ELECTROMAGNETIC DIFFRACTION BY A CIRCULAR DISK

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ABSTRACT. It is shown that the problem of the diffraction of an arbitrary electromagnetic wave by a circular disk can be reduced to the solution of independent scalar problems. These latter problems are of a type which can be reduced to the solution of a Fredholm integral equation of the second kind and are particularly suitable for solution by approximate means at low frequencies.

1. *Introduction.* The problem of electromagnetic diffraction by a disk is one which has attracted many research workers and several different methods of solution have been presented. An account is given in (1) of the publications on this problem which had appeared previous to 1953 and additional references to work published after 1953 are to be found in (2).

The first rigorous treatment of the problem seems to be that of Meixner (3) who considered the problem of a plane wave incident normal to the disk and obtained the solution as a series of spheroidal functions. A second method of solving this problem also employing spheroidal functions was presented two years later by Meixner and Andrejewski (4). Alternative rigorous formulations of the general electromagnetic diffraction problem have been presented by Nomura and Katsura (5) and Lur'e (6). In (5) the problem is reduced to the solution of an infinite set of linear algebraic equations and in (6) the general problem is reduced to the solution of Fredholm integral equations of the second kind.

A systematic method of obtaining the solution of the diffraction problem for low frequencies as a power series in (frequency × radius) has been given by Bouwkamp (cf. 1). For the case of normal incidence Bouwkamp has obtained the transmission cross-section correct to the eighth power of the above product. In (2) Bouwkamp's approach is used to calculate the transmission cross-section for oblique incidence correct to the sixth power in (frequency × radius).

The object of the present note is to show that the general electromagnetic problem for diffraction by a disk can be reduced to the solution of several independent scalar problems. These latter problems are of a type for which a general method of solution has been presented recently by the author (7). The approach of (7) consists in reducing the scalar problems to the determination of certain unknown functions which are solutions of Fredholm integral equations of the second kind. Quantities of physical interest such as the radiation pattern can be expressed directly as single integrals involving these unknown functions. The integral equations are such that they are amenable to solution by iteration and solutions can be obtained, if desired, as power series in (frequency × radius). It seems likely that the present approach to the general

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diffraction problem is likely to be more useful in determining numerical solutions than one based on an expansion using spheroidal functions.

A similar approach to the present one has been employed in (6) where the general diffraction problem is also reduced to the solution of independent scalar problems. A dual integral approach is then used to reduce these scalar problems to the solution of Fredholm integral equations of the second kind. The integral equations derived in (6) are completely different from the present ones and the relationship between the two sets of integral equations is roughly that between the integral equations derived by Jones (8) and Magnus (9) for the scalar diffraction problem.

2. *Formulation of the problem.* A cylindrical polar system of coordinates (ρ, ϕ, z) is chosen so that the disk occupies the region $z = 0$, $0 \leqslant \rho \leqslant 1$, all ϕ, and $\mathbf{i}$ is defined to be the unit vector in the z-direction. It is also convenient to define a spherical polar system (r, θ, ϕ) by $\rho = r\sin\theta, z = r\cos\theta$. Rationalized m.k.s. units will be used throughout and the medium outside the disk assumed to be free space of dielectric susceptibility ϵ_0 and magnetic permeability μ_0. Throughout the work a time dependence $\exp(-i\omega t)$ will be understood.

It is convenient to assume that the incident field can be split up into two separate fields, one of which has no electric field normal to the disk and the other with zero magnetic field normal to the disk. The boundary-value problems for these two incident fields can be treated separately and the solution for an arbitrary incident field may then be obtained by superposition.

(*a*) *Incident electric field parallel to the disk.* In this case the incident electric field will have the form $\operatorname{curl}(u\mathbf{i})$ where u is a solution of

$$(\nabla^2 + k^2)u = 0 \tag{1}$$

and $k^2 = \omega^2\epsilon_0\mu_0$. We shall assume that $u(\rho, \phi, z)$ possesses the Fourier expansion $\sum_{m=0}^{\infty} u_m(\rho, z)\cos m\phi$ and for the convenience of notation it will be assumed that the suffix m attached to any function of ρ and z implies that the function can be regarded as the mth Fourier component of a solution of (1).

