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RULES FOR SUBATOMIC DERIVATION

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Abstract. In proof-theoretic semantics the meaning of an atomic sentence is usually determined by a set of derivations in an atomic system which contain that sentence as a conclusion (see, in particular, Prawitz, 1971, 1973). The paper critically discusses this standard approach and suggests an alternative account which proceeds in terms of subatomic introduction and elimination rules for atomic sentences. A simple subatomic normal form theorem by which this account of the semantics of atomic sentences and the terms from which they are composed is underpinned, shows moreover that the proof-theoretic analysis of first-order logic can be pursued also beneath the atomic level.

§1. Introduction. Proof-theoretic semantics, in the tradition of Dummett and Prawitz, is primarily a semantics for the logical constants (cf. Kahle & Schroeder-Heister, 2006). It takes the meaning of logical constants to be determined by their natural deduction introduction rules and justifies the elimination rules of these constants in terms of reduction procedures which show that the elimination rules can be understood as inverses of the meaning determining introduction rules (cf. Prawitz, 1965, 2006; the historical reference is Gentzen, 1934, p. 189).

To be more precise, the main idea of proof-theoretic semantics, as pursued within this tradition, is to take the meaning of a logically compound sentence A of first-order logic to be given by a valid argument for that sentence which is in canonical form, that is, by an argument whose last step is an introduction the immediate subarguments of which are valid (see, e.g., Prawitz, 2006, p. 520). For example, let $A = C \supset D$. A valid canonical argument for $C \supset D$ will have the form of implication introduction:

$$\dfrac{\begin{array}{c}[C]\\ \mathcal{D}\\ D\end{array}}{C \supset D}\,(\supset\mathrm{I})$$

The rule for implication elimination

$$\dfrac{C \supset D \quad C}{D}\,(\supset\mathrm{E})$$

is then justified by the meaning determining form of a canonical argument for $C \supset D$ in terms of the inversion principle. This principle says that a proof of the conclusion of an elimination is already contained in the proofs of the premisses, when the major premiss (in the present case: $C \supset D$) is derived by an introduction. The principle is made formally precise in terms of a normal form theorem whose proof appeals to reductions for the logical connectives and quantifiers.

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The reduction for implication is as follows:

$$\dfrac{\dfrac{\begin{array}{c}[C]\\ \mathcal{D}_1\\ D\end{array}}{C \supset D} \quad \begin{array}{c}\mathcal{D}_2\\ C\end{array}}{D} \qquad \text{reduces to} \qquad \begin{array}{c}\mathcal{D}_2\\ C\\ \mathcal{D}_1\\ D\end{array}$$

The right-hand side derivation is said to be in normal form, since it does not contain an introduction of $C \supset D$ which is at the same time also the major premiss of implication elimination. The normal form theorem says, in effect, that every derivation in the natural deduction system for first-order logic in question can be brought into normal form by means of reduction procedures which appeal to such reductions. It thereby ensures that the conclusion of an elimination only says what must obtain in virtue of the meaning of the major premiss of that elimination.

It is not sufficiently clear, though, how the semantics of atomic sentences and the terms from which they are composed might be explained in proof-theoretic terms. From the point of view of proof-theoretic semantics such an account is surely a desideratum, if this approach is to match its strength against model-theoretic semantics. In this paper we first subject the standard account of the semantics of atomic sentences to a critical discussion (Section 2) and then develop an alternative proof-theoretic approach which introduces subatomic introduction and elimination rules for atomic sentences (Section 3). Section 4 contains a discussion of related work.

§2. Atomic systems. The standard proof-theoretic account of the meaning of atomic sentences is due to Prawitz (see, in particular, Prawitz, 1970, 1971, 1973). It proceeds in terms of atomic systems.[1] The main logical purpose of these systems is to determine the valid atomic formulae. Thus they perform a function in proof-theoretic semantics which is analogous to that of models in model-theoretic semantics.

An *atomic system* **S** is a pair $\langle \mathbf{L}, \mathbf{R} \rangle$, where **L** is a set of *descriptive constants* (i.e., individual, operational, and predicate constants) and individual variables and where **R** is a set of *inference rules*. The inference rules in **R** are determined as the instances of a finite number of schemata of the form

$$\dfrac{A_1 \quad A_2 \quad \cdots \quad A_n}{B}\ (\mathrm{A}),$$

where $A_1, A_2, \ldots, A_n$, and B are atomic formulae which are composed from the descriptive constants and variables in **L**. A rule which lacks premisses is called an *axiom*. For example, the familiar natural deduction rules for the introduction and elimination of identity formulae can be interpreted as rules in **R**, the introduction rule being an axiom (Prawitz, 1971, p. 261). An argument whose inferences are all instances of the rules in **R** is

[1] Prawitz calls these systems also 'atomic bases', 'elementary formal systems', or 'atomic calculi'. Since such systems have been already employed by Post (1943), he also uses the name 'Post systems'. As Prawitz (1970) observes, similar systems have been also used in the constructive setting of Lorenzen's (1955) operative approach. Systems of this kind are also used, for instance, in the theory of iterated inductive definitions (see, e.g., Martin-Löf, 1971) and in the theory of definitional reasoning developed by Hallnäs and Schroeder-Heister (see, in particular, Hallnäs & Schroeder-Heister, 1990/1991).

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a *derivation* in **S**. A *consistent* atomic system is an atomic system in which there is no derivation of the atomic sentence $\perp$ (Prawitz, 1973, p. 231).[2]

One may impose the following condition, the *parameter constraint* as we shall call it, on the rules in **R**:

> The conclusion B of a rule in **R** is to contain no parameter that does not occur in some premiss A_i (cf. Prawitz, 1971, p. 243).

An atomic system **S** can be *extended* to an atomic system **S′** by adding further rules to the rules contained in **S** (Prawitz, 1971, p. 276).[3]

The familiar sets of rules for natural deduction systems of minimal (**M**), intuitionistic (**I**), and classical (**C**) first-order logic can be extended by adding the rules of **S** resulting in *extended natural deduction systems* **M**(**S**), **I**(**S**), and **C**(**S**), respectively. When $\mathbf{S}^+$ is obtained from **S** by adding the absurdity rule $\dfrac{\perp}{A}$, the system **I**(**S**) is the same as **M**($\mathbf{S}^+$) (Prawitz, 1971, pp. 242–243).

As a further example for an atomic system we may take Prawitz's own definition of the atomic system for first-order Peano arithmetic $\mathbf{S}^A$ which is determined by:

(i). the descriptive constants 0, $'$, $=$, Q, and R where 0 is an individual constant (denoting zero), $'$ is an 1-place operational constant (denoting the successor function), $=$ is a 2-place predicate constant (denoting identity), and Q and R are 3-place predicate constants to express addition and multiplication and

(ii). the usual rules for identity, rules corresponding to Peano's third and fourth axioms, rules corresponding to the recursive definitions of addition and multiplication, and rules expressing the functional property of Q and R; for example, the rule corresponding to Peano's fourth axiom is

$$\frac{a' = b'}{a = b}$$

(Prawitz, 1971, p. 262).

