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The Lyapunov exponents of generic zero divergence three-dimensional vector fields

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Abstract. We prove that for a C^1-generic (dense G_δ) subset of all the conservative vector fields on three-dimensional compact manifolds without singularities, we have for Lebesgue almost every (a.e.) point $p \in M$ that either the Lyapunov exponents at p are zero or X is an Anosov vector field. Then we prove that for a C^1-dense subset of all the conservative vector fields on three-dimensional compact manifolds, we have for Lebesgue a.e. $p \in M$ that either the Lyapunov exponents at p are zero or p belongs to a compact invariant set with dominated splitting for the linear Poincaré flow.

1. *Introduction and statement of the results*

Given any C^1-dynamical system, the Lyapunov exponents measure the exponential behavior of the tangent map. A positive (or negative) Lyapunov exponent gives us the average exponential rate of divergence (or convergence) of two neighboring trajectories whereas zero exponents give us the absence of any kind of exponential behavior. We know by recent results of Abdenur, Bonatti and Crovisier that for any C^1 diffeomorphism, with non-zero Lyapunov exponents and with a dominated splitting with an index compatible with the non-uniformly hyperbolic splitting, we obtain rich information about geometric properties of the system, namely stable/unstable manifold theory. These geometric tools are the base of most of the central results on dynamical systems nowadays. So it is important to detect when Lyapunov exponents vanish.

A central result in this direction for discrete systems is the *Bochi–Mañé* theorem, which provides a C^1-residual set of area-preserving diffeomorphisms on surfaces where we have either Anosov systems or for Lebesgue almost every (a.e.) point zero Lyapunov exponents. This theorem was announced at the beginning of the 1980s by Mañé (1948–1995) in [**9**] but there was only available a sketch of a proof (see [**10**]). A complete proof, due to Bochi, appeared in [**4**].

Subsequently, Bochi and Viana in [**5**] extended this result to a large class of discrete systems: volume-preserving diffeomorphisms with arbitrary dimension, symplectic maps and also linear cocycles. For a survey of the theory, see [**6**].

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Motivated by these results we start the approach for the continuous-time setting by proving in [**3**] the dichotomy (dominated splitting versus zero Lyapunov exponents) for two-dimensional conservative linear differential systems. Here we explore the volume-preserving 3-flows case by following closely the strategy for the proof of the discrete case, doing the natural adaptations and developing the required techniques for perturbations of vector fields. Our first result is the analog to the Bochi–Mañé theorem for volume-preserving vector fields in three-dimensional compact manifolds without singularities.

THEOREM 1. *There exists a residual $\mathfrak{R} \subseteq \mathfrak{X}^1_\mu(M)^*$ such that, if $X \in \mathfrak{R}$, then either X is Anosov or else for Lebesgue a.e. $p \in M$ all Lyapunov exponents of X^t are zero.*

Finally, developing the ideas of the proof of Theorem 1, adding some new observations we are able to prove the following for volume-preserving vector fields in three-dimensional compact manifolds.

THEOREM 2. *There exists a dense $\mathfrak{D} \subseteq \mathfrak{X}^1_\mu(M)$ such that, if $X \in \mathfrak{D}$, then there exist X^t-invariant sets D and O whose union has full measure, so that for $p \in O$ we have zero Lyapunov exponents and D is a countable increasing union of compact invariant sets Λ_{m_n} admitting an m_n-dominated splitting for the linear Poincaré flow.*

2. *Preliminaries*

2.1. *Notation and standing assumptions.* Given a volume form ω, let μ be the probability measure associated to ω, which we call Lebesgue measure. We consider vector fields $X : M \to TM$, where M is a three-dimensional compact, connected, boundaryless C^∞ Riemannian manifold M. Given a vector field X we have an associated flow X^t which is the infinitesimal generator of X, i.e.,

$$\left.\frac{dX^t}{dt}\right|_{t=s}(p) = X(X^s(p)).$$

The flow X^t has a tangent map DX^t_p which is the solution of the non-autonomous linear differential equation $\dot{u}(t) = DX_{X^t(p)} \cdot u(t)$ called the *linear variational equation*. We denote by $\mathfrak{X}^1_\mu(M)$ the set of all C^1 vector fields that preserve μ. We assume that $\mathfrak{X}^1_\mu(M)$ is endowed with the C^1 topology. The subset of $\mathfrak{X}^1_\mu(M)$ formed by the vector fields without singularities will be denoted by $\mathfrak{X}^1_\mu(M)^*$.

2.2. *Oseledets' theorem for the linear Poincaré flow.* Oseledets' theorem [**12**] is valid in the setting of discrete-time *cocycles* and it holds in particular for any dynamical cocycle over a diffeomorphism $f : M \to M$ preserving a probability measure μ. It asserts that we have for μ-a.e. point $p \in M$ a splitting $T_pM = E^1_p \oplus \cdots \oplus E^{k(p)}_p$ (*Oseledets splitting*) and real numbers $\lambda_1(p) > \cdots > \lambda_{k(p)}(p)$ (*Lyapunov exponents*) such that

$$Df_p(E^i_p) = E^i_{f(p)} \quad \text{and} \quad \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n_p \cdot v^i\| = \lambda_i(p)$$

for any $v^i \in E^i_p \setminus \{\vec{0}\}$ and $i = 1, \ldots, k(p)$.

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Consider $X \in \mathfrak{X}^1_\mu(M)$ and the associated flow $X^t : M \to M$. Since Oseledets' theorem is an asymptotic result and $DX^r_{(\cdot)}$, for fixed r, is a uniformly bounded operator, we may replace the tangent map $DX^t_p = DX^r_{X^n(p)} \cdot DX^n_p$ by the least integer time-n map, DX^n_p, and reformulate Oseledets' theorem for flows. This theorem allows us to conclude also that, for $X \in \mathfrak{X}^1_\mu(M)$ and for μ-a.e. $p \in M$, we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \log |\det(DX^t_p)| = \sum_{i=1}^{k(p)} \lambda_i(p).\dim(E^i_p) = 0, \tag{1}$$

which is related to the sub-exponential decrease of the angles between any subspaces of the Oseledets splitting along μ-a.e. orbits.

Denote by $\mathbb{R}X(p)$ the subbundle $\{v \in T_pM : v = \eta X(p), \eta \in \mathbb{R}\}$. Since we have $DX^t_p(X(p)) = X(X^t(p))$, we conclude that one of the Oseledets subspaces is $\mathbb{R}X(p)$, and that its associated Lyapunov exponent is zero.

Our conservative setting on 3-manifolds (which implied the last equality in (1)), and the previous paragraph guarantees that $\lambda_1(p) + \lambda_3(p) = 0$. Hence either $\lambda_1(p) = -\lambda_3(p) > 0$ or both are zero. The former case for μ-a.e. $p \in M$ implies the existence of two directions E^u_p and E^s_p associated to the positive Lyapunov exponent $\lambda_1(p) = \lambda_u(p)$ and the negative one $\lambda_3(p) = \lambda_s(p)$, respectively. We denote by $\mathcal{O}(X)$ the set of Oseledets points, by $\mathcal{O}^+(X) \subseteq \mathcal{O}(X)$ the set of points with positive Lyapunov exponent and by $\mathcal{O}^0(X) \subseteq \mathcal{O}(X)$ the set of those with all Lyapunov exponents zero. We note that $\mathcal{O}^+(X) = \mathcal{O}(X) \setminus \mathcal{O}^0(X)$. When there is no ambiguity we denote $\mathcal{O}(X)$ by $\mathcal{O}$, omitting the vector field.

Let R be the set of regular points for the vector field X, i.e., $X(p) \neq 0$ for all $p \in R$. The vector field X induces a decomposition of the tangent bundle T_RM in a way that each fiber T_pM has a splitting $N_p \oplus \mathbb{R}X(p)$ where $N_p = \mathbb{R}X(p)^\perp$ is the normal subbundle for $p \in R$.

Consider the automorphism of vector bundles $DX^t : T_RM \longrightarrow T_RM$ such that $DX^t(p, v) = (X^t(p), DX^t(p) \cdot v)$ and $\Pi_{X^t(p)}$ the canonical projection on $N_{X^t(p)}$. The linear map $P^t_X(p) : N_p \longrightarrow N_{X^t(p)}$ defined by $P^t_X(p) = \Pi_{X^t(p)} DX^t(p)$ is called the *linear Poincaré flow* at p associated to the vector field X. (This notion was first introduced by Doering in [**8**] to prove the hyperbolicity of robustly transitive three-dimensional flows.)

The linear Poincaré flow is the differential of the standard *Poincaré map* $\mathcal{P}^t_X(p) : \mathcal{V}_p \subset \mathcal{N}_p \to \mathcal{N}_{X^t(p)}$, where $\mathcal{N}_{X^s(p)}$, for $s = 0, t$, is a surface contained in M whose tangent space at $X^s(p)$ is $N_{X^s(p)}$ for $s = 0, t$ and $\mathcal{V}_p$ is a small neighborhood of p. We can always guarantee the existence of a continuous time-t arrival function $\tau(p, t)(\cdot)$ from $\mathcal{V}_p$ into $\mathcal{N}_{X^t(p)}$ by using the implicit function theorem. Due to the presence of singularities $\mathcal{V}_p$ may be very small.

In our setting, if we have a point $p \in R \cap \mathcal{O}^+$, the Oseledets splitting on T_pM induces a $P^t_X(p)$-invariant splitting on N_p, say $\Pi_p(E^\sigma_p) = N^\sigma_p$ for $\sigma = u, s$. So we have $N_p = N^u_p \oplus N^s_p$. If $p \in \mathcal{O}^0$, then the P^t_X-invariant splitting will be trivial, i.e., it is just the normal subbundle. It follows directly from the sub-exponential behavior along the orbit of p of the angle $\theta_t = \measuredangle(X(X^t(p)), E^\sigma_{X^t(p)})$ $(\sigma = u, s)$ that the Lyapunov exponents of $P^t_X(p)$ associated to the subspaces N^u_p and N^s_p are respectively $\lambda_u(p) \geq 0$ and $\lambda_s(p) \leq 0$ (for a proof see Lemma 2.3.1 of [**2**]). Therefore in order to decrease the Lyapunov exponents

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associated to the tangent flow, we can, equivalently, decrease the Lyapunov exponents associated to the linear Poincaré flow.

In our conservative context we may restate the Oseledets' theorem for the linear Poincaré flow as follows.

THEOREM 2.1. (Oseledets) *Let $X \in \mathfrak{X}^1_\mu(M)$. For μ-a.e. $p \in M$ there exists the upper Lyapunov exponent $\lambda^+(X, p)$ defined by the limit*

$$\lim_{t\to+\infty} \frac{1}{t} \log \|P^t_X(p)\|$$

that is a non-negative measurable function of p. For μ-a.e. point $p \in \mathcal{O}^+$ there is a splitting of the normal bundle $N_p = N^u_p \oplus N^s_p$ which varies measurably with p and is such that:

(a) *if $v \in N^u_p \setminus \{\vec{0}\}$, then*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P^t_X(p) \cdot v\| = \lambda^+(X, p);$$

(b) *if $v \in N^s_p \setminus \{\vec{0}\}$, then*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P^t_X(p) \cdot v\| = -\lambda^+(X, p);$$

(c) *if $\vec{0} \neq v \notin N^u_p, N^s_p$, then*

$$\lim_{t\to+\infty} \frac{1}{t} \log \|P^t_X(p) \cdot v\| = \lambda^+(X, p) \quad \text{and} \quad \lim_{t\to-\infty} \frac{1}{t} \log \|P^t_X(p) \cdot v\| = -\lambda^+(X, p).$$

Remark 2.1. By Proposition 1.1 of [**8**], we conclude that any X^t-invariant and *compact* set Λ is hyperbolic for the flow if and only if the linear Poincaré flow is hyperbolic on Λ.

2.3. *Coordinates, metrics, flowboxes and measures.* According to Moser's theorem (see [**11**, Lemma 2]) given a volume form ω in M there exists an atlas $\mathcal{A} = \{\alpha_i : U_i \to \mathbb{R}^3\}$ such that $(\alpha_i)_*\omega = dx \wedge dy \wedge dz$. Moreover by compactness of M we can take $\mathcal{A}$ to be finite. Fix an atlas $\mathcal{A}$.

There exists $r > 0$ such that, for all $p \in M$, there exists $i(p)$ (take the smallest among all possible choices) such that the Riemannian ball of radius r, $B_r(p)$, is contained in $U_{i(p)}$. When we deal locally, and for example to compute our estimates, instead of using the original Riemannian norm at T_pM, we use the equivalent norm $\|v\|_p := \|(D\alpha_{i(p)})_p.v\|$ defined in $B_r(p)$.

Given two linear maps $A(t) : T_pM \to T_{X^t(p)}M$ and $B(t) : T_qM \to T_{X^t(q)}M$ we estimate the distance, say $\|A(t) - B(t)\|$, between them by using the atlas $\mathcal{A}$ and translating the base points to $\vec{0} \in \mathbb{R}^3$. Therefore, $\|A(t) - B(t)\|$ is given by

$$\|(D\alpha_{i(X^t(p))})_{X^t(p)} A(t) (D\alpha_{i(p)})^{-1}_p - (D\alpha_{i(X^t(q))})_{X^t(q)} B(t) (D\alpha_{i(q)})^{-1}_q\|. \tag{2}$$

Analogously we estimate the distance between the linear Poincaré flows based at different points. We implicitly make use of the charts defined above to translate our objects to $\mathbb{R}^3$,

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which makes the presentation clearer. For example, when we consider the usual Poincaré map $\mathcal{P}_X^t(p) : \mathcal{N}_p \to \mathcal{N}_{X^t(p)}$, we will consider the surface to be the normal bundle $N_{X^t(p)}$. In §3.2, we will consider another coordinate system which is adequate to the perturbations involved, and in the standard way, like in (2), we estimate distance between linear maps.

Given the Poincaré map $\mathcal{P}_X^t(p) : \mathcal{V}_p \subseteq N_p \to N_{X^t(p)}$, where $\mathcal{V}_p$ is chosen sufficiently small, and given $B \subseteq \mathcal{V}_p$, the self-disjoint set

$$\mathcal{F}_X^n(p)(B) := \{\mathcal{P}_X^t(p)(q) : q \in B, t \in [0, n]\}$$

is called the time-n length *flowbox* at p associated to the vector field X.

