the Sacks Density Theorem [25].
- Over the base theory $P^- + B\Sigma_2$, the following are equivalent [6, 14, 16, 16, 41, 62]:
  (1) $I\Sigma_2$;
  (2) The existence of a high incomplete r.e. degree;
  (3) The existence of a low_2 and nonlow r.e. degree;
  (4) The Sacks Jump inversion Theorem;
  (5) The existence of an r.e. minimal pair;
  (6) The existence of a maximal r.e. set;
  (7) The existence of a Friedberg numbering;
  (8) The existence of a noncuppable r.e. degree.
- Over the base theory $P^- + B\Sigma_3$, the following are equivalent:
  (1) $I\Sigma_3$;
  (2) The density of branching r.e. degrees.

As can be seen, most of the equivalences are with $I\Sigma_n$ over the base theory $P^- + B\Sigma_n$. Little is known about equivalence with $B\Sigma_{n+1}$ over the base theory $P^- + I\Sigma_n$.

QUESTION 6. *Is there a theorem in recursion theory which is equivalent to $B\Sigma_2$ over the base theory $P^- + I\Sigma_1$?*

§**4. Second Order Arithmetic and Ramsey's Theorem for pairs.** We now turn to applications of nonstandard models to reverse mathematics. The language of second order arithmetic expands that of first order by adding set variables $X, Y, Z, \ldots$ and parameters that denote subsets of the underlying universe of a structure. We allow a first order formula to include set

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parameters and variables, and use Σ^0_n and Π^0_n for formulas which in the previous sections were denoted Σ_n and Π_n, respectively (recall that only first order objects were involved previously). RCA_0 is the system consisting of $P^- + I\Sigma^0_1$, and the Δ^0_1-comprehension scheme:

$$(\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where φ is Σ^0_1 and ψ is Π^0_1. RCA_0 is the base theory for the study of reverse mathematics. Although there is a well-defined hierarchy of systems of increasing proof-theoretic strength in the literature (known as the *big five* systems), our interest in this section is at a relatively "low" level, below ACA_0 which is the system extending RCA_0 that incorporates the arithmetical comprehension scheme. Despite such a restriction, it is sufficiently strong for the study of a rich collection of problems concerning combinatorial principles. Consistent with the general theme of this paper, we devote our attention almost exclusively to nonstandard models of RCA_0.

A model $\mathcal{M}$ of RCA_0 is a structure of the form $\langle M, \mathbb{X}, 0, S, +, \times, < \rangle$ where $\mathbb{X} \subseteq 2^M$. Since $\mathcal{M} \models P^-$, ω is an initial segment of M. We say that $\mathcal{M}$ is an ω-model if $M = \omega$. The models that are of key interest to us are non-ω-models that satisfy $B\Sigma^0_2$ but not $I\Sigma^0_2$. We commence the discussion by recalling the theorem that is the subject of this section.

THEOREM 4.1 (Ramsey's Theorem for Pairs (RT^2_2)). *If h is a function mapping (unordered) pairs of numbers into $\{0, 1\}$, then there is an infinite set H and an $i \in \{0, 1\}$ such that $h(a, b) = i$ for any $a, b \in H$.*

Thus every two-coloring of pairs of numbers has an infinite monochromatic subset. We say that H is *homogeneous* for the two-coloring h. We will consider RT^2_2 and related principles, including those that it implies. Collectively we refer to these as "Ramsey-type" principles. Taking RCA_0 as the base theory, we will look at the role played by induction scheme in relation to the proof-theoretic strength of Ramsey-type principles. This statement deserves elaboration and is best explained by listing a number of basic questions: First, given a Ramsey-type principle P, does $\mathsf{RCA}_0 + \mathrm{P}$ prove an induction scheme stronger than $I\Sigma^0_1$, a scheme that is already guaranteed by RCA_0? Second, if P_1 and P_2 are two Ramsey-type principles, does $\mathsf{RCA}_0 + \mathrm{P}_1$ prove more first order theorems than $\mathsf{RCA}_0 + \mathrm{P}_2$ if P_2 follows from $\mathsf{RCA}_0 + \mathrm{P}_1$? Third, under the same hypothesis that P_2 follows from $\mathsf{RCA}_0 + \mathrm{P}_1$, is the implication strict, i.e., does P_1 follow from $\mathsf{RCA}_0 + \mathrm{P}_2$? Finally, if the answer is negative, would a stronger induction scheme accomplish the job? In what follows we discuss these questions and the techniques of nonstandard models that have been introduced to study them. We begin with the inductive strength of combinatorial principles and "work our way up" the rest of the questions. From now on, all models considered are models of RCA_0.