We assume the scattered field to be of the form

$$\mathbf{E}^s = i\omega\mu_0 \operatorname{curl}\boldsymbol{\Pi}, \quad \mathbf{H}^s = k^2\boldsymbol{\Pi} + \operatorname{grad}(\operatorname{div}\boldsymbol{\Pi}), \tag{2}$$

where the components of $\boldsymbol{\Pi}$ are of the form

$$\left.\begin{aligned} \Pi_\rho = \sum_{m=1}^{\infty} \chi_{m-1}(\rho, z)\cos m\phi, \quad \Pi_\phi = -\sum_{m=1}^{\infty} \chi_{m-1}(\rho, z)\sin m\phi, \\ \Pi_z = \sum_{m=0}^{\infty} \Pi_m(\rho, z)\cos m\phi. \end{aligned}\right\} \tag{3}$$

Equations (2) and (3) show that the boundary conditions on the disk reduce to

$$\Pi_m = -\frac{1}{i\omega\mu_0}u_m + A_m\rho^m, \quad \frac{\partial\chi_m}{\partial z} = (m+1)A_{m+1}\rho^m \quad (m \geqslant 0, z = 0, \rho \leqslant 1), \tag{4}$$

where the A_m's are undetermined constants. An equation similar to equation (4) has also been derived by Lur'e (6) for the particular case of an incident plane wave. The

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conditions of continuity off the disk require that the Π_m's and χ_m's are even and odd functions, respectively, of z and

$$\frac{\partial \Pi_m}{\partial z} = \chi_m = 0 \quad (z = 0, \rho \geqslant 1).$$

The radial and tangential components of the current density on the disk are given by J_ρ and J_ϕ respectively where

$$J_\rho = 2 \sum_{m=1}^{\infty} \left\{ k^2 \chi_{m-1} + \frac{m}{\rho} \left[\rho^{m-1} \frac{\partial}{\partial \rho} \{ \rho^{1-m} \chi_{m-1} \} + \left(\frac{\partial \Pi_m}{\partial z} \right)_{z=0} \right] \right\} \sin m\phi, \tag{5}$$

$$J_\phi = 2 \sum_{m=1}^{\infty} \left\{ k^2 \chi_{m-1} + \frac{\partial}{\partial \rho} \left[\rho^{m-1} \frac{\partial}{\partial \rho} \{ \rho^{1-m} \chi_{m-1} \} + \left(\frac{\partial \Pi_m}{\partial z} \right)_{z=0} \right] \right\} \cos m\phi + \frac{2\partial^2 \Pi_0}{\partial z \, \partial \rho}. \tag{6}$$

(*b*) *Incident magnetic field parallel to the disk.* In this case the incident magnetic field will have the form $\mathbf{H} = \operatorname{curl}(v\mathbf{i})$ where v is a solution of equation (1) and is assumed to have the Fourier expansion $\sum_{m=0}^{\infty} v_m(\rho, z) \cos m\phi$.

We assume that the scattered field has the form

$$\left.\begin{aligned} \mathbf{E}^s &= i\omega\mu_0 \operatorname{curl} \mathbf{\Pi}^* - \operatorname{grad} \frac{\partial \psi_0}{\partial z}(\rho, z) - k^2 \psi_0(\rho, z)\, \mathbf{i}, \\ \mathbf{H}^s &= k^2 \mathbf{\Pi}^* + \operatorname{grad}(\operatorname{div} \mathbf{\Pi}^*) + i\omega\epsilon_0 \operatorname{curl}[\mathbf{i}\psi_0(\rho, z)], \end{aligned}\right\} \tag{7}$$

where the components of $\mathbf{\Pi}^*$ are

$$\Pi_\rho^* = \sum_{m=1}^{\infty} \chi_{m-1}^*(\rho, z) \sin m\phi, \quad \Pi_\phi^* = \sum_{m=1}^{\infty} \chi_{m-1}^*(\rho, z) \cos m\phi, \quad \Pi_z^* = \sum_{m=1}^{\infty} \Pi_m^*(\rho, z) \sin m\phi. \tag{8}$$