Given $n_1, n_2 \in N_p$ and induced by the volume form ω we can define a pair of 2-forms by $\hat{\omega}_p(n_1, n_2) := \omega_p(X(p), n_1, n_2)$ and $\overline{\omega}_p(n_1, n_2) = \omega_p(X(p)\|X(p)\|^{-1}, n_1, n_2)$. It turns out that $(\mathcal{P}_X^t(p))^*\hat{\omega}_p = \hat{\omega}_p$. The measure $\overline{\mu}$ induced by the 2-form $\overline{\omega}$ is not necessarily $\mathcal{P}_X^t$-invariant; however the two measures $\hat{\mu}$ and $\overline{\mu}$ are equivalent. We call $\overline{\mu}$ the Lebesgue measure at normal sections or *modified area*. In fact, given $n_1, n_2 \in N_p$ we have that $(\mathcal{P}_X^t(p))^*\overline{\omega}_p(n_1, n_2) = x(t)^{-1}\overline{\omega}_p(n_1, n_2)$, where $x(t) = \|X(X^t(p))\|\|X(p)\|^{-1}$. Since the flow is volume-preserving we have $|\det P_X^t(p)| = x(t)^{-1}$. Therefore we can give an explicit expression for the infinitesimal distortion area factor of the linear Poincaré flow, which is expressed by the following simple lemma (for a proof see [**2**, Lemma 3.1.3]).

LEMMA 2.2. *Given $\nu > 0$ and $T > 0$, there exists $r > 0$ such that for any measurable set $K \subseteq B(p, r) \subseteq N_p$ we have $|\overline{\mu}(K) - x(t).\overline{\mu}(\mathcal{P}_X^t(p)(K))| < \nu$ for all $t \in [0, T]$.*

2.4. *Dominated splitting for the linear Poincaré flow.* Let Λ be an X^t-invariant subset of M. The splitting $N^1 \oplus N^2$ of the normal bundle N is an *m-dominated splitting* for the linear Poincaré flow if it is P_X^t-invariant and there is a *uniform* $m \in \mathbb{N}$ such that for any point $p \in \Lambda$ the following inequality holds:

$$\Delta(p, m) = \frac{\|P_X^m(p)|_{N_p^1}\|}{\|P_X^m(p)|_{N_p^2}\|} \leq \frac{1}{2}. \tag{3}$$

Let us say a few words about this definition. If we take $p \in \mathcal{O}^+$, then the existence of a positive Lyapunov exponent only guarantees that we will see expansion for large iterates, say for $P_X^{m(p)}(p)$, but this function $m(\cdot)$ varies from point to point, and possibly may not be bounded along the orbit of p. So the information given by the Oseledets' theorem cannot detect uniformity. If Λ has a dominated splitting, then the splitting varies continuously from point to point and we may extend it to the closure of Λ. Moreover, the decomposition is unique and the angle between the two subspaces is bounded away from zero on Λ. Next we define some useful X^t-invariant sets:

- $\Lambda_m(X) := \{p \in \mathcal{O}^+ : p$ has m-dominated splitting for the linear Poincaré flow$\}$;
- $\Gamma_m(X) := M \setminus \Lambda_m(X)$;
- $\Gamma_m^+(X) := \mathcal{O}^+(X) \setminus \Lambda_m(X)$;
- $\Gamma_m^*(X) := \{p \in \Gamma_m^+(X) : p \notin \mathrm{Per}(X^t)\}$.

The set of points in $\mathcal{O}^+$ where (3) does not hold will be denoted by $\Delta_m(X)$. Clearly, in $\Delta_m(X)$, the orbits do not have an m-dominated splitting; nevertheless for some $p \in \Delta_m(X)$ there might still exist some iterate $X^t(p)$ where (3) holds. Taking this into account we note that $\Gamma_m^+(X) = \bigcup_t X^t(\Delta_m(X))$.

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2.5. *Ergodicity of sets with dominated splitting.* Theorem 14 of [**6**] states the following.

THEOREM 2.3. (Bochi–Viana) *Let M be a compact manifold, μ the Lebesgue measure, $f \in \mathrm{Diff}^2_\mu(M)$ and Λ a compact hyperbolic set for f. Then either $\mu(\Lambda) = 0$ or $\Lambda = M$.*

We note that in the three-dimensional conservative setting without singularities, dominated splitting is tantamount to hyperbolicity.

LEMMA 2.4. *For $X \in \mathfrak{X}^1_\mu(M)^*$, if $\Lambda_m \neq \emptyset$ then $\overline{\Lambda_m}$ is a hyperbolic set.*

Proof. Since we do not have singularities and we have constant dimensions of the subbundles the dominated splitting extends to the closure of Λ_m. Of course, an m-dominated splitting implies that there exist $\hat{C} > 0$ and $0 < \sigma < 1$ such that $\Delta(p, t) \leq \hat{C}\sigma^t$ for every $p \in \overline{\Lambda_m}$ and $t > 0$. Let $\theta_t := \measuredangle(N^u_{X^t(p)}, N^s_{X^t(p)})$, which is bounded below by $\beta > 0$ for every $p \in \overline{\Lambda_m}$. Since we do not have singularities there exists $K > 1$ such that, for all $p \in M$, $K^{-1} \leq \|X(p)\| \leq K$. Since the flow is conservative we have

$$\sin(\theta_0) = \|P^t_X(p)|_{N^u_p}\|.\|P^t_X(p)|_{N^s_p}\|.\sin(\theta_t).x(t),$$

so

$$\|P^t_X(p)|_{N^s_p}\|^2 = \sin(\theta_0)\sin^{-1}(\theta_t)x(t)^{-1}.\Delta(p, t) \leq K^2\sin^{-1}(\beta)\hat{C}\sigma^t.$$

We proceed analogously for the unstable direction and we have $\overline{\Lambda_m}$ hyperbolic for the linear Poincaré flow, with constants σ and $C := \hat{C}\sin(\beta)^{-1}K^2$. Now by Remark 2.1 we conclude that $\overline{\Lambda_m}$ is hyperbolic for the flow. □

Now using the ideas of Theorem 2.3 jointly with Lemma 2.4 and also the C^2 regularity of the vector field, we are able to prove the following proposition.

PROPOSITION 2.5. *Let $X \in \mathfrak{X}^2_\mu(M)^*$ and Λ_m be an X^t-invariant set with an m-dominated splitting for the linear Poincaré flow. Then $\mu(\overline{\Lambda_m}) = 0$ or X is Anosov.*

Proof. See Proposition 2.5.1 in [**2**]. □

A conservative flow X^t is called *aperiodic* if $\mu(\mathrm{Per}(X^t)) = 0$.

LEMMA 2.6. *There exists $D \subseteq \mathfrak{X}^1_\mu(M)^*$ such that D is C^1-dense and, for $X \in D$, X^t is aperiodic, X is of class C^s ($s \geq 2$) and all its sets with dominated splitting for the linear Poincaré flow have zero or full measure.*

Proof. We take the C^s-residual given by Robinson's version of the Kupka–Smale theorem; see [**13**]. This residual set of vector fields is of class C^s and the associated flows have countable periodic points. Since $\mathfrak{X}^s_\mu(M)$ is a Baire space (with respect to C^s topology), it follows that we have a C^s-dense set D, therefore a C^1-dense set, of vector fields with countable periodic orbits on $\mathfrak{X}^s_\mu(M)$. We know that $\mathfrak{X}^s_\mu(M)$ is C^1-dense on $\mathfrak{X}^1_\mu(M)$ (see [**14**]), so D is C^1-dense on $\mathfrak{X}^1_\mu(M)$ and all vector fields in D are C^s. Finally, we use Proposition 2.5 and the lemma is proved. □

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3. *Proof of Theorem 1*

3.1. *Outline of the proof of Theorem 1.* Given $X \in \mathfrak{X}^1_\mu(M)$, let $\lambda^+(X, p)$ be the upper Lyapunov exponent which exists μ-a.e. $p \in M$ by Theorem 2.1. When there is no ambiguity we denote $\lambda^+(X, p)$ by $\lambda^+(p)$. Given an X^t-invariant set $\Gamma \subseteq M$ we define the 'entropy function' by $LE(X, \Gamma) = \int_\Gamma \lambda^+(p)\, d\mu(p)$. The next simple lemma is proved in [**4**] (see Proposition 2.1).

LEMMA 3.1. *We have*

$$LE(X, \Gamma) = \inf_{n \geq 1} \frac{1}{n} \int_\Gamma \log \|P^n_X(p)\|\, d\mu(p),$$

so it is upper semicontinuous.

Denote $LE(X, M)$ by $LE(X)$. The next proposition will be crucial to prove Theorem 1.

PROPOSITION 3.2. *Let $X \in \mathfrak{X}^2_\mu(M)^*$, with X^t aperiodic, and suppose that any hyperbolic set has zero Lebesgue measure. Let $\epsilon, \delta > 0$ be given. Then there exists a C^1 zero divergence vector field Y ϵ-C^1-close to X, such that $LE(Y) < \delta$.*

For now we assume Proposition 3.2 and prove Theorem 1. By Lemma 2.6, we have a dense set such that every X is C^2, aperiodic and with hyperbolic sets having full or zero measure. The set of conservative Anosov vector fields, denoted by $\mathbf{A}$, is open. For all $k \in \mathbb{N}$ the set $\mathbf{A}_k = \{X \in \mathfrak{X}^1_\mu(M)^* : LE(X) < k^{-1}\}$ is open, because by Lemma 3.1 LE is upper semicontinuous. By Proposition 3.2, with $\delta = k^{-1}$, we get $\mathbf{A}_k$ dense in $\mathbf{A}^c$, so the set $\mathfrak{R} = \bigcap_k \mathbf{A} \cup \mathbf{A}_k$ is a C^1-residual set. But

$$\mathfrak{R} = \mathbf{A} \cup \bigcap_k \mathbf{A}_k = \mathbf{A} \cup \{X \in \mathfrak{X}^1_\mu(M)^* : LE(X) = 0\},$$

therefore for $X \in \mathfrak{R}$ we have either that X is an Anosov vector field or

$$LE(X) = \int_M \lambda^+(p)\, d\mu(p) = 0.$$

This last equality implies that μ-a.e. $p \in M$ has zero Lyapunov exponents and Theorem 1 is proved.

To prove Proposition 3.2 we consider a large $m \in \mathbb{N}$ (depending on ϵ) and we use the fact that a.e. orbit does not have an m-dominated splitting because otherwise, by Proposition 2.5, we must have that X is Anosov. We start with a local procedure and we take $p \in \Gamma^+_m(X)$, and also $t \gg m$. By a recurrence result (see Lemma 3.16), we obtain $q \approx X^{t/2}(p)$ such that $\Delta(q, m) \geq 1/2$, say $q \in \Delta_m(X)$. In §3.4 we use the ϵ-C^1-perturbation Y of X, developed previously in §§3.2 and 3.3, to map the direction N^u_q into $N^s_{X^m(q)}$. In §3.5, we conclude that this argument allows us to prove that for most points q near p the norm of $P^t_Y(q)$ is smaller than δ. The formula for $LE(\cdot)$ in Lemma 3.1 allows us to compute a bound for $LE(Y)$ in a finite time t. Finally, in §3.6, we make this procedure global by using a Kakutani tower argument.

3.2. *A conservative flowbox theorem.* The following theorem, due to Dacorogna and Moser [**7**], will be used to obtain a conservative local change of coordinates which trivialize a vector field.

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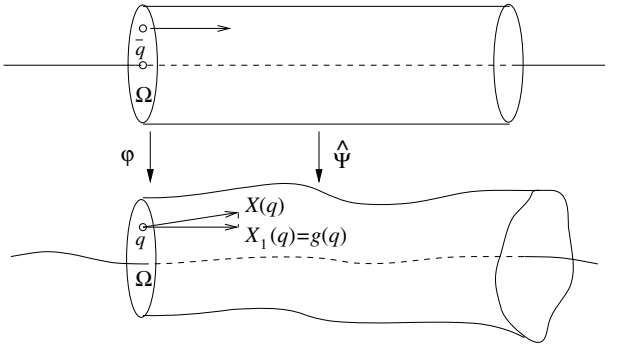


FIGURE 1. The conservative change of coordinates straightening out all orbits.

THEOREM 3.3. (Dacorogna–Moser) *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^5 boundary $\partial\Omega$ and g, $f : \overline{\Omega} \to \mathbb{R}$ positive functions of class C^s ($s \geq 2$). Then there exists a diffeomorphism $\varphi : \Omega \to \varphi(\Omega) \subseteq \mathbb{R}^n$ of class C^s which satisfies the partial differential equation*

$$\det D\varphi_q g(\varphi(q)) = \lambda f(q), \tag{4}$$

for all $q \in \Omega$ where $\lambda = \int g / \int f$. We also have $\varphi = \mathrm{Id}$ at $\partial\Omega$.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the constant vector field defined by $T(x, y, z) = (c, 0, 0)$ for some $c > 0$ and $\mathfrak{F}$, $\mathfrak{C}$ be the flowboxes $\mathcal{F}^1_X(p)(B(p, r))$ and $\mathcal{F}^1_T(\overline{p})(B(\overline{p}, r))$. We start by giving a brief and informal idea of the proof of Lemma 3.4 below. We would like to find a C^2 volume-preserving diffeomorphism $\hat{\Psi} : \mathfrak{C} \to \mathfrak{F}$ such that $X = \hat{\Psi}_* T$. Given a map $\psi : B(\overline{p}, r) \to B(p, r)$ and $w \in \mathfrak{C}$, let t_w be such that $T^{-t_w} \in B(\overline{p}, r)$. We define $\hat{\Psi}(w) := X^{t_w}(\psi \circ T^{-t_w}(w))$. It is clear that if ψ is volume-preserving, then $\hat{\Psi}$ will be a composition of three volume-preserving maps and we are done. However, some care is needed in the choice of ψ in order to be volume-preserving, and for our objective to be achievable we will use Theorem 3.3 in an appropriate way.