4.1. Σ^0_2-bounding. The first key fact to note is the following proposition:

PROPOSITION 4.2. *$P^- + I\Sigma^0_2$ proves "Every recursive two-coloring of pairs of numbers has an infinite homogeneous set".*

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Note that the existence of a homogeneous set should be viewed "externally": it is not claimed that the homogeneous set belongs to a given model of $P^- + I\Sigma^0_2$.

PROOF. Theorem 4.2 of Jockusch [30] states that every recursive two-coloring of pairs of numbers has an infinite Π^0_2-homogeneous set. A close examination shows that the construction of the homogeneous set may be carried out within the system $P^- + I\Sigma^0_2$. ⊣

DEFINITION 4.3. Given models $\mathcal{M}_i = \langle M_i, \mathbb{X}_i, S, +, \times, 0\rangle$ where $i = 0, 1$, we say that $\mathcal{M}_1$ is an M_0-*extension* of $\mathcal{M}_0$ if $M_0 = M_1$ and $\mathbb{X}_0 \subseteq \mathbb{X}_1$.

Thus an M-extension of $\mathcal{M} = \langle M, \mathbb{X}, S, +, \times, 0\rangle$ is an extension which preserves the first order universe M but may expand its second order collection $\mathbb{X}$. With extra effort, one obtains a reverse mathematics version of Proposition 4.2:

THEOREM 4.4 (Theorem 10.1 of [3]). *Every model $\mathcal{M}$ of $\mathsf{RCA}_0 + I\Sigma^0_2$ has an M-extension that satisfies in addition RT^2_2.*

Chubb, Hirst, and McNicholl [18] defined a Ramsey type coloring principle on binary trees denoted TT^1: Every finite coloring of the full binary tree has a monochromatic subtree isomorphic to the full binary tree. One notes immediately that Proposition 4.2 and Theorem 4.4 remain valid when RT^2_2 is replaced by TT^1 in the respective statements. There is a natural question on the lower bound of the inductive strength for these principles. Hirst [28] showed that for RT^2_2 the inductive power is inherently stronger than $I\Sigma^0_1$:

PROPOSITION 4.5. $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ *implies* $B\Sigma^0_2$.

PROOF. Let $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{RT}^2_2$. Suppose on the contrary that $B\Sigma^0_2$ fails in $\mathcal{M}$. Let $f : [0, a] \to M$ be $\Sigma^0_2(\mathcal{M})$-definable with unbounded range. Thus there is a recursive f' such that $\lim_s f'(s, x) = f(x)$ for each $x \in [0, a]$. Call $\{f'(s, x)|s \in M\}$ the xth-column. We can arrange f' so that $\{f'(s, x)|s \in M\} \cap \{f'(s, y)|s \in M\} = \varnothing$ for $x \neq y$, and $\bigcup_{x\in[0,a]}\{f'(s, x)|s \in M\} = M$. Define a two-coloring h such that for $u \neq v$, $h(u, v) = 0$ if u and v are on the same column and $h(u, v) = 1$ otherwise. Then there is no infinite h-homogeneous set since each column is finite and there are only finitely many columns. ⊣

The coloring h is in fact *stable* in the following sense: For each u, $\lim_v h(u, v)$ exists (and is equal to 1). Hence if we define Stable Ramsey's Theorem for Pairs (SRT^2_2) as: Every stable two colouring has an infinite homogeneous set, then $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ implies $B\Sigma^0_2$ as well. This was pointed out in [3].

One may replace RT^2_2 in Proposition 4.5 with TT^1 (see [18]): Define an a-coloring h on the nodes σ of $2^{<M}$ level by level so that $h(\sigma) = x$ if $|\sigma|$ is on the xth column. Then each color appears finitely often on $2^{<M}$ and hence the full binary tree has no monochromatic subtree isomorphic to it. Many, perhaps most, of the combinatorial principles that have been studied imply $B\Sigma^0_2$ (there are exceptions such as Weak König's Lemma WKL_0 to be

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defined later). Some are consequences of RT^2_2 and known to be weaker than RT^2_2 (see Hirschfeldt and Shore [27]). We list three here:

DEFINITION 4.6.

(i) Chain/antichain (CAC): Every infinite partially ordered set has an infinite chain or an infinite antichain.
(ii) Ascending/descending sequence (ADS): Every infinite linearly ordered set has an infinite ascending or descending subsequence.
(iii) Partition (PART): If X is an infinite linearly ordered set, with first and last points such that any partition of the set into two pieces contains exactly one part that is infinite, then the same conclusion applies to any finite partition of X.

