Ergod. Th. & Dynam. Sys., (2026), **46**, 2887–2940 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10321

Refined methods in foliated Brouwer theory

NELSON SCHUBACK †‡

† *Institut de Mathématiques de Jussieu-Paris Rive Gauche (IMJ-PRG), Sorbonne Université, Paris, France*
(e-mail: nelson.schuback@imj-prg.fr)
‡ *Faculty of Mathematics and Computer Science, Jagiellonian University, Krakow, Poland*
(e-mail: nelson.schuback@uj.edu.pl)

(*Received* 13 *February* 2026 *UTC and accepted in revised form* 22 *May* 2026 *UTC*)

Abstract. A Brouwer homeomorphism is a fixed-point free, orientation-preserving homeomorphism of the plane. A foundational result of Le Calvez [A foliated version of the Brouwer translation theorem. *Comment. Math. Helv.* **79**(2) (2004), 229–259 (in French)] establishes that every such homeomorphism f admits an oriented planar foliation $\mathcal{F}$ such that every point $x \in \mathbb{R}^2$ can be connected to its image $f(x)$ by a path positively transverse to $\mathcal{F}$. This provides a powerful framework for analyzing the dynamics of f by studying how its orbits cross the leaves of $\mathcal{F}$. In this article, we refine this framework by identifying additional qualitative dynamical information about f that is encoded in $\mathcal{F}$, which can be systematically recovered through the concept of proper transverse trajectories. Later, we investigate the possible combinatorial configurations of these proper trajectories for finite collections of orbits and characterize their simplest forms. As a key application, this refined framework is used in a subsequent work [A foliated viewpoint on homotopy Brouwer theory. Preprint, 2025, arXiv:2512.07510] to offer a new perspective on homotopy Brouwer theory, originally introduced by Handel [A fixed-point theorem for planar homeomorphisms. *Topology* **38**(2) (1999), 235–264].

Key words: surface dynamics, Brouwer homeomorphisms, planar foliations, transverse trajectories, forcing theory
2020 Mathematics Subject Classification: 37E30 (Primary); 37C86 (Secondary)

Contents

1 Introduction 2888
1.1 Description of the results 2889
2 Preliminaries 2893
2.1 Planar topology 2893
2.2 Topological planar foliations 2895

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

2.3 Foliated Brouwer theory 2899
3 Asymptotic behavior of orbits 2900
3.1 Limitation of the framework 2901
3.2 Crossed leaves and limit leaves 2902
3.3 Definition of order $\prec$ on the sets $\partial_L C_O$ and $\partial_R C_O$ 2903
3.4 Orbit induced cuts 2905
3.5 Proper transverse trajectories 2909
3.6 Equivalence classes of $\mathcal{F}$-asymptotic orbits 2912
4 Sided preorders of orbits 2915
4.1 Defining $\lesssim_L$ and $\lesssim_R$ 2916
5 Weak $\mathcal{F}$-transverse intersections 2922
6 Maximal leaf domains and critical leaves 2923
6.1 Orbits traveling across maximal leaf domains 2925
6.2 Directed planar forest of maximal leaf domains 2927
7 Proper transverse trajectories with minimal intersections 2928
7.1 Technical lemmas for the proof of Theorem B 2928
7.2 Proof of Theorem B 2933
8 Adaptive orders across maximal leaf domains 2934
8.1 Adaptive ordering over maximal leaf domains 2936
9 Proof of Theorem C 2937
Acknowledgements 2940
References 2940

1. *Introduction*

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an orientation-preserving and fixed-point free homeomorphism, often referred to as a *Brouwer homeomorphism*. Such maps are named after Brouwer, who first studied them in the early twentieth century. In his seminal work [**Bro12**], Brouwer proved the Brouwer translation theorem (BTT), which asserts that through every point of the plane passes a *Brouwer line* λ of f, that is, a proper embedding $\lambda : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that

$$f(L(\lambda)) \subset L(\lambda) \quad \text{and} \quad f^{-1}(R(\lambda)) \subset R(\lambda).$$

By the Jordan curve theorem, λ admits a left side and a right side, denoted $L(\lambda)$ and $R(\lambda)$, corresponding to the connected components of $\mathbb{R}^2 \setminus \lambda$ lying to the left and right of λ.

In particular, the BTT implies that the plane can be covered by unbounded simply connected open f-invariant subsets, on which f is conjugate to a translation. This means that every point is wandering under f and, thus, the limit set of every orbit of f is empty. In other words, every orbit of f is the image of a proper embedding of $\mathbb{Z}$ into $\mathbb{R}^2$. This fundamental description of the dynamics of Brouwer homeomorphisms became a basis for the development of Brouwer theory in the twentieth century (see [**Gui94**] for a historic account).

In the early 2000s, Le Calvez introduced a foliated version of the BTT [**LC04**] (and its equivariant form in [**LC05**]), thereby initiating the so-called foliated Brouwer theory, a powerful framework for studying Brouwer homeomorphisms.

THEOREM. (Le Calvez [**LC04**]) *There exists an oriented topological foliation $\mathcal{F}$ of the plane such that every leaf $\phi \in \mathcal{F}$ is a Brouwer line of f.*

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

A foliation $\mathcal{F}$ of the plane by Brouwer lines of f is called a *transverse foliation* of f. Although such a transverse foliation is not unique, its existence already yields a robust structure for studying the dynamics of f. Indeed, one may connect any point $x \in \mathbb{R}^2$ to its image $f(x)$ by a path $\gamma_x : [0, 1] \longrightarrow \mathbb{R}^2$ that is positively transverse to $\mathcal{F}$, i.e. γ_x intersects any leaf $\phi \in \mathcal{F}$ at most once, passing from $R(\phi)$ to $L(\phi)$. This allows one to represent the orbit $O = O(f, x)$ of x under f by an embedded line $\Gamma_O : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by the concatenation

$$\Gamma_O := \prod_{n\in\mathbb{Z}} \gamma_{f^n(x)}.$$

We refer to Γ_O as a *transverse trajectory* of the orbit O, with respect to the foliation $\mathcal{F}$. Since the choice of paths $\gamma_{f^n(x)}$ is not unique, the orbit O admits uncountably many transverse trajectories Γ_O with respect to the same foliation $\mathcal{F}$, many of which not being properly embedded. Nevertheless, all such transverse trajectories Γ_O intersect exactly the same set of leaves of $\mathcal{F}$. Although this approach loses some qualitative dynamical information of the orbit, it has proved remarkably effective for reproducing classical results in surface dynamics and for establishing new powerful criteria for detecting chaotic behavior in surface homeomorphisms isotopic to the identity, through the celebrated forcing theory [**LCT22**].

There are some problems, however, that cannot be addressed using the current methods of foliated Brouwer theory. For instance, when tackling the study of Brouwer mapping classes in homotopy Brouwer theory (see [Bav17, Han99, LR17]), it is important to know whether the transverse trajectories $\{\Gamma_O\}_{O\in\mathcal{O}}$ of a finite collection $\mathcal{O}$ of orbits of f can be chosen to be properly embedded and with minimal pairwise intersections.

In this work, we refine the methods of foliated Brouwer theory to address such problems, centered around future applications to the study of homotopy Brouwer theory.

1.1. *Description of the results.* Let us fix $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ to be a Brouwer homeomorphism, and let $\mathcal{F}$ be a transverse foliation of f. The set of all orbits of f is denoted by $\mathrm{Orb}(f)$.

To each $O \in \mathrm{Orb}(f)$, we associate the set of leaves $C_O \subset \mathcal{F}$ crossed by O, defined as

$$C_O = \{\phi \in \mathcal{F} \mid \phi \cap \Gamma_O \neq \varnothing\},$$

where Γ_O is any transverse trajectory of O. This set is independent of the choice of Γ_O.

By abuse of notation, one may treat C_O as a subset of $\mathbb{R}^2$ by identifying $C_O = \bigcup_{\phi\in C_O} \phi$. In §2.2, we show that $C_O \subset \mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$ and trivially foliated by $\mathcal{F}$. Moreover, the boundary ∂C_O is formed by a locally finite collection of leaves of $\mathcal{F}$. At last, we define two distinguished subsets $\partial_L C_O$ and $\partial_R C_O$ of the boundary ∂C_O, given by

$$\partial_L C_O := \partial C_O \cap \left(\bigcap_{\phi\in C_O} L(\phi) \right) \quad \text{and} \quad \partial_R C_O := \partial C_O \cap \left(\bigcap_{\phi\in C_O} R(\phi) \right),$$

and a total order $\prec$ among the leaves in $\partial_L C_O$ and similarly for $\partial_R C_O$ (see Figure 1).

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

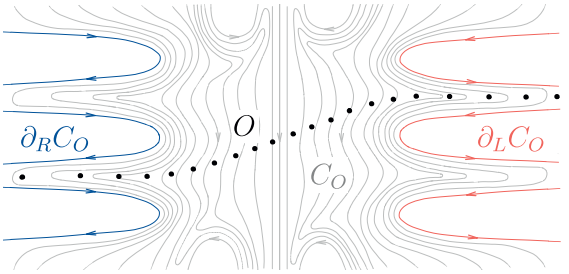


FIGURE 1. The set C_O of leaves of $\mathcal{F}$ crossed by the orbit O (light gray), with its right boundary $\partial_R C_O$ (blue) and left boundary $\partial_L C_O$ (red). Note that the terms left and right are given by the orientation of the leaves in C_O (color figures are available online).

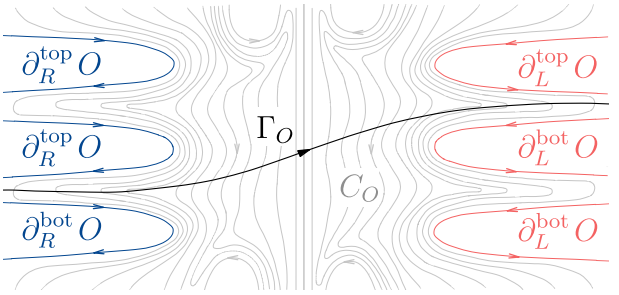


FIGURE 2. The cuts $(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O)$ and $(\partial_R^{\text{top}} O, \partial_R{}^{\text{bot}} O)$ canonically associated to the orbit O.

Let $\text{Cut}_\prec(\partial_L C_O)$ denote the set of all cuts of the totally ordered set $(\partial_L C_O, \prec)$, that is, the set of all pairs (A, B) such that $A, B \subset \partial_L C_O$ are disjoint subsets of $\partial_L C_O$ satisfying:

- $A \cup B = \partial_L C_O$;
- for any leaf $\phi \in A$ and any leaf $\phi' \in B$, it holds that $\phi \prec \phi'$.

Similarly, let $\text{Cut}_\prec(\partial_R C_O)$ denote the set of all cuts of the totally ordered set $(\partial_R C_O, \prec)$.

Definition 1.1. A proper transverse trajectory Γ_O of an orbit $O \in \text{Orb}(f)$ is a transverse trajectory which is, in addition, a proper embedding of $\mathbb{R}$ into the plane.

The following result shows that such a trajectory always exists, and that it canonically determines and is determined by a pair of cuts.

THEOREM A. *Every orbit $O \in \text{Orb}(f)$ is canonically associated to two cuts*

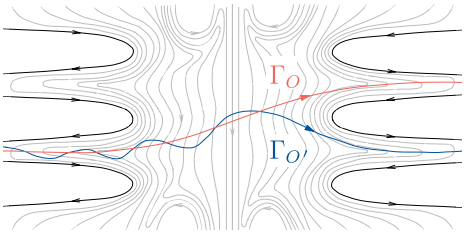
$$(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O) \in \text{Cut}_\prec(\partial_L C_O) \quad \text{and} \quad (\partial_R^{\text{top}} O, \partial_R^{\text{bot}} O) \in \text{Cut}_\prec(\partial_R C_O).$$

In addition, the orbit O admits a proper transverse trajectory Γ_O, and it must satisfy

$$\partial_L^{\text{top}} O = \partial_L C_O \cap L(\Gamma_O), \quad \partial_R^{\text{top}} O = \partial_R C_O \cap L(\Gamma_O),$$
$$\partial_L^{\text{bot}} O = \partial_L C_O \cap R(\Gamma_O), \quad \partial_R^{\text{bot}} O = \partial_R C_O \cap R(\Gamma_O).$$

For the sake of visual intuition, we adopt the convention $\partial^{\text{top}} \prec \partial^{\text{bot}}$ (see Figure 2).

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press



$$
\begin{aligned}
C_O &= C_{O'} \\
\partial_R^{\mathrm{top}} O &= \partial_R^{\mathrm{top}} O' \\
\partial_R^{\mathrm{bot}} O &= \partial_R^{\mathrm{bot}} O' \\
\partial_L^{\mathrm{top}} O &\subsetneq \partial_L^{\mathrm{top}} O' \\
\partial_L^{\mathrm{bot}} O &\supsetneq \partial_L^{\mathrm{bot}} O'
\end{aligned}
$$

FIGURE 3. Example of orbits O and O' crossing the same leaves of $\mathcal{F}$, but with distinct $\mathcal{F}$-asymptotic behavior.

As a consequence of Theorem A, we obtain a finer description of the behavior of orbits in terms of the structure of the transverse foliation $\mathcal{F}$, given by the set C_O, and the associated cuts $(\partial_L^{\mathrm{top}} O, \partial_L^{\mathrm{bot}} O)$ and $(\partial_R^{\mathrm{top}} O, \partial_R^{\mathrm{bot}} O)$, or dually, by its proper transverse trajectories. The qualitative data are referred throughout the text as the *$\mathcal{F}$-asymptotic behavior* of O.

Being able to describe the $\mathcal{F}$-asymptotic behavior refines the framework, for instance, by allowing us to distinguish orbits of f even when they cross the same leaves of $\mathcal{F}$, something not achievable before. This is illustrated in the following example (see Figure 3).

Definition 1.2. Two orbits $O, O' \in \mathrm{Orb}(f)$ are *forward $\mathcal{F}$-asymptotic*, denoted $O \overset{+}{\sim} O'$, if there exists a leaf $\phi \in C_O \cap C_{O'}$ such that $L(\phi) \cap C_O = L(\phi) \cap C_{O'}$ and, in addition,

$$\partial_L^{\mathrm{top}} O = \partial_L^{\mathrm{top}} O' \quad (\text{or equivalently,} \quad \partial_L^{\mathrm{bot}} O = \partial_L^{\mathrm{bot}} O').$$

Similarly, we define *backward $\mathcal{F}$-asymptotic*, denoted $O \overset{-}{\sim} O'$, by replacing each L for R.

By comparing the different $\mathcal{F}$-asymptotic behaviors induced by the foliation $\mathcal{F}$, we obtain two relations $\lesssim_L$ and $\lesssim_R$ on the set $\mathrm{Orb}(f)$, which describe how the orbits are arranged with respect to $\mathcal{F}$. For a definition of these relations, we refer to §4.1.

For each $\phi \in \mathcal{F}$, the relations $\lesssim_L$ and $\lesssim_R$ restrict to total preorders on the subset

$$\mathrm{Orb}(\phi) := \{O \in \mathrm{Orb}(f) \mid \phi \in C_O\},$$

and are compatible with the $\mathcal{F}$-asymptotic relations $\overset{+}{\sim}$ and $\overset{-}{\sim}$, respectively (see §4.1).

THEOREM B. *Let $\phi \in \mathcal{F}$ and let $\mathcal{O} \subset \mathrm{Orb}(\phi)$ be a finite collection of orbits crossing ϕ. Assume that $\mathcal{F}$ is generic for $\mathcal{O}$, that is, no leaf in $\mathcal{F}$ intersects more than one orbit in $\mathcal{O}$. Then, there exists a family $\{\Gamma_O\}_{O \in \mathcal{O}}$ of proper transverse trajectories of $\mathcal{O}$ that satisfy*

$$\Gamma_O \cap \Gamma_{O'} \cap \overline{L(\phi)} = \varnothing \quad \text{for all } O, O' \in \mathcal{O},\ O \neq O'.$$

Moreover, the order at which the intersection points $p_O \in \Gamma_O \cap \phi$ appear along the leaf ϕ is compatible with $\lesssim_L$, i.e., $p_O < p_{O'}$ along ϕ only if $O \lesssim_L O'$ (see Figure 4).

Finally, we introduce the notion of *weak $\mathcal{F}$-transverse intersection* between two orbits, which generalizes the notion of $\mathcal{F}$-transverse intersection defined by Le Calvez and Tal

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

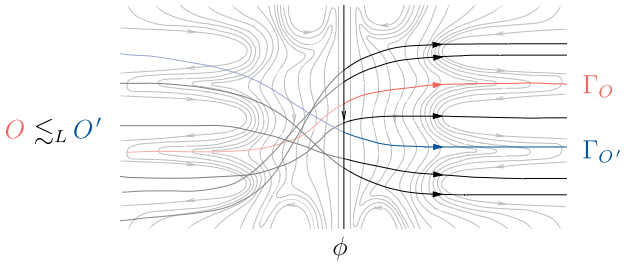


FIGURE 4. Family of proper transverse trajectories $\{\Gamma_O\}_{O\in\mathcal{O}}$ obtained in Theorem B.

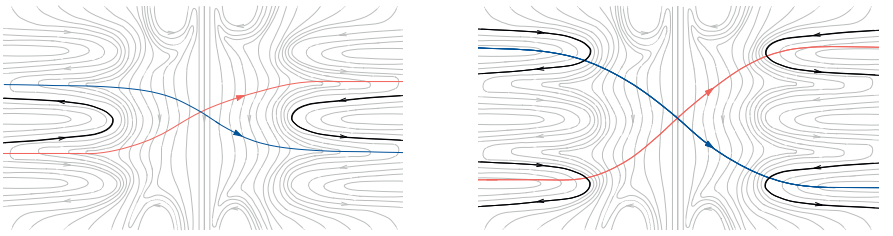


FIGURE 5. Comparison between the classic and weak $\mathcal{F}$-transverse intersections.

in [**LCT18**]. Roughly speaking, two orbits O and O' have a weak $\mathcal{F}$-transverse intersection if they are ordered by $\lesssim_L$ and $\lesssim_R$ exclusively in opposite ways (see Figure 5).

Definition 1.3. Two orbits $O, O' \in \mathrm{Orb}(f)$ are said to have a *weak $\mathcal{F}$-transverse intersection* if they satisfy $O \overset{+}{\not\sim} O'$ and $O \overset{-}{\not\sim} O'$, and they are oppositely ordered by $\lesssim_L$ and $\lesssim_R$, meaning that

$$O \lesssim_L O' \text{ and } O' \lesssim_R O \quad \text{or} \quad O' \lesssim_L O \text{ and } O \lesssim_R O'.$$

In Figure 5, we compare the classic and weak notions of $\mathcal{F}$-transverse intersection.

Before stating Theorem C, we need to introduce some terminology. Denote the open unit disk by $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. A foliation of $\mathbb{D}^2$ is said to be *nicely extended to the boundary* if each leaf of the foliation can be extended to a simple arc in the closed disk $\overline{\mathbb{D}^2}$ with distinct endpoints on the boundary $\partial\mathbb{D}^2$. For any foliation $\mathcal{F}$ of the plane and any homeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{D}^2$, we denote by $\varphi(\mathcal{F})$ the foliation of $\mathbb{D}^2$ formed by the images of the leaves of $\mathcal{F}$ under φ. Finally, a *diagram* is a finite collection of oriented line segments in $\mathbb{D}^2$ with pairwise distinct endpoints on the boundary $\partial\mathbb{D}^2$.

THEOREM C. *Let $\mathcal{O} \subset \mathrm{Orb}(f)$ be a finite collection of orbits and assume $\mathcal{F}$ is generic for $\mathcal{O}$. Then, there exists a family $\{\Gamma_O\}_{O\in\mathcal{O}}$ of proper transverse trajectories for $\mathcal{O}$ and a homeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{D}^2$ that satisfy (see Figure 6) the following properties:*

(i) *the foliation $\varphi(\mathcal{F})$ is nicely extended to the boundary;*
(ii) *the family $\{\varphi(\Gamma_O)\}_{O\in\mathcal{O}}$ is a diagram in $\mathbb{D}^2$;*
(iii) *two distinct segments $\varphi(\Gamma_O)$ and $\varphi(\Gamma_{O'})$ intersect if, and only if, the orbits O and O' have a weak $\mathcal{F}$-transverse intersection.*

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

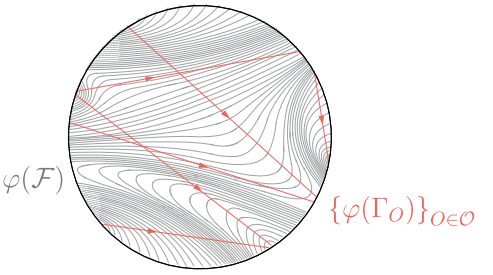


FIGURE 6. Diagram of proper transverse trajectories described by Theorem C.

2. *Preliminaries*

2.1. *Planar topology.* Let $\mathbb{R}^2$ be the Euclidean plane, equipped with the topology induced by the Euclidean metric d. An open ball of radius $r > 0$ centered at $x \in \mathbb{R}^2$ is denoted by $B(x, r)$.

A *path* $\gamma : I \longrightarrow \mathbb{R}^2$ is a continuous map defined on an interval $I \subset \mathbb{R}$. A path γ is said to be a reparameterization of another path $\gamma' : I' \longrightarrow \mathbb{R}^2$ if there exists a strictly increasing map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ that satisfies $\varphi(I) = I'$ and $\gamma = \gamma' \circ \varphi|_I$. A *loop* $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ is a continuous map defined on the unit circle $\mathbb{S}^1$. A path or a loop is said to be *simple* if it is injective. By abuse of notation, if there is no ambiguity, we may consider paths and loops as subset of the plane by identifying them with their images.

A *topological line* is a path $\ell : \mathbb{R} \longrightarrow \mathbb{R}^2$ that is injective and proper. Observe that any topological line admits a natural orientation induced by the positive orientation on $\mathbb{R}$. Similarly, we define a *half-line* as a path $\ell^+ : [0, \infty) \longrightarrow \mathbb{R}^2$ that is injective and proper. Often enough, it is convenient to consider half-lines defined on $(-\infty, 0]$ as well.