LEMMA 3.4. (Conservative flowbox theorem) *Given a vector field $X \in \mathfrak{X}^s_\mu(M)$ (for $s \geq 2$) and a non-singular point $p \in M$ (eventually periodic with period $\tau > 1$), there exists a conservative C^s diffeomorphism $\Psi : \mathfrak{F} \to \mathfrak{C}$ such that $T = \Psi_* X$.*

Proof. Assume that $p = \vec{0}$ and $X(p) \subseteq \{(x, 0, 0) : x \in \mathbb{R}\}$. Let $X_1(x, y, z)$ be the projection into the first coordinate of $X(x, y, z)$. For a small $r > 0$, we define the functions $f : B(\overline{p}, r) \to \mathbb{R}$ and $g : B(p, r) \to \mathbb{R}$ such that $f(\overline{y}, \overline{z}) := 1$ for $(0, \overline{y}, \overline{z}) \in B(\overline{p}, r)$ and $g(y, z) := X_1(0, y, z)$ for $(0, y, z) \in B(p, r)$ (see Figure 1). Since $g \in C^s$ we apply Theorem 3.3 to $\Omega = B(p, r) \subseteq \mathbb{R}^2$ so there exists a diffeomorphism $\varphi : \Omega \to \varphi(\Omega) \subseteq \mathbb{R}^2$ of class C^s satisfying the partial differential equation $g(\varphi(\overline{y}, \overline{z}))\det D\varphi_{\overline{y},\overline{z}} = \lambda$, for all $(\overline{y}, \overline{z}) \in \Omega$ where $\lambda = \int g / \int 1$, and $\varphi|_{\partial\Omega} = \mathrm{Id}$. Now we define the C^s change of coordinates by

$$\begin{aligned} \hat{\Psi} : \quad & \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^3 \\ & (\overline{x}, \overline{y}, \overline{z}) \longmapsto X^{\lambda^{-1}\overline{x}}(0, \varphi(\overline{y}, \overline{z})). \end{aligned}$$

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First we claim that

$$\det D\hat{\Psi}_{(0,\overline{y},\overline{z})} = 1 \quad \text{for all } (0, \overline{y}, \overline{z}) \in \mathbb{R} \times \Omega. \tag{5}$$

By a straightforward computation of the derivative, we obtain

$$\det(D\hat{\Psi}_{(0,\overline{y},\overline{z})}) = \begin{vmatrix} \lambda^{-1}X_1(X^0(0, y, z)) & 0 & 0 \\ \lambda^{-1}X_2(X^0(0, y, z)) & (\partial\varphi_1/\partial\overline{y})|_{(\overline{y},\overline{z})} & (\partial\varphi_1/\partial\overline{z})|_{(\overline{y},\overline{z})} \\ \lambda^{-1}X_3(X^0(0, y, z)) & (\partial\varphi_2/\partial\overline{y})|_{(\overline{y},\overline{z})} & (\partial\varphi_2/\partial\overline{z})|_{(\overline{y},\overline{z})} \end{vmatrix}.$$

Now using equation (4) of Theorem 3.3 we conclude that

$$\det(D\hat{\Psi}_{(0,\overline{y},\overline{z})}) = \lambda^{-1}X_1((0, y, z)) \det D\varphi_{(\overline{y},\overline{z})} = g(y, z)\lambda^{-1} \det D\varphi_{(\overline{y},\overline{z})} = 1,$$

and therefore (5) is proved. Let us now check that $\det D\hat{\Psi}_{(\overline{x}_0,\overline{y}_0,\overline{z}_0)} = 1$ for all $(\overline{x}_0, \overline{y}_0, \overline{z}_0) \in \mathfrak{C}$. We note that

$$\hat{\Psi}(\overline{x}, \overline{y}, \overline{z}) = X^{\lambda^{-1}\overline{x}_0}[X^{\lambda^{-1}(\overline{x}-\overline{x}_0)}((0, \varphi(\overline{y}, \overline{z})))] = X^{\lambda^{-1}\overline{x}_0}[\hat{\Psi}(\overline{x} - \overline{x}_0, \overline{y}, \overline{z})],$$

so

$$D\hat{\Psi}_{(\overline{x},\overline{y},\overline{z})} = DX^{\lambda^{-1}\overline{x}_0}_{\hat{\Psi}(\overline{x}-\overline{x}_0,\overline{y},\overline{z})} D\hat{\Psi}_{(\overline{x}-\overline{x}_0,\overline{y},\overline{z})}.$$

Evaluating $D\hat{\Psi}_{(\overline{x},\overline{y},\overline{z})}$ at $\overline{x} = \overline{x}_0$ we get

$$D\hat{\Psi}_{(\overline{x}_0,\overline{y},\overline{z})} = DX^{\lambda^{-1}\overline{x}_0}_{\hat{\Psi}(0,\overline{y},\overline{z})} D\hat{\Psi}_{(0,\overline{y},\overline{z})}.$$

We use (5) and the fact that the flow X^t is volume-preserving to conclude that $\det D\hat{\Psi}_{(\overline{x}_0,\overline{y}_0,\overline{z}_0)} = 1$. Finally, take $c := \lambda$ and consider the constant vector field $T := (\lambda, 0, 0)$. Let $(x, y, z) = \hat{\Psi}(\overline{x}, \overline{y}, \overline{z})$. By a simple computation, we obtain

$$\begin{aligned} \hat{\Psi}_* T(x, y, z) &= D\hat{\Psi}_{(\overline{x},\overline{y},\overline{z})} T(\overline{x}, \overline{y}, \overline{z}) \\ &= (X_1(X^{\lambda^{-1}\overline{x}}(0, y, z)), X_2(X^{\lambda^{-1}\overline{x}}(0, y, z)), X_3(X^{\lambda^{-1}\overline{x}}(0, y, z))) \\ &= X(\hat{\Psi}(\overline{x}, \overline{y}, \overline{z})). \end{aligned}$$

Taking $\Psi = \hat{\Psi}^{-1}$ we obtain $T = \Psi_* X$. This completes the proof of Lemma 3.4. □

3.3. *Realizable linear flows.* The next definition adapts the definition of realizable sequence given by Bochi in [**4**] and will also be central in the proof of our theorem. In broad terms we consider *modified* area-preserving linear maps acting in the normal bundle at p, $L^t(p) : N_p \to N_{X^t(p)}$, that do exactly what we want. Finally, we ask whether these maps are (γ-almost C^1) realizable as the linear Poincaré flow of Y, ϵ-C^1-close to X, computed on small transversal neighborhoods of one point.

Definition 3.1. Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$ and a non-periodic point p, we say that the modified area-preserving sequence of linear maps $L_j : N_{X^j(p)} \to N_{X^{j+1}(p)}$ for $j = 0, \ldots, n-1$ is an (ϵ, κ)-realizable linear flow of length n at p if, for all $\gamma > 0$, there exists, $r > 0$ such that, for any open set $\emptyset \neq U \subseteq B(p, r) \subseteq N_p$, there exist

(a) a measurable set $K \subseteq U$ such that $\overline{\mu}(K) > (1 - \kappa)\overline{\mu}(U)$ and

(b) a zero divergence vector field Y, ϵ-C^1-close to X, such that

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(i) $Y^t = X^t$ outside $\mathcal{F}^n_X(p)(U)$ and $DX_q = DY_q$ for every $q \in U$, $\mathcal{P}^n_X(p)(U)$, and
(ii) if $q \in K$, then $\|P^1_Y(Y^j(q)) - L_j\| < \gamma$ for $j = 0, 1, \ldots, n-1$.

In the previous definition we consider integer iterates, but there is no restriction if we consider any intermediate linear maps, like $L_j : N_{X^{t_j}(p)} \to N_{X^{t_j+1}(p)}$ with $t_j < t_{j+1}$ and $\sum_{j=0}^{n-1} t_j = n$. The point p may also be periodic, but with period larger than n. The realizability is with respect to the C^1 topology.

Next we show how to produce some elementary realizable linear flows, the linear Poincaré flow itself and also that juxtaposition of two realizable linear flows are realizable linear flows.

LEMMA 3.5. *Let $X \in \mathfrak{X}^1_\mu(M)$ and $p \in M$ a non-periodic point.*

(1) *For all $t \in \mathbb{R}$, $P^t_X(p)$ (trivial map) is (ϵ, κ)-realizable of length t for every ϵ and κ.*
(2) *Let $\{L_0, \ldots, L_{n-1}\}$ be (ϵ, κ_1)-realizable of length n at p and $\{L_n, \ldots, L_{n+m-1}\}$ be (ϵ, κ_2)-realizable of length m at $X^n(p)$. For $\kappa = \kappa_1 + \kappa_2 < 1$ the sequence of linear maps $\{L_0, \ldots, L_{n+m-1}\}$ is (ϵ, κ)-realizable.*

Proof. Claim (1) follows by choosing $Y = X$. Let us prove (2). Given $\gamma > 0$, let r_1, r_2 be the radius according to Definition 3.1 related to the realizable linear flows $\{L_0, \ldots, L_{n-1}\}$ and $\{L_n, \ldots, L_{n+m-1}\}$ respectively. We take any non-empty open set $U \subseteq B(p, r_1)$. If we have $\mathcal{P}^n_X(p)(B(p, r_1)) \subseteq B(X^n(p), r_2)$ fine; otherwise we choose a smaller $r < r_1$. Given $\nu > 0$, decrease the radius r if necessary, by using Lemma 2.2, to get $|\overline{\mu}(K) - x(t)\overline{\mu}(\mathcal{P}^n_X(p)(K))| < \nu$ for all $t \in [0, n]$ and any measurable set $K \subseteq B(p, r)$. By definition and choice of the radius $r > 0$, we have the flowbox $\mathcal{F}^{n+m}_X(p)(U)$. Again by definition, given any $U \subseteq B(p, r)$ we get a measurable $K_1 \subseteq U$ and a vector field Y_1 satisfying (a) and (b) of Definition 3.1. Also for any non-empty open subset of $B(X^n(p), r_2)$, in particular for $\mathcal{P}^n_X(p)(U)$, we get a measurable $\hat{K}_2 \subseteq \mathcal{P}^n_X(p)(U) =: \hat{U}$ and a vector field Y_2 satisfying (a) and (b) of Definition 3.1.

Now define the vector field $Y = Y_1$ in the flowbox $\mathcal{F}^n_X(p)(U)$, $Y = Y_2$ in the flowbox $\mathcal{F}^m_X(X^n(p))(\hat{U})$ and $Y = X$ elsewhere. Then Y is C^1 because by definition $(DY_1)_q = DX_q = (DY_2)_q$ for any $q \in \mathcal{P}^n_X(p)(U)$, so Y and U satisfy (i). To check (a) we define $K := K_1 \cap K_2$ where K_2 is such that $\mathcal{P}^n_X(p)(K_2) = \hat{K}_2$. By Lemma 2.2, we get $x(n)\overline{\mu}(\hat{U}) < \overline{\mu}(U) + \nu$ and also $\overline{\mu}(U \setminus K_2) < x(n)\overline{\mu}(\hat{U} \setminus \hat{K}_2) + \nu$. So we get

$$\begin{aligned}
\overline{\mu}(U \setminus K) &= \overline{\mu}(U \setminus (K_1 \cap K_2)) \\
&\leq \overline{\mu}(U \setminus K_1) + \overline{\mu}(U \setminus K_2) \\
&< \kappa_1 \overline{\mu}(U) + x(n)\overline{\mu}(\hat{U} \setminus \hat{K}_2) + \nu \\
&< \kappa_1 \overline{\mu}(U) + x(n)\kappa_2 \overline{\mu}(\hat{U}) + \nu \\
&< \kappa_1 \overline{\mu}(U) + \kappa_2 \overline{\mu}(U) + \kappa_2 \nu + \nu \\
&= \kappa \overline{\mu}(U) + (1 + \kappa_2)\nu.
\end{aligned}$$

Therefore the result follows by considering a sufficiently small ν. Finally, (ii) follows by definition. □

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Remark 3.1. Using elementary Vitali's covering arguments we obtain that we only have to prove the realizability of the linear maps $\{L_0, \ldots, L_{n-1}\}$ for $U = B(p', r')$ where $B(p', r') \subseteq B(p, r)$.

3.3.1. *Small rotations.* Next we will construct realizable linear flows of time-1 length at p, which rotate by a small angle ξ the action of the linear Poincaré flow, i.e., $L_0 := P_X^1(p)\hat{R}_\xi$ where $\hat{R}_\xi$ is a rotation of angle ξ defined by the matrix

$$\hat{R}_\xi := \begin{pmatrix} \cos(\xi) & -\sin(\xi) \\ \sin(\xi) & \cos(\xi) \end{pmatrix}$$

in a canonical base of N_p. Let R_ξ be the 3×3 matrix associated to the linear map $(x, y, z) \mapsto (x, \hat{R}_\xi(y, z))$.