The stable versions of CAC and ADS (called SCAC and SADS, respectively) and their weaker siblings are also defined in [27] (see Section 4.2.2). Over RCA_0, $\mathsf{CAC} \implies \mathsf{ADS}$ and $\mathsf{SCAC} \implies \mathsf{SADS} \implies \mathsf{PART}$ (see [27] and Lerman, Solomon and Towsner [40]. In fact the first implication is essentially "folklore"). Each of these principles implies $B\Sigma^0_2$, as a consequence of the following (Chong, Lempp, and Yang [7]):

THEOREM 4.7. $\mathsf{RCA}_0 + \mathsf{PART}$ *implies* $B\Sigma^0_2$.

PROOF. Suppose $\mathcal{M} \models \mathsf{RCA}_0$ and $B\Sigma^0_2$ fails in the model. Let $[0, a]$ and f be as in Proposition 4.5 witnessing the failure of $B\Sigma^0_2$ in $\mathcal{M}$. In particular, every number in M belongs to one and only one column. The first point to note is that one may choose a and f so that f is "bitame" Σ^0_2 on $[0, a]$ in the following sense: For any $0 < x < a$, either $\sup_{y \le x} f(y)$ is bounded, or $\sup_{x < y \le a} f(y)$ is bounded. One may visualize this as a bell shaped curve defined over $[0, a]$. Now let $\prec$ be a linear ordering on M so that 0 is the least element, a the largest element, and for any $s \neq t$, $s \prec t$ if and only if either (i) s and t belong to the same column and $s < t$, or (ii) s is in the xth column and t is in the yth column with $x < y$. Then M is partitioned into a-many parts under $\prec$, each of which is M-finite. Hence PART fails in $\mathcal{M}$. ⊣

REMARK 4.8. The notion of tame Σ^0_2 for a function was introduced by Lerman [38] in α-recursion theory. The link between bitameness and failure of $B\Sigma^0_2$ was hinted at in Slaman [60]. It is also worth noting that as the combinatorial principle becomes weaker, the proof that it implies $B\Sigma^0_2$ gets more elaborate.

It should also be noted that not every Ramsey type principle implies $B\Sigma^0_2$. Here is an example:

DEFINITION 4.9 (Rainbow Ramsey Principle RRT^2_2). If $f : [M]^2 \to M$ satisfies the condition that every number has at most 2 preimages, then there is an infinite $A \subseteq M$ on which f is injective.

Thus RRT^2_2 is a strong form of "anti-RT^2_2" theorem. Csima and Mileti [22] showed without using nonstandard models that $\mathsf{RT}^2_2 \to \mathsf{RRT}^2_2$ over RCA_0, and the implication is strict. It is known that RRT^2_2 does not imply $B\Sigma^0_2$ (Slaman, unpublished). What one wants to know is whether any of the

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principles introduced above that proves $B\Sigma^0_2$ in fact implies $I\Sigma^0_2$. As will be seen below, none is known to be so.

4.2. Π^1_1-conservation. Suppose $T_1 \supset T_2$ are theories in the language of second order arithmetic. We say that T_1 is Π^1_1-conservative over T_2 (with RCA_0 as the base theory) if every Π^1_1 sentence provable in $\mathsf{RCA}_0 + T_1$ is already provable in $\mathsf{RCA}_0 + T_2$. We first consider Π^1_1-conservation of various combinatorial principles known to be weaker than RT^2_2 (see Hirschfeldt and Shore [27]). An idea due to Harrington of showing T_1 to be Π^1_1-conservative over T_2 goes as follows: Assume that every model $\mathcal{M}$ of $\mathsf{RCA}_0 + T_2$ has an M-extension that is a model of $\mathsf{RCA}_0 + T_1$. The claim is that T_1 is Π^1_1-conservative over T_2. Suppose otherwise. Let $\forall X \exists x \varphi$ be a counterexample so that $\mathsf{RCA}_0 + T_1 \vdash \forall X \exists x \varphi$ but there is a model $\mathcal{M}$ of $\mathsf{RCA}_0 + T_2$ satisfying $\exists X \forall x \neg \varphi$. Then by assumption there is an M-extension $\mathcal{M}^*$ that is a model of $\mathsf{RCA}_0 + T_1$. But then $\mathcal{M}^* \models \exists X \forall x \neg \varphi$, which is a contradiction. Harrington used this approach to show that adding the principle WKL_0 (Weak König's Lemma) to RCA_0 does not prove new Π^1_1 sentences.

4.2.1. *The principle of cohesiveness* COH. The principle states that for any recursive set R of pairs of numbers, there is an infinite set C such that for each i, either $C \cap \{j | (i, j) \in R\}$ is finite, or $C \cap \{j | (i, j) \notin R\}$ is finite. We say that C is cohesive for R (or C is R-cohesive). Here, of course, the notions of "finite" and "infinite" are understood to be with respect to the underlying model.