In the sphere compactification of the plane $\mathbb{S}^2 = \mathbb{R}^2 \sqcup \{\infty\}$, every topological line ℓ can be extended to a simple loop on $\mathbb{S}^2$ by adding the point at infinity. According to the Jordan curve theorem [**Jor87**], the complement of any simple loop on the sphere has exactly two connected components. By the Schoenflies theorem (see [**Cai51**] for an elementary proof), each of these connected components is homeomorphic to $\mathbb{R}^2$. In our context, this means that every topological line ℓ separates the plane into two sets, each of which is homeomorphic to the plane. These sets are called the left and right sides of ℓ, denoted $L(\ell)$ and $R(\ell)$, according to the orientation of ℓ.

Now, we present a fundamental generalization of the Schoenflies theorem, which provides a criterion for extending homeomorphisms defined on subsets of the sphere to the entire sphere. This theorem is particularly useful in the context of low-dimensional topology because it allows us to handle 'nice' subsets of $\mathbb{S}^2$ with more flexibility.

Here, by *arc*, we mean a simple path $\gamma : [0, 1] \longrightarrow \mathbb{S}^2$. A *Y-set* is a subset of $\mathbb{S}^2$, which is the union of three arcs intersecting at a single point that is an endpoint of each arc. Note that the orientation of $\mathbb{S}^2$ induces a cyclic order on the arcs constituting a Y-set in a neighborhood of the intersection point.

THEOREM. (Homma–Schoenflies) [**Hom53**] *Let $X \subset \mathbb{S}^2$ be a compact, connected, and locally connected set. Then, an injective and continuous map $h : X \longrightarrow \mathbb{S}^2$ can be*

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

extended to an orientation-preserving homeomorphism of $\mathbb{S}^2$ if, and only if, the map h preserves the cyclic order of any Y-set contained in X.

Applying the Homma–Schoenflies theorem is not necessarily straightforward, as it requires checking the preservation of the cyclic order of Y-sets. However, there are many classic examples of applications of this theorem (see [**LR06**, Appendix]). Along this preliminary section, we will guide the reader through the process of applying the Homma–Schoenflies theorem as we introduce the new concepts and tools.

A family $\{A_i\}_{i\in I}$ of subsets of $\mathbb{R}^2$ is called *locally finite* if, for every compact set $K \subset \mathbb{R}^2$, we have that $\{i \in I \mid A_i \cap K \neq \varnothing\}$ is a finite set.

LEMMA 2.1. *Let $\{X_i\}_{i\in I}$ be a family of locally finite and pairwise disjoint closed subsets of the plane. Then, there exists a locally finite family $\{U_i\}_{i\in I}$ of pairwise disjoint open subsets that satisfy $X_i \subset U_i$, for every $i \in I$.*

Proof. For each $i \in I$, consider the set $Y_i := \bigcup_{j\in I} X_j \setminus X_i$. First, we show that Y_i is closed. For a given $i \in I$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in Y_i that converges to a point $x \in \mathbb{R}^2$. Observe that, since the family $\{X_j\}_{j\in I}$ is locally finite, the set $\{j \in I \mid d(x, X_j) < r\}$ is finite for any $r > 0$. Consequently, there exists an index $j_0 \in I \setminus \{i\}$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that satisfies $x_{n_k} \in X_{j_0}$ for every $k \in \mathbb{N}$. Since X_{j_0} is a closed set, we have that $x \in X_{j_0}$. Since X_{j_0} is disjoint from X_i, we have that $x \notin X_i$. This allows us to conclude that $x \in Y_i$, which proves that the set Y_i is closed.

We can now define, for each $i \in I$, the set

$$U_i := \{z \in \mathbb{R}^2 \mid d(z, X_i) < \min(d(z, Y_i), 1)\}.$$

Since X_i and Y_i are disjoint sets, we observe that

$$d(z, X_i) = 0 < \min(d(z, Y_i), 1) \quad \text{for all } z \in X_i.$$

This proves that $X_i \subset U_i$ for every $i \in I$. Next, we remark that

$$d(z, X_i) < \min(d(z, Y_i), 1) \leq d(z, X_j) \quad \text{for all } j \neq i, \quad \text{for all } z \in U_i.$$

This implies that U_i is disjoint from U_j for every $j \in I \setminus \{i\}$. Finally, note that the function

$$g_i : z \longmapsto d(z, X_i) - \min(d(z, Y_i), 1)$$

is continuous. Thus, by writing $U_i = g_i^{-1}((-\infty, 0))$, we conclude that U_i is an open set.

To conclude the proof, we need to show that the family $\{U_i\}_{i\in I}$ is locally finite. For that, we consider for each $i \in I$, the closed ball

$$B(X_i, 1) := \{z \in \mathbb{R}^2 \mid d(z, X_i) \leq 1\}.$$

Since the family $\{X_i\}_{i\in I}$ is locally finite, the family of closed balls $\{B(X_i, 1)\}_{i\in I}$ is also locally finite. From the definition of the set U_i, we can observe that

$$U_i \subset B(X_i, 1) \quad \text{for all } i \in I.$$

Therefore, we conclude that the family of open sets $\{U_i\}_{i\in I}$ also is locally finite. □

Here, we provide a first application of the Homma–Schoenflies theorem.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

Let $\{\ell_n\}_{n\in\mathbb{Z}}$ be a locally finite family of pairwise disjoint topological lines such that

$$L(\ell_{n_1}) \subset L(\ell_{n_2}) \subset L(\ell_{n_3}) \quad \text{for all } n_1 < n_2 < n_3 \in \mathbb{Z}.$$

In this context, the union of the lines in the family $\{\ell_n\}_{n\in\mathbb{Z}}$ and the point at infinity forms a compact, connected, and locally connected subset of $\mathbb{S}^2$. It is worth noting that this would not be the case if the family $\{\ell_n\}_{n\in\mathbb{Z}}$ was not locally finite. Using the Homma–Schoenflies theorem, we can obtain an orientation-preserving homeomorphism on the plane that maps each line ℓ_n to the vertical line $\{n\} \times \mathbb{R}$ oriented downwards.

For more applications, we refer the reader to [**LR17**, Appendix B].

2.2. *Topological planar foliations.* A *topological planar foliation* $\mathcal{F}$ is a partition of the plane into disjoint topological lines, called *leaves*, that admits a local parallel structure. In other words, for any point $x \in \mathbb{R}^2$, there exists a closed neighborhood $U \subset \mathbb{R}^2$ of x and a homeomorphism $\varphi : U \to [0, 1] \times [0, 1]$ such that, for every leaf $\phi \in \mathcal{F}$, each connected component of the set $\phi \cap U$ is of the form $\varphi^{-1}(\{t\} \times [0, 1])$.

In the particular case of foliations of $\mathbb{R}^2$, the condition above can be exchanged for the condition that the entire set $\phi \cap U$ is of the form $\varphi^{-1}(\{t\} \times [0, 1])$ for some $t \in [0, 1]$.

A *cross-section* of $\mathcal{F}$ over a leaf $\phi \in \mathcal{F}$ is a compact arc $\sigma \subset \mathbb{R}^2$ that intersects ϕ and is transverse to $\mathcal{F}$, meaning that the arc σ intersects each leaf of $\mathcal{F}$ at most once.

Let $\phi \in \mathcal{F}$ be a leaf and let $\{\sigma_k\}_{k\in\mathbb{Z}^*}$ be a locally-finite collection of pairwise disjoint compact cross-sections of $\mathcal{F}$ over the leaf ϕ. Assume that, for every $k > 0$, it holds that:

- the cross-sections σ_{-k} and σ_k intersect the same leaves in $\mathcal{F}$;
- the sub-arc $u_k \subset \phi$ joining σ_{-k} and σ_k satisfies $u_k \subset u_{k+1}$;
- the only leaf of $\mathcal{F}$ intersecting all cross-sections in $\{\sigma_k\}_{k\in\mathbb{N}^*}$ is the leaf ϕ;
- every leaf of $\mathcal{F}$ that intersects σ_{k+1} also intersects σ_k.

The closed set $V \subset \mathbb{R}^2$, obtained as the union of all compact sub-arcs of leaves in $\mathcal{F}$ joining the cross-sections σ_{-k} and σ_k, for some $k > 0$, is a *trivial neighborhood* of the leaf ϕ. Observe that, for any open neighborhood $U \subset \mathbb{R}^2$ of ϕ, we can consider sufficiently small cross-sections $\{\sigma_k\}_{k\in\mathbb{Z}^*}$ so that the neighborhood V is contained in U.

It is worth noting that trivial neighborhoods of leaves can be taken to be arbitrarily small. More precisely, for any leaf $\phi \in \mathcal{F}$ and any open neighborhood $U \subset \mathbb{R}^2$ of ϕ, there exists a trivial neighborhood V of ϕ such that $V \subset U$. Such a trivial neighborhood V can be obtained by taking the cross-sections in the collection $\{\sigma_k\}_{k\in\mathbb{Z}^*}$ to be sufficiently small.

By the Homma–Schoenflies theorem, for any trivial neighborhood V of a leaf $\phi \in \mathcal{F}$, we can find a homeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that satisfies the following properties:

- the leaf ϕ is mapped by φ into the vertical line $\{0\} \times \mathbb{R}$;
- the set V is mapped by φ into the set $\bigcup_{n\in\mathbb{N}^*}[-1/n, 1/n] \times [-n, n]$;
- for each leaf $\phi' \in \mathcal{F}$, the set $\varphi(\phi' \cap V)$ is a vertical line $\{t\} \times \mathbb{R}$ for some $t \in \mathbb{R}$.

As a direct application of Lemma 2.1, we have the following lemma, which will be of main importance in the developments of §3.

LEMMA 2.2. *Let $\{\phi_i\}_{i\in I}$ be a locally finite family of pairwise disjoint leaves of $\mathcal{F}$. Then, for any closed set $X \subset \mathbb{R}^2$ that is disjoint from all leaves in $\{\phi_i\}_{i\in I}$, there exists a locally*

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

finite family of pairwise disjoint closed sets $\{V_i\}_{i\in I}$ such that, for each $i \in I$, the set V_i is a trivial neighborhood of the leaf ϕ_i that is disjoint from X.

Proof. Observe that $\{\phi_i\}_{i\in I} \cup \{X\}$ forms a locally finite family of pairwise disjoint closed subsets of $\mathbb{R}^2$. By applying Lemma 2.1, we can find a locally finite family of pairwise disjoint open sets $\{U_i\}_{i\in I}$ such that, for each $i \in I$, the set U_i is a neighborhood of the leaf ϕ_i and is disjoint from X. Since trivial neighborhoods of leaves can be taken to be arbitrarily small, for each $i \in I$, we can consider a trivial neighborhood V_i of the leaf ϕ_i such that $V_i \subset U_i$. Since $\{U_i\}_{i\in I}$ is locally finite, so is the family $\{V_i\}_{i\in I}$. □

We can endow any foliation $\mathcal{F}$ with the *quotient topology*, which corresponds to the finest topology on $\mathcal{F}$ for which the quotient map $\pi_{\mathcal{F}} : \mathbb{R}^2 \longrightarrow \mathcal{F}$ that sends each point to the leaf passing through it is continuous. In this topology, the open sets of the foliation $\mathcal{F}$ are those whose preimages under $\pi_{\mathcal{F}}$ are open in $\mathbb{R}^2$. Throughout this text, we will always consider the quotient topology on foliations, unless otherwise stated.

A subset $A \subset \mathbb{R}^2$ is called *saturated* (with respect to $\mathcal{F}$) if it contains all leaves of $\mathcal{F}$ that intersect it. The restriction of a foliation $\mathcal{F}$ to a saturated subset $A \subset \mathbb{R}^2$ is denoted by $\mathcal{F}|_A$. Note that, in the case of saturated sets, we have $\mathcal{F}|_A = \pi_{\mathcal{F}}(A)$. By a slight abuse of notation, we often treat saturated sets $A \subset \mathbb{R}^2$ as subsets of the foliation $\mathcal{F}$ by identifying $A = \mathcal{F}|_A$. This is the abuse that allows us to denote saturated sets as subsets $A \subset \mathcal{F}$ or to refer to leaves as elements $\phi \in A$. This convention introduces no topological ambiguity when referring to an open saturated set or to its boundary. Indeed, from the definition of the quotient topology, a saturated set $A \subset \mathbb{R}^2$ is open if, and only if, the restriction $\mathcal{F}|_A$ is an open subset of $\mathcal{F}$. Additionally, the boundary ∂A of a saturated set A may be understood equivalently as the boundary of A inside $\mathbb{R}^2$ or as the boundary of $\mathcal{F}|_A$ inside $\mathcal{F}$, with no risk of confusion.

This next lemma states that the connectedness is also preserved through this notation.

LEMMA 2.3. *A saturated set $A \subset \mathbb{R}^2$ is connected if, and only if, $\mathcal{F}|_A$ is connected.*

Proof. ($\Longrightarrow$) Let $A \subset \mathbb{R}^2$ be a connected saturated set. Since $\pi_{\mathcal{F}}$ is continuous, the image $\pi_{\mathcal{F}}(A)$ is connected. This proves the implication.

($\Longleftarrow$) Let us assume that $A \subset \mathbb{R}^2$ is a saturated set such that $\mathcal{F}|_A$ is connected. Suppose by contradiction that A is not connected. Therefore, there exists two non-empty open subsets $U, V \subset \mathbb{R}^2$ such that $A \subset U \cup V$ and such that $U \cap A$ is disjoint from $V \cap A$. Since each leaf $\phi \in A$ is connected, then either $\phi \cap U$ or $\phi \cap V$ is empty. Since A is saturated and covered by the union $U \cup V$, we have that each leaf $\phi \in A$ is contained in either U or V. In other words, $U \cap A$ and $V \cap A$ are disjoint saturated sets that cover A. Since $\pi_{\mathcal{F}}$ is a quotient map, $\mathcal{F}|_{U\cap A} = \pi_{\mathcal{F}}(U \cap A)$ and $\mathcal{F}|_{V\cap A} = \pi_{\mathcal{F}}(V \cap A)$ are open subsets of $\mathcal{F}|_A$ that are disjoint and cover $\mathcal{F}|_A$. This contradicts the assumption that $\mathcal{F}|_A$ is connected. □

Equipped with the quotient topology, the foliation $\mathcal{F}$ becomes a simply connected and non-Hausdorff one-dimensional manifold, often called the *leaf space* of $\mathcal{F}$. As a matter of fact, the classical result due to Haefliger and Reeb [**HR57**] proves that every simply connected, non-Hausdorff 1-manifold can be realized as the leaf space of a planar

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

foliation. Note that, any simply connected (non-Hausdorff) manifold is orientable. In the one-dimensional context, this corresponds to the existence of an atlas of charts mapping the local orientation at each point of the manifold to the positive orientation of $\mathbb{R}$. By choosing an orientation, we fix a local notion of left and right sides around each point. Since, by removing a point, we separate any simply connected non-Hausdorff 1-manifold into two connected components, this local notion of left and right sides is extended globally to each connected component.

In the context of the leaf space, this means that for each leaf $\phi \in \mathcal{F}$, the set $\mathcal{F} \setminus \{\phi\}$ has exactly two connected components, one labeled as the left side $L(\phi)$ and the other as the right side $R(\phi)$. Each leaf of $\mathcal{F}$ thereby inherits an orientation and this assignment is locally coherent, since it is determined by the local orientation at each point of the leaf space. In terms of local trivializing maps of the foliation $\mathcal{F}$, this means that one can choose each trivializing map $\varphi : U \to [0, 1] \times [0, 1]$ so that the orientation of every leaf $\phi \in \mathcal{F}$ is mapped to the positive direction of the second coordinate on the set $\varphi(U \cap \phi) = \{t\} \times [0, 1]$. This is the classical definition of an oriented foliation.

Once the foliation $\mathcal{F}$ is equipped with an orientation, it inherits a naturally defined partial order, where $\phi \leq \phi'$ if and only if $L(\phi') \subset L(\phi)$. Note that pairs of leaves $\phi, \phi' \in \mathcal{F}$ which are not Hausdorff-separable must satisfy either $R(\phi) \cap R(\phi') = \varnothing$ or $L(\phi) \cap L(\phi') = \varnothing$ and, consequently, they cannot be comparable under this partial order relation. Recall that, an interval of the foliation $\mathcal{F}$ with respect to this partial order consists in a subset of $\mathcal{F}$ that contains all leaves between any two of its leaves. Observe that leaf intervals are not necessarily homeomorphic to an interval of the real line $\mathbb{R}$, as the leaf space of $\mathcal{F}$ is not necessarily Hausdorff.

A saturated set $A \subset \mathbb{R}^2$ is said to be *non-separating* if for every leaf $\phi \in A$, one of the connected components of $\mathbb{R}^2 \setminus \phi$ contains all leaves in $A \setminus \{\phi\}$.

LEMMA 2.4. *Let $A \subset \mathbb{R}^2$ be an open, connected saturated set. Then, the following hold:*

(i) *the restriction $\mathcal{F}|_A$ is an interval of the foliation $\mathcal{F}$;*
(ii) *the boundary ∂A is a locally finite and non-separating collection of leaves in $\mathcal{F}$;*
(iii) *the set $A \subset \mathbb{R}^2$ is homeomorphic to a plane.*

Proof. Proof of item (i). Suppose by contradiction that there exist $\phi, \phi' \in \mathcal{F}|_A$ such that $\phi < \phi'$, and $\phi'' \in \mathcal{F} \setminus \mathcal{F}|_A$ such that $\phi < \phi'' < \phi'$. In this case, the sets $L(\phi'')$ and $R(\phi'')$ are disjoint open sets whose union covers A, and such that $L(\phi'') \cap A$ and $R(\phi'') \cap A$ are both non-empty. This contradicts the assumption that A is connected.

Proof of item (ii). Since $A \subset \mathbb{R}^2$ is open and connected, for any leaf $\phi \in \partial A$, the set A must be contained in one of the two connected components of $\mathbb{R}^2 \setminus \phi$. Therefore, each of the remaining leaves in $\partial A \setminus \{\phi\}$ is contained in the same connected component of $\mathbb{R}^2 \setminus \phi$ that contains A. This implies that the set ∂A is non-separating.

Next, observe that any open set $U \subset \mathbb{R}^2$ that locally trivializes the foliation $\mathcal{F}$ can at most intersect two leaves in ∂A. This is because, for any three leaves of $\mathcal{F}$ intersecting U, we get that one of them must separate the other two. Let $K \subset \mathbb{R}^2$ be a compact set and let $\{U_i\}_{0<i<k}$ be a finite cover of K by open sets that locally trivialize $\mathcal{F}$. Since each set in the cover $\{U_i\}_{0<i<k}$ intersects at most two leaves in ∂A, the set K intersects at most finitely many leaves in ∂A. This implies that the collection of leaves in ∂A is locally finite.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

Proof of item (iii). Knowing that the boundary ∂A is non-separating, we can write A as the complement of the set $\bigcup_{\phi\in\partial A} K(\phi)$, where $K(\phi)$ is defined as the closure of the connected component of $\mathbb{R}^2 \setminus \phi$ that does not contain A. Observe that, by adding the point at infinity, the set $\bigcup_{\phi\in\partial A} K(\phi) \cup \{\infty\}$ becomes a compact and connected subset of $\mathbb{S}^2$. Since A is the complement of a compact and connected subset of $\mathbb{S}^2$, we conclude that it is simply connected. According to the Riemann mapping theorem, being a simply connected open subset of $\mathbb{R}^2$, the set A must be homeomorphic to the plane. This completes the proof. □

A saturated set $A \subset \mathbb{R}^2$ is said to be *trivially foliated* if the restriction $\mathcal{F}|_A$ is totally ordered by the natural order on the foliation $\mathcal{F}$.

Definition 2.5. A *leaf domain* is a saturated set that is open, connected, and trivially foliated.

Following Lemmas 2.3 and 2.4, we observe that non-empty leaf domains of $\mathcal{F}$ correspond exactly to the subsets of $\mathcal{F}$ that are homeomorphic to $\mathbb{R}$. Moreover, the boundary of a leaf domain is formed by a locally finite and non-separating collection of leaves in $\mathcal{F}$.

LEMMA 2.6. *A finite intersection of leaf domains also is a leaf domain.*

Proof. The lemma holds because, in a simply connected non-Hausdorff 1-manifold, the intersection of two subsets homeomorphic to $\mathbb{R}$ is either empty or homeomorphic to $\mathbb{R}$. □

The boundary of a leaf domain $A \subset \mathcal{F}$ admits two important subsets, defined as

$$\partial_R A := \partial A \cap \left(\bigcap_{\phi\in A} R(\phi) \right) \quad \text{and} \quad \partial_L A := \partial A \cap \left(\bigcap_{\phi\in A} L(\phi) \right).$$

These sets are disjoint by definition and their union may not cover the entire boundary ∂A. The set A lies on the left side of each leaf in $\partial_R A$ and on the right side of every leaf in $\partial_L A$.

The sets $\partial_R A$ and $\partial_L A$ are each totally ordered by a relation $\prec$, defined as follows.

- Fix a leaf of reference $\phi_* \in A$. For any two leaves $\phi, \phi' \in \partial_L A$ (respectively $\partial_R A$), we define $\phi \prec \phi'$ if and only if there exist two disjoint paths $\gamma, \gamma' : [0, 1] \longrightarrow \mathbb{R}^2$ with endpoints $\gamma(1) \in \phi$, $\gamma'(1) \in \phi'$, and $\gamma(0), \gamma'(0) \in \phi_*$, ordered $\gamma(0) \prec \gamma'(0)$ along the leaf ϕ_*.

We remark that $\prec$ is well defined and independent of the choice of reference leaf $\phi_* \in A$. Later, in §3.3, we provide a more formal alternative definition of this order relation.

A path $\gamma : I \longrightarrow \mathbb{R}^2$ defined on an interval $I \subset \mathbb{R}$ is said to be *positively transverse* to an oriented foliation $\mathcal{F}$ if the map $\pi_{\mathcal{F}} \circ \gamma : I \longrightarrow \mathcal{F}$ is strictly increasing with respect to the natural order on $\mathcal{F}$. In other words, a positively transverse path intersects each leaf of $\mathcal{F}$ at most once and it does so crossing each leaf from right to left.

Each positively transverse path γ defines a saturated set $C_\gamma \subset \mathbb{R}^2$, which is the union of all leaves in $\mathcal{F}$ that are intersected by γ. From the definition of positively transverse

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

paths, it follows that the set C_γ (seen as a subset of the foliation $\mathcal{F}$) is homeomorphic to an interval of the real line $\mathbb{R}$. In particular, this proves the following elementary lemma.