First-order arithmetic, traditionally construed, cannot be formulated in terms of atomic systems, since the induction axioms cannot receive the shape of the rules contained in $\mathbf{S}^A$. The minimal system of first-order Peano arithmetic $\mathbf{P}_M$ can be obtained by adding the *rule of induction* to **M**($\mathbf{S}^A$):

$$\frac{A(0) \qquad \overset{[A(a)]}{A(a')}}{A(t)}$$

This rule can be applied with the provisio that a, which is the proper parameter of the application, must not occur in the assumptions upon which $A(a')$ is dependent with the exception of those assumptions of the form $A(a)$ which are closed by the application of the rule (Prawitz, 1971, p. 262). The intuitionistic and the classical systems of first-order arithmetic, that is, $\mathbf{P}_I$ and $\mathbf{P}_C$, are obtained in an analogous way from **I**($\mathbf{S}^A$) and **C**($\mathbf{S}^A$), respectively.

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[2] We replace Prawitz's symbol for absurdity Λ by the more standard $\perp$.

[3] In view of such extensions, Prawitz suggests that we might think of atomic systems as "approximations of our knowledge about the atomic sentences" (Prawitz, 1971, p. 277).

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Now, according to Prawitz, the *meaning (or the content) of an atomic sentence* is determined by a set of arguments for that sentence, that is, by a set of derivations in **S** which contain that sentence as a conclusion (Prawitz, 1973, p. 231). However, due to the form of the rules in atomic systems, Prawitz's account invites a couple of objections which, in my opinion, expose a serious shortcoming in his approach.

First, take any instance of Prawitz's rule scheme (A), for example, the following axiomatic instances:

$$\overline{Fab} \qquad \overline{a = a}$$

How do the terms which occur in these atomic sentences contribute to their meaning? Model-theoretic semantics provides a clear answer to this question, but there is no answer to it on Prawitz's account. Indeed, according to his explanation of the meaning of atomic sentences, the terms do not seem to provide any semantical contribution to the atomic compounds whatsoever.

Next, consider the following instance of Prawitz's rule scheme:

$$\frac{Fab}{Gc}$$

There seems to be no structural connection between the premiss and the conclusion at all. How then is the meaning of the former relevant to the meaning of the latter? Recall that, in view of the familiar subformula principle (see 3.14), there is an intuitive and clear-cut structural connection between the premisses and the conclusions in the natural deduction rules for the logical connectives and quantifiers.

Furthermore, on Prawitz's account, instances of the intuitionistic absurdity rule

$$\frac{\bot}{A}$$

may serve (like, e.g., in the system $\mathbf{S}^+$ mentioned above) to determine the meaning of its atomic conclusion. But what interesting could such instances teach us about the meaning of atomic sentences and the terms from which they are composed?

Moreover, consider the following pair of rules:

$$\frac{Fa \quad Gb}{Ga} \qquad \frac{Fa \quad Gb}{Fb}$$

Like the previous examples, these rules do not satisfy the subformula principle, but in contrast to the previous rules they clearly satisfy the parameter constraint, as in both rules all parameters in the conclusion occur also in some premiss. However, can instances of the rule scheme which are restricted by the parameter constraint explain the semantical contribution of the components of an atomic sentence to the meaning of that sentence? Furthermore, how could this constraint be used to explain the inferential connection between premisses and conclusions? (Note in this connection that an exceptionless acceptance of the parameter constraint would exclude, e.g., axioms.) Moreover, can there sensibly be inferential connections between distinct atomic sentences which do not appeal to logically compound formulae?

Finally, what is on Prawitz's account the meaning of terms (i.e., of individual constants and elementary predicates)? Is it their denotation?[4] But if so, what about the meaning of

[4] This way of explaining the meaning of terms seems to be suggested by Prawitz's remarks according to which the individual constants determine an individual domain (Prawitz, 1973, p. 231). See also his remarks in the next quotation in which he considers properties rather than denotational extensions of elementary predicates.

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denotationless terms and of atomic sentences which contain them? More fundamentally, should proof-theoretic semanticists content themselves with a denotational account of the meaning of terms?

The above criticisms can be summed up as follows: Atomic systems do not provide a satisfactory proof-theoretic account of (i) the meaning of atomic sentences, (ii) the meaning of the terms from which they are composed, (iii) the semantic relations between atomic sentences and the terms from which they are composed, and of (iv) the inferential relations which obtain between atomic sentences.

A question of a more general nature is: How does one arrive at the rules which instantiate Prawitz's rule scheme? Prawitz's remarks on atomic systems (or calculi) in Prawitz (1970) might seem to suggest an answer:

> Although a calculus is a mathematical object, it may represent an empirical situation. The individual terms may represent observable objects, the predicate symbols may represent observable properties, the rules of the form $\Rightarrow A$ or the form $A \Rightarrow \bot$ where A is a sentence may represent truths and falsehoods that have been established by observation, and the other rules may correspond to inductive generalizations or employed hypotheses (Prawitz, 1970, p. 103).[5]

These intuitive suggestions are, to my mind, not entirely satisfactory, at least, for the following reasons: First, they exclude denotationless terms and truths (falsehoods) that contain such terms. Second, the proposal concerning axioms suggests that, for example, identity sentences like $a = a$ can be established by observation which seems to defy strong intuitions to the contrary. Furthermore, it is difficult to see in which sense the "other rules" in an atomic calculus could be taken to correspond to inductive generalizations or hypotheses in view of the fact that, on the face of it, they lack quantified formulae and auxiliary assumptions, respectively. However, we shall not discuss general questions of this sort in this paper. We make these remarks merely to indicate that Prawitz's intuitive explanation of these rules is not unproblematic.

§3. Subatomic systems. We have just argued that atomic systems do neither provide a sufficiently satisfactory proof-theoretic account of the semantics of atomic sentences and the terms from which they are composed nor of the inferential relations which obtain between atomic sentences.

In view of the fact that atomic systems also serve for other purposes in proof theory and proof-theoretic semantics, a more satisfactory account seems to be desirable. For instance, atomic systems implicitly serve to account for the constructive meaning of the logical constants and the quantifiers in terms of the familiar Brouwer-Heyting-Kolmogorov (BHK) explanation, they are involved in the standard proof-theoretic explanation of validity (in terms of introduction rules or, alternatively in terms of elimination rules in so far as validity is relativized to atomic systems; cf. Prawitz, 1971, and Schroeder-Heister, 2006), they are used (*qua* ordinary productions) in the theory of iterated inductive definitions (see

[5] In Prawitz (1970) the notation $\Rightarrow A$ and $A \Rightarrow \bot$ is used for $\overline{A}$ and $\dfrac{A}{\bot}$, respectively. More generally, the rules of an atomic calculus correspond to the rule scheme (A) and have the form $A_1, A_2, \ldots, A_n \Rightarrow B$, where $n \geq 0$ and A_i $(i = 1, 2, \ldots, n)$ and B are atomic formulae, with $\bot$ counting as an atom (Prawitz, 1970, p. 102).

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Martin-Löf, 1971), and they serve to explain the meaning of production clauses in the theory of definitional reasoning (see, in particular, Hallnäs & Schroeder-Heister, 1990/1991).