Stephan and Jockusch [31] (see also [3]) showed that there is always a C that is low_2 relative to R. The result and its proof hold in any model of $\mathsf{RCA}_0 + I\Sigma^0_2$. In fact one has a reverse mathematics interpretation of this theorem which follows from Theorem 4.4:

THEOREM 4.10. *Let $\mathcal{M}$ be a countable model of $\mathsf{RCA}_0 + I\Sigma^0_2$. Then there is an M-extension $\mathcal{M}^*$ such that $\mathcal{M}^* \models \mathsf{RCA}_0 + I\Sigma^0_2 + \mathsf{COH}$.*

A corollary of Theorem 4.10 is that $\mathsf{RCA}_0 + I\Sigma^0_2 + \mathsf{COH}$ is Π^1_1-conservative over $\mathsf{RCA}_0 + I\Sigma^0_2$. It should be pointed out that the existence of a set C that is low_2 relative to R requires $I\Sigma^0_2(R)$ in general. In fact, there are models in which *proper* low_2 sets do not exist (i.e., a set C such that $C' >_T \varnothing'$ and $C'' \leq_T \varnothing''$):

THEOREM 4.11. *Let $\mathcal{M} \models P^- + B\Sigma^0_2 + \neg I\Sigma^0_2$ with ω as a Σ^0_2 cut. Suppose that every definable subset of ω is coded on ω. Then there is no proper low_2 set. In particular, there is a recursive array R that has no low_2 R-cohesive set.*

PROOF. Relativizing Theorem 9 of Chong and Mourad [9], we have $\omega \oplus \varnothing'$ to be minimal over $\varnothing'$. Suppose C is proper low_2. If $C <_T \varnothing'$, then by Theorem 3.16 which applies to Δ_2 sets, $C' \equiv_T \omega \oplus \varnothing'$. By Theorem 4.1 of Chong and Yang [17], $C'' \not\leq_T \varnothing''$.

If $C \not\leq_T \varnothing'$, then $C' >_T \varnothing'$. Since $C' <_T \varnothing''$, relativizing Theorem 4.1 of Li [42] we have $C' \equiv_T C^*$ for some $\Sigma_1(\varnothing')$ set C^*. Now C^* is either regular or nonregular. If it is regular, then it is recursive in $\omega \oplus \varnothing'$. Since $C^* >_T \varnothing'$, it in fact computes $\omega \oplus \varnothing'$. On the other hand, if it is nonregular, then it also computes $\omega \oplus \varnothing'$. Applying Theorem 4.1 of [17] again, we conclude that

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$(C^*)' >_T \varnothing''$ and hence C is not proper low_2. For the other conclusion, let R be a recursive array that has no cohesive set recursive in $\varnothing'$. ⊣

Thus if one takes a countable model $\mathcal{M}$ satisfying the conditions of Theorem 4.11, then it is not possible to imitate the construction used in the proof of Theorem 4.10, to work "internally" in $\mathcal{M}$ to obtain an M-extension that satisfies $\mathsf{RCA}_0 + B\Sigma^0_2 + \mathsf{COH}$. This explains why in the proof of the following theorem (cf. Chong, Slaman, and Yang [11]), the forcing construction that produces an R-cohesive path on a tree preserving $B\Sigma^0_2$ is carried out "externally". A model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2$ is *topped* if there is a second order element in $\mathcal{M}$ which computes every other second order element in the model.

THEOREM 4.12. *Let $\mathcal{M}$ be a countable topped model of $\mathsf{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$. Then $\mathcal{M}$ has an M-extension that is topped, satisfies $\mathsf{RCA}_0 + B\Sigma^0_2$ and is in addition a model of COH. Hence $B\Sigma^0_2 + \mathsf{COH}$ is Π^1_1-conservative over $B\Sigma^0_2$.*

A byproduct of Theorem 4.12 is that $\mathsf{RCA}_0 + B\Sigma^0_2 + \mathsf{COH}$ does not prove Σ^0_2 induction. This is perhaps not surprising since $\mathsf{RCA}_0 + \mathsf{COH}$ does not imply $B\Sigma^0_2$ (see [27]). As in Theorem 4.12, Conidis and Slaman [21] showed that RRT^2_2 is Π^1_1-conservative over $B\Sigma^0_2$ (this was also proved independently by Wei Wang (unpublished)). In fact, in [21] a stronger combinatorial principle 2-RAN was introduced. It asserts that given an X there is a 2-random set relative to X. It was shown that 2-RAN is Π^1_1-conservative over $B\Sigma^0_2$. Csima and Mileti [22] had earlier shown that every 2-random real bounds a rainbow for a recursive instance of RRT^2_2.