Equations (7) and (8) show that the boundary conditions on the disk reduce to

$$\Pi_m^* = -\frac{1}{i\omega\epsilon_0} \frac{\partial v_m}{\partial z} + B_m \rho^m, \quad \frac{\partial \psi_0}{\partial z} = -\frac{1}{i\omega\epsilon_0} \frac{\partial v_0}{\partial z} + B_0,$$

$$\frac{\partial \chi_m^*}{\partial z} = (m+1) B_{m+1} \rho^m - \frac{1}{i\omega\epsilon_0 \rho^{m+1}} \frac{\partial}{\partial \rho} \left(\rho^{m+1} \frac{\partial v_{m+1}}{\partial z} \right) \quad \text{on} \quad z = 0, \ 0 \leqslant \rho \leqslant 1.$$

The B_m are arbitrary constants.

The conditions of continuity off the disk require that the Π_m^* are even functions of z and the χ_m^* and ψ_0 are odd functions of z and

$$\frac{\partial \Pi_m^*}{\partial z} = \chi_m^* = \psi_0 = 0 \quad \text{on} \quad z = 0, \ \rho \geqslant 1.$$

The radial and tangential components of the current density are

$$J_\rho = -2 \sum_{m=1}^{\infty} \left\{ k^2 \chi_{m-1}^* + \frac{m}{\rho} \left[\rho^{m-1} \frac{\partial}{\partial \rho} (\rho^{1-m} \chi_{m-1}^*) + \left(\frac{\partial \Pi_m^*}{\partial z} \right)_{z=0} \right] \right\} \cos m\phi + 2i\omega\epsilon_0 \frac{\partial \psi_0}{\partial \rho}, \tag{9}$$

$$J_\phi = 2 \sum_{m=1}^{\infty} \left\{ k^2 \chi_{m-1}^* + \frac{\partial}{\partial \rho} \left[\rho^{m-1} \frac{\partial}{\partial \rho} (\rho^{1-m} \chi_{m-1}^*) + \left(\frac{\partial \Pi_m^*}{\partial z} \right)_{z=0} \right] \right\} \sin m\phi. \tag{10}$$

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3. *Determination of the scattered field.* Using the suffix notation of the previous section we define two sets of functions $\phi_m^{(i)}(\rho, z)$ and $\psi_m^{(i)}(\rho, z)$ $(i = 1, 2, \ldots)$ satisfying the radiation condition at infinity and such that

$$\left.\begin{aligned} &\phi_m^{(i)}(\rho, z) = \phi_m^{(i)}(\rho, -z), \quad \psi_m^{(i)}(\rho, z) = -\psi_m^{(i)}(\rho, -z), \\ &\frac{\partial \phi_m^{(i)}}{\partial z} = \psi_m^{(i)} = 0 \quad \text{on} \quad z = 0, \rho \geqslant 1, \\ &\phi_m^{(i)} = f_m^{(i)}(\rho), \quad \frac{\partial \psi_m^{(i)}}{\partial z} = g_m^{(i)}(\rho) \quad \text{on} \quad z = 0, \rho \leqslant 1. \end{aligned}\right\} \tag{11}$$

The functions $f_m^{(i)}(\rho)$ and $g_m^{(i)}(\rho)$ $(i = 1, 2, \ldots)$ are as yet any arbitrary functions of ρ.

The general boundary-value problems for $\phi_m^{(i)}$ and $\psi_m^{(i)}$ are of the type treated in (7) and may be reduced to the solution of

$$W_m^{(i)}(\rho) = \frac{1}{2\pi\rho^m}\frac{d}{d\rho}\int_0^\rho \frac{f_m^{(i)}(t)\, t^{m+1}\cosh k(\rho^2 - t^2)^{\frac{1}{2}}}{(\rho^2 - t^2)^{\frac{1}{2}}}\, dt$$
$$- \int_0^1 W_m^{(i)}(t)\,[L_1^{(m)}(\rho, t) + L_2^{(m)}(\rho, t)]\, dt \quad (0 \leqslant \rho \leqslant 1), \tag{12}$$

$$U_m^{(i)}(\rho) = \frac{1}{2\pi\rho^m}\int_0^\rho \frac{g_m^{(i)}(t)\, t^{m+1}\cosh k(\rho^2 - t^2)^{\frac{1}{2}}}{(\rho^2 - t^2)^{\frac{1}{2}}}\, dt$$
$$+ \int_0^1 U_m^{(i)}(t)\,[M_1^{(m)}(\rho, t) + M_2^{(m)}(\rho, t)]\, dt \quad (0 \leqslant \rho \leqslant 1). \tag{13}$$