In this section we suggest an alternative approach to the proof-theoretic semantics of atomic sentences which is based on normalizing introduction and elimination rules for atomic sentences. Our concern will be primarily with the plain language of first-order logic (with identity), that is, with a language which does not contain function symbols. The reasons for this decision are threefold. The plain language is the elementary basis of its, for example, arithmetical extensions, it is central from the point of view of formal semantics, and the atomic rules for sentences which contain function symbols can, in principle, be integrated into the account to be given in this section. Moreover, as our focus will be on atomic systems and atomic rules which cannot capture the rule of induction, we shall not discuss arithmetical extensions.[6]

3.1. The language. For the purpose of the formulation of the subatomic account we assume that a first-order language with individual (or nominal) constants $\mathcal{L}$ has been defined in the usual manner. We let $Atm\ (\mathcal{C}, \mathcal{P})$ be the nonempty set of atomic sentences (nominal constants, elementary predicates) of that language. We do not take the identity predicate to be elementary. Moreover, we define the sets $Atm(\alpha)$ and $Atm(\varphi^n)$ as follows: For any constant α from $\mathcal{C}$, $Atm(\alpha) = \{A \in Atm$: A contains at least one occurrence of $\alpha\}$; and for any elementary predicate φ^n from $\mathcal{P}$, $Atm(\varphi^n) = \{A \in Atm$: A contains an occurrence of $\varphi^n\}$.

3.2. Subatomic systems. A *subatomic system* $\mathcal{S}$ is a pair $\langle \mathcal{I}, \mathcal{R} \rangle$ whose first member is a triple $\langle \mathcal{C}, \mathcal{P}, \upsilon \rangle$ where $\mathcal{C}$ and $\mathcal{P}$ are already familiar and where υ is a function which assigns to any constant α in $\mathcal{C}$ (respectively to any elementary predicate φ^n in $\mathcal{P}$) a subset of $Atm(\alpha)$ (respectively $Atm(\varphi^n)$). In the present context we call the values of υ *term assumptions* for α (respectively φ^n).[7] We write, for example, $\alpha\{\varphi^1\alpha, \chi^2\alpha\alpha, \psi^3\alpha\beta\gamma\}$ to express that the constant α is associated with the term assumptions $\{\varphi^1\alpha, \chi^2\alpha\alpha, \psi^3\alpha\beta\gamma\}$. Similarly, we write $\chi^2\{\chi^2\alpha\alpha, \chi^2\alpha\beta, \chi^2\gamma\alpha\}$ to express that the 2-place elementary predicate χ^2 is associated with the term assumptions $\{\chi^2\alpha\alpha, \chi^2\alpha\beta, \chi^2\gamma\alpha\}$. There are various ways in which term assumptions can be interpreted intuitively. For example, we may view term assumptions as the (potentially defeasible) "elementary information" which a subject (or a group of subjects) associates with names and predicates, respectively. Alternatively, we may conceive of them in a subject independent manner.

The second member of $\mathcal{S} = \langle \mathcal{I}, \mathcal{R} \rangle$ is a set of subatomic rules for the introduction and elimination of atomic sentences. The introduction rule takes the following shape:

$$\frac{\varphi^n\{\ldots, \varphi^n\alpha_1 \ldots \alpha_n, \ldots\} \quad \begin{array}{c}\alpha_1\{\ldots, \varphi^n\alpha_1 \ldots \alpha_n, \ldots\} \\ \vdots \\ \alpha_n\{\ldots, \varphi^n\alpha_1 \ldots \alpha_n, \ldots\}\end{array}}{\varphi^n\alpha_1 \ldots \alpha_n}\ (at\mathrm{I})$$

[6] As a note to this last paragraph we mention that, in the presence of quantifiers in the language, function symbols (e.g., the successor function x') can be always removed in favor of predicates (e.g., the predicate Pxy to be read as "x is the predecessor of y"). This familiar insight is made, e.g., in Gentzen, 1934 IV.3 where it is used in a cut elimination–based consistency proof for an induction-free formal system of arithmetic.

[7] In effect, $\mathcal{I} = \langle \mathcal{C}, \mathcal{P}, \upsilon \rangle$ is defined in exactly the same way as in the model-theoretic setting of Więckowski (2010).

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Thus, an atomic sentence can be derived in case it is contained in the intersection of the term assumptions of all the terms from which that sentence is composed. We call the term assumptions which figure in the introduction rule *introductive term assumptions*.

The elimination rule for atomic sentences is as follows:

$$\frac{\varphi^n\alpha_1 \ldots \alpha_n}{\begin{array}{ll}\varphi^n\{\varphi^n\alpha_1 \ldots \alpha_n\} & \alpha_1\{\varphi^n\alpha_1 \ldots \alpha_n\} \\ & \vdots \\ & \alpha_n\{\varphi^n\alpha_1 \ldots \alpha_n\}\end{array}} (at\mathrm{E})$$

According to this rule, an atomic sentence is eliminated in terms of a transition from that sentence to the term assumptions for the terms from which it is composed, where the assumptions are singletons which contain the eliminated atomic sentence. We call the term assumptions which figure in (atE) *eliminative term assumptions*.

In a subatomic system we have two ways of combining the subatomic rules: we can have successive applications of (atE) and (atI) and of (atI) and (atE), respectively.

Note that, unlike the eliminative term assumptions, introductive term assumptions may contain more than a single atomic sentence. Note, moreover, that we may read (atI) and (atE) also as rules for the elimination and the introduction of term assumptions, respectively.

3.3. Constraints on subatomic systems. We require that the sets of term assumptions (in symbols: Υ, Υ', ...) be nonempty. This constraint is not a mere technical stipulation, as it allows for an intuitive interpretation of the subatomic rules (see 3.4 below). (Note that, by their definitions, the rules (atI) and (atE) do contain the atomic sentence $\perp$ neither as a premiss nor as a conclusion.)

3.4. Informal remarks. In contrast to the familiar rules of natural deduction and the rules in Prawitz's atomic systems, subatomic rules serve—in view of their shape—so to speak, for the introduction (respectively elimination) of truths rather than for truth preservation. So in this sense these rules are not logical rules.[8]

Perhaps, the intuitive content of the subatomic rules can be illustrated (in a subject-dependent manner) as follows: A genealogist is inquiring whether Mary is married to John. To find out whether 'Mary is married to John' is true, he inspects all the available data about Mary (curriculum vitae, memoirs, etc.), all the available data about John, and the documents relevant to the marriage of Mary and John (e.g., church books, registry office documents). Only if the marriage lists and the data about Mary and John (i.e, the information which he has associated with 'Mary', 'John', and the predicate '. . . is married to . . .') say that Mary is married to John, he will be prepared to conclude that 'Mary is married to John' is true. Not all the information which the genealogist has associated with these two names and the predicate in the process of surveying the relevant documents will be relevant to accomplish the task of his inquiry (e.g., 'Mary owns Fido' or 'Susan is married to Paul'). From the truth of 'Mary is married to John' our genealogist may extract the relevant information which he has associated with 'Mary', 'John', and the predicate '. . . is married to . . .' in this particular inquiry. (The reasoning behind this illustration is an instance of the derivation scheme (atI) + (atE); see 3.6 below.)

The discussion of subatomic rules and the meaning of nonlogical terms and atomic sentences is relegated to Subsection 3.9.

[8] Of course, (atE) + (atI) inference steps performed on a given atomic sentence will be truth preserving in the same trivial way in which a reiteration of a given atom would be truth preserving.

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3.5. Extensions of subatomic systems. A subatomic system $\mathcal{S}$ can be extended to a subatomic system $\mathcal{S}'$ by adding further rules to the rules contained in $\mathcal{S}$. We admit empty extensions taking a subatomic system to be its own extension.