LEMMA 2.7. *Let $I \subset \mathbb{R}$ be an open interval and let $\gamma : I \longrightarrow \mathbb{R}^2$ be a path positively transverse to the foliation $\mathcal{F}$. Then, the saturated set C_γ is a leaf domain of $\mathcal{F}$.*

Conversely, for any leaf domain $A \subset \mathcal{F}$ and any pair of points $x, y \in A$ that do not lie on the same leaf, there exists a positively transverse path $\gamma : I \longrightarrow \mathbb{R}^2$ that satisfies either $\gamma(0) = x$ and $\gamma(1) = y$ or $\gamma(0) = y$ and $\gamma(1) = x$.

We present two examples of applications of the Homma–Schoenflies theorem.

(1) For any path $\gamma : [0, 1] \longrightarrow \mathbb{R}^2$ positively transverse to a foliation $\mathcal{F}$, there exists an orientation-preserving homeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that satisfies the following properties:

- the path γ is mapped by φ into the horizontal arc $[0, 1] \times \{0\}$;
- for $i \in \{0, 1\}$, the leaf $\phi_{\gamma(i)} \in \mathcal{F}$ passing through $\gamma(i)$ is mapped by φ into the vertical line $\{i\} \times \mathbb{R}$ oriented downwards;
- there exists a closed set $V \subset \mathbb{R}^2$ containing γ that is mapped by φ into the square $[0, 1] \times [-1, 1]$ such that, for any leaf $\phi \in \mathcal{F}$ intersecting V, the set $\phi \cap V$ is mapped by φ into a vertical arc of the form $\{t\} \times [-1, 1]$ for some $t \in [0, 1]$.

(2) For any topological line $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ positively transverse to $\mathcal{F}$, there exists an orientation-preserving homeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that satisfies the following properties:

- the topological line Γ is mapped by φ into the horizontal line $\mathbb{R} \times \{0\}$;
- there exists a closed set $V \subset \mathbb{R}^2$ containing Γ that is mapped by φ into the horizontal strip $\mathbb{R} \times [-1, 1]$ and such that, for any leaf $\phi \in \mathcal{F}$ intersecting V, the set $\phi \cap V$ is mapped by φ into a vertical arc of the form $\{t\} \times [-1, 1]$ oriented downwards.

2.3. *Foliated Brouwer theory.* Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a *Brouwer homeomorphism*, that is, an orientation-preserving and fixed-point free homeomorphism of the plane.

The orbit of a point $x \in \mathbb{R}^2$ under f is defined as the set

$$\mathcal{O}(f, x) = \{f^n(x) \mid n \in \mathbb{Z}\} \subset \mathbb{R}^2.$$

Sometimes, it may be useful to denote the set of orbits of f by $\mathrm{Orb}(f)$. As proved in [**Bro12**], every orbit of f is wandering and, thus, it has an empty limit set. This means that every orbit of f is the image of a proper embedding of $\mathbb{Z}$ into the plane.

A *Brouwer line* for f is an oriented topological line λ on $\mathbb{R}^2$ that satisfies

$$f(L(\lambda)) \subset L(\lambda) \quad \text{and} \quad f^{-1}(R(\lambda)) \subset R(\lambda),$$

where $L(\lambda)$ and $R(\lambda)$ denote the left and right sides of λ. The classical *Brouwer translation theorem* [**Bro12**] asserts that through every point of the plane passes a Brouwer line for f.

THEOREM. (Le Calvez [**LC04**]) *There exists an oriented topological foliation $\mathcal{F}$ of the plane such that every leaf $\phi \in \mathcal{F}$ is a Brouwer line of f.*

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

Any foliation of the plane by Brouwer lines of f is called a *transverse foliation* of f. It is worth noting that a Brouwer homeomorphism admits several transverse foliations, which may exhibit different structures and non-homeomorphic leaf spaces, for instance.

Let us fix a transverse foliation $\mathcal{F}$ of a Brouwer homeomorphism f. In this context, one can show that every point $x \in \mathbb{R}^2$ may be connected to its image $f(x)$ by a path γ_x that is positively transverse to the foliation $\mathcal{F}$. By concatenating the paths $\gamma_{f^n(x)}$ for all $n \in \mathbb{Z}$, we obtain

$$\Gamma := \prod_{n\in\mathbb{Z}} \gamma_{f^n(x)},$$

which is a positively transverse path connecting the entire orbit $\mathcal{O}(f, x)$, called a *transverse trajectory* of $O(f, x)$ (with respect to the foliation $\mathcal{F}$). Since $\mathcal{F}$ is fixed, we often omit its mention when referring to transverse trajectories. Note that each orbit $O(f, x)$ admits uncountably many transverse trajectories, depending on the choice of $\{\gamma_{f^n(x)}\}_{n\in\mathbb{Z}}$.

Independently of the choice of transverse paths $\{\gamma_{f^n(x)}\}_{n\in\mathbb{Z}}$, the leaves that are intersected by the transverse trajectory Γ_O of $O = O(f, x)$ are determined by the orbit itself. In fact, the set $C_O \subset \mathcal{F}$ of leaves *crossed* by the orbit O can be equivalently defined as

$$C_O = \{\phi \in \mathcal{F} \mid \phi \cap \Gamma_O \neq \varnothing\} = \{\phi \in \mathcal{F} \mid O \text{ intersects } R(\phi) \text{ and } L(\phi)\}.$$

We remark that, if the choice of paths $\{\gamma_{f^n(x)}\}_{n\in\mathbb{Z}}$ is locally finite on the plane, then the transverse trajectory Γ_O is a topological line and it is called a *proper transverse trajectory* of the orbit O. The converse is also true.

Let $\mathcal{O} = \{O_1, \ldots, O_r\}$ be a finite collection of orbits of f. Throughout this text, we may treat $\mathcal{O}$ as a subset of $\mathbb{R}^2$ by identifying it with $\mathcal{O} = \bigcup_{i=1}^{r} O_i$. A transverse foliation $\mathcal{F}$ of f is said to be *generic* for the collection $\mathcal{O}$ if each leaf $\phi \in \mathcal{F}$ intersects at most one orbit in $\mathcal{O}$. This condition is indeed generic, as we explain in the following.

Assume that a leaf $\phi \in \mathcal{F}$ intersects two orbits in $\mathcal{O}$, at points $x, x' \in \phi$. Let B be a closed ball centered at x that is sufficiently small so that $B \cap \mathcal{O} = \{x\}$ and $f(B) \cap B = \varnothing$. We can perturb $\mathcal{F}$ inside the ball B in such a way that x no longer lies on the leaf passing by x'. Since B satisfies $f(B) \cap B = \varnothing$, the leaves of $\mathcal{F}$ continue to be Brouwer lines of f after the perturbation. Now, since $\mathcal{O}$ is a finite collection of orbits of f and each orbit of f is a closed discrete subset of $\mathbb{R}^2$, we have that for any compact set $K \subset \mathbb{R}^2$, the set $K \cap \mathcal{O}$ is finite. Therefore, by repeating the perturbation process described above on a locally finite family of disjoint balls covering $\mathcal{O}$, we can obtain a transverse foliation $\mathcal{F}'$ of f that is generic for $\mathcal{O}$.

3. *Asymptotic behavior of orbits*

Since f is a Brouwer homeomorphism, every orbit of f goes to infinity, in the sense that, for any $x \in \mathbb{R}^2$, the set $O = \bigcup_{n\in\mathbb{Z}} f^n(x)$ is the image of a proper embedding of $\mathbb{Z}$ into $\mathbb{R}^2$. Without an additional structure on the system, we cannot precisely describe the way that

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

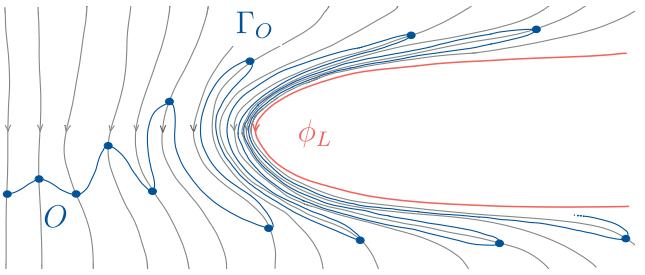


FIGURE 7. Weird asymptotic behavior that would make the existence of proper transverse trajectories impossible.

the orbits of f escape to infinity. Recently, the most efficient way to describe the asymptotic behavior of orbits of f has been through transverse foliations and transverse trajectories.

As explained in §2.3, once we consider a planar foliation $\mathcal{F}$ transverse to f, each orbit O of f can be described through its transverse trajectory Γ_O. The information carried by an arbitrary transverse trajectory Γ_O is equivalent to the information carried by the set

$$C_O := \{\phi \in \mathcal{F} \mid \phi \cap \Gamma_O \neq \varnothing\}.$$

The set C_O is independent of the choice of transverse trajectory Γ_O, and it corresponds to the set of leaves $\phi \in \mathcal{F}$ such that both sides $R(\phi)$ and $L(\phi)$ are intersected by the orbit O. Until now, this has been the information used to describe the asymptotic behavior of orbits of Brouwer homeomorphisms in the framework of foliated Brouwer theory.

3.1. *Limitation of the framework.* Let $x \in \mathbb{R}^2$ be a point in the plane and let $O = O(f, x)$. For each $n \in \mathbb{Z}$, let $\phi_n \in \mathcal{F}$ be the leaf $\mathcal{F}$ containing the point $f^n(x)$. Note that $(\phi_n)_{n\in\mathbb{Z}}$ is a bi-infinite sequence of leaves in C_O that is increasing with respect to the natural order of the foliation $\mathcal{F}$, meaning, $L(\phi_{n+1}) \subset L(\phi_n)$ for all $n \in \mathbb{Z}$. Although O must escape to infinity, the sequence $(\phi_n)_{n\in\mathbb{Z}}$ may accumulate at certain leaves of $\mathcal{F}$. Since $(\phi_n)_{n\in\mathbb{Z}}$ is increasing, there can be two types of leaves accumulated by the sequence $(\phi_n)_{n\in\mathbb{Z}}$. Either a leaf $\phi_L \in \mathcal{F}$, which is a limit leaf of the forward sequence $(\phi_n)_{n\geq 0}$, or a leaf $\phi_R \in \mathcal{F}$, which is a limit leaf of the backward sequence $(\phi_n)_{n\leq 0}$. It is worth noting, since $\mathcal{F}$ is not necessarily Hausdorff, each of these sequences may admit multiple limit leaves.

A priori, each point $f^n(x) \in O$ can be located anywhere along its underlying leaf ϕ_n.

Hence, in case where the sequence $(\phi_n)_{n\geq 0}$ converges to a leaf ϕ_L, the orbit O could possibly exhibit a weird asymptotic behavior with respect to ϕ_L (see Figure 7) which impedes the existence of properly embedded transverse trajectories of O. Such behavior would pose a significant limitation to the control of topological properties of transverse trajectories.

In this section, we overcome this limitation by proving that every orbit of f indeed admits proper transverse trajectories (see Theorem A). Moreover, we show that a proper transverse trajectory Γ_O associated to an orbit O encodes additional information about the

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

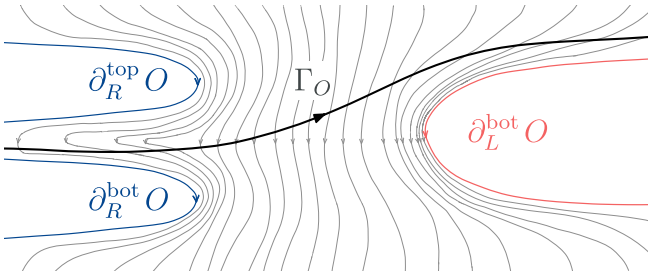


FIGURE 8. Description of the information encoded by proper transverse trajectories.

asymptotic behavior of O that is not captured by C_O alone, but rather by C_O and four distinguished subsets of ∂C_O, namely $\partial_L^{\text{top}} O$, $\partial_L^{\text{bot}} O$, $\partial_R^{\text{top}} O$, and $\partial_R^{\text{bot}} O$ (see Figure 8).

3.2. *Crossed leaves and limit leaves.* For each orbit $O \in \text{Orb}(f)$, we define the set

$$C_O := C(\mathcal{F}, O) = \{\phi \in \mathcal{F} \mid \phi \cap \Gamma_O \neq \varnothing\},$$

where Γ_O represents any transverse trajectory of O. By abuse of notation, we treat C_O as a saturated subset of $\mathbb{R}^2$ (see §2.2). According to Lemmas 2.7 and 2.4, the saturated set $C_O \subset \mathbb{R}^2$ is a leaf domain of $\mathcal{F}$ and, consequently, it is homeomorphic to a trivially foliated plane, and its boundary ∂C_O is a locally finite and non-separating collection of leaves in $\mathcal{F}$. The set C_O is often referred to as the *leaf domain* of the orbit O leaves in C_O are said to be *crossed* by the orbit O.

Since the set C_O is a leaf domain, its boundary ∂C_O admits two distinguished subsets, namely the *left-limit* and *right-limit* leaves of C_O, which are defined as follows:

$$\partial_L C_O := \partial C_O \cap \left(\bigcap_{\phi \in C_O} L(\phi) \right) \quad \text{and} \quad \partial_R C_O := \partial C_O \cap \left(\bigcap_{\phi \in C_O} R(\phi) \right).$$

By fixing a basepoint $x \in O$ and considering the bi-infinite sequence $(\phi_n)_{n\in\mathbb{Z}}$ formed by the leaves $\phi_n \in \mathcal{F}$ passing through the points $f^n(x)$, we note that the forward sequence $(\phi_n)_{n\geq 0}$ converges simultaneously to all leaves in $\partial_L C_O$ (with respect to the quotient topology on $\mathcal{F}$), while the backward sequence $(\phi_n)_{n\leq 0}$ converges to all leaves in $\partial_R C_O$. This can be deduced from the fact that, for any leaf $\phi \in C_O$, there exists $n \geq 0$ such that $L(\phi_{-n}) \subset L(\phi) \subset L(\phi_n)$. Observe that the sets $\partial_L C_O$ and $\partial_R C_O$ are disjoint by definition, and their union may not cover the entire boundary set ∂C_O (see Figure 1).

Remark 3.1. For any pair of orbits $O, O' \in \text{Orb}(f)$, the following properties hold.

(i) *Common leaf versus Separated.* The orbits O and O' respect a natural dichotomy: either their C_O and $C_{O'}$ intersect, in which case every leaf $\phi \in C_O \cap C_{O'}$ is called a *common leaf* of O and O'; or sets C_O and $C_{O'}$ are disjoint, in which case we say the orbits O and O' are *separated*, since they are separated by the leaf $\phi \in \partial C_O$ that bounds the connected component of $\mathbb{R}^2 \setminus C_O$, which contains $C_{O'}$ (see Figure 9).

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

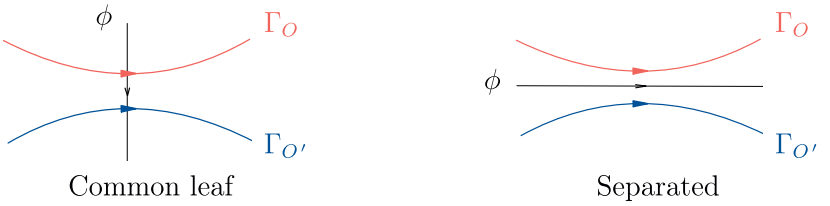


FIGURE 9. The two possible configurations for pairs of orbits of f with respect to $\mathcal{F}$.

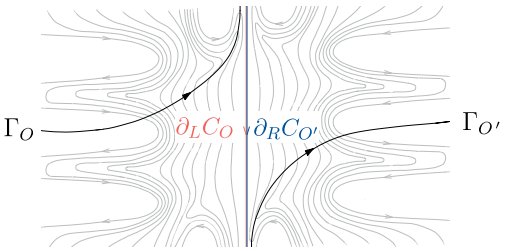


FIGURE 10. Illustration of the property (ii) from Remark 3.1.

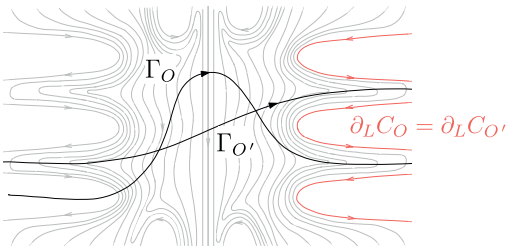


FIGURE 11. Illustration of the property (iii) from Remark 3.1.

(ii) If $\partial_L C_O \cap \partial_R C_{O'} \neq \varnothing$, then $C_O \cap C_{O'} = \varnothing$ and $\partial_L C_O \cap \partial_R C_{O'} = \{\phi\}$ (Figure 10).
(iii) If $\partial_L C_O \cap \partial_L C_{O'} \neq \varnothing$, then $C_O \cap C_{O'} \neq \varnothing$ and $\partial_L C_O = \partial_L C_{O'}$ (Figure 11).

3.3. *Definition of order $\prec$ on the sets $\partial_L C_O$ and $\partial_R C_O$.* Fix $O \in \mathrm{Orb}(f)$ and a basepoint $x \in O$. Since ∂C_O is a locally finite collection of leaves disjoint from the closed set $\phi_0 \cup O$, we can apply Lemma 2.2 to affirm that there exists a locally finite family $\{V_\phi\}_{\phi \in \partial C_O}$ of pairwise disjoint closed sets such that, for each $\phi \in \partial C_O$, the set V_ϕ is a trivial neighborhood of ϕ disjoint from $\phi_0 \cup O$.

For each $n \in \mathbb{Z}$, let $\phi_n \in \mathcal{F}$ be the leaf passing through the point $f^n(x)$ and define

$$J_n := \{\phi \in \partial_L C_O \cup \partial_R C_O \mid V_\phi \cap \phi_n \neq \varnothing\}$$

(see Figure 12).

Observe that, by construction, $J_n \subset \partial_L C_O$ when $n > 0$ and $J_n \subset \partial_R C_O$ when $n < 0$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

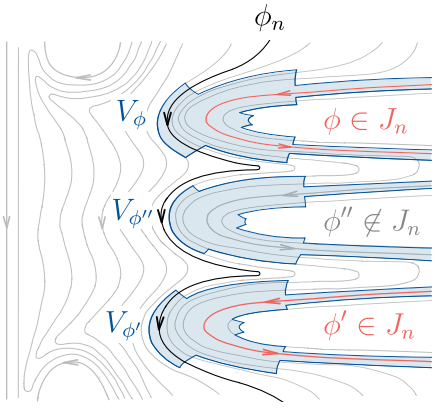


FIGURE 12. Definition of the sets J_n.

LEMMA 3.2. *The sequence of sets $(J_n)_{n\geq 0}$ forms a filtration of $\partial_L C_O$, in the sense that:*

(i) $J_0 = \varnothing$;

(ii) $J_n \subset J_{n+1}$ *for any* $n \geq 0$;

(iii) $\partial_L C_O = \bigcup_{n\geq 0} J^n$.

Similarly, the sequence of sets $(J_n)_{n\leq 0}$ forms a filtration of the set $\partial_R C_O$.

Proof. We prove the lemma for the sequence $(J_n)_{n\geq 0}$; the proof for $(J_n)_{n\leq 0}$ is analogous.

Since the leaf ϕ_0 is disjoint from each set in the family $\{V_\phi\}_{\phi\in\partial_L C_O}$, it holds that $J_0 = \varnothing$. Next, as the sequence of leaves $(\phi_n)_{n\geq 0}$ is increasing in $\mathcal{F}$, if some leaf ϕ_n intersects V_ϕ for a given integer $n \geq 0$, then the leaf ϕ_{n+1} also intersects V_ϕ. In other words, it holds that

$$J_n \subset J_{n+1} \quad \text{for all } n \geq 0.$$

Finally, note that for each leaf $\phi \in C_O$, there exists some $n > 0$ such that $L(\phi_n) \subset L(\phi)$. Therefore, for each $\phi \in \partial_L C_O$, there exists the smallest positive integer $N_\phi > 0$ satisfying

$$\phi_{N_\phi} \cap V_\phi \neq \varnothing.$$

This implies that $\partial_L C_O = \bigcup_{n\geq 0} J^n$ and, thus, we conclude the proof of the lemma. □

First, note that for each $n \in \mathbb{Z}$ and $\phi \in J_n$, the set $\alpha_n(\phi) := \phi_n \cap V_\phi$ is a non-empty compact sub-arc of the leaf ϕ_n. This gives us, for each $n \in \mathbb{Z}$, a family $A_n := \{\alpha_n(\phi)\}_{\phi\in J_n}$ of pairwise disjoint sub-arcs of the leaf ϕ_n. Next, note that the orientation of ϕ_n induces a natural total order on the set A_n, which is defined as follows. For any two arcs $\alpha, \alpha' \in A_n$, we write $\alpha \prec_n \alpha'$ if, along the oriented leaf ϕ_n, the arc α' is positioned after α. Through the bijection $\phi \longmapsto \alpha_n(\phi)$, we import this order to J_n. Given two leaves $\phi, \phi' \in J_n$, we say that

$$\phi \prec_n \phi' \quad \text{if } \alpha_n(\phi) \prec_n \alpha_n(\phi')$$

(see Figure 13).

Observe that $\prec_n$ and $\prec_{n+1}$ are compatible with the inclusion $J_n \subset J_{n+1}$ for any $n \geq 0$. This means that, if two leaves $\phi, \phi' \in J_n$ satisfy $\phi \prec_n \phi'$, then they also satisfy $\phi \prec_{n+1} \phi'$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

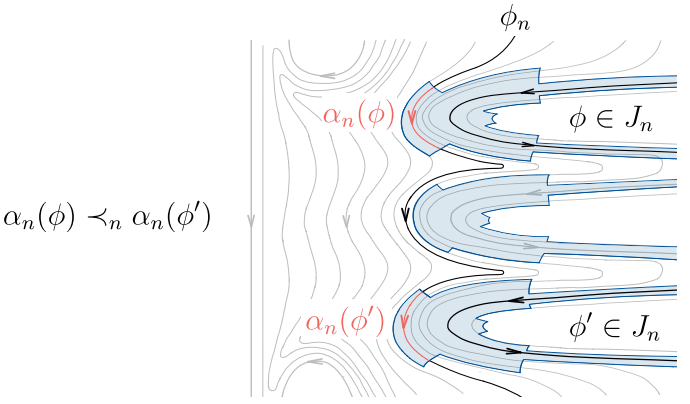


FIGURE 13. Definition of the order $\prec_n$ on the set J_n.