4.2.2. CAC *and weaker principles.* It is shown in [11] that every topped model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2$ has an M-extension that is in addition a model of P, where P is either ADS or CAC. It follows that, with RCA_0 as the base theory, both CAC and ADS are Π^1_1-conservative over $B\Sigma^0_2$. The proof proceeds by first showing that the M-extension property holds for the stable versions SCAC and SADS of these principles, which we now define:

DEFINITION 4.13.

(i) (SADS) If $\langle X, \leq \rangle$ is an infinite linearly ordered structure such that for any $y \in X$, either $\{y | y \leq x\}$ or $\{x | x \geq y\}$ is finite, then X has an infinite subset that is either an increasing or descending sequence.

(ii) (SCAC) Assume that $\langle X, \leq \rangle$ is infinite and partially ordered such that for all $y \in X$, either all but finitely many $x \in X$ are greater than y or incomparable with y. Then there is an infinite subset Y of X that is either linearly ordered under $\leq$, or the elements of Y are pairwise incomparable. The same conclusion holds when "greater" is replaced by "less".

The key idea for the construction of an M-extension that satisfies SADS or SCAC that also preserves $B\Sigma^0_2$ comes from [27] where it is shown that a low solution for a recursive linear ordering (in the case of SADS) or partially ordered set (for SCAC) always exists. The existence of a "low solution" over

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a model of $\mathsf{RCA}_0 + B\Sigma^0_2$ turns out to be true, although with a different construction. The role of a low set is apparent from the following proposition, which enables iteration of the construction:

PROPOSITION 4.14. *Let* $\mathcal{M} = \langle M, \mathbb{X}, 0, S, +, \times, < \rangle$. *If* $\mathcal{M} \models \mathsf{RCA}_0 + B\Sigma^0_2$ *and* $G \subset M$ *is low relative to a* $Y \in \mathbb{X}$, *then* $\mathcal{M}[G] \models \mathsf{RCA}_0 + B\Sigma^0_2$.

PROOF. $B\Sigma^0_2(G)$ holds if and only if $B\Sigma^0_1(G')$ holds. The latter is equivalent to the statement that $B\Sigma^0_1(Y')$ holds, which is true since $G' \equiv_T Y'$. But $Y \in \mathbb{X}$ implies that $B\Sigma^0_1(Y')$ holds. ⊣

It follows that if one starts with a topped countable model $\mathcal{M}_0$ of $\mathsf{RCA}_0 + B\Sigma^0_2$, then one may construct by a process of iteration a sequence $\{\mathcal{M}_n\}_{n<\omega}$, where $\mathcal{M}_n = \langle M_n, \mathbb{X}_n, 0, S, +, \times, < \rangle$, of models of $\mathsf{RCA}_0 + B\Sigma^0_2$ such that $\mathcal{M}_{n+1}$ is an M_n-extension and solves an instance of SCAC or SADS with a set that is low relative to the (finitely many) sets in $\mathbb{X}_n$ used as parameters to define the instance. The union of the $\mathcal{M}_n$'s is the desired model:

THEOREM 4.15. *Let* $\mathcal{M}$ *be a countable topped model of* $\mathsf{RCA}_0 + B\Sigma^0_2$. *Then for* P = SCAC *or* SADS, *there is an* M*-extension of* $\mathcal{M}$ *that is a model of* $\mathsf{RCA}_0 + B\Sigma^0_2 + \mathrm{P}$.

One concludes from Theorem 4.15 that both SADS and SCAC are Π^1_1-conservative over $B\Sigma^0_2$. Applying the equivalence of ADS with COH + SADS and CAC with SCAC + ADS over RCA_0 ([27]), one derives the Π^1_1-conservation of these two principles over $B\Sigma^0_2$. Since PART is a consequence of SADS, the same conclusion holds for PART as well.

Taking a step back to Σ^0_2 induction, we note as for COH that applying Theorem 4.4 one is able to conclude that RT^2_2—and hence every principle provable from it—is Π^1_1-conservative over $I\Sigma^0_2$. What we know concerning $B\Sigma^0_2$ in this regard is less complete. The major open problem concerns RT^2_2 and its close associate SRT^2_2 (see Question 9).