3.6. A subatomic inversion principle. The rule (atE) can be conceived of as the *inverse* of (atI), since the conclusion obtained by (atE) does not state more than what must have already obtained in the case in which the maximum premiss of the elimination (i.e., the atomic sentence which is simultaneously both the conclusion of an application of (atI) and a premiss of an application of (atE)) was inferred by (atI):

$$\dfrac{\dfrac{\varphi^n\{\ldots,\varphi^n\alpha_1\ldots\alpha_n,\ldots\}\qquad \begin{array}{c}\alpha_1\{\ldots,\varphi^n\alpha_1\ldots\alpha_n,\ldots\}\\ \vdots\\ \alpha_n\{\ldots,\varphi^n\alpha_1\ldots\alpha_n,\ldots\}\end{array}}{\varphi^n\alpha_1\ldots\alpha_n}\,(at\mathrm{I})}{\dfrac{\varphi^n\{\varphi^n\alpha_1\ldots\alpha_n\}\qquad \begin{array}{c}\alpha_1\{\varphi^n\alpha_1\ldots\alpha_n\}\\ \vdots\\ \alpha_n\{\varphi^n\alpha_1\ldots\alpha_n\}\end{array}}{\varphi^n\alpha_1\ldots\alpha_n}\,(at\mathrm{E})}$$

In this derivation all the atomic sentences contained in the introductive term assumptions get lost in the eliminative assumptions except for the atomic sentence derived. These sentences are those which are irrelevant for the derivation. Intuitively, the atomic sentences which are not derivable from the term assumptions as defined by the first element of the subatomic system $\mathcal{S} = \langle \mathcal{I}, \mathcal{R}\rangle$ can be conceived of as the defeasible information associated with the terms (by the subject).

In view of the fact that nothing new is obtained by such subsequent applications of (atI) and (atE), it is natural to view such derivations as dispensable. It can be verified by inspection of the subatomic rules that the above derivation *reduces* to its eliminative term assumptions:

$$\varphi^n\{\varphi^n\alpha_1\ldots\alpha_n\}\qquad \begin{array}{c}\alpha_1\{\varphi^n\alpha_1\ldots\alpha_n\}\\ \vdots\\ \alpha_n\{\varphi^n\alpha_1\ldots\alpha_n\}\end{array}$$

In this way the rules (atI) and (atE) satisfy what we shall call the *subatomic inversion principle*:

> The term assumptions in an application of (atE) are already contained in the term assumptions of the premisses when the major premiss (i.e., the atomic sentence) is inferred by an application of (atI).

In other words, the eliminative term assumptions are already contained in the introductive term assumptions of successive (atI) and (atE) derivations for a given atomic sentence.

3.7. Subatomic normal form. We say that a subatomic derivation is in *subatomic normal form*, in case it contains no maximum atomic sentence. The following scheme captures the *normal form of subatomic derivations*:

$$\dfrac{\dfrac{\dfrac{\varphi^n\alpha_1\ldots\alpha_n}{}\,(at\mathrm{E})\qquad \begin{array}{c}\alpha_1\{\varphi^n\alpha_1\ldots\alpha_n\}\\ \vdots\\ \alpha_n\{\varphi^n\alpha_1\ldots\alpha_n\}\end{array}}{\varphi^n\alpha_1\ldots\alpha_n}\,(at\mathrm{I})}{\vdots}$$

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It will be convenient to abbreviate the scheme as follows:

$$\frac{\varphi^n\alpha_1 \ldots \alpha_n}{\Upsilon}\,(at\mathrm{E})$$
$$\frac{\Upsilon}{\varphi^n\alpha_1 \ldots \alpha_n}\,(at\mathrm{I})$$

Adapting Prawitz's familiar terminology (cf. Prawitz, 1965), we call Υ the *subatomic minimum* part of a normal subatomic derivation and the parts above and below it the *subatomic analytic* and the *subatomic synthetic* part, respectively. The subatomic analytic and the subatomic synthetic parts can be empty.

The idea expressed in the subatomic inversion principle can be stated more precisely in terms of the following *subatomic normal form theorem*:

THEOREM 3.1. *If an atomic sentence A (other than* $\perp$*) or the term assumptions* Υ *are derivable from the assumptions* Γ *(which may be either atomic sentences or term assumptions) by subatomic rules, then there is a normal subatomic derivation of A or* Υ *from* Γ.

Proof. The proof is immediate. □

The subatomic normal form theorem justifies a subatomic system as a whole—and hence the rules for the derivation of atomic sentences—by ensuring that all maximum atomic sentences can be removed from a derivation in this system. There is no analogous justification for Prawitz's atomic systems.

3.8. The subterm principle. In view of the subatomic normal form theorem, we may say that normal subatomic derivations satisfy the *subterm principle*:

> The top and the bottom sentences in a normal subatomic derivation are constituted from the terms with which the assumptions in Υ are associated.

Note that the parameter constraint discussed in Section 2 can be used to state a similar principle for atomic systems. In the present case, though, the relationship between the terms and the atomic sentences is both syntactical (by drawing on the syntactical components of the sentence) and rule governed (by drawing on the inferential relationship between term assumptions and atomic sentences).

3.9. Meaning. The observations on the subatomic inversion principle and subatomic normal form suggest, in a way similar to their analogues for the familiar natural deduction systems, that subatomic introduction rules can be conceived of as determining the *meaning of atomic sentences*.

We take an atomic sentence of the form $\varphi^n\alpha_1 \ldots \alpha_n$ to mean that there is a subatomic derivation of that sentence from the introductive term assumptions associated with the predicate φ^n and the constants $\alpha_1, \ldots, \alpha_n$. This meaning explanation states the sufficient conditions for the derivation of $\varphi^n\alpha_1 \ldots \alpha_n$. So, as in the case of the connectives and quantifiers, the elimination of atomic sentences is justified by the meaning determining introduction rule for atomic sentences in view of the fact that the subatomic inversion principle ensures that the conclusion of an atom elimination only says what must obtain in virtue of the meaning of the premiss of that elimination.

We take the *meaning of a term*, that is, a nominal constant or an elementary predicate, to be determined by the term assumptions associated with it in the minimum parts of derivations which are in subatomic normal form. Here we may take the *stable* meaning of a

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term to be determined by the *eliminative* term assumptions associated with the term in the minimum parts of such derivations; eliminative term assumptions do not contain defeasible information. In addition, we may allow that the meaning of elementary predicates be also determined by other predicates in terms of universally quantified formulae which perform the role of meaning postulates. (The topmost formula in the right branch of the derivation in Figure 3 below can serve as an example.)

In a certain sense we may view (atI) as manifesting the Fregean *compositionality principle* (roughly, the meaning of a sentence is determined by the meanings of its components) and (atE) as manifesting his *context principle* (roughly, words have meaning only in the context of a sentence). The introduction rule determines the meaning of the atomic sentence in terms of the term assumptions associated with the terms from which it is composed. The elimination rule, in turn, can be seen as stating that the meaning of the terms is captured by the term assumptions derived from the atomic sentence which is understood as their meaning determining context.

Finally, a remark on meaning and absurdity. On Prawitz's account (according to which $\mathbf{I}(\mathbf{S})$ is the same as $\mathbf{M}(\mathbf{S}^{+})$) the intuitionistic absurdity rule can be contained as a rule in an atomic system. By contrast, the rules for subatomic derivation exclude instances which contain the absurdity symbol $\bot$ in their premisses or conclusions. So, unlike on Prawitz's account, we cannot define the natural deduction system for intuitionistic logic in two different ways. We obtain $\mathbf{I}(\mathcal{S})$ only by adding the rules of $\mathcal{S}$ to the rules of $\mathbf{I}$. As a consequence, on the present account of the meaning of atomic sentences, the intuitionistic absurdity rule cannot—as a matter of principle rather than stipulation—be interpreted as an introduction rule for atomic sentences; it therefore cannot serve to determine their meaning.