This ensures that the set $\partial_L C_O = \bigcup_{n>0} J_n$ inherits a total order $\prec$ that extends the sets in the filtration $(J_n)_{n\geq 0}$, in the sense that, for any $\phi, \phi' \in \partial_L C_O$, we define the relation

$$\phi \prec \phi' \quad \text{if there exists } n > 0 \text{ such that } \phi \prec_n \phi'.$$

Note that the set $\partial_R C_O$ inherits an analogous total order $\prec$ from the filtration $(J_n)_{n\leq 0}$.

Remark 3.3. There are two important properties of the order $\prec$ that we want to highlight.

(i) The definition of $\prec$ does not change if the trivial neighborhoods in the family $\{V_\phi\}_{\phi\in\partial C_O}$ are replaced by smaller ones. This ensures that our construction is independent of the choice of trivial neighborhoods, if they are taken small enough;

(ii) By taking sufficiently small cross-sections $\sigma_\phi \subset V_\phi$ over each leaf $\phi \in \partial_L C_O$, and then extending them until they reach the leaf ϕ_0, we can obtain a family $\{\gamma_\phi\}_{\phi\in\partial_L C_O}$ of pairwise disjoint paths $\gamma_\phi : [0, 1] \longrightarrow \mathbb{R}^2$ positively transverse to the foliation $\mathcal{F}$ that joins the points $\gamma_\phi(0) \in \phi_0$ and $\gamma_\phi(1) \in \phi$. This allows us to express the order $\prec$ in terms of the order of the endpoints $\{\gamma_\phi(0)\}_{\phi\in\partial_L C_O}$ along the leaf ϕ_0, meaning that

$$\phi \prec \phi' \iff \gamma_\phi(0) \prec \gamma_{\phi'}(0) \text{ according to the orientation of } \phi_0.$$

3.4. *Orbit induced cuts.* Let $\mathrm{Cut}_\prec(\partial_L C_O)$ be the set of cuts of $(\partial_L C_O, \prec)$, that is, the set of all pairs (A, B) such that $A, B \subset \partial_L C_O$ are disjoint subsets of $\partial_L C_O$ satisfying:

- $A \cup B = \partial_L C_O$;
- for any leaf $\phi \in A$ and any leaf $\phi' \in B$, it holds that $\phi \prec \phi'$.

Similarly, let $\mathrm{Cut}_\prec(\partial_R C_O)$ denote the set of all cuts of the totally ordered set $(\partial_R C_O, \prec)$.

PROPOSITION 3.4. *Every orbit $O \in \mathrm{Orb}(f)$ admits two canonically defined cuts*

$$(\partial_L^{\mathrm{top}} O, \partial_L^{\mathrm{bot}} O) \in Cut_\prec(\partial_L C_O) \quad \textit{and} \quad (\partial_R^{\mathrm{top}} O, \partial_R^{\mathrm{bot}} O) \in Cut_\prec(\partial_R C_O).$$

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

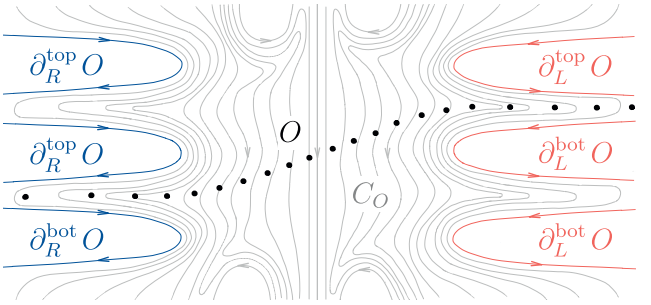


FIGURE 14. The cuts $(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O)$ and $(\partial_R^{\text{top}} O, \partial_R^{\text{bot}} O)$ canonically associated to the orbit O.

It is worth remarking that the sets $\partial_L^{\text{top}} O$, $\partial_L^{\text{bot}} O$, $\partial_R^{\text{top}} O$, and $\partial_R^{\text{bot}} O$ depend not only on O, but also on the foliation $\mathcal{F}$. However, we often omit the dependence on $\mathcal{F}$ in the notation.

The leaf domain C_O and the cuts $(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O)$ and $(\partial_R^{\text{top}} O, \partial_R^{\text{bot}} O)$ describe the asymptotic behavior of the orbit O with respect to $\mathcal{F}$ (Figure 14). The qualitative data associated with these objects are strictly greater than those given by the set C_O alone. Throughout this work, we refer to the qualitative data as the *$\mathcal{F}$-asymptotic behavior* of O.

Later, in §3.5, we show that the cuts $(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O)$ and $(\partial_R^{\text{top}} O, \partial_R^{\text{bot}} O)$ can be interpreted by proper transverse trajectories of O (see Theorem A). These proper transverse trajectories completely describe the $\mathcal{F}$-asymptotic behavior of the orbit O, differently from arbitrary (non-proper) trajectories, which only encode the information carried by C_O.

Proof of Proposition 3.4. Fix an orbit $O \in \text{Orb}(f)$, a basepoint $x \in O$, a family $\{V_\phi\}_{\phi \in \partial C_O}$ of trivial-neighborhoods, the leaves $(\phi_n)_{n\in\mathbb{Z}}$, and the sets $(J_n)_{n\in\mathbb{Z}}$ as defined in §3.3.

Observe that, by cutting each leaf ϕ_n at the point $f^n(x)$, we obtain two half-lines

$$\phi_n^+ : [0, \infty) \longrightarrow \mathbb{R}^2 \quad \text{and} \quad \phi_n^- : (-\infty, 0] \longrightarrow \mathbb{R}^2,$$

with endpoints $\phi_n^+(0) = \phi_n^-(0) = f^n(x)$ and satisfying $\phi_n^+, \phi_n^- \subset \phi_n$. Recall that each set in the family $\{V_\phi\}_{\phi \in \partial C_O}$ is disjoint from the orbit O. Consequently, for each $n \in \mathbb{Z}$ and $\phi \in J^n$, the arc $\alpha_n(\phi) := \phi_n \cap V_\phi$ must lie entirely within one of the two half-lines, either ϕ_n^+ or ϕ_n^-. This allows us to define, for any $n \in \mathbb{Z}$, a cut $(J_n^-, J_n^+) \in \text{Cut}_\prec(J_n)$ given by the sets

$$J_n^- := \{\phi \in J_n \mid \alpha_n(\phi) \subset \phi_n^-\} \quad \text{and} \quad J_n^+ := \{\phi \in J_n \mid \alpha_n(\phi) \subset \phi_n^+\}.$$

CLAIM 3.5. *For each leaf $\phi \in \partial_L C_O$, there exists $M_\phi > 0$ which is the smallest positive integer such that, for any $n \geq M_\phi$, the following two equivalences hold*

$$\phi \in J_n^+ \iff \phi \in J_{n+1}^+,$$
$$\phi \in J_n^- \iff \phi \in J_{n+1}^-.$$

Similarly, for each leaf $\phi \in \partial_R C_O$, there exists $M_\phi < 0$ satisfying an analogous property.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

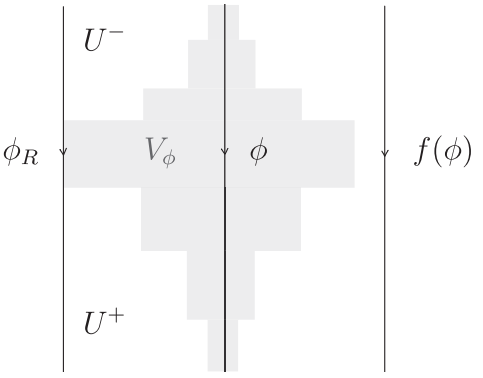


FIGURE 15. Setting in the proof of Claim 3.5 after applying the Homma–Schoenflies theorem.

Proof of Claim 3.5. Fix a leaf $\phi \in \partial_L C_O$ and denote by $\phi_R \in C_O$ the right-most leaf that intersects the trivial-neighborhood V_ϕ. The open set $L(\phi_R) \cap R(\phi) \setminus V_\phi$ has exactly two connected components, named U^+ and U^- according to the following equivalences:

$$\phi \in J_n^+ \iff f^n(x) \in U^+;$$
$$\phi \in J_n^- \iff f^n(x) \in U^-.$$

To simplify the argument, we can profit from Homma–Schoenflies theorem to suppose, up to conjugacy, that the topological lines ϕ_R, ϕ, and $f(\phi)$ are verticals on the plane oriented downwards; the set U^- lies on the upper half-plane; the set U^+ lies on the lower half-plane; and that f maps the points of ϕ into $f(\phi)$ as a horizontal (see Figure 15). We represent V_ϕ disjoint from $f(\phi)$ because V_ϕ is considered small enough; however, our proof does not depend on this property and it is possible that V_ϕ intersects $f(\phi)$.

Now, we suppose by contradiction that the claim of the lemma does not hold. More precisely, we suppose that there exists an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ satisfying

$$\phi \in J_{n_k}^- \bigcap J_{n_k+1}^+.$$

According to the equivalence that defines U^- and U^+, for every $k \in \mathbb{N}$, it holds that

$$z_k := f^{n_k}(x) \in U^+ \quad \text{and} \quad f(z_k) \in U^-.$$

Since f is a Brouwer homeomorphism, $(z_k)_{k\in\mathbb{N}}$ has no accumulation points on the plane. As the set U^+ is bounded in the horizontal coordinate and contained in the lower half-plane, the sequence $(z_k)_{k\in\mathbb{N}}$ is unbounded from below on the vertical coordinate. Up to taking a subsequence, we may suppose that $(z_k)_{k\in\mathbb{N}}$ is strictly decreasing on the vertical coordinate. Consequently, the family $\{\alpha_k\}_{k\in\mathbb{N}}$ of horizontal arcs α_k that join z_k to a point $y_k \in \phi$ is, by construction, locally finite (see Figure 16).

Let $K \subset \mathbb{R}^2$ be the compact arc of zero vertical coordinate connecting ϕ_R and $f(\phi)$. From the hypothesis given by the Homma–Schoenflies theorem, for any $k \in \mathbb{N}$, we have that the point $y_k \in \phi$ and its image $f(y_k) \in f(\phi)$ have exactly the same horizontal coordinate. Consequently, the sequence $(f(y_k))_{k\in\mathbb{N}}$ lies on the lower half-plane, while

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

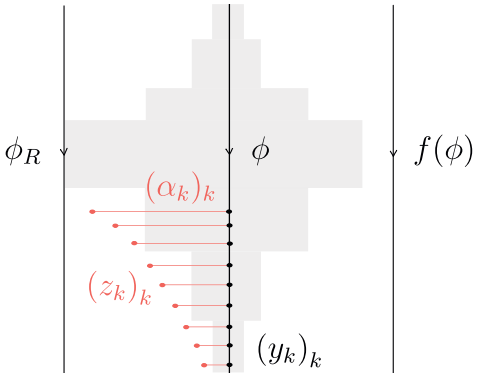


FIGURE 16. Illustration of the family $\{\alpha_k\}_{k\in\mathbb{N}}$ of horizontal arcs in the proof of Claim 3.5.

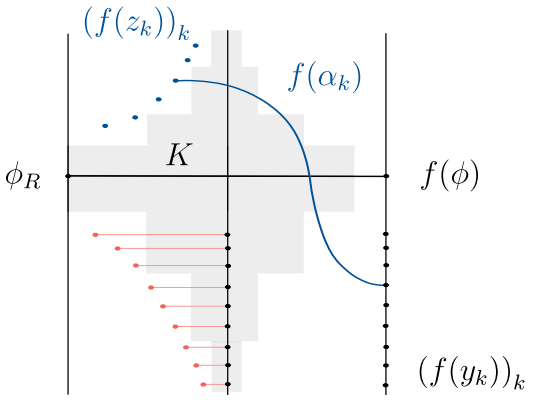


FIGURE 17. Illustration of the family $\{f(\alpha_k)\}_{k\in\mathbb{N}}$ of horizontal arcs in the proof of Claim 3.5.

$(f(z_k))_{k\in\mathbb{N}}$ lies on the upper half-plane. Since the arcs in the family $\{\alpha_k\}_{k\in\mathbb{N}}$ are contained in the set $\overline{L(\phi_R)\cap R(\phi)}$, their images $\{f(\alpha_k)\}_{k\in\mathbb{N}}$ are contained in $\overline{L(\phi_R)\cap R(f(\phi))}$. Note that, since $f(\alpha_k)$ joins the point $f(y_k)$ to $f(z_k)$, which are respectively in the lower and upper half-planes, each arc in the family $\{f(\alpha_k)\}_{k\in\mathbb{N}}$ must intersect the compact set K. In other words, $\{f(\alpha_k)\}_{k\in\mathbb{N}}$ is not a locally finite family of arcs (see Figure 17).

However, since f is a homeomorphism and $\{\alpha_k\}_{k\in\mathbb{N}}$ is locally finite, $\{f(\alpha_k)\}_{k\in\mathbb{N}}$ should be, in fact, locally finite. This gives us a contradiction. □

Due to Claim 3.5, we can now define the following sets:

$$\partial_L^{\text{top}} O := \{\phi \in \partial_L C_O \mid \text{ there exists } M > 0 \text{ for all } n > M, \phi \in J_n^+\};$$
$$\partial_L^{\text{bot}} O := \{\phi \in \partial_L C_O \mid \text{ there exists } M > 0 \text{ for all } n > M, \phi \in J_n^-\}.$$

By construction, we have that $(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O) \in \text{Cut}_{\prec}(\partial_L C_O)$. We can define the other two sets $\partial_R^{\text{top}} O$ and $\partial_R^{\text{bot}} O$ in a similar way, which thus concludes the proof of the proposition. □

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

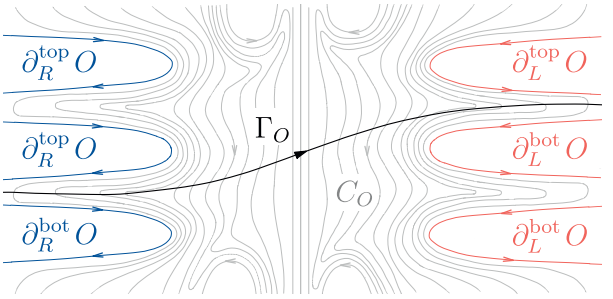


FIGURE 18. Depiction of how a proper transverse trajectory Γ_O of an orbit O recovers the cannonically associated cuts $(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O)$ and $(\partial_R^{\text{top}} O, \partial_R^{\text{bot}} O)$, as stated in Theorem A.

3.5. *Proper transverse trajectories.* We now prove Theorem A, which we restate.

THEOREM A. *Every orbit $O \in \text{Orb}(f)$ admits a proper transverse trajectory. Moreover, any proper transverse trajectory Γ_O associated to the orbit O satisfies the equalities*

$$\partial_L^{\text{top}} O = \partial_L C_O \cap L(\Gamma_O), \quad \partial_R^{\text{top}} O = \partial_R C_O \cap L(\Gamma_O),$$
$$\partial_L^{\text{bot}} O = \partial_L C_O \cap R(\Gamma_O), \quad \partial_R^{\text{bot}} O = \partial_R C_O \cap R(\Gamma_O)$$

(see Figure 18).

Proof. Consider an orbit $O \in \text{Orb}(f)$, a basepoint $x \in O$, a family $\{V_\phi\}_{\phi \in \partial C_O}$ of trivial neighborhoods, the leaves $(\phi_n)_{n\in\mathbb{Z}}$, and the sets $(J_n)_{n\in\mathbb{Z}}$ as defined in §3.3. Recall that in the proof of Lemma 3.2 and Proposition 3.4, we defined, for each leaf $\phi \in \partial_L C_O$, two integers $N_\phi > 0$ and $M_\phi > 0$, which are the smallest positive integers satisfying

$$\phi_n \cap V_\phi \neq \varnothing \quad \text{for all } n \geq N_\phi,$$
$$\phi \in J_n^+ \iff \phi \in J_{n+1}^+ \quad \text{for all } n \geq M_\phi.$$

Observe that, for $n \geq M_\phi$, the following two scenarios *cannot occur* (see Figure 19):

$$\text{(a) } \alpha_n(\phi) \subset \phi_n^+ \text{ and } \alpha_{n+1}(\phi) \subset \phi_{n+1}^-; \qquad \text{(b) } \alpha_n(\phi) \subset \phi_n^- \text{ and } \alpha_{n+1}(\phi) \subset \phi_{n+1}^+.$$

This means that, for each $n \geq M_\phi$, there exists a path $\gamma_{n,\phi} : [0, 1] \longrightarrow \mathbb{R}^2$ positively transverse to $\mathcal{F}$ that joins $f^n(x)$ to $f^{n+1}(x)$ and is disjoint from the neighborhood V_ϕ.

The first step of the proof is to show that, by modifying the trivial neighborhoods in the family $\{V_\phi\}_{\phi\in\partial C_O}$, we can ensure that $N_\phi = M_\phi$ for all $\phi \in \partial C_O$. Indeed, to show this, one should simply observe that each leaf $\phi \in \partial_L C_O$ also admits the trivial neighborhood

$$V'_\phi := V_\phi \cap \overline{L(\phi^{M_\phi})}. \tag{3.1}$$

Thus, by redefining the original V_ϕ to be instead the newly defined trivial neighborhood V'_ϕ, we automatically get the property that $N_\phi = M_\phi$. Therefore, we conclude that after this redefinition, we know that for each $n \geq 0$ and $\phi \in \partial_L C_O$, the points $f^n(x)$ and $f^{n+1}(x)$ can be connected by a path $\gamma_{n,\phi}$ positively transverse to $\mathcal{F}$ that is disjoint from V_ϕ.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

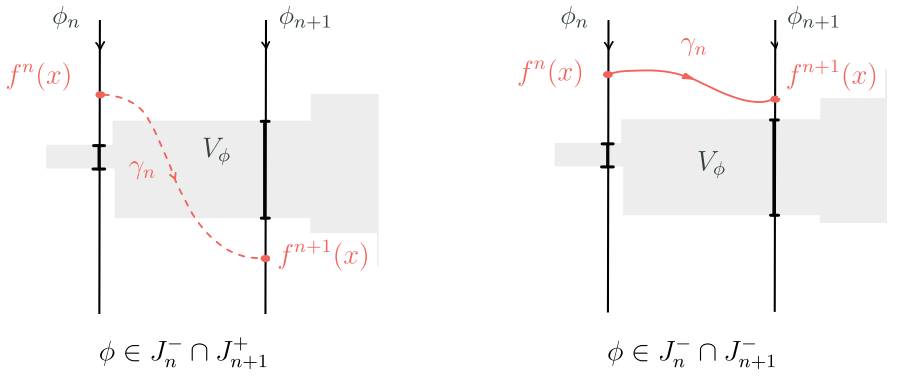


FIGURE 19. Illustration of what cannot occur (on the left) and what actually occurs (on the right).

Now, the second step of the proof is to show that, for each $n \geq 0$, we can always construct the paths $\gamma_{n,\phi}$ so that they are the same among all leaves $\phi \in \partial_L C_O$.

For every $n \geq 0$, we define a partition of the set $A^n = \{\alpha_n(\phi)\}_{\phi \in J_n}$ given by

$$A_n^+ := \{\alpha_n(\phi) \in A^n \mid \phi \in J_n^+\} \quad \text{and} \quad A_n^- := \{\alpha_n(\phi) \in A^n \mid \phi \in J_n^-\}.$$

The set A^n is ordered according to the orientation of the leaf ϕ_n. Observe that any element in A_n^+ is greater than any element in A_n^-. Now, since the family $\{V_\phi\}_{\phi \in \partial_L C_O}$ is locally finite, any compact sub-arc of ϕ_n intersects finitely many arcs of A^n. Consequently, if $A_n^+ \neq \varnothing$, then there exists a minimal element $\alpha_n^{\min}$ in A_n^+. Similarly, if $A_n^- \neq \varnothing$, then there exists a maximal element $\alpha_n^{\max}$ in A_n^-. In the case where $A_n^+ = \varnothing$ or $A_n^- = \varnothing$, such arcs are considered empty. Note that the connected component of $\phi_n \setminus (\alpha_n^{\max} \cup \alpha_n^{\min})$ that contains $f^n(x)$ intersects no trivial neighborhood in $\{V_\phi\}_{\phi \in J^n}$. For each $n \geq 0$, let $V_n^{\max}$ and $V_n^{\min}$ be the trivial neighborhoods in the family $\{V_\phi\}_{\phi \in J^n}$ containing $\alpha_n^{\max}$ and $\alpha_n^{\min}$, respectively.

For each $n \geq 0$, we define the set $D_n := \overline{L(\phi_n) \cap R(\phi_{n+1})}$. After the redefinition of the trivial neighborhoods as $V'_\phi := V_\phi \cap L(\phi^{M_\phi})$ in (3.1), only two scenarios are possible:

(1) either $V_n^{\max} = V_{n+1}^{\max}$;
(2) or the intersection $V_{n+1}^{\max} \cap D_n$ is contained in ϕ_{n+1}.

These scenarios are respectively illustrated in Figure 20. Observe that, if the trivial neighborhood $V_{n+1}^{\max}$ is distinct from $V_n^{\max}$, then it must stop at the leaf ϕ_{n+1}. This is also a consequence of the redefinition in (3.1), as it implies that $V_\phi \subset \overline{L(\phi^{M_\phi})}$.

We have completely analogous alternatives for the sets $V_n^{\min}$ for all $n \geq 0$.

Either way, for each $n \geq 0$, we have that the points $f^n(x)$ and $f^{n+1}(x)$ are contained in the same connected component, namely D_n^* (see Figure 21), of the set

$$D_n \setminus (V_n^{\max} \cup V_n^{\min}).$$

This further implies that, for each $n \geq 0$, the points $f^n(x)$ and $f^{n+1}(x)$ can be connected via a path $\gamma_n : [0, 1] \longrightarrow \mathbb{R}^2$ that is positively transverse to $\mathcal{F}$ and contained in D_n^*. Therefore, since $\gamma_n \subset D_n^*$, it follows that γ_n is disjoint from every set in the family $\{V_\phi\}_{\phi \in \partial_L C_O}$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

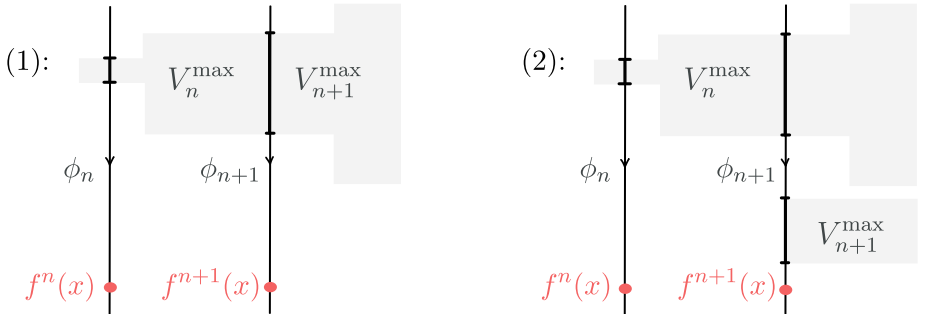


FIGURE 20. Illustration of scenarios (1) and (2) after redefining $V'_\phi := V_\phi \cap \overline{L(\phi^{M_\phi})}$.