LEMMA 3.6. *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$ and a fixed time $T = 1$. There exists an angle ξ (not depending on p and $\xi = O(\epsilon)$) such that, for every $\gamma > 0$, there exists $r > 0$ (depending on p) and a zero divergence vector field Y, ϵ-C^1-close to X, such that:*

(a) *$Y - X$ is supported in the flowbox $\mathcal{F}_X^1(p)(B(p, r))$; and*

(b) *for all $q \in B(p, r\sqrt{1-\kappa})$ we have $\|P_Y^1(q) - P_X^1(p)\hat{R}_\xi\| < \gamma$.*

Proof. We take $p \in M$ to be a non-periodic point. Let $C := \max\{\|DX_p^1\| : p \in M\}$ and suppose that this constant is valid for any vector field ϵ-C^1-close to X. Using Lemma 3.4 we obtain a C^2 conservative diffeomorphism $\Psi : \mathfrak{F} \to \mathfrak{C}$ and $T = (c, 0, 0)$. Take a uniform bound $\Theta > 0$ for the norm of the first and second derivatives of Ψ computed in time-1 thin flowboxes and suppose that this constant is also valid for any vector field ϵ-C^1-close to X. We take the angle ξ such that

$$\xi < \frac{\epsilon(1 - \sqrt{1 - \kappa/2})}{4\Theta^2}. \tag{6}$$

Take any $\gamma > 0$. In order to obtain part (a) we note that for any $\alpha < 1$ with $\alpha \approx 1$ there exists $r > 0$ such that $X^{[0,\alpha]}(q) \cap N_{X^1(p)} = \emptyset$, for all $q \in B(p, r)$. Fix $\alpha > 0$ and let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $g(t) = 0$ for $t < 0$, $g(t) = t$ for $t \in [1 - \alpha, 2\alpha - 1]$, and $g(t) = \alpha$ for $t \geq \alpha$ and $\dot{g} \leq 2$. Now for all $q = (0, y, z) \in B(p, r)$ we decrease r in order to have

$$|y|, |z| < \min\left\{\frac{c\epsilon}{\Theta^2 \ddot{g}}, \frac{\epsilon}{2\Theta}\right\}. \tag{7}$$

For such an $r > 0$ let $G : \mathbb{R} \to [0, 1]$ be a C^∞ function such that $G(\rho) = 1$ for $\rho \leq r\sqrt{1 - \kappa/2}$, $G(\rho) = 0$ for $\rho \geq r$ and $\dot{G} \leq 2[(1 - \sqrt{1 - \kappa/2})r]^{-1}$. Let $\rho = \sqrt{y^2 + z^2}$ and consider the rotation flow $R_{\xi g(t) G(\rho)}(0, y, z)$ acting on N_p, which we denote by $\mathcal{R}_t(q)$, and is defined by $\mathcal{R}_t(0, y, z) = (0, \hat{R}_{\xi g(t) G(\rho)}(y, z))$. Denote the time derivative by $\dot{\mathcal{R}}_t$. By a simple computation, we obtain that

$$\dot{\mathcal{R}}_t \mathcal{R}_t^{-1}(0, y, z) = \xi \dot{g}(t) G(\rho)(0, -z, y). \tag{8}$$

We consider the flow $T^t = (ct, 0, 0)$ associated to T and for $q = (0, y, z) \in B(p, r)$ we define $\Upsilon(t, q) := T^t(\mathcal{R}_t(q))$. Let $H(t, q) := (t, \mathcal{R}_t(q))$ and $F(t, \mathcal{R}_t(q)) := T^t(\mathcal{R}_t(q)) =$

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$\Upsilon(t,q)$ so we obtain $\Upsilon(t,q) = F \circ H(t,q)$. We take time derivatives at $t=s$:

$$\begin{aligned}\frac{d}{dt}\Upsilon(t,q)\Big|_{t=s} &= \frac{d}{dt}T^t(\mathcal{R}_t(q))\Big|_{t=s} = DF_{H(s,q)}\cdot DH_s \\ &= \begin{pmatrix}\partial_1 F & \partial_2 F\end{pmatrix}_{H(s,q)}\begin{pmatrix}\partial_1 H \\ \partial_2 H\end{pmatrix}_s \\ &= \left(T(T^s(\mathcal{R}_s))DT^s_{\mathcal{R}_s(q)}\right)\begin{pmatrix}1 \\ \dot{\mathcal{R}}_s(q)\end{pmatrix} \\ &= T(T^s\circ\mathcal{R}_s(q)) + DT^s_{\mathcal{R}_s(q)}\dot{\mathcal{R}}_s(q).\end{aligned}$$

The vector field Z is defined in flowbox coordinates by

$$Z(\cdot) = T(\cdot) + DT^s_{\mathcal{R}_s(q)}\cdot\dot{\mathcal{R}}_s(\mathcal{R}_{-s}\cdot T^{-s})(\cdot).$$

We use (8) and we obtain that the C^1-perturbation is given by $Z = T + P$ with

$$P(x,y,z) = \xi f(x,y,z).A(x,y,z), \tag{9}$$

where $f(x,y,z)$ is the scalar function $\dot{g}(x/c)G(\sqrt{y^2+z^2})$ and $A(x,y,z)$ is the linear map $(0,-z,y)$. It is straightforward to see that $\operatorname{div} Z = 0$. We also have that the support of the perturbation P is $B(p,r)\times[0,c\alpha]\subset\mathfrak{C}$. Now we are going to estimate the C^1-norm of P. Note that by (7) we have $\|P\|_{C^0}\le\epsilon/\Theta$. To compute the C^1-norm we take derivatives in (9) and by the product rule we have

$$\begin{aligned}DP &= \xi[\nabla f\cdot A(x,y,z) + f.A] \\ &= \xi\left[\left(\ddot{g}\left(\frac{x}{c}\right)Gc^{-1}, \dot{g}\left(\frac{x}{c}\right)\dot{G}\frac{y}{\sqrt{y^2+z^2}}, \dot{g}\left(\frac{x}{c}\right)\dot{G}\frac{z}{\sqrt{y^2+z^2}}\right)\cdot A(x,y,z) + f.A\right].\end{aligned}$$

We use again (7) to bound the terms $\ddot{g}(x/c)G(\rho)c^{-1}z$ and $\ddot{g}(x/c)G(\rho)c^{-1}y$. Let us see that the other terms are unaffected by the choice of $r>0$ small. We take, for example,

$$\dot{g}(x/c)\dot{G}\frac{y}{\sqrt{y^2+z^2}}z$$

and polar coordinates $(y,z) = (\rho\cos(\beta), \rho\sin(\beta))$; then we have

$$\xi\dot{g}(x/c)\dot{G}\frac{\rho\cos(\beta)}{\rho}\rho\sin(\beta) \le \xi\dot{G}\frac{2\rho^2}{\rho} \le \xi\frac{4\rho}{(1-\sqrt{1-\kappa/2})r} \le \xi\frac{4}{(1-\sqrt{1-\kappa/2})} < \frac{\epsilon}{\Theta^2}.$$

For the other three analogous terms we proceed in the same way and we obtain that $\|DP\|_{C^0} < \epsilon/\Theta^2$. Note that we are allowed to take y, z close to zero without interfering with the size of the perturbation. This is a key property of the C^1 topology. For $q\in B(p, r\sqrt{1-\kappa})$ we have $Z^1(\Psi(q)) = (c, R_{\xi\alpha}(\Psi(q)))$, so $P^1_Z(\Psi(q)) = \hat{R}_{\xi\alpha}\approx\hat{R}_\xi$ (just choose r very small). Hence we obtain the analog of part (b) for the trivial vector field. Now it is straightforward to see that part (b) follows by choosing r very small (depending on the constants γ, C and Θ), which is possible by what we have seen above. Finally, we estimate the C^1-norm of P_1. Using (7) and the choice of ξ in (6) we obtain that $\|Y-X\|_{C^1} = \|P_1\|_{C^1}\le\epsilon$ and Lemma 3.6 is proved. □

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LEMMA 3.7. *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$ and a fixed time $T = 1$. There exists an angle ξ (not depending on p) such that, for every $\gamma > 0$, there exists $r > 0$ (depending on p) and a zero divergence vector field Y, ϵ-C^1-close to X, such that:*

(a) *$Y - X$ is supported in $\mathcal{F}_X^{-1}(p)(B(p, r)) = \{\mathcal{P}_X^t(p)(q) : q \in B(p, r), t \in [-1, 0]\}$;*

(b) *for all $q \in \mathcal{P}_X^{-1}(p)(B(p, \sqrt{1-\kappa}r))$ we have $\|P_Y^1(q) - \hat{R}_\xi P_X^1(X^{-1}(p))\| < \gamma$.*

Proof. We proceed like in Lemma 3.6, this time for X^{-t} finding a change of coordinates $\hat{\Psi}(x, y, z) = X^{-\lambda^{-1}x}(0, \varphi(y, z))$. Then we consider $R^{-1}_{\xi g(t) G(\rho)}$, $t > 0$ and we find Z. We define $Z = \Psi_* Y$ and we get

$$P_Y^1(q) = [P_Y^{-1}(Y^1(q))]^{-1} \approx [P_X^{-1}(p)\hat{R}_\xi^{-1}]^{-1} = \hat{R}_\xi [P_X^{-1}(p)]^{-1} = \hat{R}_\xi P_X^1(X^{-1}(p)),$$

and the arguments are equal to the ones used in the proof of Lemma 3.6. □

Now we use the two previous lemmas to build some useful realizable linear flows.

LEMMA 3.8. *Given $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$, a non-periodic point p and a fixed time $T = 1$. Then there exists an angle ξ (not depending on p) such that $L_0 = P_X^1(p)\hat{R}_\xi$ and $L_0 = \hat{R}_\xi P_X^1(p)$ are (ϵ, κ)-realizable linear flows of length one at p.*

Proof. We prove that $L_0 = P_X^1(p)\hat{R}_\xi$ is (ϵ, κ)-realizable. Let $\gamma > 0$. By Remark 3.1, we may choose the open set U to be a ball, say $B(p', r') \subseteq B(p, r)$. Now we apply Lemma 3.6 and we get a zero divergence vector field Y, ϵ-C^1-close to X such that $Y - X$ is supported inside the flowbox $\mathcal{F}_X^1(p')(B(p', r'))$ and for every $q \in B(p', r'\sqrt{1-\kappa})$ we have $\|P_Y^1(q) - P_X^1(p')\hat{R}_\xi\| < \gamma$. Note that, since $r > 0$ can be taken very small, the arrival time at $N_{X^1(p)}$ for points in $B(p, r)$ is almost one.

Taking $K = \overline{B}(p', r'\sqrt{1-\kappa}) \subseteq U$ we get

$$\frac{\overline{\mu}(K)}{\overline{\mu}(U)} = \frac{\pi(1-\kappa)r'^2}{\pi . r'^2} = 1 - \kappa$$

so part (a) follows. The support of the perturbation is contained in the flowbox. For the perturbation P defined in Lemma 3.6 we have $DX_q = DY_q$ for any $q \in B(p', r') \cup \mathcal{P}_X^1(p)(B(p', r'))$. Therefore (i) in Definition 3.1 is true. Finally, (ii) follows from (b) of Lemma 3.6 and the continuity of the linear Poincaré flow.

For $L_0 = \hat{R}_\xi P_X^1(p)$ we proceed analogously, now using Lemma 3.7. □

LEMMA 3.9. *Given $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$ and a non-periodic point p, there exists an angle ξ such that, for $|\xi_i| < \xi$, $i = 1, 2$,*

$$N_p \xrightarrow{P_X^1(p)\hat{R}_{\xi_1}} N_{X^1(p)} \xrightarrow{P_X^r(p)} N_{X^{1+r}(p)} \xrightarrow{\hat{R}_{\xi_2} P_X^1(X^{1+r}(p))} N_{X^{r+2}(p)}$$

is an (ϵ, κ)-realizable linear flow of length $r + 2$ at p.

Proof. Take $\gamma > 0$. By Lemma 3.8, for $\kappa_1 < \kappa$ we get ξ such that $P_X^1(p)\hat{R}_{\xi_1}$ and $\hat{R}_{\xi_2} P_X^1(X^{1+r}(p))$ are (ϵ, κ_1)-realizable. By Lemma 3.5(1), the trivial map P_X^r is (ϵ, κ_1)-realizable. Now if $\kappa_1 = \kappa/3$, then we use Lemma 3.5(2) and we obtain the (ϵ, κ)-realizability. □

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3.3.2. *Large rotations.* Now we will find conditions under which we can rotate by large angles. In the previous section we were able to rotate by time 1, so what happens if we increase time? We want to rotate by an angle 2π; thus we take a time m such that $\xi m = 2\pi$. But ξ is in general very small, so m must be very large. Note that the choice of m may affect the norm of the perturbation because $\|\Psi\|$, for Ψ given by Lemma 3.4, depends on m and in general increases with m. One of the reasons for this fact is that, for $v \in N_p$, we may have $\measuredangle(X^m(v), X(X^m(p)))$ very small. Furthermore the dynamics along the orbit may also obstruct the construction of a small norm perturbation. Let us consider a situation on which this last problem is minimized, say when we have simultaneously:

(a) no domination for each bundle, i.e., $P_X^t(p)$ is 'almost conformal' for all $t \in [0, m]$;

(b) $\measuredangle(N^u_{X^t}(p), N^s_{X^t}(p)) \approx \pi/2$, for all $t \in [0, m]$.

Even if we have properties (a) and (b), our perturbations may not have a small C^1-norm, because of the aforementioned problem of the decay into $\mathbb{R}X$ of the normal directions. If this problem does not occur, then under conditions (a) and (b) we could perform large rotations with just one single perturbation. In fact, this is what we did when we carried out the development of perturbations for linear differential systems (see [**3**, Lemma 3.5]). In general, we will concatenate several time-1 small rotations; however this concatenation worsens κ. In [**4**], this problem is bypassed by using a nested rotation lemma (Lemma 3.7 in [**4**]) and here we will adapt this procedure.

Let $\mathcal{E}(p) \subseteq N_p$ be an ellipse centered in p. As in [**4**], given an ellipse we call

$$E := \sqrt{\frac{\text{major axis}}{\text{minor axis}}}$$

its *eccentricity*. We also consider $J \in SL(2, \mathbb{R})$ the map such that $J(\mathcal{E}(p))$ is a disk. For $r > 0$ we define the ellipse $\mathcal{E}(p, r) := J^{-1}(B(p, r))$. Given any ellipse $\mathcal{E}(p, r)$ and $J \in SL(2, \mathbb{R})$ such that $J(\mathcal{E}(p, r))$ is a disk, we denote by $\hat{E}_\xi$ the elliptical rotation defined by $J^{-1}\hat{R}_\xi J$.

LEMMA 3.10. *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, a fixed time $T = 1$ and $E \geq 1$. There exists $\hat{\epsilon} > 0$ (not depending on p) such that, for every $\gamma > 0$, there exists $r > 0$ (depending on p) with the following properties.*

If $\mathcal{E}(p, r)$ is an ellipse with eccentricity at most E, $\operatorname{diam}(\mathcal{E}(p, r)) < \epsilon$, $\hat{E}_\xi$ is a rotation of the ellipse $\mathcal{E}(p, r)$ and $\|P^1_X(p) - P^1_X(p)\hat{E}_\xi\| < \hat{\epsilon}$, then there exists a C^1 zero divergence vector field Y, ϵ-C^1-close to X such that:

(a) *$Y - X$ is supported in the flowbox $\mathcal{F}^1_X(p)(\mathcal{E}(p, r))$; and*

(b) *for all $q \in \mathcal{E}(p, r\sqrt{1-\kappa})$ we have $\|P^1_Y(q) - P^1_X(p)\hat{E}_\xi\| < \gamma$.*

Proof. The proof is the same as for Lemma 3.6, but the angle ξ depends also on E. We use Lemma 3.4 and we consider the flowbox $\mathcal{F}^1_T(p)(\mathcal{E}(p, r))$. Let $J \in SL(2, \mathbb{R})$ be such that $J(\mathcal{E}(p, r))$ is a disk. Now we take the flowbox $\mathcal{F}^1_T(p)(J(\mathcal{E}(p, r)))$ and we define the elliptical rotation by $E_{\xi g(t)G(\rho)} := J^{-1}R_{\xi g(t)G(\rho)}J$. Since $E = \|J\| = \|J^{-1}\|$, ξ should be smaller than that in Lemma 3.6, which is obtained by taking $\hat{\epsilon} \approx 0$. □

The next simple lemma says that if we fix a small ellipse in a small ball $B(p, r) \subseteq N_p$ and consider its arrival into $N_{X^1(p)}$, then this set is almost the image by the linear Poincaré

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flow at p of the same ellipse modulo translations. A similar statement is proved in Lemma 3.6 of [**4**].