3.10. Subatomic extensions of natural deduction systems. Natural deduction systems for minimal, intuitionistic, and classical logic (i.e., $\mathbf{M}$, $\mathbf{I}$, and $\mathbf{C}$, respectively) can be extended to *subatomically extended systems of natural deduction*, that is, $\mathbf{M}(\mathcal{S})$, $\mathbf{I}(\mathcal{S})$, and $\mathbf{C}(\mathcal{S})$, by adding the rules in $\mathcal{S}$ to the rules in $\mathbf{M}$, $\mathbf{I}$, and $\mathbf{C}$, respectively. We require the conclusions of the application of the intuitionistic and the classical absurdity rules to be atomic sentences and different from $\bot$.

In a subatomically extended system, the rules in $\mathcal{S}$ operate on atomic sentences and term assumptions, whereas the rules in the systems of natural deduction operate on formulae. In addition to the two kinds of derivation which are present in a subatomic system (see 3.2), we can now use (atE) also to derive term assumptions from an atomic sentence which has been derived in terms of the elimination rule for a logical operator, and we can use (atI) to derive an atomic sentence from term assumptions which is to serve as a premiss of an application for the introduction rule or, alternatively, an elimination rule for a logical operator. For illustration see the figures below.

3.11. An equivalence. The subatomically extended system of natural deduction $\mathbf{M}(\mathcal{S})$ is equivalent to Prawitz's extended system of natural deduction $\mathbf{M}(\mathbf{S})$ provided $\mathbf{M}(\mathcal{S})$ is supplemented with a set of meaning postulates $\mathbf{P}$ which correspond, in a way to be made precise below, to the rules from $\mathbf{S}$ in $\mathbf{M}(\mathbf{S})$ which are not axioms. More, precisely, the following *equivalence theorem* can be shown to hold:

THEOREM 3.2. *When A is a sentence of $\mathcal{L}$ which does not contain an occurrence of the identity symbol and Γ is a set of $\mathcal{L}$-sentences, then A is derivable in $\mathbf{M}(\mathbf{S})$ from Γ just in case it is derivable in $\mathbf{M}(\mathcal{S})+\mathbf{P}$ from Γ.*

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Proof. To verify this we make use of two stipulations. First, we stipulate that to any instance of the rule scheme (A) from **S** in **M**(**S**) which has the shape of an axiom rule and an atomic sentence B as its conclusion

$$\frac{}{B}$$

there corresponds an instance of the subatomic introduction rule (atI) from $\mathcal{S}$ in **M**($\mathcal{S}$)+**P** with B as its conclusion and with term assumptions for each term in B containing that atom. As for the second stipulation, we first observe that any instance of the rule scheme (A) from **S** in **M**(**S**) which does not have the shape of an axiom and which has atomic sentences $A_1(a_1), \ldots, A_n(a_n)$ as its premisses and $B(b)$ as its conclusion (where the constants need not be distinct) can be regarded as corresponding to an instance of (A) which has atomic formulae $A_1(x_1), \ldots, A_n(x_n)$ as its premisses and $B(y)$ as its conclusion (the variables need not be distinct) with the variables $x_1, \ldots, x_n$ replaced by the constants $a_1, \ldots, a_n$, respectively, and the variable y replaced by b. We then stipulate that for any instance of

$$\frac{A_1(x_1) \quad \ldots \quad A_n(x_n)}{B(y)}$$

in **S** there is a corresponding meaning postulate of the form $(\forall x_1) \ldots (\forall x_n)\ (\forall y)\ (A_1(x_1) \wedge \ldots \wedge A_n(x_n) \supset B(y))$ and we define **P** in **M**($\mathcal{S}$) + **P** to be the set of these postulates. Finally, we note that the rules for the logical operators are the same in both systems. □

In the proofs for the extended intuitionistic and classical systems we proceed by analogous stipulations. The derivation in Figure 3 (see 3.12) is an example for a derivation in **M**($\mathcal{S}$) + **P** when the open assumption $(\forall x)(\forall y)(\varphi^2 xy \supset \chi^1 x)$ is taken to be a postulate in **P**.

3.12. General normal form. Let $\mathcal{D}$ be a derivation in **M**($\mathcal{S}$) which is in *general normal form*.[9] The following scheme captures the form of the branches in $\mathcal{D}$:

$$\begin{array}{l} \dfrac{\mathcal{D}_{(logE)}}{\varphi^n \alpha_1 \ldots \alpha_n}\ (log\text{E}) \\[2ex] \dfrac{\varphi^n \alpha_1 \ldots \alpha_n}{\Upsilon}\ (at\text{E}) \\[2ex] \dfrac{\Upsilon}{\varphi^n \alpha_1 \ldots \alpha_n}\ (at\text{I}) \\[2ex] \dfrac{\varphi^n \alpha_1 \ldots \alpha_n}{\mathcal{D}_{(logI)}}\ (log\text{I}) \end{array}$$

Here we use (logI) and (logE) to refer to the introduction and elimination rules for the logical connectives and quantifiers of **M**($\mathcal{S}$); and we use $\mathcal{D}_{(logE)}$ and $\mathcal{D}_{(logI)}$ to refer to the analytic part (or the E-part) and synthetic part (or the I-part) of the branches in $\mathcal{D}$, respectively.

We take the minimum part of the branches in $\mathcal{D}$ to contain atomic sentences and term assumptions which are subject to the subatomic rules. Thus the minimum part comprises the subatomic analytic (synthetic, minimum) part (see 3.7). In the systems **I**($\mathcal{S}$) and **C**($\mathcal{S}$) the minimum part may also contain the atomic sentence $\perp$ which is subject to a (logE) rule. The (subatomic) analytic and the (subatomic) synthetic parts of branches in derivations

[9] We stick to the familiar definitions of the technical terms (e.g., branch, path) as presented in the standard reference Prawitz (1965). Deviations from Prawitz's terminology will be made explicit.

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which are in general normal form can be empty (cf. Prawitz, 1965, pp. 41, 53 and Prawitz, 1971, pp. 249–250).

In applications of (logI) and (logE) which appeal to atomic formulae rather than atomic sentences such as (∀I) and (∃E), the scheme for the form of branches in derivations which are in general normal form takes the usual shape.

An example. The derivation in $\mathbf{M}(\mathcal{S})$ displayed below is in general normal form:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall z)\chi^1 z}{\chi^1\alpha}\,(\forall\mathrm{E})}{\chi^1\{\chi^1\alpha\}\qquad \alpha\{\chi^1\alpha\}}\,(at\mathrm{E})}{\chi^1\alpha}\,(at\mathrm{I})}{(\exists z)\chi^1 z}\,(\exists\mathrm{I})$$

(Fig. 1)

We may now prove the following *general normal form theorem*:

THEOREM 3.3. *If A is derivable in $\boldsymbol{M}(\mathcal{S})$ from Γ, then there is a derivation of A from Γ in $\boldsymbol{M}(\mathcal{S})$ which is in general normal form.*

Proof. The proof uses the standard methods proposed in Prawitz (1965) and combines the proof of the usual normal form theorem for $\mathbf{M}$ and the proof for the subatomic normal form theorem for the subatomic system $\mathcal{S}$ in question. □

The result holds, *mutatis mutandis*, also for the systems $\mathbf{I}(\mathcal{S})$ and $\mathbf{C}(\mathcal{S})$ (cf. Prawitz, 1965, pp. 50–51). Derivations in $\mathbf{M}(\mathcal{S})$ which are in general normal form may contain redundant applications of (atE), (∨E), or (∃E). We say that a derivation in $\mathbf{M}(\mathcal{S})$ is in *full general normal form* if it does not contain such redundant steps (cf. Prawitz, 1965, p. 50 and Prawitz, 1971, p. 249).