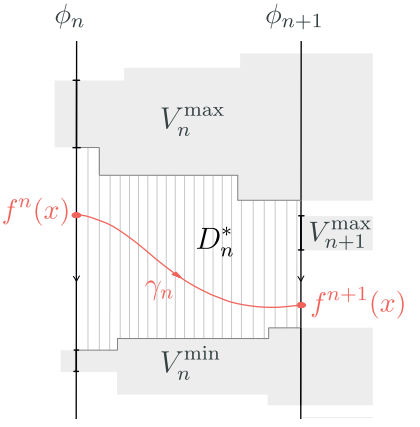


FIGURE 21. Illustration describing the properties of the set D_n^*.

By repeating the same construction with respect to the set $\partial_R C_O$, we end up with a family of positively transverse paths $\{\gamma_n\}_{n\in\mathbb{Z}}$, where each γ_n connects $f^n(x)$ to $f^{n+1}(x)$, and is disjoint from V_ϕ for any leaf $\phi \in \partial_L C_O \cup \partial_R C_O$.

Finally, the last step is to prove that the family $\{\gamma_n\}_{n\in\mathbb{Z}}$ is locally finite. For that, define

$$W := \bigcup_{n\geq 0} \overline{L(\phi_{-n}) \cap R(\phi_n)}$$

and observe that every point $z \in W$ has an open neighborhood $U_z \subset \mathbb{R}^2$ that intersects finitely many sets in the family $\{D_n\}_{n\in\mathbb{Z}}$. In particular, each neighborhood U_z intersects finitely many paths in the family $\{\gamma_n\}_{n\in\mathbb{Z}}$. Let $K \subset \mathbb{R}^2$ be a compact set and denote

$$K^* := K \setminus \bigcup_{\phi\in\partial_L C_O\cup\partial_R C_O} \operatorname{int}(V_\phi).$$

Note that K^* is a compact set contained in W satisfying $\gamma_n \cap K = \gamma_n \cap K^*$ for all $n \in \mathbb{Z}$. Consider a cover of K^* of the form $\{U_z\}_{z\in K^*}$. By taking a finite sub-cover, we conclude that K^* intersects at most finitely many transverse paths in $\{\gamma_n\}_{n\in\mathbb{Z}}$. Hence, K

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

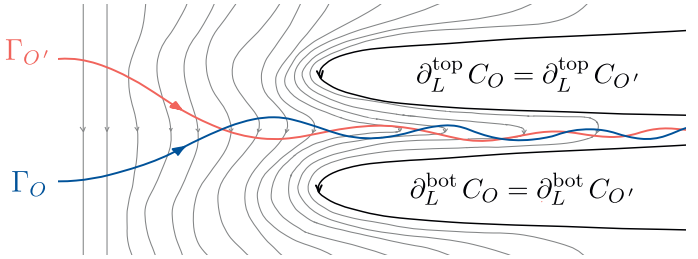


FIGURE 22. Existence of 'forever intersecting' proper trajectories for pairs of $\mathcal{F}$-asymptotic orbits.

intersects finitely many paths in $\{\gamma_n\}_{n\in\mathbb{Z}}$. This shows that $\{\gamma_n\}_{n\in\mathbb{Z}}$ is a locally finite family and thus

$$\Gamma_O := \prod_{n\in\mathbb{Z}} \gamma_n$$

is a properly embedded transverse trajectory associated to O.

The fact that Γ_O satisfies $\partial_L^{\text{top}} O = \partial_L C_O \cap L(\Gamma_O)$ and $\partial_L^{\text{bot}} O = \partial_L C_O \cap R(\Gamma_O)$ follows directly from the construction. The analogous properties for $\partial_R^{\text{top}} O$ and $\partial_R^{\text{bot}} O$ hold as well. To see that these equalities hold for any proper transverse trajectory Γ_O, it suffices to note that they hold automatically if the trivial neighborhoods in the family $\{V_\phi\}_{\phi\in\partial C_O}$ are disjoint from Γ_O. Since, in the proper case, $\Gamma_O \subset \mathbb{R}^2$ is a closed set disjoint from ∂C_O, Lemma 2.2 ensures that such a family $\{V_\phi\}_{\phi\in\partial C_O}$ exists. This concludes the proof. □

3.6. *Equivalence classes of $\mathcal{F}$-asymptotic orbits.* As presented in Definition 1.2, we can relate orbits exhibiting similar $\mathcal{F}$-asymptotic behavior via equivalence relations $\overset{+}{\sim}$ and $\overset{-}{\sim}$ on the set orb. Below, we interpret these relations in terms of proper transverse trajectories.

LEMMA 3.6. *For any $O, O' \in \text{Orb}(f)$, the following statements are equivalent:*

(i) *the orbits O and O' are forward $\mathcal{F}$-asymptotic, that is, $O \overset{+}{\sim} O'$;*
(ii) *there exist proper transverse trajectories Γ_O and $\Gamma_{O'}$ of O and O', respectively, such that $\Gamma_O([t,\infty)) \cap \Gamma_{O'}([t,\infty)) \neq \varnothing$ for every $t \in \mathbb{R}$ (Figure 22).*

We can obtain a similar result for the relation $\overset{-}{\sim}$ by replacing $[t,\infty)$ with $(-\infty, t]$.

Lemma 3.6 admits a dual version which is slightly stronger, presented in Lemma 3.7.

LEMMA 3.7. *For any $O, O' \in \text{Orb}(f)$ satisfying $C_O \cap C_{O'} \neq \varnothing$, the following are equivalent:*

(i) *the orbits O and O' are not forward $\mathcal{F}$-asymptotic, that is, $O \overset{+}{\not\sim} O'$;*
(ii) *for any pair of proper transverse trajectories Γ_O and $\Gamma_{O'}$ of O and O', respectively, there exists $\phi_L \in C_O \cap C_{O'}$ such that $\Gamma_O \cap \Gamma_{O'} \cap \overline{L(\phi_L)} = \varnothing$ (Figure 23).*

We obtain a similar result for non-backward $\mathcal{F}$-asymptotic orbits by replacing each L with R.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

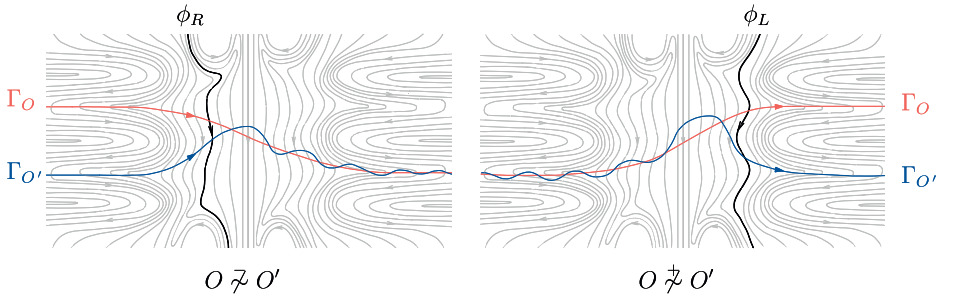


FIGURE 23. Forced divergence for any pair of proper transverse trajectories associated to non- $\mathcal{F}$-asymptotic orbits.

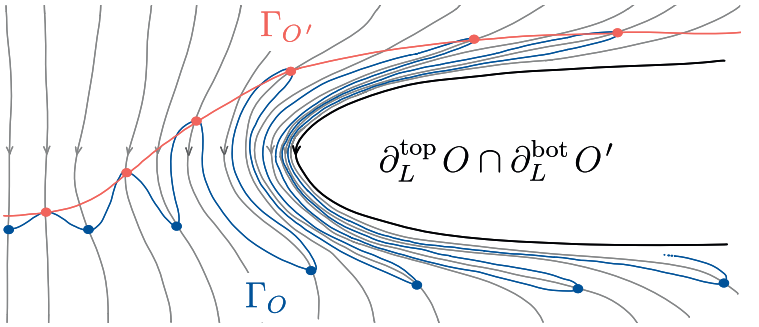


FIGURE 24. Illustration of the example described in Remark 3.8.

Remark 3.8. It is important to remark that the results of Lemmas 3.6 and 3.7 are no longer true if we consider non-proper transverse trajectories as in the following example.

We suppose that $O, O' \in \text{Orb}(f)$ are two orbits admitting a leaf $\phi \in \partial_L^{\text{top}} O \cap \partial_L^{\text{bot}} O'$. These orbits are clearly not forward $\mathcal{F}$-asymptotic. However, as one can see in Figure 24, there still exist non-proper transverse trajectories Γ_O and $\Gamma_{O'}$ associated to O and O', respectively, that satisfy the property $\Gamma_O([t, \infty)) \cap \Gamma_{O'}([t, \infty)) \neq \varnothing$ for any $t \in \mathbb{R}$.

Proof of Lemma 3.6. (i) $\Longrightarrow$ (ii): From statement (i), we have $C_O \cap C_{O'} \neq \varnothing$ and $\partial_L C_O = \partial_L C_{O'}$. Let $\phi_* \in C_O \cap C_{O'}$ and let $(\phi_n)_{n\geq 0}$ be the increasing sequence of leaves in $\mathcal{F}$ formed by the leaves intersecting the set $(O \cup O') \cap \overline{L(\phi_*)}$. Note that some leaves of $(\phi_n)_{n\geq 0}$ may intersect simultaneously both orbits O and O'. Next, according to Lemma 2.2, there exists a family of sufficiently small trivial neighborhoods $\{V_\phi\}_{\phi \in \partial_L C_O}$ of the leaves in $\partial_L C_O$ that is pairwise disjoint and disjoint from the closed set $\phi_0 \cup O \cup O'$.

Similarly to §3.3, we have a filtration $(J_n)_{n\geq 0}$ of the set $\partial_L C_O$ by sets of the form $J_n = \{\phi \in \partial_L C_O \mid V_\phi \cap \phi_n\}$. Moreover, for each $n \geq 0$ such that $O \cap \phi_n \neq \varnothing$, we have a cut

$$(J_n^-(O), J_n^+(O)) \in \text{Cut}_\prec(J_n).$$

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

However, for the values of $n \geq 0$ such that $O \cap \phi_n = \varnothing$, we define $m(n) > n$ as the smallest integer such that $O \cap \phi_{m(n)} \neq \varnothing$, and we set

$$J_n^-(O) := J_n \cap J_{m(n)}^-(O) \quad \text{and} \quad J_n^+(O) := J_n \cap J_{m(n)}^+(O).$$

Performing the same adaptation for the cuts induced by the orbit O', we end up with two cuts $(J_n^-(O), J_n^+(O))$ and $(J_n^-(O'), J_n^+(O'))$ of the ordered set $(J_n, \prec)$ for every $n \geq 0$.

Since $O \overset{+}{\sim} O'$ for each leaf $\phi \in \partial_L C_O$, there exists $M_\phi > 0$ such that

$$\begin{aligned} \phi \in \partial_L^{\text{top}} O &\implies \phi \in J_n^-(O) \cap J_n^-(O') \quad \text{for all } n \geq M_\phi, \\ \phi \in \partial_L^{\text{bot}} O &\implies \phi \in J_n^+(O) \cap J_n^+(O') \quad \text{for all } n \geq M_\phi. \end{aligned}$$

By the same redefinition of the family $\{V_\phi\}_{\phi \in \partial_L C_O}$ presented in (3.1), we can ensure that each leaf $\phi \in \partial_L C_O$ satisfies $\phi \notin J_n$ for all $n \prec M_\phi$ and, consequently, for any $n \geq 0$, we have

$$\partial_L^{\text{top}} O = J_n^-(O) = J_n^-(O') \quad \text{and} \quad \partial_L^{\text{bot}} O = J_n^+(O) = J_n^+(O').$$

For each $n \geq 0$, let p_n and p'_n be the points where the leaf ϕ_n intersects O and O', respectively. In the case where ϕ_n intersects only one of these orbits, we consider $p_n = p'_n$. Following arguments similar to those in the proof of Theorem A, we obtain for each $n \geq 0$, two positively transverse paths $\gamma_n, \gamma'_n : [0, 1] \longrightarrow \mathbb{R}^2$ that are disjoint from all sets in the family $\{V_\phi\}_{\phi \in \partial_L C_O}$ and such that $\gamma_O^n(0) = p_n$, $\gamma_{O'}^n(0) = p'_n$, $\gamma_O^n(1) = p_{n+1}$, and $\gamma_{O'}^n(1) = p'_{n+1}$. Observe that p_n, p'_n, p_{n+1}, and p'_{n+1} belong to the same connected component of the set

$$\overline{L(\phi_n) \cap R(\phi_{n+1})} \setminus \bigcup_{\phi \in \partial_L C_O} \text{int}(V_\phi).$$

Therefore, for any pair of open sets $U_n, U_{n+1} \subset \mathbb{R}^2$ satisfying $\phi_n \subset U_n$ and $\phi_{n+1} \subset U_{n+1}$, the same arguments of §3.5 allow us to construct γ_n and γ'_n in such a way that they coincide outside the neighborhoods U_n and U_{n+1}. Moreover, in the case where $p_n = p'_n$ and $p_{n+1} = p'_{n+1}$, we can assume that $\gamma_n = \gamma'_n$. In all cases, we have $\gamma_n \cap \gamma'_n \neq \varnothing$ for all $n \geq 0$. The argument of §3.5 proves that the families $\{\gamma_n\}_{n \geq 0}$ and $\{\gamma'_n\}_{n \geq 0}$ are both locally finite. Thus, they can be concatenated, and then extended, into proper transverse trajectories Γ and Γ' associated respectively to the orbits O and O'. These proper transverse trajectories satisfy statement (ii).

(ii) $\implies$ (i). Since Γ and Γ' intersect each other, there exists a leaf $\phi_0 \in C_O \cap C_{O'}$. Let $(\phi_n)_{n \geq 0}$ be the increasing sequence of leaves in $\mathcal{F}$ formed by the leaves meeting the set $(O \cup O') \cap \overline{L(\phi_0)}$. Since Γ and Γ' are proper lines, the set $\Gamma \cup \Gamma' \subset \mathbb{R}^2$ is closed. Thus, by applying Lemma 2.2, we know there exists a family $\{V_\phi\}_{\phi \in \partial_L C_O}$ of pairwise disjoint trivial neighborhoods of $\partial_L C_O$ that satisfies $V_\phi \cap (\phi_0 \cup \Gamma \cup \Gamma') = \varnothing$ for any $\phi \in \partial_L C_O$.

For each $n \geq 0$, we consider $J_n = \{\phi \in \partial_L C_O \mid V_\phi \cap \phi_n\}$. As in (i) $\implies$ (ii), consider the sets $J_n^-(O)$, $J_n^+(O)$, $J_n^-(O')$, and $J_n^+(O')$. Note that, as each V_ϕ is disjoint from $\Gamma \cup \Gamma'$, hypothesis (ii) implies that, for all $n \geq 0$, we have $J_n^+(O) = J_n^+(O')$ and $J_n^-(O) = J_n^-(O')$. This shows that the orbits O and O' induce the same cut

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

$(\partial_L^{\mathrm{top}} O, \partial_L^{\mathrm{bot}} O)$ on $\partial_L C_O$ and, consequently, we conclude that O and O' are forward $\mathcal{F}$-asymptotic. □

Remark 3.9. The argument (i) $\Longrightarrow$ (ii) within the proof of Lemma 3.6 can be adapted to any finite subset $\mathcal{O} \subset \mathrm{Orb}(f)$ satisfying

$$O \overset{+}{\sim} O' \quad \text{for all } O, O' \in \mathcal{O}.$$

If each leaf of $\mathcal{F}$ intersects at most one orbit in $\mathcal{O}$, this adaptation proves Lemma 7.2.

Proof of Lemma 3.7. (i) $\Longrightarrow$ (ii). Suppose that $O \overset{+}{\not\sim} O'$. According to Lemma 3.6, for any pair of proper transverse trajectories Γ_O and $\Gamma_{O'}$ of O and O', there exists $M > 0$ such that

$$\Gamma_O([M, \infty)) \cap \Gamma_{O'}([M, \infty)) = \varnothing.$$

Since Γ_O and $\Gamma_{O'}$ are topological lines, the intersection $\Gamma_O \cap \Gamma_{O'}$ is a closed subset of $\mathbb{R}^2$. Therefore, the set $\Gamma_O{}^{-1}(\Gamma_O \cap \Gamma_{O'})$ is a closed subset of $\mathbb{R}$ bounded from above by M and, consequently, it admits a maximal element $t_{\max}$.

Note that the leaf $\phi_{\max} \in \mathcal{F}$ passing through the point $\Gamma_O(t_{\max})$ belongs to $C_O \cap C_{O'}$. Since $C_O \cap C_{O'}$ is an open set, there exists a leaf $\phi_L \in \mathcal{F}$ satisfying $L(\phi_L) \subset L(\phi_{\max})$ and sufficiently close to $\phi_{\max}$ so that it also belongs to $C_O \cap C_{O'}$. This leaf proves (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i). This implication follows trivially by Lemma 3.6. □

4. *Sided preorders of orbits*

In this section, we develop the tools necessary to compare orbits which cross a common leaf and exhibit distinct $\mathcal{F}$-asymptotic behaviors.

We are motivated by the setting below.

Let $O, O' \in \mathrm{Orb}(f)$ be orbits crossing a common leaf, i.e. $C_O \cap C_{O'} \neq \varnothing$. Suppose that O and O' are neither forward nor backward $\mathcal{F}$-asymptotic. Thus, according to Lemma 3.7, for any proper transverse trajectories Γ_O and $\Gamma_{O'}$ associated with O and O', respectively, there exists a pair of leaves $\phi_R, \phi_L \in C_O \cap C_{O'}$ satisfying $\phi_R < \phi_L$ such that

$$\Gamma_O \cap \Gamma_{O'} \cap \overline{L(\phi_L)} = \varnothing \quad \text{and} \quad \Gamma_O \cap \Gamma_{O'} \cap \overline{R(\phi_R)} = \varnothing.$$

In this setting, two possible configurations arise (see Figure 25).

(a) The points $p_L(O) \in \Gamma_O \cap \phi_L$ and $p_L(O') \in \Gamma_{O'} \cap \phi_L$ are ordered along the leaf ϕ_L in the same order as the points $p_R(O) \in \Gamma_O \cap \phi_R$ and $p_R(O') \in \Gamma_{O'} \cap \phi_R$ along ϕ_R.

(b) The points $p_L(O) \in \Gamma_O \cap \phi_L$ and $p_L(O') \in \Gamma_{O'} \cap \phi_L$ are ordered along the leaf ϕ_L in the opposite order as the points $p_R(O) \in \Gamma_O \cap \phi_R$ and $p_R(O') \in \Gamma_{O'} \cap \phi_R$ along ϕ_R.

In §4.1, we show that the configuration into which these orbits O and O' fall depends solely on their $\mathcal{F}$-asymptotic behavior, not on the particular choices of proper transverse trajectories Γ_O and $\Gamma_{O'}$. This is not the case if $O \overset{+}{\sim} O'$ or $O \overset{-}{\sim} O'$.

Observe that, in configuration (b), the trajectories Γ and Γ' are forced to intersect each other in between the leaves ϕ_R and ϕ_L. In configuration (a), Γ and Γ' could be disjoint.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

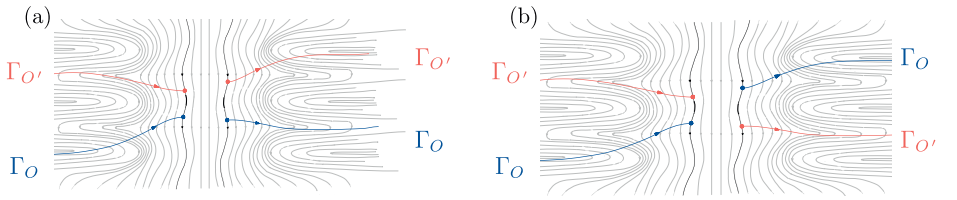


FIGURE 25. Two possible configurations at infinity for pairs of orbits O and O' that cross a common leaf but are neither forward $\mathcal{F}$-asymptotic nor backward $\mathcal{F}$-asymptotic.

4.1. *Defining $\lesssim_L$ and $\lesssim_R$.* In this section, we will define two relations $\lesssim_L$ and $\lesssim_R$ on the set of orbits $\mathrm{Orb}(f)$ that satisfy the following properties.

- For each $\phi \in \mathcal{F}$, the relations $\lesssim_L$ and $\lesssim_R$ are total preorder on the set
$$\mathrm{Orb}(\phi) := \{O \in \mathrm{orb} \mid \phi \in C_O\}.$$
- For any pair of orbits $O, O' \in \mathrm{Orb}(f)$, the following equivalences hold:
$$O \overset{+}{\sim} O' \iff O \lesssim_L O' \quad \text{and} \quad O' \lesssim_L O,$$
$$O \overset{-}{\sim} O' \iff O \lesssim_R O' \quad \text{and} \quad O' \lesssim_R O.$$

Before, we note that $C_O \cap C_{O'}$ is a (possibly empty) leaf domain of $\mathcal{F}$ (see Lemma 2.6). Moreover, limit sets $\partial_L(C_O \cap C_{O'})$ and $\partial_R(C_O \cap C_{O'})$ are totally ordered by a relation $\prec$, just as for $\partial_L C_O$ and $\partial_R C_O$ (see §3.3). This allows us to introduce the next definition.

Definition 4.1. For any $O, O' \in \mathrm{Orb}(f)$, we define $\lesssim_R$ and $\lesssim_L$ as in Figure 26.