4.3. Combinatorial principles and Σ^0_2 induction. Theorem 4.15 tells us more about the strength of a principle than the conservation of sentences. If one starts with a $B\Sigma^0_2$ model $\mathcal{M}$ satisfying the hypothesis of the theorem, then the resulting M-extension will not satisfy $I\Sigma^0_2$ either. The import of this is that $\mathsf{RCA}_0 + \mathsf{CAC}$ and therefore all the principles it implies do not prove Σ^0_2 induction.

Despite the technical challenge it presented, there was little available to hint at a conjecture on the inductive strength of RT^2_2 with respect to $I\Sigma^0_2$. The only available data that might shed some light was the breakthrough of Seetapun (Seetapun and Slaman [52]) separating RT^2_2 from Ramsey's Theorem for triples RT^3_2, which states that any coloring of unordered triples of numbers into two colors has an infinite homogeneous set (a set all of whose triples have the same color). A proof *a la* nonstandard models goes as follows:

THEOREM 4.16. $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \not\vdash \mathsf{RT}^3_2$.

PROOF. Theorem 5.7 of Jockusch [30], see [3]) implies that any model of $\mathsf{RCA}_0 + \mathsf{RT}^3_2$ satisfies $I\Sigma^0_n$ for all $n \geq 1$ (hence a model of ACA_0). On the other hand, Theorem 4.4 holds for a model of $\mathsf{RCA}_0 + I\Sigma^0_2 + \neg B\Sigma^0_3$, ⊣

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Theorem 4.16 implies that the inductive strength of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ lies within the range $B\Sigma^0_2$ to $I\Sigma^0_2$. It turns out that $I\Sigma^0_2$ is not provable in the system $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ (Theorem 4.20). We consider first its stable version:

THEOREM 4.17 (Chong, Slaman, and Yang [12]). *There is a model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2 + \mathsf{SRT}^2_2$ all of whose second order elements are low.*

An immediate reaction to the above theorem might be that it contradicts the result of Downey, Hirschfeldt, Lempp, and Solomon [23] that there is a recursive stable colouring of pairs with no low homogeneous set. Their argument applies the $\mathbf{0}''$-priority method and in the context of reverse recursion theory, may be implemented in any model of $P^- + I\Sigma_2$. It follows that no model of $\mathsf{RCA}_0 + I\Sigma^0_2 + \mathsf{SRT}^2_2$ can have all of its second order elements to be low. The key to resolving this apparent conflict, is that the model used for Theorem 4.17 satisfies only $B\Sigma^0_2$ and is therefore necessarily nonstandard. The result of [23] and its generalization to models of $P^- + I\Sigma_2$ point to the possibility of looking at $B\Sigma_2$ models and their second order counterparts for a solution.

It is known ([3] and [7]) that SRT^2_2 is equivalent over RCA_0 to the following principle:

D^2_2: For every Δ^0_2-set A, either A or $\overline{A}$ contains an infinite subset.

Hence for our purpose, the model $\mathcal{M}$ to be constructed for Theorem 4.17 will be a model of D^2_2. The proof of the theorem is rather intricate. We give an outline of the main ideas below.

PROOF. We take $\mathcal{M}_0$ to be a second order version of the reflection model introduced in Subsection 2.4.3, with the additional property of arithmetical saturation. Thus ω is a Σ^0_2-cut of $\mathcal{M}_0$, and every real definable over $\mathcal{M}_0$ is coded on ω (in $\mathcal{M}_0$).

There are two major steps to the construction. The first is to show that given a Δ^0_2-set A, there is an M_0-infinite low set G such that either $G \subseteq A$ or $G \subseteq \overline{A}$. For convenience let us call members of A *red* and members of $\overline{A}$ *blue*. The second step is to argue that the construction may be iterated over the join of the finitely many low sets obtained earlier. This turns out to be a fairly delicate operation requiring more than a straightforward induction as we shall explain. Nevertheless, the discussion will focus on the first step and only briefly touch on the second whose proof involves another level of complexity.

The reason for making G low is that by Proposition 4.14 one may conclude that $\mathcal{M}_0[G] \models \mathsf{RCA}_0 + B\Sigma^0_2$. The preservation of $B\Sigma^0_2$ in the generic extension paves the way for the iteration operation. To make G low, one forces the Σ^0_1-theory of G so that G' is recursive in $\varnothing'$. We divide the Σ^0_1-sentences into ω-many disjoint blocks using $\varnothing'$. As an example, let B_0 be the first block of sentences. The strategy adopted to force the Σ^0_1-theory of G with respect to B_0 comprises two parts. The first is to find a red or blue $\mathcal{M}_0$-finite set D that *positively Cohen forces* a maximal subset of B_0. This means that a sentence in B_0 of the form $\exists x \varphi(\underline{G})$ where φ is bounded is either satisfied by interpreting D for $\underline{G}$ or no extension of D with the same color

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can do so. The combinatorics involved in achieving this is embedded in an $\mathcal{M}_0$-finite object called a (maximal) *exit tree* which is recursively enumerated and whose existence (i.e., its maximality) is guaranteed by the reflection property of $\mathcal{M}_0$. In fact the exit tree may be computed by $\varnothing'$ within the initial segment $\mathcal{I}$ of M_0 where $\langle I, 0, S, +, \times, <\rangle \prec_{\Sigma_1} \mathcal{M}_0$ is a model of PA and B_0 is coded in $\mathcal{I}$.