The functions $L_1^{(m)}$, $L_2^{(m)}$; $M_1^{(m)}$, $M_2^{(m)}$ are defined in equations (10), (17), (35) and (36) of (7) if α in those equations is replaced by k. The functions $W_m^{(i)}(\rho)$ and $U_m^{(i)}(\rho)$ are defined by

$$\left(\frac{\partial \phi_m^{(i)}}{\partial z}\right)_{z=0} = 4\rho^{m-1}\frac{d}{d\rho}\int_\rho^1 \frac{t^{1-m}\, W_m^{(i)}(t)\cosh k(t^2 - \rho^2)^{\frac{1}{2}}\, dt}{(t^2 - \rho^2)^{\frac{1}{2}}} \quad (0 \leqslant \rho \leqslant 1), \tag{14}$$

$$(\psi_m^{(i)})_{z=0} = -4\rho^m \int_\rho^1 \frac{t^{-m} U_i^{(m)}(t)\cosh k(t^2 - \rho^2)^{\frac{1}{2}}\, dt}{(t^2 - \rho^2)^{\frac{1}{2}}} \quad (0 \leqslant \rho \leqslant 1). \tag{15}$$

Thus if equations (12) and (13) can be solved for $W_m^{(i)}$ and $U_m^{(i)}$ the values of $\partial\phi_m^{(i)}/\partial z$ and $\psi_m^{(i)}$ can be determined everywhere on $z = 0$ and hence $\phi_m^{(i)}$ and $\psi_m^{(i)}$ can be determined from Rayleigh's formulae. It can also be shown (7) that as $r \to \infty$

$$\phi_m^{(i)} \sim \frac{e^{ikr}}{r} C_m^{(i)}(\theta), \quad \psi_m^{(i)} \sim \frac{e^{ikr}}{r} D_m^{(i)}(\theta),$$

where
$$C_m^{(i)}(\theta) = \frac{2(2\pi k)^{\frac{1}{2}}(\sin\theta)^m a}{(i)^m(\cos\theta)^{m-\frac{1}{2}}}\int_0^1 W_m^{(i)}(v)\, v^{\frac{1}{2}} I_{m-\frac{1}{2}}(kv\cos\theta)\, dv, \tag{16}$$

$$D_m^{(i)}(\theta) = \frac{-2(2\pi k)^{\frac{1}{2}}\sin^m\theta}{(i)^{m+1}(\cos\theta)^{m-\frac{1}{2}}}\int_0^1 U_m^{(i)}(v)\, v^{\frac{1}{2}} I_{m+\frac{1}{2}}(kv\cos\theta)\, dv. \tag{17}$$

The kernels of equations (12) and (13) tend to zero with k and expansions in powers of k are derived in (7). For the particular case when $f_m^{(i)}$ and $g_m^{(i)}$ are both equal to the mth Fourier coefficient of a plane wave the first two terms in the Neumann series for

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$W_m^{(i)}$ and $U_m^{(i)}$ are obtained in (3) for $m > 0$. For $m = 0$ several more terms in the Neumann series for these functions have been derived in (6). Thus for a given $f_m^{(i)}(\rho)$ or $g_m^{(i)}(\rho)$ the boundary-value problem for $\phi_m^{(i)}$ or $\psi_m^{(i)}$ can be reduced to the solution of a Fredholm integral equation of a type amenable to solution by iteration or by numerical means.