PROPOSITION 4.2. *The relations $\lesssim_L$ and $\lesssim_R$ satisfy the following properties.*

(i) *Both relations $\lesssim_L$ and $\lesssim_R$ are total preorders on the set* $\mathrm{Orb}(\phi)$ *for every $\phi \in \mathcal{F}$.*

(ii) *For any pair of orbits $O, O' \in \mathrm{Orb}(f)$, the following equivalences hold:*
$$O \overset{+}{\sim} O' \iff O \lesssim_L O' \quad \textit{and} \quad O' \lesssim_L O,$$
$$O \overset{-}{\sim} O' \iff O \lesssim_R O' \quad \textit{and} \quad O' \lesssim_R O.$$

Remark 4.3. The result of Proposition 4.2 is equivalent to say that $\lesssim_L$ and $\lesssim_R$ induce total orders on the quotient sets $\mathrm{Orb}(\phi)/\overset{+}{\sim}$ and $\mathrm{Orb}(\phi)/\overset{-}{\sim}$, respectively. More precisely:

- for $[O]^+, [O']^+ \in \mathrm{Orb}(\phi)/\overset{+}{\sim}$ we have $[O]^+ \lesssim_L [O']^+$ whenever $O \lesssim_L O'$;
- for $[O]^-, [O']^- \in \mathrm{Orb}(\phi)/\overset{-}{\sim}$ we have $[O]^- \lesssim_R [O']^-$ whenever $O \lesssim_R O'$.

Proof of Proposition 4.2. Before we prove item (i), observe that any two orbits O and O' that are comparable under $\lesssim_L$ (or $\lesssim_R$) must satisfy $C_O \cap C_{O'} \neq \varnothing$. The converse statement proves that $\lesssim_L$ and $\lesssim_R$ are total relations on the subsets $\mathrm{Orb}(\phi) \subset \mathrm{Orb}(f)$ for every $\phi \in \mathcal{F}$.

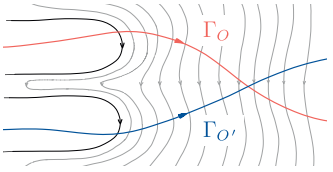
Proof of item (i). Let $O, O' \in \mathrm{Orb}(f)$ be two orbits satisfying the condition $C_O \cap C_{O'} \neq \varnothing$. Note that each of the orbits O and O' is allowed to cross at most two leaves in $\partial(C_O \cap C_{O'})$, one contained in $\partial_R(C_O \cap C_{O'})$ and one in $\partial_L(C_O \cap C_{O'})$. Moreover,

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

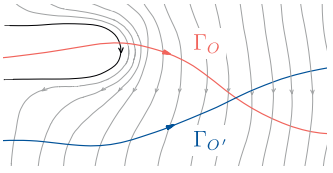
$O \lesssim_R O'$

if the orbits O and O' satisfy $C_O \cap C_{O'} \neq \varnothing$ and one of the following conditions:

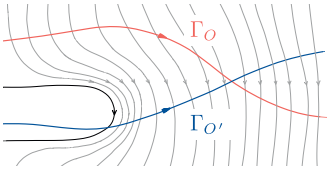
(R1) There exist $\phi, \phi' \in \partial_R(C_O \cap C_{O'})$, with $\phi \prec \phi'$, such that $\phi \in C_O$, $\phi' \in C_{O'}$.



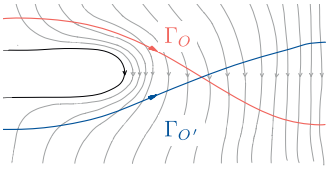
(R2) There exists a leaf $\phi \in \partial_R(C_O \cap C_{O'})$ such that $\phi \in C_O \cap \partial_R^{\text{top}} O'$.



(R3) There exists a leaf $\phi' \in \partial_R(C_O \cap C_{O'})$ such that $\phi' \in C_{O'} \cap \partial_R^{\text{bot}} O$.



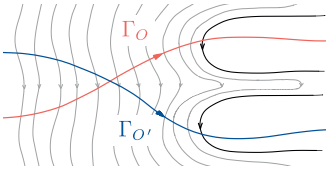
(R4) $\partial_R(C_O \cap C_{O'}) = \partial_R C_O = \partial_R C_{O'}$ and, moreover, $\partial_R^{\text{top}} O \subseteq \partial_R^{\text{top}} O'$.



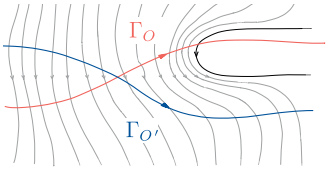
$O \lesssim_L O'$

if the orbits O and O' satisfy $C_O \cap C_{O'} \neq \varnothing$ and one of the following conditions:

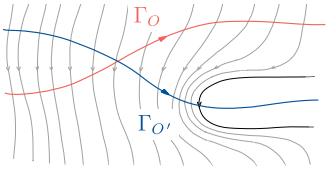
(L1) There exist $\phi, \phi' \in \partial_L(C_O \cap C_{O'})$, with $\phi \prec \phi'$, such that $\phi \in C_O$, $\phi' \in C_{O'}$.



(L2) There exists a leaf $\phi \in \partial_L(C_O \cap C_{O'})$ such that $\phi \in C_O \cap \partial_L^{\text{top}} O'$.



(L3) There exists a leaf $\phi' \in \partial_L(C_O \cap C_{O'})$ such that $\phi' \in C_{O'} \cap \partial_L^{\text{bot}} O$.



(L4) $\partial_L(C_O \cap C_{O'}) = \partial_L C_O = \partial_L C_{O'}$ and, moreover, $\partial_L^{\text{top}} O \subseteq \partial_L^{\text{top}} O'$.

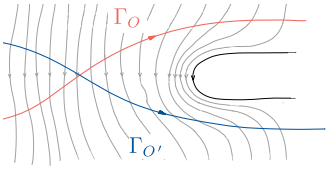


FIGURE 26. Definition of $\lesssim_R$ and $\lesssim_L$.

no leaf in $\partial(C_O \cap C_{O'})$ can be crossed simultaneously by O and O'. Based on these observations, we conclude that the orbits O and O' must satisfy one of the following.

(1) *The orbits O and O' cross distinct leaves in* $\partial_L(C_O \cap C_{O'})$: there exists $\phi, \phi' \in \partial_L(C_O \cap C_{O'})$, with $\phi \neq \phi'$, such that $\phi \in C_O$ and $\phi' \in C_{O'}$.

(2) *The orbit O' does not cross any leaf in* $\partial_L(C_O \cap C_{O'})$, *but O does:* the set $\partial_L(C_O \cap C_{O'})$ coincides with $\partial_L C_{O'}$ and, moreover, C_O intersects $\partial_L C_{O'}$.

(3) *The orbit O does not cross any leaf in* $\partial_L(C_O \cap C_{O'})$, *but O' does:* the set $\partial_L(C_O \cap C_{O'})$ coincides with $\partial_L C_O$ and, moreover, $C_{O'}$ intersects $\partial_L C_O$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

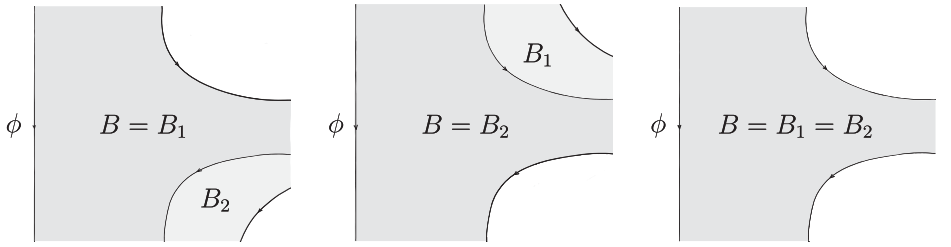


FIGURE 27. Three possible configurations between the sets B, B_1 and B_2 in the proof of Proposition 4.2.

(4) *The orbits O and O' do not cross any leaf in $\partial_L(C_O \cap C_{O'})$*: the set $\partial_L(C_O \cap C_{O'})$ coincides with both $\partial_L C_O$ and $\partial_L C_{O'}$.

Similar observations apply to the behavior of the orbits O and O' relative to $\partial_R(C_O \cap C_{O'})$.

By observing that whenever O and O satisfy property *(k)* for some $k \in \{1, 2, 3, 4\}$, they also satisfy $O \lesssim_L O'$ or $O' \lesssim_L O$ according to the respective condition (Lk). This proves that the relation $\lesssim_L$ is total on every subset of the form $\mathrm{Orb}(\phi)$ for $\phi \in \mathcal{F}$. The same reasoning applies to the relation $\lesssim_R$ using the analogous versions of conditions (1), (2), (3), and (4).

Note that the reflexivity of $\lesssim_L$ and $\lesssim_R$ is evident, as every orbit $O \in \mathrm{Orb}(f)$ trivially satisfies conditions (L4) and (R4) with itself. To conclude the proof of item (i), we show that $\lesssim_L$ is transitive on the set $\mathrm{Orb}(\phi)$, for every $\phi \in \mathcal{F}$. The proof for $\lesssim_R$ is analogous.

Now, we prove the transitivity of $\lesssim_L$. Let $\phi \in \mathcal{F}$, and let O, O', $O'' \in \mathrm{Orb}(\phi)$ be three orbits satisfying $O \lesssim_L O'$ and $O' \lesssim_L O''$. First, we consider the following sets:

$$B := C_O \cap C_{O'} \cap C_{O''} \cap L(\phi),$$
$$B_1 := C_O \cap C_{O'} \cap L(\phi),$$
$$B_2 := C_{O'} \cap C_{O''} \cap L(\phi).$$

These sets are non-empty, and satisfy $B \subset B_1$ and $B \subset B_2$. Moreover, they admit three possible configurations (see Figure 27), which are identified in the following claim.

CLAIM 4.4. *The set B must be equal to B_1 or B_2, with possibly $B = B_1 = B_2$.*

Proof of Claim 4.4. Indeed, suppose by contradiction that B is different from B_1 and B_2. In this case, there exists leaves $\phi_1, \phi_2 \in \partial_L B$ such that $\phi_1 \in B_1$ and $\phi_2 \in B_2$. Note that if these leaves ϕ_1 and ϕ_2 coincide, then the orbits O, O', and O'' have to intersect the leaf ϕ_1. This would imply that $\phi_1 \in B$, which is a contradiction. Meanwhile, if $\phi_1 \neq \phi_2$, this means that O' intersects both sets $L(\phi_1)$ and $L(\phi_2)$, which is also a contradiction. □

Proof under the assumption that $B = B_1 \neq B_2$. In this case, we have $\partial_L B = \partial_L(C_O \cap C_{O'})$ and there exists a leaf $\phi_2 \in \partial_L(C_O \cap C_{O'})$ that is crossed by both orbits O' and O''. Hence, since the orbits O and O' satisfy $O \lesssim_L O'$, we fall into two possible cases (Figure 28).

(a) *The orbit O do not cross any leaf in $\partial_L(C_O \cap C_{O'})$.*
In this case, we have $\partial_L C_O = \partial_L(C_O \cap C_{O'})$ and ϕ_2 must belong to $\partial_L^{\mathrm{bot}} O$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

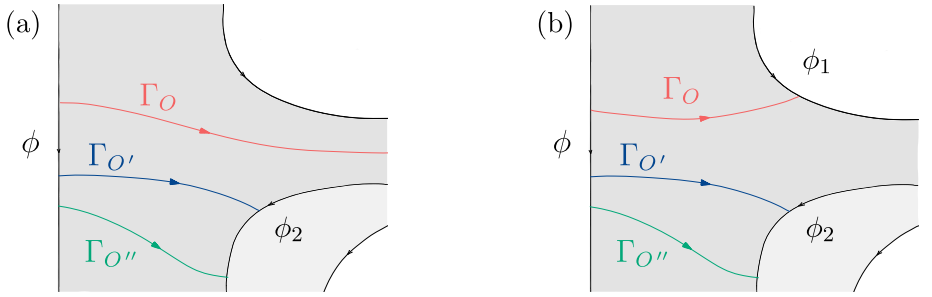


FIGURE 28. Cases (a) and (b) under the assumption that $B = B_1 \neq B_2$ in the proof of Proposition 4.2.

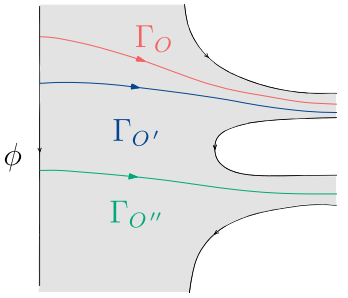


FIGURE 29. Subcase (1) under the assumption that $B = B_1 = B_2$ in the proof of Proposition 4.2.

(b) *The orbit O cross some leaf in $\partial_L(C_O \cap C_{O'})$.*

In this case, there exists $\phi_1 \in \partial_L(C_O \cap C_{O'})$, with $\phi_1 \prec \phi_2$, such that $\phi_1 \in C_O$.

In case (a), we have $O \lesssim_L O''$ through condition (L3), and in case (b), through condition (L1). We remark that the proof under the assumption $B = B_2 \neq B_1$ is completely analogous.

Proof under the assumption that $B = B_1 = B_2$. In this situation, we have that $\partial_L B = \partial_L(C_O \cap C_{O'}) = \partial_L(C_{O'} \cap C_{O''}) = \partial_L(C_O \cap C_{O''})$. We separate the proof into four different subcases, depending on whether O, O', or O'' cross some leaf in $\partial_L B$.

Subcase (1): none of the orbits O, O', and O'' crosses a leaf in the set $\partial_L B$ (see Figure 29). In this case, we have that $\partial_L B = \partial_L C_O = \partial_L C_{O'} = \partial_L C_{O''}$. This means that both $O \lesssim_L O'$ and $O' \lesssim_L O''$ are guaranteed by condition (L4). Consequently, we have the triple inclusion $\partial_L^{\text{top}} O \subset \partial_L^{\text{top}} O' \subset \partial_L^{\text{top}} O''$, which implies $O \lesssim_L O''$ by condition (L4).

Subcase (2): one orbit among O, O', and O'' crosses a leaf in the set $\partial_L B$ (see Figure 30). In this case, we assume that O crosses a leaf in $\partial_L B$, while O' and O'' do not. The remaining possible combinations are analogous, with the case where O' crosses a leaf in $\partial_L B$ needing a slight modification in the arguments. Here, we have that $\partial_L B = \partial_L C_{O'} = \partial_L C_{O''}$, with some leaf $\phi_1 \in \partial_L B$ being crossed by the orbit O, i.e. $\phi_1 \in C_O$. In this context, $O \lesssim_L O'$ is given by condition (L2), implying that $\phi_1 \in \partial_L^{\text{top}} O'$. Meanwhile, $O' \lesssim_L O''$ is given by condition (L4), implying that $\partial_L^{\text{top}} O' \subset \partial_L^{\text{top}} O''$. This shows that ϕ_1 belongs to $\partial_L^{\text{top}} O''$; thus, proving that the inequality $O \lesssim_L O''$ holds via condition (L2).

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

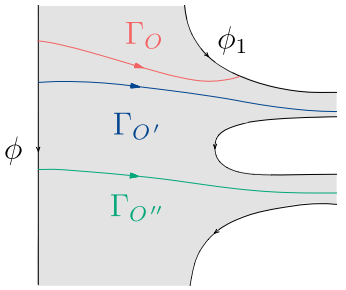


FIGURE 30. Subcase (2) under the assumption that $B = B_1 = B_2$ in the proof of Proposition 4.2.

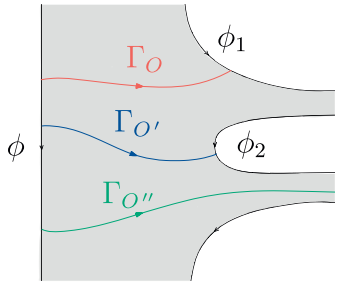


FIGURE 31. Subcase (3) under the assumption that $B = B_1 = B_2$ in the proof of Proposition 4.2.

Subcase (3): two orbits among O, O', and O'' cross a leaf in the set $\partial_L B$ (see Figure 31). Now, assume that both O and O' cross leaves in $\partial_L B$, while O'' does not. The remaining possibilities are analogous, with some necessary modifications when O and O'' are the ones crossing leaves in $\partial_L B$. Here, we have $\partial_L B = \partial_L C_{O''}$, with two leaves $\phi_1, \phi_2 \in \partial_L C_{O''}$ satisfying $\phi_1 \in C_O$ and $\phi_2 \in C_{O'}$. By the initial assumption, the leaves ϕ_1 and ϕ_2 belong to $\partial_L(C_O \cap C_{O'})$ and are distinct. Thus, the inequality $O \lesssim_L O'$ is given by condition (L1), implying that $\phi_1 \prec \phi_2$. Meanwhile, $O' \lesssim_L O''$ is given by condition (L3), implying that $\phi_2 \in \partial_L^{\text{top}} O''$. This shows that ϕ_1 belongs to $\partial_L^{\text{top}} O''$; thus, implying $O \lesssim_L O''$ through condition (L2).

Subcase (4): each of the orbits O, O', and O'' cross a leaf in the set $\partial_L B$ (see Figure 32). In this case, there exist $\phi_1, \phi_2, \phi_3 \in \partial_L B$ satisfying $\phi_1 \in C_O, \phi_2 \in C_{O'}$, and $\phi_3 \in C_{O''}$. By the initial assumption, $\partial_L B = \partial_L(C_O \cap C_{O'}) = \partial_L(C_{O'} \cap C_{O''}) = \partial_L(C_O \cap C_{O''})$, with the three leaves $\phi_1, \phi_2, \phi_3 \in \partial_L B$ being distinct. Thus, both $O \lesssim_L O'$ and $O' \lesssim_L O''$ are given by condition (L1), meaning that $\phi_1 \prec \phi_2 \prec \phi_3$. This implies $O \lesssim_L O''$ via condition (L1) and, thus, concludes the proof of transitivity of $\lesssim_L$ on $\text{Orb}(\phi)$. This also ends the proof of item (i).

Proof of item (ii). We prove the first equivalence and the second one follows analogously.

($\Longrightarrow$) Let $O, O' \in \text{Orb}(f)$ be two forward $\mathcal{F}$-asymptotic orbits. Then, O and O' satisfy condition (L4) in both senses, which implies that $O \lesssim_L O'$ and $O' \lesssim_L O$.

($\Longleftarrow$) Conversely, let $O, O' \in \text{Orb}(f)$ be two orbits such that $O \lesssim_L O'$ and $O' \lesssim_L O$. Observe that the only condition in the definition of $\lesssim_L$ admitting symmetry is

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

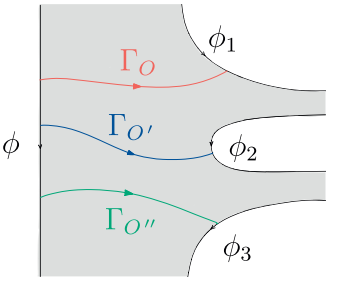


FIGURE 32. Subcase (4) under the assumption that $B = B_1 = B_2$ in the proof of Proposition 4.2.

condition (L4). Thus, the orbits O and O' must satisfy condition (L4) in both senses, meaning that

$$C_O \cap C_{O'} \neq \varnothing, \quad \partial_L(C_O \cap C_{O'}) = \partial_L C_O = \partial_L C_{O'}, \quad \partial_L^{\mathrm{top}} O = \partial_L^{\mathrm{top}} O'.$$

Note that the equalities $\partial_L C_O = \partial_L C_{O'}$ and $\partial_L^{\mathrm{top}} O = \partial_L^{\mathrm{top}} O'$ imply that $\partial_L^{\mathrm{bot}} O = \partial_L^{\mathrm{bot}} O'$. This concludes that the orbits O and O' are forward $\mathcal{F}$-asymptotic. □

Remark 4.5. The relations $\lesssim_L$ and $\lesssim_R$ may fail to be transitive on the entire set $\mathrm{Orb}(f)$. To see that, it suffices to consider the simple example where $O, O', O'' \in \mathrm{Orb}(f)$ satisfy

$$C_O \cap C_{O'} \neq \varnothing, \quad C_{O'} \cap C_{O''} \neq \varnothing, \quad C_O \cap C_{O''} = \varnothing.$$

In this case, transitivity fails because O and O'' are not comparable under $\lesssim_L$ and $\lesssim_R$.

Remark 4.6. The configuration of proper transverse trajectories can provide information about the ordering of the associated orbits with respect to $\lesssim_L$ and $\lesssim_R$.

Let $\phi \in \mathcal{F}$ be a leaf, and suppose that two orbits $O, O' \in \mathrm{Orb}(\phi)$ admit respective proper transverse trajectories Γ_O and $\Gamma_{O'}$ satisfying the following conditions:

- the trajectories Γ_O and $\Gamma_{O'}$ are disjoint on the left side of ϕ, i.e. $\Gamma_O \cap \Gamma_{O'} \cap \overline{L(\phi)} = \varnothing$;
- the intersection points $p_O \in \Gamma_O \cap \phi$ and $p_{O'} \in \Gamma_{O'} \cap \phi$ are ordered as $p_O < p_{O'}$ according to the orientation of the leaf ϕ.

In this scenario, the orbits O and O' are forced to satisfy the inequality $O \lesssim_L O'$. However, this alone does not provide enough information to conclude whether $O' \lesssim_L O$ also holds or not.

We highlight that the existence of the proper transverse trajectories Γ_O and $\Gamma_{O'}$, as described above, is assumed at this stage. Theorem B later guarantees the existence of such proper transverse trajectories under a generic condition on the foliation $\mathcal{F}$.

Remark 4.7. If Γ_O and $\Gamma_{O'}$ are not proper, they fail entirely to convey information about how the orbits O and O' are related under $\lesssim_L$ and $\lesssim_R$. For example, consider the configuration in Figure 33, in which the orbits O and O' do not satisfy $O \lesssim_L O'$, yet they admit non-proper transverse trajectories Γ_O and $\Gamma_{O'}$ satisfying

$$\Gamma_O \cap \Gamma_{O'} \cap \overline{L(\phi)} = \varnothing$$

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

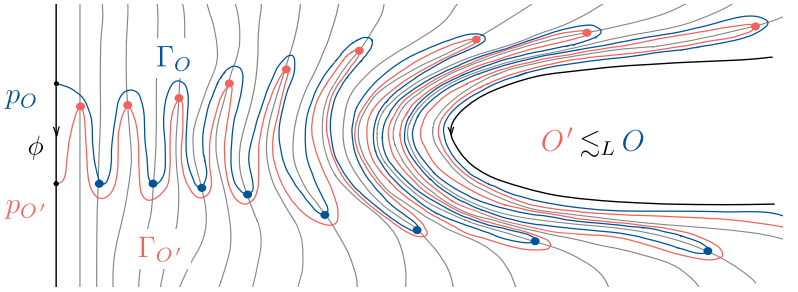


FIGURE 33. Illustration of the example described in Remark 4.7.

with intersection points of $p_O \in \Gamma_O \cap \phi$ and $p_{O'} \in \Gamma_{O'} \cap \phi$ ordered as $p_O \prec p_{O'}$ along the leaf ϕ.