The derivation in Figure 1, for instance, can be brought into full general normal form by eliminating the redundant application of (atE):

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall z)\chi^1 z}{\chi^1\alpha}\,(\forall\mathrm{E})}{\chi^1\{\chi^1\alpha\}\qquad \alpha\{\chi^1\alpha\}}\,(at\mathrm{E})}{\chi^1\alpha}\,(at\mathrm{I})}{(\exists z)\chi^1 z}\,(\exists\mathrm{I}) \qquad \text{reduces to} \qquad \dfrac{\dfrac{(\forall z)\chi^1 z}{\chi^1\alpha}\,(\forall\mathrm{E})}{(\exists z)\chi^1 z}\,(\exists\mathrm{I})$$

(Fig. 2)

Another example. The following derivation in $\mathbf{M}(\mathcal{S})$ with open assumptions is also in full general normal form:

$$\dfrac{\dfrac{\varphi^2\{\varphi^2\alpha\alpha,\varphi^2\alpha\beta,\varphi^2\alpha\gamma\}\qquad \beta\{\varphi^2\alpha\beta,\varphi^2\beta\alpha,\chi^1\beta\}}{\varphi^2\alpha\beta}\,(at\mathrm{I}) \qquad \dfrac{\alpha\{\varphi^2\alpha\beta,\varphi^2\alpha\gamma,\chi^1\alpha,\psi^3\alpha\beta\gamma\}\qquad \dfrac{(\forall x)(\forall y)(\varphi^2 xy\supset\chi^1 x)}{(\forall y)(\varphi^2\alpha y\supset\chi^1\alpha)}\,(\forall\mathrm{E})}{\varphi^2\alpha\beta\supset\chi^1\alpha}\,(\forall\mathrm{E})}{\chi^1\alpha}\,(\supset\mathrm{E})}{\chi^1\{\chi^1\alpha\}\qquad \alpha\{\chi^1\alpha\}}\,(at\mathrm{E})$$

(Fig. 3)

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The subatomic introduction belongs to the synthetic part of the path. (Interpretation: α is 'John', β is 'Mary', φ^2 is '. . . is the husband of . . .', and χ^1 is '. . . is male'.) The derivation of the term assumptions by means of (atE) establishes that the elementary information $\chi^1\alpha$ associated with α is not defeasible. From this we may conclude that $\chi^1\alpha$ is associated with χ^1 in the subatomic system.

3.13. Conservative extension. We observe, as a corollary to the general normal form theorem, that a derivation of either an atomic sentence A (other than $\bot$) from a set of term assumptions or of term assumptions from A in $\mathbf{M}(\mathcal{S})$ ($\mathbf{I}(\mathcal{S})$, $\mathbf{C}(\mathcal{S})$) which is in full general normal form is a derivation in $\mathcal{S}$. Hence the systems are conservative extensions over the subatomic system.

Obviously, the system $\mathbf{M}(\mathcal{S}) + \mathbf{P}$ is a conservative extension over $\mathbf{M}(\mathcal{S})$, as any derivation $\mathbf{M}(\mathcal{S}) + \mathbf{P}$ which does not appeal to the postulates in $\mathbf{P}$ is a derivation in $\mathbf{M}(\mathcal{S})$. Like remarks apply to the intuitionistic and classical systems.

3.14. The subexpression principle. According to the familiar *subformula principle* for **M** and **I**:

> Each formula in a fully normal derivation in **M** or **I** of A from Γ is a subformula of A or of some formula in Γ.

Mutatis mutandis for **C**. Due to the presence of atomic systems, this principle does obviously not hold (independently of whether the parameter constraint is satisfied or not) for the systems **M**(**S**), **I**(**S**), and **C**(**S**) without a restriction for atomic formulae (Prawitz, 1971, p. 251). However, in view of the subterm principle for subatomic systems (see 3.8), for subatomically extended systems we may explain the structural connection between atomic sentences in a derivation which is in full general normal form in a more fine-grained manner in terms of the following principle which is a corollary to the general normal form theorem (see 3.12):

> *Subexpression principle*. Each formula in a full general normal derivation in $\mathbf{M}(\mathcal{S})$ or $\mathbf{I}(\mathcal{S})$ of A from Γ is a subformula of A or of some formula in Γ; in case A is an atomic sentence other than $\bot$, it is constituted from the terms with which the assumptions in Υ are associated.

Mutatis mutandis for $\mathbf{C}(\mathcal{S})$.

3.15. Subatomic identity systems. Since on the present account the identity predicate does not count as an elementary predicate and is therefore not contained in a subatomic system, an alternative account of the meaning of identity sentences is needed. To avoid a confusion with the usual account which uses the axiomatic introduction rule

$$\frac{}{a = a}\ (=\mathrm{I})$$

we shall employ the symbol $\ddot{=}$ for the notion of identity as used in subatomically extended systems with identity.

Given a subatomically extended system of natural deduction $\mathbf{M}(\mathcal{S})$ ($\mathbf{I}(\mathcal{S})$, $\mathbf{C}(\mathcal{S})$), we may extend that system further to a *subatomically extended system with identity* $\mathbf{M}(\mathcal{S})^{\ddot{=}}$, ($\mathbf{I}(\mathcal{S})^{\ddot{=}}$, $\mathbf{C}(\mathcal{S})^{\ddot{=}}$) by adding introduction and elimination rules for identity ($\ddot{=}$) to the rules of the subatomically extended system in question.

The introduction rule for identity sentences takes the following shape:

$$\frac{\varphi_1^n(\alpha) \leftrightarrow \varphi_1^n(\beta) \qquad \varphi_2^n(\alpha) \leftrightarrow \varphi_2^n(\beta) \qquad \ldots \qquad \varphi_k^n(\alpha) \leftrightarrow \varphi_k^n(\beta)}{\alpha \ddot{=} \beta}\ (\ddot{=}\mathrm{I})$$

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Here $\varphi_i^n(\beta)$ is an atomic sentence which is just like $\varphi_i^n(\alpha)$ except for containing occurrences of the constant β at every place at which $\varphi_i^n(\alpha)$ contains an occurrence of α. The elementary predicates φ_i^n ($\ddot{=}$I) with $1 \leq i \leq k$ are finite in number. In a subject-dependent setting these predicates may be taken to be all the predicates available to the subject. Intuitively, this rule reflects the idea that an equality obtains between two terms just in case they satisfy the same elementary predications.