LEMMA 3.11. *Let $X^t : M \longrightarrow M$ be a C^1-flow, $\zeta \in]0, 1[$ (near 1), $E \geq 1$. There exists $r > 0$ such that, for all ellipses $\mathcal{E}(q, \tilde{r}) \subseteq B(p, r) \subseteq N_p$ with eccentricity at most E we have*

$$P_X^1(p)(\mathcal{E}(q, \zeta\tilde{r}) - q) + \mathcal{P}_X^1(p)(q) \subseteq \mathcal{P}_X^1(p)(\mathcal{E}(q, \tilde{r})) \subseteq P_X^1(p)(\mathcal{E}(q, (2-\zeta)\tilde{r}) - q) + \mathcal{P}_X^1(p)(q).$$

Since $P_X^t(p)$ is modified area-preserving, we measure the non-conformality using its norm $\|P_X^t(p)\|$ in the following way. Suppose that $d \geq a$ are the major axis and the minor axis (respectively) of the ellipse $\mathcal{E}(p) = P_X^t(p)(B(p, 1))$. Then the eccentricity of $\mathcal{E}(p)$ is $E = \sqrt{d/a}$. Since $\|P_X^t(p)\| = d$ and by volume-preserving we have $a^{-1} = d.x(t)$, we conclude that $E = \sqrt{d/a} = \sqrt{d^2 x(t)} = d\sqrt{x(t)} = \|P_X^t(p)\|\sqrt{x(t)}$. The next lemma is a version of Lemma 3.7 of [**4**] and says that bounded eccentricity is crucial to concatenate many elliptical small rotations keeping κ controlled.

LEMMA 3.12. *Given $X \in \mathfrak{X}_\mu^2(M)$, $\epsilon > 0$, $0 < \kappa < 1$ and $E \geq 1$, there exists $\hat{\epsilon} > 0$ satisfying the following. Let $p \in M$ be a non-periodic point and suppose that for some $n \in \mathbb{N}$ we have $\|P_X^j(p)\| \leq E\sqrt{x^{-1}(j)}$ for $j = 1, \ldots, n$. If $L_j : N_{X^j(p)} \to N_{X^{j+1}(p)}$ for $j = 0, \ldots, n-1$ is a sequence of linear maps satisfying*

(a) $L_{j-1} \cdots L_0(B(p, 1)) = P_X^j(p)(B(p, 1))$ *for* $j = 1, \ldots, n$, *and*

(b) $\|P_X^1(X^j(p)) - L_j\| < \hat{\epsilon}$ *for* $j = 0, 1, \ldots, n-1$,

then $\{L_0, L_1, \ldots, L_{n-1}\}$ is an (ϵ, κ)-realizable linear flow at p.

Proof. Let us start by fixing some constants. We choose $\kappa_0 < \kappa$ by taking $\lambda \in]0, 1[$ near 1 such that $\lambda^{4n}(1 - \kappa_0) > 1 - \kappa$. We take $\zeta \in]0, 1[$, such that $\zeta \in]\lambda, 1[$ and $2 - \zeta \in]1, \lambda^{-1}[$. Let $\hat{\epsilon}$ be given by Lemma 3.10 depending on ϵ, E and κ_0. Using part (a) and hypothesis $\|P_X^j(p)\| \leq E\sqrt{x^{-1}(j)}$ (for each j) we consider $\hat{E}_j$ the rotation of the ellipse $P_X^j(p)(B(p, 1))$. Clearly $L_j := P_X^1(X^j(p))\hat{E}_j$ satisfy part (b) and these are the linear maps we will be interested to (ϵ, κ)-realize. Let $\gamma > 0$ be given. By Lemma 3.11 applied n times and shrinking the radii a little bit at each step, there exists $r_1 > 0$ such that for each j and all ellipses $\mathcal{E}(q, \tilde{r}) \subseteq B(X^j(p), r_1) \subseteq N_{X^j(p)}$ with eccentricity $\leq E$ we have:

- $P_X^1(X^j(p))(\mathcal{E}(q, \zeta\tilde{r}) - q) + \mathcal{P}_X^1(X^j(p))(q) \subseteq \mathcal{P}_X^1(X^j(p))(\mathcal{E}(q, \tilde{r}))$;
- $\mathcal{P}_X^1(X^j(p))(\mathcal{E}(q, \tilde{r})) \subseteq P_X^1(X^j(p))(\mathcal{E}(q, (2-\zeta)\tilde{r}) - q) + \mathcal{P}_X^1(X^j(p))(q)$.

Take $r < r_1$. Now we will define the vector field Y and the measurable set K as in Definition 3.1. By Remark 3.1, we consider $U = B(p', r') \subseteq B(p, r)$. For each j and denoting $\mathcal{P}_X^j(p)(p')$ by p'_j we define a sequence of ellipses $\mathcal{E}_s^j \subseteq N_{X^j(p)}$ all of eccentricity at most E by $\mathcal{E}_s^0 = B(p', sr')$ for $s \in]0, 1]$ and $\mathcal{E}_s^j = P_X^j(p)(B(p', sr') - p') + p'_j$. Decreasing r if necessary, these ellipses are in the conditions of Lemma 3.11 so for each j we also have $\mathcal{E}_{s\lambda^{-1}}^{j+1} \supset \mathcal{E}_{s(2-\zeta)}^{j+1} \supseteq \mathcal{P}_X^1(X^j(p))(\mathcal{E}_s^j) \supseteq \mathcal{E}_{s\zeta}^{j+1} \supseteq \mathcal{E}_{s\lambda}^{j+1}$.

For each j we apply Lemma 3.10 to p'_j, κ_0, $\mathcal{E}_s^j$ and $\hat{E}_j$, with $s = \lambda^n$. So, there exists a vector field Y_j such that:

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(i) $Y_j - X$ is supported in the flowbox $\mathcal{F}^1_X(p'_j)(\mathcal{E}^j_{\lambda^n})$; and

(ii) for every $q_j \in \mathcal{E}^j_{\lambda^n\sqrt{1-\kappa_0}}$ we have $\|P^1_Y(q_j) - P^1_X(p'_j)\hat{E}_j\| < \gamma$.

By (i), the Y_j have disjoint supports so we define $Y := \sum_{j=0}^{n-1} Y_j$. Define also $K := \overline{\mathcal{E}}^0_{\lambda^{2n}\sqrt{1-\kappa_0}} = \overline{B}(p', \lambda^{2n}\sqrt{1-\kappa_0}r')$. Hence we obtain

$$\frac{\overline{\mu}(K)}{\overline{\mu}(U)} = \frac{\pi(\lambda^{2n}\sqrt{1-\kappa_0}\, r')^2}{\pi r'^2} = \lambda^{4n}(1-\kappa_0) > 1-\kappa.$$

Let us see that when we iterate we have a nested sequence, i.e., for all $q \in K$ and each j we have $Y^j(q) \in \mathcal{E}^j_{\lambda^n\sqrt{1-\kappa_0}}$. We have $\mathcal{P}^1_Y(p')(\overline{\mathcal{E}}^0_s) \subseteq \overline{\mathcal{E}}^1_{s(2-\zeta)} \subset \overline{\mathcal{E}}^1_{s\lambda^{-1}}$, so for every j we obtain $\mathcal{P}^j_Y(p')(\overline{\mathcal{E}}^0_s) \subset \overline{\mathcal{E}}^j_{s\lambda^{-j}} \subseteq \overline{\mathcal{E}}^j_{s\lambda^{-n}}$. Hence for $s = \lambda^{2n}\sqrt{1-\kappa_0}$ we get $\mathcal{P}^1_Y(p')(K) \subset \overline{\mathcal{E}}^j_{\lambda^n\sqrt{1-\kappa_0}}$, and the orbit of q will always be inside the domain of each of these rotations. Finally, to prove that for all $q \in K$ we have $\|P^1_Y(Y^j(q)) - L_j\| < \gamma$ we use (ii) and we go back and decrease r once more, if necessary. □

3.4. *Blending Oseledets directions along an orbit segment.* Given $p \in \Gamma^+_m(X)$, we suppose that for $c \gg 1$ and $0 \le t + r \le m$ we have

$$\Delta(X^t(p), r) = \frac{\|P^r_X(X^t(p))|_{N^s_{X^t(p)}}\|}{\|P^r_X(X^t(p))|_{N^u_{X^t(p)}}\|} \ge c.$$

Therefore the dynamics sends vectors near $N^u_{X^t(p)}$ into vectors near $N^s_{X^{t+r}(p)}$ during that period. The next simple lemma, whose proof may be found in [**4**] (see Lemma 3.9), clarifies this behavior. Denote by $n^\sigma_t \in N^\sigma_{X^t(p)}$ two unit vectors, for $\sigma = u, s$.

LEMMA 3.13. *Given an angle ξ, there exists $c > 1$ such that, if $\Delta(X^t(p), r) > c$, then there exists $v \in N_{X^t(p)} \setminus \{\vec{0}\}$ such that $\measuredangle(v, n^u_t) < \xi$ and $\measuredangle(P^r_X(X^t(p)) \cdot v, n^s_{t+r}) < \xi$.*

The next lemma gives us the conditions under which we may apply Lemma 3.12. For a proof of the next lemma see [**2**, Lemma 4.0.11].

LEMMA 3.14. *Let $\xi > 0$ and $d > 1$ be given. There exists $E > 1$ such that: if, for all $t > 0$, $\measuredangle(N^u_{X^t(p)}, N^s_{X^t(p)}) > \xi$ and*

$$d^{-1} \le \frac{\|P^t_X(p)|_{N^s_p}\|}{\|P^t_X(p)|_{N^u_p}\|} \le d$$

then, for all $t > 0$, we have $\|P^t_X(p)\| \le E\sqrt{x(t)^{-1}}$.

Now we are able to mix the Oseledets subspaces by small perturbations along orbits with lack of hyperbolicity.

LEMMA 3.15. *Let $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$ and $0 < \kappa < 1$. There exists $m \in \mathbb{N}$ such that for every*

$$p \in \Delta_m(X) = \left\{ p \in \mathcal{O}^+ : \frac{\|P^m_X(p)|_{N^s_p}\|}{\|P^m_X(p)|_{N^u_p}\|} \ge \frac{1}{2} \right\}$$

there exists an (ϵ, κ)-realizable linear flow such that $L^m(N^u_p) = N^s_{X^m(p)}$.

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Proof. First we set up the constants. Take $\xi > 0$ the minimum of the angles satisfying simultaneously Lemmas 3.8 and 3.9 and depending on X, ϵ and $\kappa/2$. Take $C := \max\{\|DX^{\pm 1}\| : p \in M\}$ and c given by Lemma 3.13 depending on the angle ξ. It will also be useful to take $c > C^2$. Lemma 3.14 gives us $E > 1$ depending on ξ and $d = 2c^2$. Let $\hat{\epsilon} > 0$ depending on X, ϵ, κ and E given by Lemma 3.12. Let $\beta > 0$ be such that, for $\xi_0 < \beta$, $\|R_{\xi_0} - \mathrm{Id}\| \leq C^{-1}E^{-2}\hat{\epsilon}$. Finally, we take a very large $m \in \mathbb{N}$ satisfying $m \geq 2\pi/\beta$. We divide the proof into three parts.

Part I – A small angle between the Oseledets subspaces. We suppose that

$$\text{for some } r \in [0, m] \text{ we have } \measuredangle(N^u_{X^r(p)}, N^s_{X^r(p)}) < \xi. \tag{10}$$

We take advantage of this fact and we define a realizable linear flow of length one in the following way. If $r < m - 1$ the linear map is based at $X^r(p)$ and is defined by $L_0 := P^1_X(X^r(p))R_\xi$, and if $r > m - 1$ the linear map is based at $X^{r-1}(p)$ and is defined by $L_0 := R_\xi P^1_X(X^{r-1}(p))$. Now we use Lemma 3.8 and concatenate from right and left, if necessary, with trivial maps by using Lemma 3.5(1). We obtain $L^m(N^u_p) = N^s_{X^m(p)}$.

Part II – Locally N^s dominates N^u. Now we suppose that

$$\text{for some } 0 \leq r + t \leq m \text{ we have } \Delta(X^t(p), r) > c. \tag{11}$$

We use Lemma 3.13 and there exists a vector $v \in N_{X^t(p)}$ such that $\measuredangle(v, n^u_t) < \xi$ and $\measuredangle(P^r_X(X^t(p)) \cdot v, n^s_{t+r}) < \xi$. Since ξ is small, we apply Lemma 3.8 at $X^t(p)$ and at $X^{t+r}(p)$. By the choice of c above, we get $r > 2$ and so we have disjoint perturbations. Therefore, our first rotation allows us to send $N^u_{X^t(p)}$ onto $v.\mathbb{R}$, the dynamics helps us and maps this direction into $P^r_X(X^t(p)) \cdot v$ in time r, and finally another rotation sends $P^r_X(X^t(p)) \cdot v.\mathbb{R}$ onto $N^s_{X^{t+r}(p)}$. Now we use Lemma 3.5 and concatenate the three realizable linear flows, say rotation–trivial–rotation, by using Lemma 3.9 and we get $L^m(N^u_p) = N^s_{X^m(p)}$.

Part III – Conformal behavior. Finally, we suppose that we do not have both (10) and (11). We set up the conditions of Lemma 3.14. Since $\Delta(p, m) \geq \frac{1}{2}$ and (11) is false we conclude that

$$\Delta(X^r(p), t) = \Delta(X^{t+r}(p), m - t - r)^{-1}\Delta(p, m)\Delta(p, r)^{-1} \geq (2c^2)^{-1}.$$

Therefore, since $d = 2c^2$, we get

$$d^{-1} \leq \frac{\|P^t_X(X^r(p))|_{N^s_{X^r(p)}}\|}{\|P^t_X(X^r(p))|_{N^u_{X^r(p)}}\|} \leq d \quad \text{for all } r, t : 0 \leq r + t \leq m.$$

In particular for $r = 0$ we have that $\measuredangle(N^u_{X^t(p)}, N^s_{X^t(p)}) > \xi$, for all $t \in [0, m]$. We use Lemma 3.14 and conclude that $\|P^t_X(p)\| \leq E\sqrt{x(t)^{-1}}$ for all $t \in [0, m]$.