We shall now make the following definitions

$$f_1^{(m)}(\rho) = u_m(\rho, 0), \quad f_2^{(m)}(\rho) = \left\{\frac{\partial v_m}{\partial z}(\rho, z)\right\}_{z=0}, \quad f_3^{(m)}(\rho) = \rho^m;$$

$$g_1^{(m)}(\rho) = \frac{1}{\rho^{m+1}}\frac{\partial}{\partial \rho}\left\{\rho^{m+1}\frac{\partial v_{m+1}}{\partial z}(\rho, z)\right\}_{z=0}, \quad g_2^{(0)}(\rho) = \left(\frac{\partial v_0(\rho, z)}{\partial z}\right)_{z=0}, \quad g_3^{(m)}(\rho) = \rho^m.$$

From these definitions and the boundary conditions of § 2 it follows immediately that

$$\left.\begin{aligned} \Pi_m &= -\frac{1}{i\omega\mu_0}\phi_m^{(1)} + A_m\phi_m^{(3)}, \quad \chi_m = (m+1)A_{m+1}\psi_m^{(3)} \quad (m \geqslant 0), \\ \Pi_m^* &= -\frac{1}{i\omega\epsilon_0}\phi_m^{(2)} + B_m\phi_m^{(3)}, \quad \chi_m^* = (m+1)B_{m+1}\psi_m^{(3)} - \frac{1}{i\omega\epsilon_0}\psi_m^{(1)}, \\ \psi_0 &= -\frac{1}{i\omega\epsilon_0}\psi_2^{(0)} + B_0\psi_3^{(0)}. \end{aligned}\right\} \tag{18}$$

The arbitrary constants are determined by the conditions that at the edge of the disk J_ρ must vanish and J_ϕ must be $O\{(1-\rho^2)^{-\frac{1}{2}}\}$. The edge conditions and equations (5), (6), (9) and (10) then give

$$\left.\begin{aligned} A_m &= \frac{1}{i\omega\mu_0}\frac{W_m^{(1)}(1)}{\{W_m^{(3)}(1) - mU_{m-1}^{(3)}(1)\}} \quad (m \geqslant 1), \quad A_0 = \frac{1}{i\omega\mu_0}\frac{W_0^{(1)}(1)}{W_0^{(3)}(1)}, \\ B_m &= \frac{1}{i\omega\epsilon_0}\frac{\{U_m^{(1)}(1) - W_m^{(2)}(1)\}}{\{mU_{m-1}^{(3)}(1) - W_m^{(3)}(1)\}} \quad (m \geqslant 1), \quad B_0 = \frac{1}{i\omega\epsilon_0}\frac{U_0^{(2)}(1)}{U_0^{(3)}(1)}. \end{aligned}\right\} \tag{19}$$

Thus the general boundary-value problem has been reduced to the solution of the Fredholm integral equations (12) and (13) and for an arbitrary incident field this will involve the solution of each of the two equations for three different 'forcing' or inhomogeneous terms. The solutions $\phi_m^{(3)}$ and $\psi_m^{(3)}$ will be the same for all incident fields but the others will vary with the incident field. For a general incident field the iterative solution of the equations will be very laborious and it seems that it may be necessary to resort to numerical means in general. The far field pattern and hence the transmission cross-section can be expressed explicitly as integrals involving the solutions of equations (12) and (13).

For the particular case of an incident plane wave the functions $\phi_m^{(1)}$ and $\phi_m^{(2)}$ differ by a constant factor, and $\psi_0^{(2)}$ can be obtained in terms of $\psi_0^{(1)}$. For an incident plane wave $u_m(\rho, 0)$ is of the form $J_m(\lambda\rho)$, where λ is a constant, and hence $\phi_m^{(3)}$ can be obtained from $\phi_m^{(1)}$ by a limiting process. Similarly $\psi_m^{(3)}$ can be obtained from $\psi_m^{(1)}$ and hence, for an incident plane wave, only one solution for each integral equation has to be obtained. The first few terms in a power series in k can be deduced directly from results derived in (7) and (10) and in this way the transmission coefficients obtained in (2, 6) can be

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rederived. The derivation of additional terms in the power series for the transmission coefficient is extremely laborious and it would seem to be advisable to use a high-speed computer for this purpose.

Lur'e (6) has also reduced the electromagnetic boundary-value problem to the solution of certain Fredholm integral equations. The kernels of his equations, particularly for lower values of m, are somewhat more complicated than the ones of equations (12) and (13). Their expansion in powers of k also present some difficulties. The determinations of the arbitrary constants in Lur'e's solution requires the evaluation of the integrals of the solutions of his integral equations.

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