The shape of the elimination rules is as follows:

$$\frac{\alpha \ddot{=} \beta \qquad \varphi_i^n(\alpha)}{\varphi_i^n(\beta)}\ (\ddot{=}\text{E}) \qquad\qquad \frac{\alpha \ddot{=} \beta \qquad \varphi_i^n(\beta)}{\varphi_i^n(\alpha)}\ (\ddot{=}\text{E})$$

We take ($\ddot{=}$I) to determine the meaning of identity sentences. ($\ddot{=}$E) can be justified in terms of the following reductions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi_1^n(\alpha)]}{\mathcal{D}_{1_1}}}{\varphi_1^n(\beta)}(\supset\text{I}) \quad \dfrac{\dfrac{[\varphi_1^n(\beta)]}{\mathcal{D}_{1_2}}}{\varphi_1^n(\alpha)}(\supset\text{I})}{\varphi_1^n(\alpha) \supset \varphi_1^n(\beta) \;\&\; \varphi_1^n(\beta) \supset \varphi_1^n(\alpha)}(\&\text{I})}{\varphi_1^n(\alpha) \leftrightarrow \varphi_1^n(\beta)}(\leftrightarrow\text{I}) \quad \ldots \quad \dfrac{\dfrac{\dfrac{\dfrac{[\varphi_k^n(\alpha)]}{\mathcal{D}_{k_1}}}{\varphi_k^n(\beta)}(\supset\text{I}) \quad \dfrac{\dfrac{[\varphi_k^n(\beta)]}{\mathcal{D}_{k_2}}}{\varphi_k^n(\alpha)}(\supset\text{I})}{\varphi_k^n(\alpha) \supset \varphi_k^n(\beta) \;\&\; \varphi_k^n(\beta) \supset \varphi_k^n(\alpha)}(\&\text{I})}{\varphi_k^n(\alpha) \leftrightarrow \varphi_k^n(\beta)}(\leftrightarrow\text{I})}{\alpha \ddot{=} \beta}(\ddot{=}\text{I}) \qquad \varphi_i^n(\alpha)}{\varphi_i^n(\beta)}(\ddot{=}\text{E})$$

reduces to

$$\begin{array}{c} \varphi_i^n(\alpha) \\ \mathcal{D}_{i_j} \\ \varphi_i^n(\beta) \end{array}$$

The derivation which is just like the previous unreduced one except for having $\varphi_i^n(\beta)$ as the second premiss of the last step reduces to

$$\begin{array}{c} \varphi_i^n(\beta) \\ \mathcal{D}_{i_j} \\ \varphi_i^n(\alpha) \end{array}$$

In contrast to the rule for identity introduction (=I) which can be contained as an axiom in Prawitz's atomic systems, ($\ddot{=}$I) allows to define identity in a natural way by invoking the atomic truths which are composed from elementary predicates and the constants which are contained in identity sentences. The rule ($\ddot{=}$I) thus avoids the need to appeal to axioms. In a sense, it is natural to view this rule as being more fundamental than its axiomatic counterpart, as it can be intuitively taken to mirror the kind of reasoning which leads us to accept instances of (=I) as axioms.

Let L be a first-order language with $=$ whose alphabet comprises the descriptive constants of an atomic system $\mathbf{S} = \langle \mathbf{L}, \mathbf{R} \rangle$, variables, the logical connectives, and quantifiers. Let A be a sentence of L. We call A' a *corresponding sentence* of $\mathcal{L}$ if it is just like A except for containing occurrences of $\ddot{=}$ at all places where A contains occurrences of $=$. (A and A' are the same sentence in case A does not contain an occurrence of $=$.) Let Γ be a set of L-sentences. We call Γ' the set of corresponding $\mathcal{L}$-sentences in case each element of Γ' is a corresponding sentence of an element in Γ. The following theorem which is an analogue to Theorem 3.2 can be established:

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THEOREM 3.4. *An L-sentence A is derivable in* $\boldsymbol{M(S)}$ *from a set of L-sentences* Γ *exactly if its corresponding* $\mathcal{L}$*-sentence* A' *is derivable from the set of corresponding sentences* Γ' *in* $\boldsymbol{M(\mathcal{S})^{\ddot{=}}+P}$.

Proof. The proof is by induction on A and makes use of the two stipulations in the proof of Theorem 3.2 and the further stipulation that for any instance of (=I) in **S** there is a corresponding instance of ($\ddot{=}$I) in $\mathbf{M}(\mathcal{S})^{\ddot{=}}$+**P**. □

The equivalence proofs for the intuitionistic and the classical identity systems proceed in an analogous manner. We may establish the following *general normal form theorem for identity systems*:

THEOREM 3.5. *If A is derivable in* $\boldsymbol{M(\mathcal{S})^{\ddot{=}}}$ *from* Γ*, then there is a derivation of A from* Γ *in* $\boldsymbol{M(\mathcal{S})^{\ddot{=}}}$ *which is in general normal form.*

Proof. The proof uses Theorem 3.3 and the reductions for $\ddot{=}$ stated above. □

The result holds, *mutatis mutandis*, for $\mathbf{I}(\mathcal{S})^{\ddot{=}}$ and $\mathbf{C}(\mathcal{S})^{\ddot{=}}$. Theorem 3.5 justifies subatomically extended natural deduction systems with identity—in particular, the rules for the derivation of identity sentences—by ensuring that all maximum identity sentences can be removed from a derivation. There is no equally natural justification for the usual identity rules on Prawitz's account. This theorem also explains the semantical relation between identity sentences and the terms from which they are composed in a proof-theoretic manner. No explanation of this sort is available on Prawitz's atomic systems approach.

As a corollary to the general normal form theorem for identity systems we note that a derivation of either (i) an atomic sentence A (other than $\bot$) from a set of term assumptions or (ii) of term assumptions from A or (iii) of a sentence B from a set of assumptions Γ in $\mathbf{M}(\mathcal{S})^{\ddot{=}}$ ($\mathbf{I}(\mathcal{S})^{\ddot{=}}$, $\mathbf{C}(\mathcal{S})^{\ddot{=}}$) which has full general normal form for identity systems is a derivation in $\mathbf{M}(\mathcal{S})$ ($\mathbf{I}(\mathcal{S})$, $\mathbf{C}(\mathcal{S})$). Hence the identity systems are conservative extensions over the extended natural deduction systems. Likewise for identity systems which are extended with postulates.

§4. Related work. Gabbay & Gabbay (2009) develop a term sequent logic for the untyped lambda calculus. Derivability in this logic corresponds to the familiar reductions on lambda-terms (α-conversion, β-reduction, and η-expansion). As a main result cut elimination is shown for term sequents which makes it possible to study the reductions on terms by proof-theoretic methods. The intuitive idea behind the proposal is put as follows:

> Term sequents generalise sequent calculus; the intuition is *proof-theory for term-formers (as well as for logical connectives, as usual)* (Gabbay & Gabbay, 2009, p. 87).

A term sequent has the shape $\Gamma \vdash \Theta$:- $\mathtt{t}$, where Γ can be viewed as a set of assumptions, $\mathtt{t}$ as a lambda-term, and Θ as a rough normal form of a lambda-term that reduces to $\mathtt{t}$ (Gabbay & Gabbay, 2009, p. 90). The concern is with the proof theory of operations which form lambda-terms (abstraction and application) and reductions on lambda-terms rather than with the proof theory of atomic sentences of a first-order language and the nonlogical terms which constitute them.