Take $\xi_0, \xi_1, \ldots, \xi_{m-1}$ with $\xi_j < \beta$ for each j and such that $\sum_{j=0}^{m-1} \xi_j = \measuredangle(N^u_p, N^s_p)$. We define

$$\begin{aligned} L_j : \quad N_{X^j(p)} &\longrightarrow N_{X^{j+1}(p)} \\ v &\longmapsto P^{j+1}_X(p)R_{\xi_j}[P^j_X(p)]^{-1} \cdot v. \end{aligned}$$

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Let us see that we are in the hypotheses of Lemma 3.12. Since by definition for each j we have $L_{j-1}\cdots L_0 = P_X^j(p)R_{\sum_{i=0}^{j-1}\xi_i}$ we obtain part (a) of Lemma 3.12. Now we have

$$\begin{aligned}\|P_X^1(X^j(p)) - L_j\| &\le \|P_X^1(X^j(p)) - P_X^{j+1}(p)R_{\xi_j}[P_X^j(p)]^{-1}\| \\ &= \|P_X^1(X^j(p))[\mathrm{Id} - P_X^j(p)R_{\xi_j}[P_X^j(p)]^{-1}]\| \\ &\le \|P_X^1(X^j(p))\|\|P_X^j(p)[\mathrm{Id} - R_{\xi_j}][P_X^j(p)]^{-1}]\| \\ &\le \|P_X^1(X^j(p))\|\|P_X^j(p)\|\|[P_X^j(p)]^{-1}\|\|\mathrm{Id} - R_{\xi_j}\| \\ &\le CE\sqrt{x^{-1}(j)}E\sqrt{x(j)}\|\mathrm{Id} - R_{\xi_j}\|.\end{aligned}$$

In last inequality we use $\|P_X^{-t}\| \le E\sqrt{x(t)}$. Therefore we obtain

$$\|P_X^1(X^j(p)) - L_j\| \le CE^2\|\mathrm{Id} - R_{\xi_j}\| \le \hat{\epsilon}$$

and part (b) of Lemma 3.12 is true. By this lemma, we have the realizability; therefore

$$L^m(N_p^u) = L_{m-1}\cdots L_0(N_p^u) = P_X^m(p)R_{\sum_{j=0}^{m-1}\xi_j}(N_p^u) = P_X^m(p).N_p^s = N^s_{X^m(p)},$$

which proves the lemma. □

3.5. *Lowering the norm – local procedure.* We begin this section by adapting two lemmas to our setting: Lemma 3.12 from [**4**] and Lemma 4.5 from [**5**]. The first lemma (for a proof see [**2**, Lemma 5.0.13]) gives us information about *when* we have a recurrence to a positive measure set. The second lemma is an elementary result which relates the original norm with a new norm.

LEMMA 3.16. *Let $X^t : M \to M$ be a measurable μ-invariant flow, $\Delta \subseteq M$ a positive measure set, $\Gamma = \bigcup_t X^t(\Delta)$ and $\gamma > 0$. There exists a measurable function $T : \Gamma \to \mathbb{R}$ such that, for μ-a.e. $p \in \Gamma$, all $t \ge T(p)$ and every $\tau \in [0, 1]$, there exists some $s \in [0, t]$ satisfying $|s/t - \tau| < \gamma$ and $X^s(p) \in \Delta$.*

Consider $p, q := X^t(p) \in \Gamma$ and the map $P : N_p \to N_q$ with matrix written relative to the *Oseledets basis* (given by $\{n_p^u, n_p^s\}$ and $\{n_q^u, n_q^s\}$) as

$$P = \begin{pmatrix} a^{uu} & a^{us} \\ a^{su} & a^{ss} \end{pmatrix}.$$

Let $\|P\|_{\max} = \max\{|a^{uu}|, |a^{us}|, |a^{su}|, |a^{ss}|\}$.

LEMMA 3.17. *We have*
(a)

$$\|P\| \le 4\frac{1}{\sin \measuredangle(N_p^u, N_p^s)}\|P\|_{\max},$$

(b)

$$\|P\|_{\max} \le \frac{1}{\sin \measuredangle(N_q^u, N_q^s)}\|P\|.$$

Now we are able to decrease the norm under a small perturbation.

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LEMMA 3.18. *Let $X \in \mathfrak{X}^2_\mu(M)$, with X^t aperiodic and such that all hyperbolic sets have zero measure. Let $\epsilon, \delta > 0$, $0 < \kappa < 1$. There exists a measurable function $T : M \to \mathbb{R}$ such that, for μ-a.e. $p \in M$ and every $t \geq T(p)$, there exists an (ϵ, κ)-realizable linear flow at p with length t such that $\|L^t(p)\| \leq e^{t\delta}$.*

Proof. First we take $m \in \mathbb{R}$ large enough given by Lemma 3.15 and depending on $X, \epsilon, \kappa/2$. Abbreviate $\Gamma = \Gamma^+_m(X)$, $\Delta = \Delta_m(X)$ and note that X^t aperiodic implies that $\mu(\Gamma^*_m(X)) = \mu(\Gamma)$. We have $\mu(\mathcal{O}^0 \cup \Gamma) = 1$, because otherwise there exists a hyperbolic set with positive measure which contradicts our hypothesis. We suppose that $\mu(\Gamma) > 0$ because if $\mu(\Gamma) = 0$ then for μ-a.e. point $p \in M$ we have $\lambda^+(p) = 0$ and the proof is over because a trivial map does the work. Remember that $\Gamma = \bigcup_t X^t(\Delta)$. For μ-a.e. $p \in \Gamma$ the Oseledets' theorem gives us $Q(p)$ such that for all $t \geq Q(p)$ we have:

(1) $(1/t) \log \|P^t_X(p) \cdot n^u\| < \lambda^+(p) + \delta$ for all $n^u \in N^u_p \setminus \{\vec{0}\}$;

(2) $(1/t) \log \|P^t_X(p) \cdot n^s\| < -\lambda^+(p) + \delta$ for all $n^s \in N^s_p \setminus \{\vec{0}\}$;

(3)
$$\log \frac{1}{\sin \measuredangle(N^u_{X^t(p)}, N^s_{X^t(p)})} < t\delta.$$

By using Lemma 3.16 with $\tau = 1/2$, we get recurrence to Δ approximately in the middle of the orbit segment. However, to get good estimates to the norm of the linear map L^t, points in the orbit and after this time must also satisfy (1) and (2). Let $B_n := \{p \in \Gamma : Q(p) \leq n\}$ for $n \in \mathbb{N}$. Of course $B_n \subseteq B_{n+1}$ and $\mu(\Gamma \setminus B_n) \to_{n\to\infty} 0$. Consider now a family of sets defined by $C_0 := \emptyset$ and $C_n := \bigcup_t X^t(\Delta \cap X^{-m}(B_n))$. Clearly $C_n \to_{n\to\infty} \Gamma$, so the measurable function $T : \Gamma \to \mathbb{R}$ will be μ-a.e. defined on each $C_n \setminus C_{n-1}$ for $n \in \mathbb{N}$. Taking $c > \max\{\log \|DX^1_p\| : p \in M\}$ yields the Lyapunov exponents of any $p \in \mathcal{O}$ less than c. For p we have non-zero Lyapunov exponents so we have the Oseledets one-dimensional subspaces N^u_p and N^s_p. Let $\gamma = \min\{1/6, \delta/c\}$. Now we use Lemma 3.16, replacing Δ by $\Delta \cap X^{-m}(B_n)$ and Γ by $\bigcup_t X^t(\Delta \cap X^{-m}(B_n))$. By this lemma, for each n, there exists a measurable function $T_n : C_n \to \mathbb{R}$ such that for μ-a.e $p \in C_n$, and for all $t \geq T_n(p)$, there exists some $s \in [0, t]$ satisfying $X^s(p) \in \Delta \cap X^{-m}(B_n)$ and $|s/t - \frac{1}{2}| < \gamma$. Now we define a sufficiently large $T(p)$ for $p \in C_n \setminus C_{n-1}$:

$$T(p) \geq \max\left\{T_n(p), \frac{m}{\gamma}, 6Q(p), \frac{1}{\delta} \log \frac{4}{\sin \measuredangle(N^u_p, N^s_p)}\right\}. \tag{12}$$

Let $p \in C_n \setminus C_{n-1}$ and $t \geq T(p)$; since $t \geq T(p) \geq T_n(p)$, we obtain $X^s(p) \in \Delta$. So by Lemma 3.15 we define an $(\epsilon, \kappa/2)$-realizable linear flow $L_1 : N_{X^s(p)} \longrightarrow N_{X^{s+m}(p)}$, sending $N^u_{X^s(p)}$ into $N^s_{X^{s+m}(p)}$. Now we concatenate from right and left with trivial maps and by Lemma 3.5 we obtain an (ϵ, κ)-realizable linear flow defined by

$$N_p \xrightarrow{L_0} N_{X^s(p)} \xrightarrow{L_1} N_{X^{s+m}(p)} \xrightarrow{L_2} N_{X^t(p)}$$

with $L_0 = P^s_X(p)$ and $L_2 = P^{t-m-s}_X(X^{s+m}(p))$. Now we estimate $\|L^t(p)\|$, and for that we consider the linear maps relative to a suitable unitary basis $\{n^u_{X^r(p)}, n^s_{X^r(p)}\}$ for $r \in [0, t]$ which is invariant for the linear Poincaré flow, so they have the form

$$L_2 = \begin{pmatrix} c^{uu} & 0 \\ 0 & c^{ss} \end{pmatrix}, \quad L_1 = \begin{pmatrix} b^{uu} & b^{us} \\ b^{su} & b^{ss} \end{pmatrix}, \quad L_0 = \begin{pmatrix} a^{uu} & 0 \\ 0 & a^{ss} \end{pmatrix}.$$

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The key observation is that $b^{uu} = 0$. Consider the product matrix:

$$L^t(p) = L_2 \cdot L_1 \cdot L_0 = \begin{pmatrix} 0 & a^{uu} b^{us} c^{ss} \\ a^{ss} b^{su} c^{uu} & a^{ss} b^{ss} c^{ss} \end{pmatrix}.$$

CLAIM 3.1. *For $p \in C_n \setminus C_{n-1}$ and $t \geq T(p)$ we have:*
(a) $\log|a^{uu}|, \log|c^{uu}| < \frac{1}{2} t(\lambda^+(p) + 4\delta)$*; and*
(b) $\log|a^{ss}|, \log|c^{ss}| < \frac{1}{2} t(-\lambda^+(p) + 4\delta)$.

Proof. To prove that $\log|a^{uu}| < \frac{1}{2} t(\lambda^+(p) + 4\delta)$ we first note that $s > t(1/2 - \gamma) > t/3 \geq T(p)/3 \geq Q(p)$, so by Oseledets' theorem we have that $\log|a^{uu}| = \log|P_X^s(p) \cdot n_p^u| < s(\lambda^+(p) + \delta)$. Since $\gamma\lambda^+(p) < \gamma c \leq \delta$ and $\gamma < 1/2$ we obtain

$$\begin{aligned} s(\lambda^+(p) + \delta) &< t(1/2 + \gamma)(\lambda^+(p) + \delta) < t(\lambda^+(p)/2 + \delta/2 + \lambda^+(p)\gamma + \gamma\delta) \\ &< t(\lambda^+(p)/2 + \delta/2 + \delta + \delta/2) < \tfrac{1}{2} t(\lambda^+(p) + 4\delta) \end{aligned}$$

and the inequality follows. Now to prove that $\log|c^{uu}| < \frac{1}{2} t(\lambda^+(p) + 4\delta)$ we consider the fact that $X^s(p) \in X^{-m}(B_n)$, therefore $X^{s+m}(p) \in B_n$ and by definition of B_n, $Q(X^{s+m}(p)) \leq n$, so we will have the approximation rate given by Oseledets' theorem if $t - m - s > n$. By (12) for $t \geq T(p)$, we have $-m/t \geq -\gamma$. Since $-s/t > -\frac{1}{2} - \gamma$ and $-\gamma \geq -1/6$ we obtain

$$t - m - s = t\left(1 - \frac{m}{t} - \frac{s}{t}\right) > t\left(\frac{1}{2} - 2\gamma\right) > \frac{t}{6} \geq Q(p) \geq n.$$

So $t - m - s$ will be sufficiently large to use (1), hence

$$\begin{aligned} \log|c^{uu}| &= \log|P_X^{t-m-s}(X^{s+m}(p)) \cdot n^u_{X^{s+m}(p)}| < (t - m - s)(\lambda^+(p) + \delta) \\ &< t(1 - m/t - s/t)(\lambda^+(p) + \delta) < t(\gamma + 1/2)(\lambda^+(p) + \delta) \\ &= t(\gamma\lambda^+(p) + \gamma\delta + \lambda^+(p)/2 + \delta/2) \\ &< t(\delta + \delta/2 + \lambda^+(p)/2 + \delta/2) = \tfrac{1}{2} t(\lambda^+(p) + 4\delta). \end{aligned}$$

We note that part (b) is analogous to part (a) and the claim is proved. □

Now we return to the proof of Lemma 3.18 and estimate $\|L_1\|_{\max}$. First note that

$$s + m > t(1/2 - \gamma + m/t) > t(1/2 - \gamma) > t/6 > Q(p) \geq n,$$

so by (3) we have $\sin^{-1} \measuredangle(N^u_{X^{s+m}(p)}, N^s_{X^{s+m}(p)}) < e^{(s+m)\delta} < e^{t\delta}$. Since L_1 is (ϵ, κ)-realizable we conclude that $\|L_1 - P_X^m(X^s(p))\|$ is small. Therefore, since $t > T(p) \geq m/\gamma$ and $\gamma c \leq \delta$, we have $\|L_1\| \leq e^{mc} \leq e^{t\gamma c} \leq e^{t\delta}$. By Lemma 3.17(b), we get

$$\|L_1\|_{\max} \leq \sin^{-1} \measuredangle(N^u_{X^{s+m}(p)}, N^s_{X^{s+m}(p)}) \|L_1\| \leq e^{2t\delta}.$$

Now we give estimates for each of the entries of the product matrix:

$$\begin{aligned} |a^{uu} b^{us} c^{ss}| &\leq e^{\frac{1}{2}t(\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(-\lambda^+(p)+4\delta)} = e^{6t\delta}; \\ |a^{ss} b^{su} c^{uu}| &\leq e^{\frac{1}{2}t(-\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(\lambda^+(p)+4\delta)} = e^{6t\delta}; \\ |a^{ss} b^{ss} c^{ss}| &\leq e^{\frac{1}{2}t(-\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(-\lambda^+(p)+4\delta)} \leq e^{-t\lambda^+(p)+6t\delta} \leq e^{6t\delta}. \end{aligned}$$

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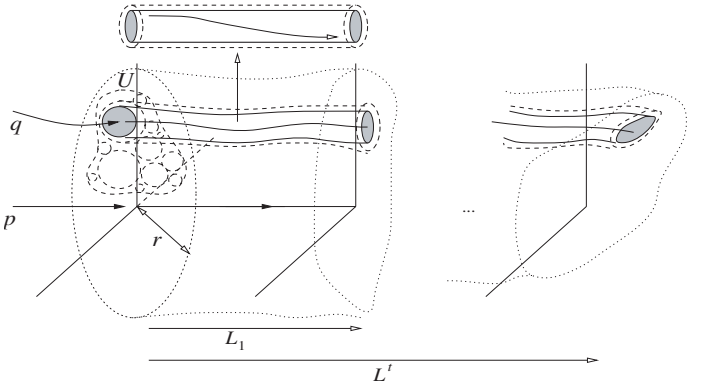


FIGURE 2. Realizing vector fields given a linear map L^t.