5. *Weak $\mathcal{F}$-transverse intersections*

As presented in Definition 1.2, by comparing the $\mathcal{F}$-asymptotic behavior of orbits, we can identify orbits that are ordered in contradictory ways by the relations $\lesssim_L$ and $\lesssim_R$. This gives rise to the notion of *weak $\mathcal{F}$-transverse intersection* between orbits (see Figure 34).

Proper transverse trajectories of orbits having a weak $\mathcal{F}$-transverse intersection are forced to intersect. Indeed, assume $O, O' \in \mathrm{orb}$ have a weak $\mathcal{F}$-transverse intersection. Then, for any pair of proper transverse trajectories Γ_O and $\Gamma_{O'}$ of O and O', respectively, there exist $\phi_R, \phi_L \in C_O \cap C_{O'}$ satisfying

$$L(\phi_L) \subset L(\phi_R) \quad \text{and} \quad \Gamma_O \cap \Gamma_{O'} \cap \overline{R(\phi_R) \cap L(\phi_L)} = \varnothing.$$

Finally, as noted in Remark 4.6, Γ_O and $\Gamma_{O'}$ intersect ϕ_R and ϕ_L in opposite orders, reflecting the fact that O and O' are oppositely ordered by $\lesssim_L$ and $\lesssim_R$. This arrangement forces Γ_O and $\Gamma_{O'}$ to intersect in the region between ϕ_L and ϕ_R (configuration (b) in Figure 25).

Remark 5.1. To any reader well versed in the framework of forcing theory for surface homeomorphisms, the name *weak $\mathcal{F}$-transverse intersection* may sound familiar. Indeed, this notion strictly generalizes the concept of $\mathcal{F}$-transverse intersections introduced by Le Calvez and Tal in [**LCT18**], which plays a central role in forcing theory.

To confirm that, it suffices to note that the notion of $\mathcal{F}$-transverse intersection in [**LCT18**] can be equivalently defined (see Figure 35) as follows. Two orbits $O, O' \in \mathrm{Orb}(f)$ are said to have an *$\mathcal{F}$-transverse intersection* if they satisfy each of the following conditions:

- $C_O \cap C_{O'} \neq \varnothing$;
- there exist leaves $\phi_L, \phi_R \in C_O$ and $\phi'_L, \phi'_R \in C_{O'}$ satisfying $\phi_L, \phi'_L \in \partial_L(C_O \cap C_{O'})$ and $\phi_R, \phi'_R \in \partial_R(C_O \cap C_{O'})$ that are oppositely oriented by $\prec$, in the sense that

$$(\phi_L \prec \phi'_L \wedge \phi'_R \prec \phi_R) \vee (\phi'_L \prec \phi_L \wedge \phi_R \prec \phi'_R).$$

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

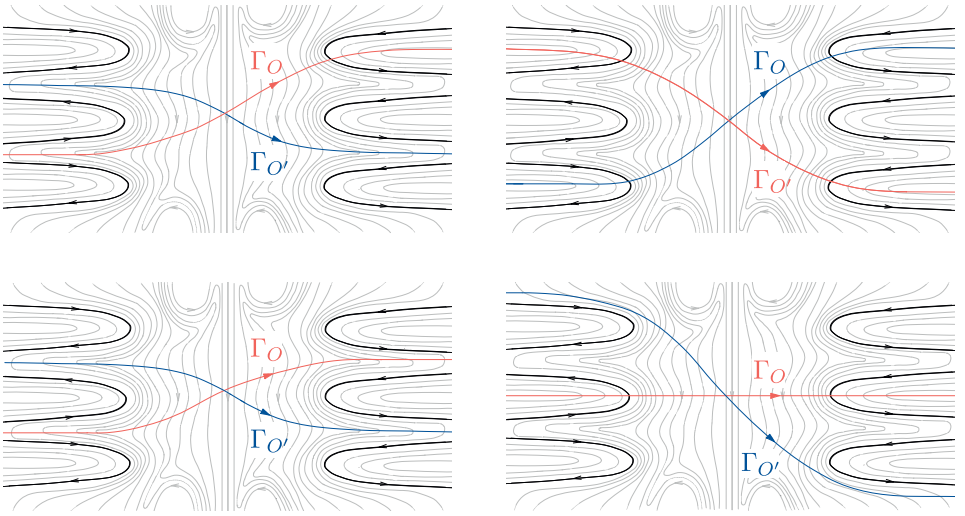


FIGURE 34. Some examples of orbits O and O' exhibiting a weak $\mathcal{F}$-transverse intersection.

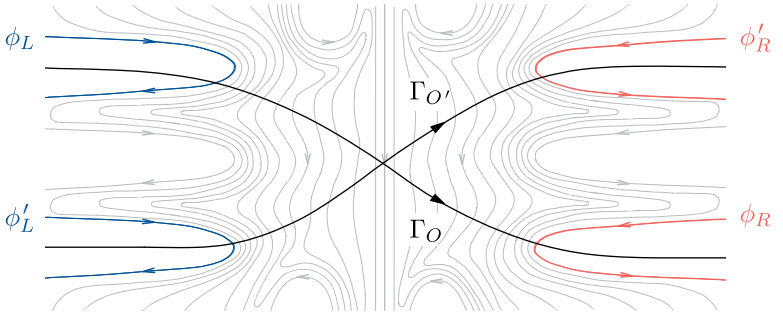


FIGURE 35. Two orbits O and O' exhibiting a $\mathcal{F}$-transverse intersection in the sense of [LCT18].

6. *Maximal leaf domains and critical leaves*

In this section, we show how a finite collection of orbits $\mathcal{O} \subset \mathrm{Orb}(f)$ induces a finite decomposition of the set $\bigcup_{O\in\mathcal{O}} C_O \subset \mathcal{F}$ into a collection of leaf domains and critical leaves.

Let $\mathcal{O} \subset \mathrm{Orb}(f)$ be a finite collection of orbits. Recall that the boundary ∂C_O is formed by a locally finite collection of leaves. Thus, the union $\bigcup_{O\in\mathcal{O}} \partial C_O$ is also a locally finite collection of leaves, since $\mathcal{O}$ is a finite set. In this context, we define the following definitions.

Definition 6.1. (Maximal leaf domains and critical leaves)

- A saturated set $D \subset \mathbb{R}^2$ is said to be a *maximal leaf domain* of $\mathcal{O}$ if it is a connected component of $\mathbb{R}^2 \setminus \bigcup_{O\in\mathcal{O}} \partial C_O$ that satisfies $D \subset \bigcup_{O\in\mathcal{O}} C_O$.
- A leaf $\phi \in \mathcal{F}$ is said to be a *critical leaf* of $\mathcal{O}$ if it is a leaf contained in $\bigcup_{O\in\mathcal{O}} \partial C_O$ that is also contained in the set $\bigcup_{O\in\mathcal{O}} C_O$.

The sets of all maximal leaf domains and critical leaves of $\mathcal{O}$ are denoted by $\mathcal{D}_\mathcal{O}$ and $\mathcal{W}_\mathcal{O}$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

PROPOSITION 6.2. *The following statements are true.*

(i) *Every $D \in \mathcal{D}_{\mathcal{O}}$ is a leaf domain of $\mathcal{F}$ (see Definition 2.5).*

(ii) *For each $D \in \mathcal{D}_{\mathcal{O}}$, there exists a non-empty subset $\mathcal{O}(D) \subset \mathcal{O}$ such that*

$$D \subset \bigcap_{O \in \mathcal{O}(D)} C_O \quad \text{and} \quad \overline{D} \cap \bigcup_{O \in \mathcal{O} \setminus \mathcal{O}(D)} C_O = \varnothing.$$

(iii) *The sets of maximal leaf domains $\mathcal{D}_{\mathcal{O}}$ and critical leaves $\mathcal{W}_{\mathcal{O}}$ are finite.*

Roughly speaking, maximal leaf domains of $\mathcal{O}$ constitute open connected sets of leaves that are crossed by exactly the same (at least one) orbits in $\mathcal{O}$, while critical leaves are the leaves that are crossed by at least one orbit in $\mathcal{O}$ and separate two maximal leaf domains.

Proof. Proof of item (i). Let $D \in \mathcal{D}_{\mathcal{O}}$ be a maximal leaf domain of $\mathcal{O}$. Since $\bigcup_{O \in \mathcal{O}} \partial C_O$ is a closed set, each connected component of $\mathbb{R}^2 \setminus \bigcup_{O \in \mathcal{O}} \partial C_O$ is open. In particular, D is an open and connected saturated subset of $\mathbb{R}^2$ and, consequently, it is formed by infinitely many leaves. Consider two distinct leaves $\phi, \phi' \in D$. Since $D \subset \bigcup_{O \in \mathcal{O}} C_O$, there exists an orbit $O \in \mathcal{O}$ such that $\phi \in C_O$. If ϕ' does not belong to C_O as well, then there exists $\phi'' \in \partial C_O$ that separates ϕ from ϕ' on the plane. This contradicts the definition of maximal leaf domains. Therefore, the leaf ϕ' also belongs to C_O. This implies either $L(\phi) \subset L(\phi')$ or $L(\phi') \subset L(\phi)$. We then conclude that D is a leaf domain of $\mathcal{F}$ (see Definition 2.5).

Proof of item (ii). According to the argument presented in the proof of item (i), for any pair of leaves $\phi, \phi' \in D$ and any orbit $O \in \mathcal{O}$, it holds that $\phi \in C_O$ if, and only if, $\phi' \in C_O$. Therefore, there exists a subset $\mathcal{O}(D) \subset \mathcal{O}$ such that, for all $\phi \in D$ and $O \in \mathcal{O}$, it holds

$$\phi \in C_O \iff O \in \mathcal{O}(D).$$

This proves that $D \subset \bigcap_{O \in \mathcal{O}(D)} C_O$. Now, since $\bigcup_{O \in \mathcal{O} \setminus \mathcal{O}(D)} C_O$ is open, we have that

$$\left(\overline{D} \cap \bigcup_{O \in \mathcal{O} \setminus \mathcal{O}(D)} C_O\right) \subset \left(\overline{D \cap \bigcup_{O \in \mathcal{O} \setminus \mathcal{O}(D)} C_O}\right) = \varnothing.$$

The set $\mathcal{O}(D)$ is non-empty because D is contained in the set $\bigcup_{O \in \mathcal{O}} C_O$. This proves item (ii).

Proof of item (iii). According to Lemma 2.6, for any pair of orbits $O, O' \in \mathcal{O}$, the set $C_O \cap C_{O'}$ is a leaf domain of $\mathcal{F}$ (possibly empty). In particular, $C_O \cap C_{O'}$ is connected. Consequently, the set $C_O \cap \partial C_{O'}$ contains at most two leaves. Moreover, note that

$$\mathcal{W}_{\mathcal{O}} = \bigcup_{O, O' \in \mathcal{O}} C_O \cap \partial C_{O'}.$$

Since $\mathcal{O}$ is a finite set, we conclude that the set $\mathcal{W}_{\mathcal{O}}$ is also finite. This further implies that, for each orbit $O \in \mathcal{O}$, the set $C_O \setminus \mathcal{W}_{\mathcal{O}}$ has finitely many connected components. Since each maximal leaf domain in $\mathcal{D}_{\mathcal{O}}$ is a connected component of the set $C_O \setminus \mathcal{W}_{\mathcal{O}}$ for some $O \in \mathcal{O}$, we conclude that $\mathcal{D}_{\mathcal{O}}$ is also a finite set. This concludes the proof of item (iii). □

Consider the example in Figure 36, which follows the conventions below.

- Transverse trajectories of orbits in $\mathcal{O}$ are represented in red.
- Leaves in $\mathcal{W}_{\mathcal{O}} \subset \bigcup_{O \in \mathcal{O}} \partial C_O$, called critical leaves, are represented in black.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

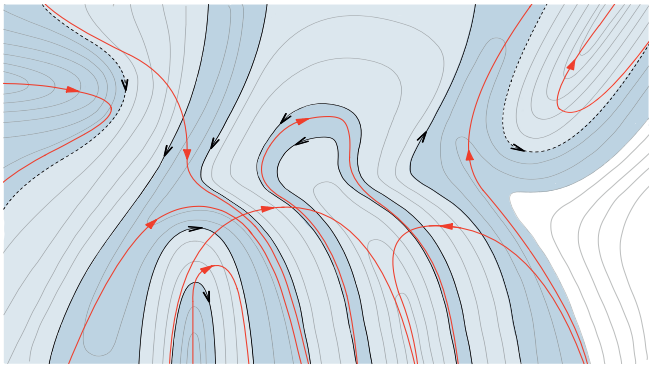


FIGURE 36. Example of a decomposition in maximal leaf domains and critical leaves.

- Leaves in $\bigcup_{O\in\mathcal{O}} \partial C_O \setminus \mathcal{W}_\mathcal{O}$ are represented by a dashed line in black for distinction.
- Leaf domains in $\mathcal{D}_\mathcal{O}$ are represented in light and dark shades of blue, to easily differentiate adjacent leaf domains.
- Regions in $\mathbb{R}^2 \setminus \bigcup_{O\in\mathcal{O}} C_O$ are represented in white.

Remark 6.3. In the following, we list some key properties of this decomposition.

(i) The set $\mathcal{W}_\mathcal{O} \cup \bigcup_{D\in\mathcal{D}_\mathcal{O}} D$ may not cover the entire plane, as it coincides with $\bigcup_{O\in\mathcal{O}} C_O$.

(ii) Each $D \in \mathcal{D}_\mathcal{O}$ consists in a saturated set $D \subset \mathbb{R}^2$ homeomorphic to a trivially foliated plane, and its boundary ∂D is a non-separating and locally finite collection of leaves.

(iii) The leaves in a maximal leaf domain $D \in \mathcal{D}_\mathcal{O}$ are crossed by exactly the same orbits in $\mathcal{O}$. Specifically, they are crossed by every orbit in $\mathcal{O}(D)$ and by no orbit in $\mathcal{O} \setminus \mathcal{O}(D)$.

(iv) Each critical leaf $\phi \in \mathcal{W}_\mathcal{O}$ is crossed by some orbit in $\mathcal{O}$, but never by all of them.

(v) For each critical leaf $\phi \in \mathcal{W}_\mathcal{O}$, there exist $D, D' \in \mathcal{D}_\mathcal{O}$ that satisfy $\overline{D} \cap \overline{D'} = \{\phi\}$.

(vi) If a leaf $\phi \in \mathcal{F}$ satisfies $\overline{D} \cap \overline{D'} = \{\phi\}$ and is not crossed by any orbit in $\mathcal{O}$, then ϕ is not a critical leaf of $\mathcal{O}$ (as represented by the dashed black lines in the example above).

(vii) Two maximal leaf domains $D, D' \in \mathcal{D}_\mathcal{O}$ can be crossed by exactly the same orbits in $\mathcal{O}$, meaning $\mathcal{O}(D) = \mathcal{O}(D')$, and still be distinct. This is illustrated in the example above.

6.1. *Orbits traveling across maximal leaf domains.* To each $O \in \mathcal{O}$, we associate an integer $m_O > 0$ and a sequence $(D_i)_{1\le i\le m_O}$ in $\mathcal{D}_\mathcal{O}$ (see Figure 37) such that:

- for any $i \in \{1, \dots, m_O\}$, the maximal leaf domain D_i is contained in the set C_O.
- For any $i \in \{1, \dots, m_O - 1\}$, there exists $\phi_i \in \mathcal{W}_\mathcal{O}$ such that $\overline{D_i} \cap \overline{D_{i+1}} = \{\phi_i\}$;
- the set C_O is decomposed as $C_O = \bigcup_{i=1}^{m_O} D_i \cup \bigcup_{i=1}^{m_O-1} \{\phi_i\}$;

We denote the first and last terms of this sequence by $D_\alpha(O) := D_1$ and $D_\omega(O) := D_{m_O}$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

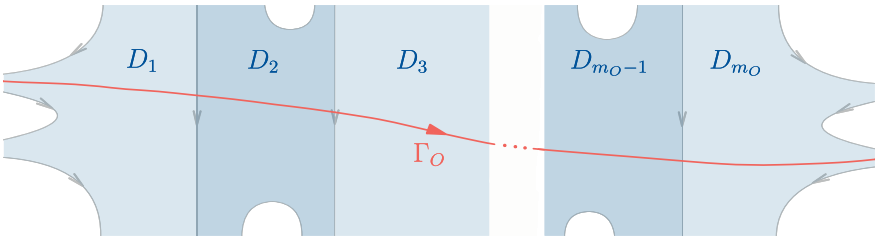


FIGURE 37. Sequence of leaf domains $(D_i)_{1\le i\le m_O}$ visited by an orbit O.

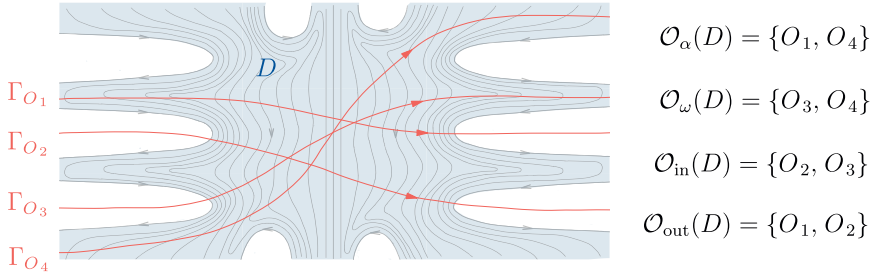


FIGURE 38. Example to illustrate the sets $\mathcal{O}_\alpha(D)$, $\mathcal{O}_\omega(D)$, $\mathcal{O}_{\text{in}}(D)$ and $\mathcal{O}_{\text{out}}(D)$.

Conversely, for each maximal leaf domain $D \in \mathcal{D}_\mathcal{O}$, we define the sets of orbits

$$\mathcal{O}_\alpha(D) := \{O \in \mathcal{O} \mid D_\alpha(O) = D\} \quad \text{and} \quad \mathcal{O}_\omega(D) := \{O \in \mathcal{O} \mid D_\omega(O) = D\}.$$

Their complementary sets are denoted as

$$\mathcal{O}_{\text{in}}(D) := \mathcal{O}(D) \setminus \mathcal{O}_\alpha(D) \quad \text{and} \quad \mathcal{O}_{\text{out}}(D) := \mathcal{O}(D) \setminus \mathcal{O}_\omega(D).$$

Intuitively speaking, $\mathcal{O}_\omega(D)$ and $\mathcal{O}_\alpha(D)$ comprise orbits in $\mathcal{O}$ that eventually stay in D under forward and backward iterations, respectively, while $\mathcal{O}_{\text{in}}(D)$ and $\mathcal{O}_{\text{out}}(D)$ comprise orbits in $\mathcal{O}$ that eventually enter and eventually exit D (see Figure 38).

Since each $D \in \mathcal{D}_\mathcal{O}$ is a leaf domain of $\mathcal{F}$, we can define the following boundary subsets:

$$\partial_R D := \partial D \cap \left(\bigcap_{\phi \in D} R(\phi) \right) \quad \text{and} \quad \partial_L D := \partial D \cap \left(\bigcap_{\phi \in D} L(\phi) \right).$$

Recall that both sets $\partial_R D$ and $\partial_L D$ are ordered by the relation $\prec$ defined in §3.3. These sets will help us to describe the behavior of the orbits $\mathcal{O}$ inside each $D \in \mathcal{D}_\mathcal{O}$.

Observe that, for any $D \in \mathcal{D}_\mathcal{O}$, the following properties hold:

$$\begin{aligned} \partial_L D &= \partial_L C_O \quad \text{for all } O \in \mathcal{O}_\omega(D), \\ \partial_R D &= \partial_R C_O \quad \text{for all } O \in \mathcal{O}_\alpha(D). \end{aligned}$$

Hence, each orbit $O \in \mathcal{O}_\omega(D)$ induces a cut $(\partial_L^{\text{top}} O, \partial_L^{\text{bot}} O)$ of the ordered set $(\partial_L D, \prec)$, and each $O \in \mathcal{O}_\alpha(D)$ induces a cut $(\partial_R^{\text{top}} O, \partial_R^{\text{bot}} O)$ of the ordered set $(\partial_R D, \prec)$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

For each leaf $\phi \in \mathcal{F}$, let $\mathcal{O}(\phi) \subset \mathcal{O}$ be the subset of orbits that cross ϕ. Observe that, for any $D \in \mathcal{D}_{\mathcal{O}}$, the subsets of orbits $\mathcal{O}_{\text{in}}(D)$ and $\mathcal{O}_{\text{out}}(D)$ can be decomposed as

$$\mathcal{O}_{\text{in}}(D) = \bigcup_{\phi \in \partial_R D} \mathcal{O}(\phi) \quad \text{and} \quad \mathcal{O}_{\text{out}}(D) = \bigcup_{\phi \in \partial_L D} \mathcal{O}(\phi).$$

In other words, the orbits in $\mathcal{O}_{\text{in}}(D)$ must enter D through a leaf in $\partial_R D$, while the orbits in $\mathcal{O}_{\text{out}}(D)$ must exit D through a leaf in $\partial_L D$. If this was not the case, there would exist leaves in D that are not crossed by exactly the same orbits in $\mathcal{O}$.

Remark 6.4. Observe that, for any $D \in \mathcal{D}_{\mathcal{O}}$, the sets $\mathcal{O}_{\omega}(D)$ and $\mathcal{O}_{\text{out}}(D)$ are stable under the equivalence relation $\overset{+}{\sim}$. More precisely, for any $O, O' \in \mathcal{O}(D)$ satisfying $O \overset{+}{\sim} O'$, we have

$$O \in \mathcal{O}_{\omega}(D) \iff O' \in \mathcal{O}_{\omega}(D),$$
$$O \in \mathcal{O}_{\text{out}}(D) \iff O' \in \mathcal{O}_{\text{out}}(D).$$

Similarly, the sets $\mathcal{O}_{o}mega(D)$ and $\mathcal{O}_{\text{out}}(D)$ are stable under the equivalence relation $\overset{-}{\sim}$.