The second part is to force a Π^0_1-outcome for each sentence in B_0 not positively forced by D. This is achieved by generating from the construction an $\mathcal{M}_0$-infinite recursive tree "above" D (meaning every number in the tree is greater than any number in D) from which all future numbers of the generic set G will be drawn.

Now the construction is carried out in ω-many stages, so that at stage n it handles the nth block B_n of Σ^0_1-sentences by defining a red or blue D_n above $D_{n'}$ for each $n' < n$. The generic set G will then be the union of all the D_n's that are red, or all those that are blue, depending on which choice makes G infinite.

The entire construction is $\varnothing''$-recursive, and a closer examination shows that questions about the construction of G that require a $\varnothing''$-oracle to answer concern subsets of ω and such sets are coded on ω by the arithmetical saturation of $\mathcal{M}_0$. This leads to the conclusion that G' is $\varnothing'$-recursive and hence G is low.

The second step in the construction of $\mathcal{M}$ concerns iterating the above. Each Δ^0_2-set A_k with parameters will require an $\mathcal{M}_0$-infinite red or blue generic G_k. A crucial property of the reflection model that was exploited to define the G above was the existence of a cofinal sequence of Σ^0_1-elementary substructures in $\mathcal{M}_0$ that are models of PA. This property is not preserved upon relativization to G. The solution is to implement a construction reminiscent of iterated forcing, building G (which we now call G_0) "in anticipation" of what future generic sets $G_k, 1 \leq k < \omega$ would do. For this purpose, a principle called *bounded monotone enumeration* (BME) is introduced that captures the essence of a "k-dimensional exit tree" for Σ^0_1-sentences with k-set constants $\underline{G}_0, \ldots, \underline{G}_{k-1}$, for each $k < \omega$. The idea is to ensure that each time a G_k is constructed, BME is preserved to enable the next stage of iteration. Finally, we let $\mathcal{M} = \langle M_0, \{G_k\}, 0, S, +, \times, <\rangle_{k<\omega}$. ⊣

Theorem 4.17 separates stable Ramsey's Theorem for pairs from Ramsey's Theorem for pairs. This follows from a result of Jockusch [30] which states that there is a recursive two-coloring of pairs with no Σ^0_2 homogeneous set. The proof uses a finite injury construction which can be implemented in any (first order) model of $P^- + B\Sigma_2$ and therefore in any model of $\mathsf{RCA}_0 + B\Sigma^0_2$, leading to the following conclusion:

COROLLARY 4.18. $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 \not\vdash \mathsf{RT}^2_2$.

This corollary leaves open the challenging question:

QUESTION 7. *Does* SRT^2_2 *imply* RT^2_2 *in every* ω*-model of arithmetic*? *More generally, does* $\mathsf{RCA}_0 + I\Sigma^0_2$ *separate these two principles*?

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The principle Weak König's Lemma WKL_0 states that every infinite binary tree has an infinite path. With some extra work one may improve Theorem 4.17 to show

THEOREM 4.19. $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 + \mathsf{WKL}_0 \not\vdash \mathsf{RT}^2_2$.

We come now to the final result that we discuss in this survey paper. It is known ([3,4]) that over RCA_0, RT^2_2 is equivalent to $\mathsf{COH} + \mathsf{SRT}^2_2$, and neither COH nor SRT^2_2 implies $I\Sigma^0_2$. It turns out that there is also an inherent limit on the first order strength of RT^2_2:

THEOREM 4.20 (Chong, Slaman, and Yang [13]). *There is a model $\mathcal{M}$ of $\mathsf{RCA}_0 + \mathsf{RT}^2_2 + B\Sigma^0_2$ in which $I\Sigma^0_2$ fails. Hence $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ does not prove Σ^0_2 induction.*