This implies the inequality $\|L^t(p)\|_{\max} < e^{6t\delta}$. By Lemma 3.17(a), we have

$$\|L^t(p)\| \le 4\frac{1}{\sin \measuredangle(N_p^u,\, N_p^s)}\|L^t(p)\|_{\max}.$$

But

$$t \ge T(p) \ge \frac{1}{\delta}\log\frac{4}{\sin \measuredangle(N_p^u,\, N_p^s)} \quad \text{so} \quad \frac{4}{\sin \measuredangle(N_p^u,\, N_p^s)} \le e^{t\delta}$$

and we get $\|L^t(p)\| \le e^{7t\delta}$. Replacing δ by $\delta/7$ we conclude that $\|L^t(p)\| \le e^{t\delta}$ and the lemma is proved. □

3.5.1. *Realizing vector fields.* Let $X \in \mathfrak{X}^2_\mu(M)^*$, with X^t aperiodic and also with all hyperbolic sets with zero Lebesgue measure. Given $\epsilon, \delta > 0$ and $0 < \kappa < 1$, we suppose that m is large enough to satisfy Lemma 3.15. By Lemma 3.18, there exists a measurable function $T : M \to \mathbb{R}$ such that, for μ-a.e. $p \in M$ and for every $t \ge T(p)$, there exists an (ϵ, κ)-realizable linear flow at p with length t such that $\|L^t(p)\| \le e^{t\delta}$. This means that, for all $\gamma > 0$, there exists $r(p, t) > 0$ such that for all open sets $U \subseteq B(p, r)$ there exist a measurable set $K \subseteq U$ such that $\overline{\mu}(K) > (1 - \kappa)\overline{\mu}(U)$ and a zero divergence C^1 vector field Y, ϵ-C^1-close to X, such that $Y = X$ outside the flowbox $\mathcal{F}_X^t(p)(U)$, and if $q \in K$, then $\|P_Y^t(q) - L^t\| < \gamma$. Since $\|L^t(p)\| \le e^{t\delta}$ we conclude that $\|P_Y^t(q)\| \le e^{\delta t} + \gamma$ for all $q \in K$ (where $\gamma \approx 0$). The vector field Y *realizes* the abstract map $L^t(p)$, i.e., $P_Y^t(q) \approx L^t(p)$ for a large percentage of points $q \in U \subset N_p$ (see Figure 2).

3.6. *Lowering the norm – global procedure*

3.6.1. *Sections of flows and special flows.* Now we use the local construction of realizable linear flows with a small norm in order to decrease the function $LE(\cdot)$ for a zero divergence vector field Y that is C^1-close to X. First we recall the definition of a special flow built under a ceiling function h. Consider a measure space Σ, a map $R : \Sigma \to \Sigma$, a measure $\tilde{\mu}$ defined in Σ and an integrable function $h : \Sigma \to \mathbb{R}^+$, with $h(x) \ge \alpha > 0$ for all

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$x \in \Sigma$ and $\int_\Sigma h(x)\, d\tilde{\mu}(x) = 1$. We consider the product space $M_h \subseteq \Sigma \times \mathbb{R}$ where M_h is the set below the graph of $h(x)$. The flow

$$\begin{aligned} S^s : \quad & \Sigma \times \mathbb{R} \longrightarrow \Sigma \times \mathbb{R} \\ & (x, r) \longmapsto (R^n(x), r + s - \textstyle\sum_{i=0}^{n-1} h(R^i(x))), \end{aligned}$$

where $n \in \mathbb{Z}$ is uniquely defined by $\sum_{i=0}^{n-1} h(R^i(x)) \le r + s < \sum_{i=0}^{n} h(R^i(x))$, is called a *special flow*. Informally speaking S^s moves any point $(x, r) \in M_h$ to $(x, r + s)$ at time-1 speed until it hits the graph of h, after which the point returns to the base Σ and continues its journey. It follows from the Ambrose–Kakutani theorem (see [**1**]) that any aperiodic flow is isomorphic to some special flow. This theorem is applicable to the flow of Proposition 3.2. The isomorphism is given by $W : M \to M_h$. The measure $\mu^* = W_*\mu$ is decomposed into the product of Lebesgue measure in $\mathbb{R}$ and an R-invariant measure $\tilde{\mu}$ in Σ, i.e.,

$$\int_{M_h} f(x, s)\, d\mu^* = \int_\Sigma \int_0^{h(x)} f(x, s)\, ds\, d\tilde{\mu}(x).$$

So we have a simplified representation of our flow $X^s(p)$. Hereafter we assume that our flow has this representation. Given a special flow over a section Σ the set $Q = \bigcup_t X^t(\Sigma)$ is called the *Kakutani castle* and the *tower of height i*, which is denoted by T_i, is the set below the graph of $h(B_i)$ where $B_i = \{x \in \Sigma : h(x) = i\}$ so $T_i = X^{[0,i]}(B_i)$. Next we consider a lemma which is the flow version of Lemma 4.1 of [**4**]. For a proof of the next lemma see [**2**, Lemma 6.1.2].

LEMMA 3.19. *Let $X^t : M \to M$ be a μ-preserving aperiodic flow. For every positive measure set $U \subseteq M$ and every $h \in \mathbb{R}$, there exists a $\tilde{\mu}$-positive measure section $B \subseteq U$ such that $X^{[0,h]}(B)$ is a flowbox and B is maximal (i.e., no set containing B and with larger measure has the same properties as B).*

For a μ-generic point p, Lemma 3.18 gives us $T(p)$, which, in general, is very large. Hence Lemma 3.19 will be very useful to avoid the overlapping of perturbations.

Consider X in the hypotheses of Proposition 3.2. For all Y that are ϵ-C^1-close to X we define $C := \max\{\|P_Y^1(p)\| : p \in M\}$. We take $\kappa = \delta^2$. Using the function given by Lemma 3.18 we define $Z_h := \{p \in M : T(p) \le h\}$. Of course, $\mu(M \setminus Z_h) \to_{h\to\infty} 0$ so for h sufficiently large we have

$$\mu(M \setminus Z_h) < \delta^2. \tag{13}$$

We intend to build a special flow with a ceiling function with height not less than h and section inside Z_h. Since for large h the set Z_h has almost full measure, by Lemma 3.19, we get a $\tilde{\mu}$-positive measure set $B \subseteq Z_h$. If $x \in B$, then $h(x) \ge h$, and since $B \subseteq Z_h$, we have $h \ge T(x)$, so we are in the hypotheses of Lemma 3.18. Let $\hat{Q}$ be the castle with base B. We have $\hat{Q} \supseteq Z_h$ in measure theoretical sense, so by (13) we get the inequality $\mu(\hat{Q}^c) \le \delta^2$. Define the subcastle $Q \subseteq \hat{Q}$ by excluding the towers of $\hat{Q}$ with height larger than $3h$. Adapting Lemma 4.2 of [**4**] (see [**2**, Lemma 6.2.1] for all the details) we obtain

$$\mu(\hat{Q} \setminus Q) < 3\delta^2. \tag{14}$$

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3.6.2. *The zero divergence vector field Y, ϵ-C^1-close to X.* Now we make use of the realizability of vector fields and the properties of special flows to construct a conservative vector field Y inside the subcastle Q by gluing a finite number of local perturbations supported on self-disjoint flowboxes. We note that the measures $\tilde{\mu}$ and $\overline{\mu}$ are equivalent. In the next lemma we follow Lemma 4.14 of [**5**].

LEMMA 3.20. *Given $\gamma > 0$, there exists Y, ϵ-C^1-close to X, a castle U for Y^t and a subcastle K for Y^t such that:*

(a) *the castle U is open;*
(b) *$\mu(U \setminus Q) < \gamma$ and $\mu(Q \setminus U) < \gamma$;*
(c) *$\mu(U \setminus K) < \kappa(1 + \gamma)$;*
(d) *$Y^t(U) = X^t(U)$ and $Y^t = X^t$ outside the castle U;*
(e) *if q is in the base of K and $h(q)$ is the height of the tower of K that contains q, then*

$$\|P_Y^{h(q)}(q)\| \leq e^{\delta h(q)} + \gamma.$$

Proof. The castle Q is a measurable set and since μ is Borel regular, there exists a compact $J \subseteq Q$ such that

$$\mu(Q \setminus J) < \gamma\mu(\hat{Q})/2. \tag{15}$$

The compact J is an X^t-castle with the same structure as Q (i.e., preserving the same dynamics of bases and towers as the castle Q). Now we choose an open castle V such that $J \subseteq V$ with

$$\mu(V \setminus J) < \gamma\mu(\hat{Q}), \tag{16}$$

and also with the same structure of Q and J. For every point p_1 in $J \cap B$ we have $h(p_1) \geq h$. Since $(J \cap B) \subseteq Z_h$ we have $T(p_1) \leq h$, and therefore $T(p_1) \leq h \leq h(p_1)$. So for all $t_1 \geq T(p_1)$, and for γ fixed, there exists a radius $r_1(p_1, t_1)$ (decrease r_1 if one leaves the open castle V) such that for almost (related with $\kappa = \delta^2$) all points in $U_1 = B(p_1, r_1) \subseteq N_{p_1}$, more precisely for all points $q \in K_1 \subseteq U_1$, we have a vector field Y_1 supported in a small flowbox containing the orbit segment $X^{[0,t_1]}(p_1)$ such that $\|P_{Y_1}^{t_1}(q)\| \leq e^{\delta t_1} + \gamma$. Now we continue by choosing the p_i and by Vitali's arguments we fill up J with a union U of self-disjoint open flowboxes in order to obtain

$$\mu(J \setminus U) \leq \gamma\mu(J)/2. \tag{17}$$

The set U is an X^t-castle with its section (base of the castle) the union of the U_i. So for each i we get a vector field Y_i supported in a small flowbox containing the orbit segment $X^{[0,t_i]}(p_i)$, ϵ-C^1-close to X and such that for all $q \in K_i \subseteq U_i$ we have $\|P_{Y_i}^{t_i}(q)\| \leq e^{\delta t_i} + \gamma$. Now we define $Y = Y_i$ inside each flowbox and $Y = X$ outside. Since these flowboxes are pairwise disjoint, the vector field is well defined and it is ϵ-C^1-close to X. Note that V is also a castle for Y^t, and U is also a Y^t-subcastle of the Y^t-castle V (having for base the union of all the U_i). We take K to be the Y^t-subcastle with a section the union of all the K_i. By construction of U, we get parts (a), (d) and (e). Now we prove part (b). By (16) and recalling that $V \supseteq U$ and $J \subseteq Q$ we get $\mu(U \setminus Q) < \mu(V \setminus J) < \gamma\mu(\hat{Q}) \leq \gamma$. To prove that $\mu(Q \setminus U) < \gamma$ we use (15) and (17) and we conclude that $\mu(Q \setminus U) \leq \mu(Q \setminus J) + \mu(J \setminus U) < \gamma\mu(\hat{Q}) < \gamma$. Finally, for part (c) we observe that part (b) implies that $\mu(U) < \mu(Q) + \mu(U \setminus Q) < 1 + \gamma$, we use the inequality $\mu(U \setminus K) < \kappa\mu(U)$ and we get $\mu(U \setminus K) < \kappa(1 + \gamma)$. □

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3.6.3. *Computing $LE(Y)$.* We take $t = h\delta^{-1}$ (we may assume that this is an integer). By Lemma 3.1, we obtain

$$LE(Y) \leq \int_M \frac{1}{t} \log \|P_Y^t(p)\| \, d\mu(p).$$

By what we did above for orbit segments inside the castle K and starting in the base, we guarantee small upper Lyapunov exponent. Therefore we define the set of points whose orbit stays for a long time in K by $G := \{p \in M : Y^s(p) \in K \ \forall\, s \in [0, t]\}$ and denote by G^c its complementary set.

LEMMA 3.21. *For $p \in G$ we have $\|P_Y^t(p)\| < e^{t(1+6\log C)\delta}$ for some $C > 0$.*

Proof. Let $p \in G$. We split the orbit segment $X^{[0,t]}(p)$ by return times at B_K (the section of the castle K), say $t = b + r_n + \cdots + r_2 + r_1 + a$, where all $X^a(p)$, $X^{r_1+a}(p)$, $X^{r_2+r_1+a}(p)$, $\ldots$, $X^{\sum_{i=1}^n r_i + a}(p)$ are in the base B_K. It is clear that $a, b, r_i \in]0, 3h]$ except when $p \in B_K$ ($a = 0$) and $X^t(p) \in B_K$ ($b = 0$). Note that

$$\begin{aligned}\|P_Y^t(p)\| = \|P_Y^{b+\sum_{i=1}^n r_i + a}(p)\| &\leq \|P_Y^b(X^{\sum_{i=1}^n r_i + a}(p))\| \\ &\quad \times \|P_Y^{r_n}(X^{\sum_{i=1}^{n-1} r_i + a}(p))\| \times \cdots \times \|P_Y^{r_1}(X^a(p))\| \times \|P_Y^a(p)\|.\end{aligned}$$

But these maps are based at points in B_K so we recall Lemma 3.20(e) and for $C := \max\{\|DX_p^1\| : p \in M\}$ (suppose that this constant is valid for any vector field ϵ-C^1-close to X) we get

$$\|P_Y^t(p)\| \leq C^{3h} e^{\sum_{i=1}^n r_i \delta} C^{3h} \leq e^{(b+\sum_{i=1}^n r_i + a)\delta} C^{6h} \leq e^{t\delta} C^{6\delta t} \leq e^{t(1+6\log C)\delta}. \qquad \square$$

For points in G^c we use inequality (14), Lemma 3.20 and some elementary observations (see [**5**, Lemma 4.16]) to prove the following lemma.