They also discuss a notion of equality which obtains between two lambda-terms "when there is a chain of reductions linking them" (Gabbay & Gabbay, 2009, p. 102). This notion, *lambda-equality* $\approx$, is made precise by extending the term sequent logic with equality term

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sequents of the form $\Gamma \vdash \mathtt{t}_1 :: \mathtt{t}_2$. The sequent rule for equality term sequents states that a sequent of this form can be derived from a sequent of the form $\Gamma \vdash \mathtt{t}_1 :\!\!-^* \mathtt{t}_2$ in which $:\!\!-^*$ stands for the transitive-symmetric closure of :-. The aforementioned chaining idea behind lambda-equality is captured by the following rule:

$$\frac{\Gamma \vdash \mathtt{t}_1 :: \mathtt{t}_2 \quad \Gamma \vdash \mathtt{t}_2 :: \mathtt{t}_3 \quad \ldots \quad \Gamma \vdash \mathtt{t}_n :: \mathtt{t}_{n+1}}{\Gamma \vdash \mathtt{t}_1 \approx \mathtt{t}_{n+1}} (\approx R)$$

These remarks suffice to appreciate that lambda-equality and the notion of identity discussed in this paper are based on quite different ideas. Lambda-equality corresponds to $\alpha\beta\eta$-equality and is based on $\alpha\beta\eta$-reductions on lambda-terms. By contrast, $\ddot{=}$ is a variant of Leibniz-equality and thus captures the idea that, roughly, an equality obtains between two terms just in case they satisfy the same predicates.

Read (2004) (see also Kremer, 2007) proposes a natural deduction introduction rule for identity which, like the present paper, makes use of the idea of Leibniz-equality (Read, 2004, p. 116):

$$\frac{\begin{array}{c}(Fa)\\ Fb\end{array}}{a = b} (=\mathrm{I}) \qquad \frac{a = b \quad A(a)}{A(b)} (=\mathrm{E})$$

The provisio for (=I) is that F does not occur (as a predicate variable) in any assumption other than Fa. And in (=E) $A(a)$ and $A(b)$ need not be atomic. In contrast to these rules, ($\ddot{=}$I) and ($\ddot{=}$E) do not appeal to predicate variables nor do they admit logically complex predicates. Moreover, Read's justification of (=E) crucially involves applications of second-order ($\exists$E) and ($\exists$I) (Read, 2004, p. 117), whereas the justification of ($\ddot{=}$E) in terms of Theorem 3.5 does not appeal to second-order quantification.

In Francez et al. (2010) and further (as yet unpublished) work Francez, Dyckhoff, and Ben-Avi develop a proof-theoretic semantics for subsentential phrases of a fragment of English within the framework of type-logical grammar (TLG, see Moortgat, 1997). The fragment considered consists of sentences such as 'Every girl smiles', 'Some girl is a student', or 'Every girl loves some boy' which are composed from intransitive or transitive verbs, nouns, determiners ('every', 'some'), and determiner phrases (e.g., 'every girl', 'some boy'). In addition to such sentences, the fragment contains also (arbitrarily nested) relative clauses composed of the relative pronouns 'who' and 'whom'.

On this account the meaning of subsentential phrases (e.g., determiners) is explained in terms of the meaning of the sentences in which the phrases occur. Roughly, a sentence is assigned two kinds of meaning (or semantic value). The *contributed* meaning of a sentence is taken to be a function from sets of (open) assumptions to canonical derivations of the sentence from these sets within a natural deduction system. Its *contributing* meaning, by contrast, is a function which is like this one except for having the set of all the derivations of the sentence in the system as its range.

The meaning of subsentential phrases is obtained by recursively decomposing the meanings of sentences into function-argument structures. The decomposition proceeds according to the TLG which assigns syntactic categories to the phrases. For example, the meaning of 'Every girl smiles' is decomposed into the meanings of 'every girl' (function) and 'smiles' (argument) and the meaning of 'every girl' into the meanings of 'every' (function) and 'girl' (argument). The idea is that the subsentential phrase which contributes the function to the function-argument structure obtains its contributed meaning by abstraction over the argument's contributing meaning. The recursive decomposition comes to an end when it arrives at a function-argument structure with a "pure argument". In the above

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example the intransitive verb ‘smiles’ and the noun ‘girl’ are pure arguments. The reason for the termination is that the meanings of pure arguments are obtained from the meanings of *ground pseudosentences* which are taken to be “given externally” to the system. Since ground pseudosentences just are the atomic sentences of the formal language for the fragment, we may claim that the meaning of atomic sentences is left unexplored.

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BIBLIOGRAPHY

Francez, N., Dyckhoff, R., & Ben-Avi, G. (2010). Proof-theoretic semantics for subsentential phrases. *Studia Logica*, **94**, 381–401.

Gabbay, M., & Gabbay, M. J. (2009). Term sequent logic. *Electronic Notes in Theoretical Computer Science*, **246**, 87–106.

Gentzen, G. (1934). Untersuchungen über das logische Schließen I, II. *Mathematische Zeitschrift*, **39**, 176–210, 405–431.

Hallnäs, L., & Schroeder-Heister, P. (1990/1991). A proof-theoretic approach to logic programming: I. Clauses as rules, II. Programs as definitions. *Journal of Logic and Computation*, **1**, 261–283, 635–660.

Kahle, R., & Schroeder-Heister, P., editors. (2006). *Proof-Theoretic Semantics*. Special issue of *Synthese*, Vol. **148**, No. 3, pp. 501–749.

Kremer, M. (2007). Read on identity and harmony—A friendly correction and simplification. *Analysis*, **67**, 157–159.

Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik* (second edition). Berlin: Springer.

Martin-Löf, P. (1971). Hauptsatz for the intuitionistic theory of iterated inductive definitions. In Fenstad, J. E., editor. *Proceedings of the Second Scandinavian Logic Symposium (Oslo 1970)*. Amsterdam, The Netherlands: North-Holland, pp. 197–215.

Moortgat, M. (1997). Categorial type logics. In van Benthem, J., & ter Meulen, A., editors. *Handbook of Logic and Language*. Amsterdam, The Netherlands: North-Holland, pp. 93–177.

Post, E. (1943). Formal reductions of the general combinatorial decision problem. *American Journal of Mathematics*, **65**, 157–215.

Prawitz, D. (1965). *Natural Deduction. A Proof-Theoretical Study*. Stockholm, Sweden: Almquist & Wiksell. Reprinted by Dover Publications, Mineola, New York, 2006.

Prawitz, D. (1970). Constructive semantics. In *Proceedings of the First Scandinavian Logic Symposium (Åbo 1968)*. Filosofiska Studier utgivna av Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet, nr 8, Uppsala, pp. 96–114.

Prawitz, D. (1971). Ideas and results in proof theory. In Fenstad, J. E., editor. *Proceedings of the Second Scandinavian Logic Symposium (Oslo 1970)*. Amsterdam, The Netherlands: North-Holland, pp. 235–309.

https://doi.org/10.1017/S175502031000033X Published online by Cambridge University Press

Prawitz, D. (1973). Towards a foundation of a general proof theory. In Suppes, P., editor. *Logic, Methodology, and Philosophy of Science IV*, Vol. 74. Amsterdam, The Netherlands: North-Holland, pp. 225–250.

Prawitz, D. (2006). Meaning approached via proofs. *Synthese*, **148**, 507–524.

Read, S. (2004). Identity and harmony. *Analysis*, **64**, 113–119.

Schroeder-Heister, P. (2006). Validity concepts in proof-theoretic semantics. *Synthese*, **148**, 525–571.

Więckowski, B. (2010). Associative substitutional semantics and quantified modal logic. *Studia Logica*, **94**, 105–138.

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