To prove the theorem, we again take the reflection model $\mathcal{M}_0$ as the ground model. The idea is to apply the fact that RT^2_2 is equivalent to $\mathsf{COH} + \mathsf{SRT}^2_2$ over RCA_0, and build $\mathcal{M}$ over $\mathcal{M}_0$ by interlacing the constructions in Theorem 4.12 and Theorem 4.17. Thus one starts with a solution G for R-cohesiveness, followed by a solution over G of an instance of D^2_2. There is however a wrinkle with the approach. First, since not every recursive two coloring of pairs has a Σ^0_2 homogeneous set, it is not possible to build $\mathcal{M}$ consisting entirely of low generic sets, even with an abundant supply of codes in $\mathcal{M}_0$. Second, the construction in Theorem 4.12 to produce an R-cohesive G is highly noneffective. This makes any solution of an instance of D^2_2 by way of a set that is low relative to G will be arithmetically undefinable. With a more complex construction involving a G'-recursive approximation, coupled with applying Lemma 2.7 relativized to G, one can resolve these difficulties.

We hasten to add that not every Ramsey type combinatorial principle that implies $B\Sigma^0_2$ is known to be weaker than $I\Sigma^0_2$. Recall the definition of TT^1 given in the paragraph following Theorem 4.3. Corduran, Groszek, and Mileti [20] have shown that TT^1 is strictly stronger than $B\Sigma^0_2$. However, its precise inductive strength is open:

QUESTION 8. *Does $\mathsf{RCA}_0 + \mathsf{TT}^1$ prove $I\Sigma^0_2$?*

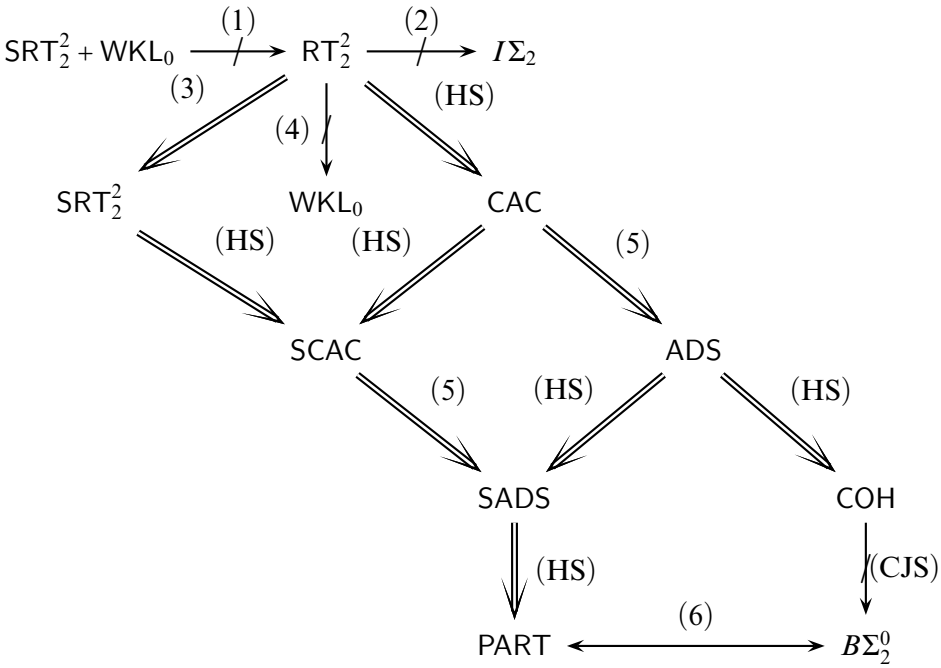
Theorems 4.17 and 4.20 do not address the question of Π^1_1-conservation of SRT^2_2 or RT^2_2 over $B\Sigma^0_2$. It is tempting to suggest that every model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2$ has an M-extension preserving $\mathsf{RCA}_0 + B\Sigma^0_2$ and satisfying SRT^2_2 or RT^2_2. The elaborate way in which Theorems 4.17 and 4.20 are proved seems to hint, at least, that such an M-extension need not exist. In particular, it is not clear how starting with a "no frills" model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$, such as one that is not endowed with any codes except those promised in Lemma 2.7, and whose second order elements consist only of recursive sets, one could construct an M-extension with the desired property.

QUESTION 9. *Is SRT^2_2 or RT^2_2 Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$?*

Our conjecture is that the answer is negative, and may be hidden somewhere within the principle BME.

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We end with a diagram that summarizes what is known conclusively about the relative strengths of Ramsey type combinatorial principles discussed in this paper:



In the diagram, a strict implication is indicated by $\Rightarrow$; implications labeled (HS) and (CJS) are proved respectively in [27] and [3]; (1) is Theorem 4.19; (2) is Theorem 4.20; (3) is Corollary 4.18; (4) is proved in [43]; (5) is proved in [40]; (6) is Theorem 4.7.

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