LEMMA 3.22. *Let $\gamma = \delta^2 h^{-1}$ as in Lemma 3.20; then $\mu(U \cup \Gamma_m^*(X) \setminus G) < 12\delta$.*

Now using Lemma 3.22 we obtain $\mu(G^c) < 12\delta$. Finally, we finish the proof of Proposition 3.2 and consequently Theorem 1:

$$\begin{aligned}LE(Y) = \inf_{n\geq 1} \int_M \frac{1}{n} \log \|P_Y^n(p)\| \, d\mu(p) &\leq \int_M \frac{1}{t} \log \|P_Y^t(p)\| \, d\mu(p) \\ &\leq \int_G \frac{1}{t} \log \|P_Y^t(p)\| \, d\mu(p) + \int_{G^c} \frac{1}{t} \log \|P_Y^t(p)\| \, d\mu(p) \\ &\leq (1 + 6\log C)\delta\mu(G) + \log C\,\mu(G^c) = (1 + 18\log C)\delta.\end{aligned}$$

Now we substitute δ by $\delta/(1 + 18\log C)$ along the proof and Proposition 3.2 is proved.

4. *Proof of Theorem 2*

4.1. *Adapting the proof of Theorem 1.* We begin by noting that, since X^t is aperiodic, the measure of all singularities is zero. Moreover, when we estimate the C^1-norm of the perturbation P, defined in (9), the choice of $r(p)$ in (7) guarantees that, even near singularities (when $c \approx 0$), the perturbation can be done. Furthermore, if in Theorem 3.3 we take Ω with C^∞ boundary and g, f also C^∞, the diffeomorphism φ, provided by

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Dacorogna and Moser, is also C^∞. So our conservative flowbox theorem guarantees a C^∞ conservative change of coordinates Ψ. Note that the perturbation P, defined in (9), is also C^∞; moreover we know by [**14**] that $\mathfrak{X}^\infty_\mu(M)$ is C^1-dense in $\mathfrak{X}^1_\mu(M)$. The following proposition is similar to Proposition 3.2. The main difference is *where* the computation of the entropy function is done.

PROPOSITION 4.1. *Let $X \in \mathfrak{X}^\infty_\mu(M)$ and $\epsilon, \delta > 0$. There exists $m \in \mathbb{N}$ and a zero divergence C^∞ vector field Y, ϵ-C^1-close to X, that equals X outside the open set $\Gamma_m(X)$ and such that $LE(Y, \Gamma_m(X)) < \delta$.*

Proof. For a fixed $m \in \mathbb{N}$ we have $p \in \Gamma^+_m(X) \setminus \Gamma^*_m(X)$ if p is periodic, has positive Lyapunov exponent and belongs to $\Gamma_m(X)$. We consider the following simple claim. For a proof see [**3**, Lemma 3.1].

CLAIM 4.1. *Given $\delta > 0$, there exists $m \in \mathbb{N}$ such that $\mu(\Gamma^+_m(X) \setminus \Gamma^*_m(X)) < \delta$.*

Take $m \in \mathbb{N}$ satisfying both Lemma 3.15 and Claim 4.1 and let $T : \Gamma^*_m(X) \to \mathbb{R}$ be similar to the function of Lemma 3.18. Let $Z_h = \{p \in \Gamma^*_m(X) : T(p) \le h\}$. Of course $\mu(\Gamma^*_m(X) \setminus Z_h) \to_{h\to\infty} 0$ so we take h such that $\mu(\Gamma^*_m(X) \setminus Z_h) < \delta^2 \mu(\Gamma^*_m(X))$. Now we increase h, if necessary, and use Oseledets' theorem to get for $p \in \mathcal{O}^0(X)$ the inequality

$$\|P^t_X(p)\| < e^{t\delta} \quad \text{for all } t \ge h. \tag{18}$$

Of course $X^t : \Gamma^*_m(X) \to \Gamma^*_m(X)$ is an aperiodic flow. We follow the construction of §3.6.2 and finally we compute $LE(Y, \Gamma^*_m(X))$. Analogously we define the set $G := \{p \in \Gamma^*_m(X) : Y^s(p) \in K, \forall\, s \in [0, t]\}$. Denote $(1/t) \log \|P^t_{(\cdot)}(p)\|$ by $A^t_p(\cdot)$. Then

$$\begin{aligned} LE(Y, \Gamma_m(X)) &\le \int_{\Gamma_m(X)} A^t_p(Y)\, d\mu \\ &\le \int_{\Gamma_m(X)\setminus(U\cup\Gamma^+_m(X))} A^t_p(Y)\, d\mu + \int_{U\cup\Gamma^+_m(X)\setminus G} A^t_p(Y)\, d\mu \\ &\quad + \int_G A^t_p(Y)\, d\mu. \end{aligned}$$

By (18) and since $Y = X$ outside U, we obtain

$$\int_{\Gamma_m(X)\setminus(U\cup\Gamma^+_m(X))} A^t_p(Y)\, d\mu \le \int_{\Gamma_m(X)\setminus\Gamma^+_m(X)} A^t_p(X)\, d\mu \le \delta.$$

Let $C := \max\{\|P^1_X(p)\| : p \in M\}$. Using Claim 4.1 and Lemma 3.22 we conclude that

$$\int_{U\cup\Gamma^+_m(X)\setminus G} A^t_p(Y)\, d\mu \le 13\delta \log C.$$

Finally, at G, our construction allows us to obtain

$$\int_G A^t_p(Y)\, d\mu(p) \le \delta$$

and the proposition is proved. □

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4.2. *End of the proof of Theorem 2.* Let $\tilde{X} \in \mathfrak{X}^1_\mu(M)$ and $\tilde{\epsilon} > 0$ be given. We will prove that there exists $Y \in \mathfrak{X}^1_\mu(M)$, $\tilde{\epsilon}$-C^1-close to X, satisfying the conclusions of Theorem 2. For $\epsilon = \tilde{\epsilon}/2$, there exists $X \in \mathfrak{X}^\infty_\mu(M)$, ϵ-C^1-close to $\tilde{X}$. It suffices to prove Theorem 2 for the vector field X and $\epsilon > 0$.

Proof of Theorem 2. Let $X \in \mathfrak{X}^\infty_\mu(M)$ and $\epsilon > 0$. We will find Y, ϵ-C^1-close to X, and a partition $M = D \cup O$ into Y^t-invariant sets such that for all $p \in O$ we have zero Lyapunov exponents and D is a countable increasing union of compact invariant sets Λ_{m_n} admitting an m_n-dominated splitting for the linear Poincaré flow. We define the sequence $\{X_n\}_{n\geq 0} \in \mathfrak{X}^\infty_\mu(M)$, $m_n \in \mathbb{N}$, and eventually $\epsilon_n > 0$ for $n \geq 0$.

Take $X_0 = X$, $\theta > 1$ (near 1) and $\delta_n \to_{n\to 0} 0$. If $\int_{\Gamma_m(X)} \lambda^+(X)\, d\mu = 0$ for some $m \in \mathbb{N}$, then we are finished by taking $Y = X$, $D = \Lambda_m(X)$ and O a full measure subset of $\Gamma_m(X)$. Otherwise for some $m = m_0$ and $X = X_0$ we have $\int_{\Gamma_{m_0}(X_0)} \lambda^+(X_0)\, d\mu > 0$. Let $\epsilon_0 \in (0, \epsilon/2)$ be such that

$$\int_{\Gamma_{m_0}(X_0)} \lambda^+(Z)\, d\mu \leq \theta \int_{\Gamma_{m_0}(X_0)} \lambda^+(X_0)\, d\mu,$$

for all Z that are $2\epsilon_0$-C^1-close to X_0 and $Z = X_0$ outside $\Gamma_{m_0}(X_0)$. Clearly, ϵ_0 always exists because $LE(\cdot, \Gamma_{m_0}(X_0))$ is upper semicontinuous and $\Gamma_{m_0}(X_0)$ is invariant for both X_0^t and Z^t. Recursively, knowing X_{n-1}, m_{n-1} and $\epsilon_{n-1} \in (0, \epsilon 2^{-n})$, we define $X_n \in \mathfrak{X}^\infty_\mu(M)$, $m_n \in \mathbb{N}$ and eventually $\epsilon_n > 0$.

By Proposition 4.1, there exists $m_n \in \mathbb{N}$ and a perturbation of X_{n-1}, $X_n \in \mathfrak{X}^\infty_\mu(M)$, ϵ_{n-1}-C^1-close to X_{n-1}, with $X_n = X_{n-1}$ outside $\Gamma_{m_n}(X_{n-1})$ and such that

$$\int_{\Gamma_{m_n}(X_{n-1})} \lambda^+(X_n)\, d\mu < \delta_n.$$

Suppose that $m_n \geq m_{n-1}$. Now $\Gamma_{m_n}(X_n) \subseteq \Gamma_{m_n}(X_{n-1}) \subseteq \Gamma_{m_{n-1}}(X_{n-1})$. If we have that $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n)\, d\mu = 0$, then we are finished by taking $Y = X_n$, $D = \Lambda_{m_n}(Y)$ and O a full measure subset of $\Gamma_{m_n}(Y)$. Otherwise if $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n)\, d\mu > 0$ we choose $\epsilon_n \in (0, \epsilon_{n-1}/2)$ such that $B(X_n, 2\epsilon_n) \subseteq B(X_{n-1}, \epsilon_{n-1})$ and also

$$\int_{\Gamma_{m_n}(X_n)} \lambda^+(Z)\, d\mu \leq \theta \int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n)\, d\mu,$$

for all Z that are $2\epsilon_n$-C^1-close to X_n and $Z = X_n$ outside $\Gamma_{m_n}(X_n)$.

We continue this procedure and if for some $n \in \mathbb{N}$ we obtain $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n)\, d\mu = 0$ we are over; otherwise the sequence $\{X_n\}_{n\geq 0}$ converges C^1 to some $Y \in \mathfrak{X}^1_\mu(M)$; moreover since $\epsilon_n < \epsilon/2^n$ we have Y that is ϵ-C^1-close to X.

Let $D = \bigcup_n \Lambda_{m_n}(X_n)$. Since $\Lambda_{m_n}(X_n) \supseteq \Lambda_{m_{n-1}}(X_{n-1})$ and $Y = X_n$ at $\Lambda_{m_n}(X_n)$, Y^t has an m_n-dominated splitting at $\Lambda_{m_n}(X_n)$.

Let $\Gamma := D^c = \bigcap_n \Gamma_{m_n}(X_n)$; clearly $\Gamma \subseteq \Gamma_{m_n}(X_n)$. To finish the proof of Theorem 2 we now check that $\int_\Gamma \lambda^+(Y)\, d\mu = 0$.

Note that $Y \in B(X_n, 2\epsilon_n)$ for all $n \in \mathbb{N}$. So we have

$$\int_\Gamma \lambda^+(Y)\, d\mu < \int_{\Gamma_{m_n}(X_n)} \lambda^+(Y)\, d\mu \leq \theta \int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n)\, d\mu \leq \theta\delta_n \xrightarrow[n\to\infty]{} 0.$$

We conclude that we have zero Lyapunov exponents in a full measure subset O of Γ and Theorem 2 is proved. □

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Now we consider the reason why Theorem 2 is stated for a *dense* subset instead of a *residual* subset. In [**5**], the authors developed a strategy to obtain a residual subset; unfortunately, this strategy does not apply in our case. Let us see why. They start with a C^1 system X which is a continuity point of the entropy function. Then they define the 'jump'(see [**5**, p. 1467]) of this semicontinuous function which is an integral over $\Gamma_\infty(X) := \bigcap_m \Gamma_m(X)$. Being a continuity point implies that the 'jump' is zero. So, by definition of 'jump', $\mu(\Gamma_\infty(X)) = 0$ or $\lambda^+(p) = 0$ for μ-a.e. point $p \in \Gamma_\infty(X)$ and the statements of Theorem 2 are verified. In order to estimate a lower bound for the 'jump' we perturb the original vector field X like we did to prove Theorem 2. But our conservative flowbox theorem may not be applied to X, unless X is of class C^2, so this argument only works for $X \in \mathfrak{X}^2_\mu(M)$. However, this set equipped with C^1 topology is not a Baire space, so in general residual sets are meaningless. Nonetheless, let us consider the following hypothesis again in the three-dimensional context.

Hypothesis A. Every vector field in $\mathfrak{X}^2_\mu(M)$ satisfies the following property: for every X^t-invariant set Λ_m with an m-dominated splitting for the linear Poincaré flow, either $\mu(\overline{\Lambda}_m) = 0$ or X is Anosov.

Under this hypothesis we prove an analog to Proposition 3.2, this time over $\mathfrak{X}^2_\mu(M)$ and consequently we obtain the global dichotomy.

THEOREM 4.2. *Under Hypothesis A there exists a residual $\mathfrak{R} \subseteq \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathfrak{R}$ then we have that X is Anosov or for μ-a.e. $p \in M$ we have zero Lyapunov exponents.*

Finally, we recall that in [**5**] the higher dimensional version of the *Bochi–Mañé* theorem is proved. The approach, like in the two-dimensional case, relies on perturbing the dynamical system by sending directions in E^u into E^s, in order to blend different expansion rates. If $\dim(E^u) = n$ it is shown that these perturbations cause a decay on the norm of the nth exterior product of the perturbed linear systems, which is sufficient to obtain the dichotomy. The multidimensional counterpart of Theorem 2 will be obtained by considering the perturbation framework developed in this paper, jointly with the strategy developed in [**5**], and will appear in a forthcoming work.

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