6.2. *Directed planar forest of maximal leaf domains.* Let $\mathcal{D}_{\mathcal{O}}$ and $\mathcal{W}_{\mathcal{O}}$ be the sets of maximal leaf domains and critical leaves of a given finite collection of orbits $\mathcal{O} \subset \text{Orb}(f)$. These sets naturally define an oriented graph $(\mathcal{D}_{\mathcal{O}}, \mathcal{W}_{\mathcal{O}})$, where the set of vertices is given by $\mathcal{D}_{\mathcal{O}}$, and the set of oriented edges consists of all ordered pairs of maximal leaf domains $(D, D') \in \mathcal{D}_{\mathcal{O}} \times \mathcal{D}_{\mathcal{O}}$ for which there exists $\phi \in \mathcal{W}_{\mathcal{O}}$ satisfying $\overline{D} \cap \overline{D'} = \{\phi\}$ and $D \subset R(\phi)$ or, equivalently, $D' \subset L(\phi)$.

Each oriented edge (D, D') in $(\mathcal{D}_{\mathcal{O}}, \mathcal{W}_{\mathcal{O}})$ is denoted $D \rightsquigarrow D'$.

LEMMA 6.5. *The graph $(\mathcal{D}_{\mathcal{O}}, \mathcal{W}_{\mathcal{O}})$ is an oriented planar forest, with each tree component corresponding to a connected component of the set $\bigcup_{O \in \mathcal{O}} C_O$.*

Proof. For each maximal leaf domain $D \in \mathcal{D}_{\mathcal{O}}$, let $p_D \in \mathbb{R}^2$ be a point in the region formed as a union of leaves in D. For any $D, D' \in \mathcal{D}_{\mathcal{O}}$ satisfying $D \rightsquigarrow D'$, denote by $\phi_{D,D'} \in \mathcal{W}_{\mathcal{O}}$ the critical leaf in between D and D', meaning, $\overline{D} \cap \overline{D'} = \{\phi_{D,D'}\}$. Observe that since

$$\phi_{D,D'} \in \partial_L D \quad \text{and} \quad \phi_{D,D'} \in \partial_R D',$$

we can connect the point p_D to $p_{D'}$ via a positively transverse path $\gamma_{D,D'} : [0, 1] \longrightarrow \mathbb{R}^2$. By associating each core region $D \in \mathcal{D}_{\mathcal{O}}$ with the point p_D and each oriented edge $D \rightsquigarrow D'$ with the path $\gamma_{D,D'}$, we obtain a planar embedding of $(\mathcal{D}_{\mathcal{O}}, \mathcal{W}_{\mathcal{O}})$ (see Figure 39).

This planar embedding shows us that the oriented graph $(\mathcal{D}_{\mathcal{O}}, \mathcal{W}_{\mathcal{O}})$ is an oriented forest, that is, a disjoint union of oriented trees. For that, it suffices to prove that each connected component of the planar graph is no longer connected if one of its edges is deleted. Indeed, the path $\gamma_{D,D'}$ is the unique edge of the planar graph intersecting the leaf $\phi_{D,D'}$. Therefore, by deleting $\gamma_{D,D'}$, we disconnect the connected component of the planar graph that contains the vertices p_D and $p_{D'}$ into two components, each lying on a different side

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

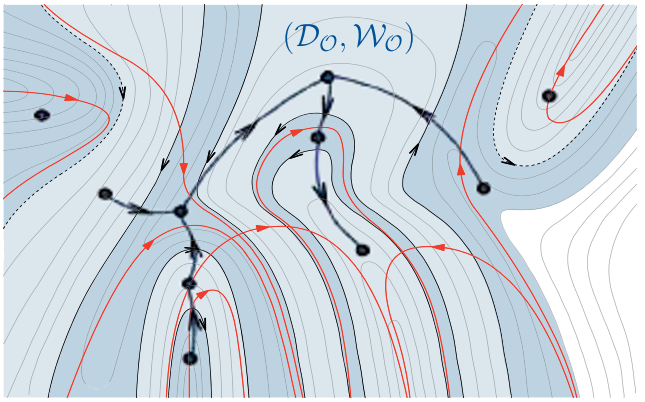


FIGURE 39. Directed planar forest of maximal leaf domains $(\mathcal{D}_{\mathcal{O}}, \mathcal{W}_{\mathcal{O}})$ of the example in Figure 35.

of the leaf $\phi_{D,D'}$. This allows us to conclude that each connected component of $(\mathcal{D}_{\mathcal{O}}, \mathcal{W}_{\mathcal{O}})$ is an oriented tree corresponding to a connected component of $\bigcup_{O\in\mathcal{O}} C_O$. □

7. *Proper transverse trajectories with minimal intersections*

In this section, we prove Theorem B. Recall that the transverse foliation $\mathcal{F}$ is said to be *generic* for $\mathcal{O}$ if no leaf in $\mathcal{F}$ intersects more than one orbit in $\mathcal{O}$. This condition is indeed generic, as any transverse foliation can be perturbed to satisfy it (see §2.3).

THEOREM B. *Let $\phi \in \mathcal{F}$ and let $\mathcal{O} \subset \mathrm{Orb}(\phi)$ be a finite collection of orbits crossing ϕ. Assume that $\mathcal{F}$ is generic with respect to $\mathcal{O}$. Then, there exists a family $\{\Gamma_O\}_{O\in\mathcal{O}}$ of proper transverse trajectories associated with the orbits in $\mathcal{O}$ that satisfy*

$$\Gamma_O \cap \Gamma_{O'} \cap \overline{L(\phi)} = \varnothing \quad \text{for all } O, O' \in \mathcal{O},\ O \neq O'$$

(see Figure 40).

Moreover, the order in which the intersection points $p_O \in \Gamma_O \cap \phi$ appear along the leaf ϕ is compatible with the relation $\lesssim_L$, in the sense that $p_O < p_{O'}$ along the leaf ϕ only if $O \lesssim_L O'$.

A result similar to Theorem B holds for the right side of the leaf ϕ, with the same assumptions and conditions, but using the relation $\lesssim_R$ instead of $\lesssim_L$.

7.1. *Technical lemmas for the proof of Theorem B.*

Before we prove Theorem B, we need four technical lemmas. In each of these lemmas, we assume that the transverse foliation $\mathcal{F}$ is generic for the finite collection of orbits $\mathcal{O} \subset \mathrm{Orb}(f)$ in question.

7.1.1. *Technical Lemma 1: flexibility to obtain pairwise disjoint transverse paths.*

LEMMA 7.1. *Let $\phi, \phi' \in \mathcal{F}$ be two distinct leaves that satisfy $L(\phi') \subset L(\phi)$. Moreover, let $\mathcal{O} \subset \mathrm{Orb}(\phi) \cap Orb(\phi')$ be a finite collection of orbits crossing both leaves ϕ and ϕ'. Then, for any indexing $\mathcal{O} = \{O_1, \ldots, O_r\}$, there exists a family $\{\gamma_i\}_{1\le i\le r}$ of pairwise*

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

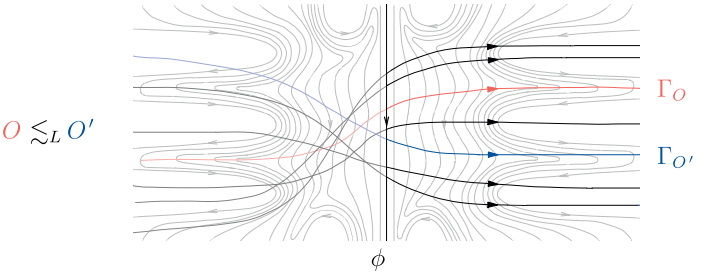


FIGURE 40. Family of proper transverse trajectories $\{\Gamma_O\}_{O\in\mathcal{O}}$ obtained in Theorem B.

disjoint paths $\gamma_i : [0, 1] \longrightarrow \mathbb{R}^2$ *positively transverse to* $\mathcal{F}$ *such that, for every* $1 \leq i \leq r$, *the following hold.*

- *The endpoints satisfy* $\gamma_i(0) \in \phi$ *and* $\gamma_i(1) \in \phi'$.
- *The fragment of orbit* $\overline{L(\phi) \cap R(\phi')} \cap O_i$ *is contained in* γ_i.
- *The endpoints are ordered* $\gamma_i(0) < \gamma_{i+1}(0)$ *according to the orientation of* ϕ.

Proof. Consider an arbitrary indexing of the set of orbits $\mathcal{O} = \{O_1, \ldots, O_r\}$. Observe that, by means of the Homma–Schoenflies theorem, up to conjugacy, we can assume that:

- $\phi = \{0\} \times \mathbb{R}$ and $\phi' = \{1\} \times \mathbb{R}$, both oriented downwards;
- the square $[0, 1]^2$ is vertically foliated by $\mathcal{F}$;
- for each $i \in \{1, \ldots, r\}$, we have that $O_i \cap \overline{L(\phi) \cap R(\phi)} \subset [0, 1] \times \{1/i\}$.

The third point only holds because of the generic condition imposed on $\mathcal{F}$ with respect to $\mathcal{O}$, as it ensures that each leaf of $\mathcal{F}$ intersects at most one orbit in $\mathcal{O}$. For each $i \in \{1, \ldots, r\}$, let $\gamma_{O_i} : [0, 1] \longrightarrow \mathbb{R}^2$ be the horizontal segment given by $\gamma_{O_i}(t) = (t, 1/i)$. This concludes the proof, as the family $\{\gamma_{O_i}\}_{1\leq i\leq r}$ satisfies the conditions of the lemma. □

7.1.2. *Technical Lemma 2: transverse half-lines for classes of* $\mathcal{F}$*-asymptotic orbits.*

LEMMA 7.2. *Let* $\phi \in \mathcal{F}$ *and let* $\mathcal{O} \subset \mathrm{Orb}(\phi)$ *be a finite set of orbits crossing the leaf* ϕ. *Then, for every class of forward* $\mathcal{F}$*-asymptotic orbits* $[O]^+ \in \mathcal{O}/\overset{+}{\sim}$, *there exists a half-line* $\Gamma^+ : [0, \infty) \longrightarrow \mathbb{R}^2$ *that is positively transverse to* $\mathcal{F}$ *and satisfies the following conditions.*

- *The endpoint* $\Gamma^+(0)$ *lies on the leaf* ϕ.
- *The forward orbit* $\overline{L(\phi)} \cap O$ *is contained in* Γ^+ *for all* $O \in [O]^+$.

Proof. The proof follows from the proof of Lemma 3.6 (see Remark 3.9). □

7.1.3. *Technical Lemma 3: disjoint transverse half-lines for* $\mathcal{F}$*-asymptotic classes.*

LEMMA 7.3. *Let* $\phi \in \mathcal{F}$ *and let* $\mathcal{O} \subset \mathrm{Orb}(\phi)$ *be a finite set of orbits crossing the leaf* ϕ. *We enumerate* $\mathcal{O}/\overset{+}{\sim} = \{[O_1]^+, \ldots, [O_r]^+\}$ *so that* $[O_i]^+ \lesssim_L [O_{i+1}]^+$ *for all* $i \in \{1, \ldots, r - 1\}$. *Then, there exists a family* $\{\Gamma_i^+\}_{1\leq i\leq r}$ *of pairwise disjoint half-lines*

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

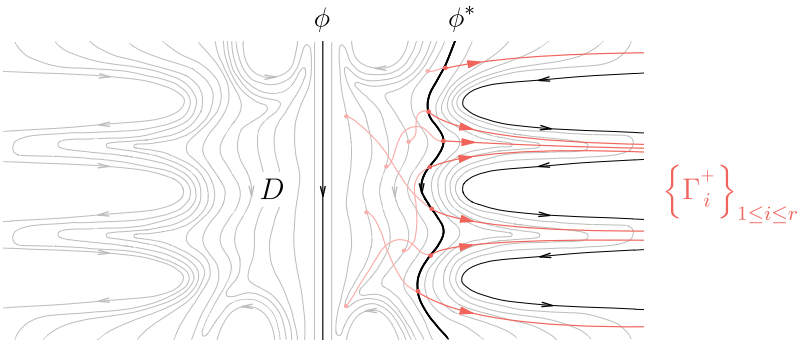


FIGURE 41. Illustration of the family $\{\Gamma_i^+\}_{1\le i\le r}$ in Case (1) within the proof of Lemma 7.3.

$\Gamma_i^+ : [0,\infty) \longrightarrow \mathbb{R}^2$ *positively transverse to* $\mathcal{F}$ *such that, for every* $1 \le i \le r$*, the following hold.*

- *The endpoints* $\Gamma_i^+(0)$ *lie on the leaf* ϕ*.*
- *The forward orbit* $\overline{L(\phi)} \cap O$ *is contained in* Γ_i^+ *for all orbits* $O \in [O_i]^+$*.*
- *The endpoints are ordered* $\Gamma_i^+(0) < \Gamma_{i+1}^+(0)$ *according to the orientation of* ϕ*.*

Proof. Note that we can enumerate $\mathcal{O}/\overset{+}{\sim} = \{[O_1]^+, \dots, [O_r]^+\}$ in an increasing order according to $\lesssim_L$ because this relation induces a total order on $\mathcal{O}/\overset{+}{\sim}$ (see Proposition 4.2). Let $\mathcal{D}_{\mathcal{O}}$ be the decomposition maximal leaf domains induced by $\mathcal{O}$ and let $\mathcal{W}_{\mathcal{O}}$ be the set of critical leaves of $\mathcal{O}$ (see Definition 6.1). Since the leaf ϕ is crossed by all orbits in $\mathcal{O}$, we know that ϕ is not a critical leaf of $\mathcal{O}$. Hence, there exists a maximal leaf domain $D \in \mathcal{D}_{\mathcal{O}}$ such that $\phi \in D$ and, consequently, $\mathcal{O}(D) = \mathcal{O}$. We separate the proof into two cases.

Case (1): assuming that $\mathcal{O}_{\text{out}}(D) = \varnothing$. According to Lemma 7.2, each class $[O_i]^+ \in \mathcal{O}/\overset{+}{\sim}$ admits a half-line $\Gamma_i^+ : [0,\infty) \longrightarrow \mathbb{R}^2$ that is positively transverse to $\mathcal{F}$, has its endpoint $\Gamma_i^+(0) \in \phi$, and satisfies

$$\overline{L(\phi) \cap O} \subset \Gamma_i^+ \quad \text{for all } O \in [O_i]^+.$$

According to Lemma 3.6, for any $i, j \in \{1, \dots, r\}$ with $i \neq j$, there exists $t > 0$ so that

$$\Gamma_i^+([t,\infty)) \cap \Gamma_j^+([t,\infty)) = \varnothing.$$

Since $\mathcal{O}$ is finite, this implies that for each $i \in \{1, \dots, r\}$, there exists $\phi_i \in \mathcal{F}$ such that

$$\Gamma_i^+ \cap \Gamma_j^+ \cap \overline{L(\phi_i)} = \varnothing \quad \text{for all } j \neq i.$$

Observe that, in the present setting, it holds that $\mathcal{O} = \mathcal{O}_\omega(D)$. Thus, we have $\phi_i \in D$ for every $i \in \{1, \dots, r\}$. Note that we can consider a leaf $\phi^* \in D$ that satisfies $L(\phi^*) \subset L(\phi_i)$ for every $i \in \{1, \dots, r\}$ (see Figure 41).

By reparameterizing the half-lines in $\{\Gamma_i^+\}_{1\le i\le r}$, we can obtain some $t^* > 0$ such that the point $\Gamma_i^+(t^*)$ lies on the leaf ϕ^* for every $i \in \{1, \dots, r\}$. We remark that, since the half-lines in the family $\{\Gamma_i^+\}_{1\le i\le r}$ are pairwise disjoint on the left side $\overline{L(\phi^*)}$, it follows

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

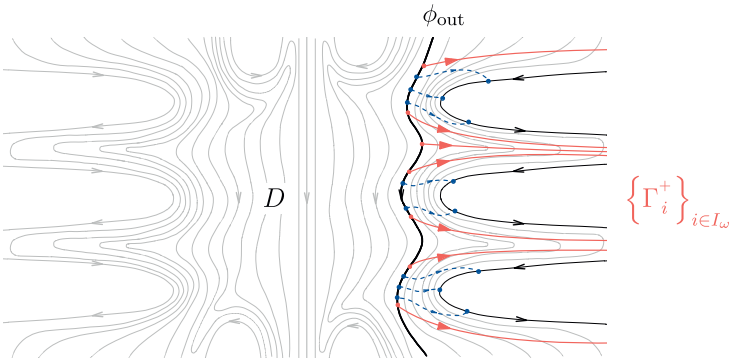


FIGURE 42. Illustration of the families $\{\Gamma_i^+\}_{1\le i\le r}$ and $\{\gamma_i\}_{i\in I_{\text{out}}}$ in Case (2) within the proof of Lemma 7.3.

from the property described in Remark 4.6 that the points in the family $\{\Gamma_i^+(t^*)\}_{1\le i\le r}$ must be ordered as

$$\Gamma_i^+(t^*) < \Gamma_{i+1}^+(t^*) \quad \text{along the leaf } \phi^* \quad \text{for all } i \in \{1,\dots,r-1\}.$$

Using Lemma 7.1, we can replace the arcs $\{\Gamma_i^+([0,t^*])\}_{1\le i\le r}$ so that they become pairwise disjoint, while still preserving the endpoint $\Gamma_i^+(t^*)$ and satisfying

$$\overline{L(\phi)\cap O\cap R(\phi_i)} \subset \Gamma_i^+([0,t^*]) \quad \text{for all } O\in[O_i]^+.$$

This concludes the proof under the assumption that $\mathcal{O}_{\text{out}}(D)=\varnothing$.

Case (2): assuming that $\mathcal{O}_{\text{out}}(D)\neq\varnothing$. First, consider a leaf $\phi_{\text{out}}\in D$ such that every leaf contained in $D\cap L(\phi_{\text{out}})$ is disjoint from the set $\bigcup_{O\in\mathcal{O}_{\text{out}}(D)} O$. Moreover, we can assume that $L(\phi_{\text{out}})\subset L(\phi)$.

Consider the partition of the set $\{1,\dots,r\}$ into the following two sets:

$$I_\omega = \{1\le i\le r \mid [O_i]^+\in \mathcal{O}_\omega(D)/\overset{+}{\sim}\},$$
$$I_{\text{out}} = \{1\le i\le r \mid [O_i]^+\in \mathcal{O}_{\text{out}}(D)/\overset{+}{\sim}\}.$$

Note that we can apply the result proved in the previous case ($\mathcal{O}_{\text{out}}(D)=\varnothing$) to the classes of orbits $[O_i]^+\in\mathcal{O}_\omega(D)/\overset{+}{\sim}$ and the leaf ϕ_{out}. This yields a family $\{\Gamma_i^+\}_{i\in I_\omega}$ of pairwise disjoint half-lines positively transverse to $\mathcal{F}$ (see Figure 42) such that, for all $i\in I_\omega$, the following conditions hold:

- the endpoint $\Gamma_i^+(0)$ lies on the leaf ϕ_{out};
- the half-orbit $\overline{L(\phi_{\text{out}})\cap O}$ is contained in Γ_i^+ for all $O\in[O_i]^+$;
- the points $\{q_i := \Gamma_i^+(0)\}_{i\in I_\omega}$ are ordered increasingly along ϕ_{out}, that is,

$$q_i < q_j \quad \text{along the leaf } \phi_{\text{out}} \quad \text{for all } i<j\in I_\omega.$$

Meanwhile, the set I_{out} admits itself a partition into sets of the form

$$I_{\text{out}}(\phi_L) = \{i\in I_{\text{out}} \mid [O_i]^+\in\mathcal{O}(\phi_L)/\overset{+}{\sim}\}, \quad \text{where } \phi_L\in\partial_L D.$$

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

In other words, for each leaf $\phi_L \in \partial_L D$, we have a set $I_{\text{out}}(\phi_L)$ formed by the indices of the classes of orbits $[O_i]^+ \in \mathcal{O}_{\text{out}}(D)/\overset{+}{\sim}$ that are leaving the maximal leaf domain D through the leaf $\phi_L \in \partial_L D$. Now, for each leaf $\phi_L \in \partial_L D$, we can consider a family of points $\{p_i\}_{i \in I_{\text{out}}(\phi_L)}$ lying on the leaf ϕ_L and ordered increasingly along the leaf ϕ_L, i.e.

$$p_i \prec p_j \quad \text{along the leaf } \phi_L \quad \text{for all } i < j \in I_{\text{out}}(\phi_L).$$

Next, we consider a family of points $\{q_i\}_{i \in I_{\text{out}}}$ lying on the leaf ϕ_{out} in such a way that the whole family $\{q_i\}_{1 \le i \le r}$ is ordered increasingly along ϕ_{out} (see Figure 42).

Note that for each $i \in I_{\text{out}}$, we can construct a path $\gamma_i : [0, 1] \longrightarrow \mathbb{R}^2$ that is positively transverse to $\mathcal{F}$ and connects the point q_i to the point p_i. By constructing these paths one at a time, we can ensure that the resulting family $\{\gamma_i\}_{i \in I_{\text{out}}}$ satisfies the following properties:

- $\gamma_i \cap \gamma_j = \varnothing$ for any $i, j \in I_{\text{out}}$ with $i \neq j$;
- $\gamma_i \cap \Gamma_j^+ = \varnothing$ for any $i \in I_{\text{out}}$ and $j \in I_\omega$.

Using Lemma 7.1, we can extend all half-lines in the family $\{\Gamma_i^+\}_{i \in I_\omega}$ and all paths in the family $\{\gamma_i\}_{i \in I_{\text{out}}}$ so that their endpoints $\Gamma_i^+(0)$ and $\gamma_i(0)$ lie on the leaf ϕ, while preserving pairwise disjointness and ensuring that no orbit point lying between ϕ_{out} and ϕ is missed.

Finally, by repeating this same construction for each $\phi_L \in \partial_L D$, we extend the paths in the family $\{\gamma_i\}_{i \in I_{\text{out}}}$ until we get a family $\{\Gamma_i^+\}_{i \in I_{\text{out}}}$ of pairwise disjoint half-lines satisfying

$$\overline{L(\phi) \cap O} \subset \Gamma_i^+ \quad \text{for all } O \in [O_i]^+.$$

This family, together with the family $\{\Gamma_i^+\}_{i \in I_\omega}$ that we already had before, gives us a family of half-lines $\{\Gamma_i^+\}_{1 \le i \le r}$ that satisfies the conditions of the lemma. This concludes the proof. □

7.1.4. *Technical Lemma 4: blowing-up proper transverse trajectories.*

LEMMA 7.4. *Let $\phi \in \mathcal{F}$ and let $\mathcal{O} \subset \text{Orb}(\phi)$ be a finite set of orbits crossing the leaf ϕ. Consider a class $[O]^+ \in \mathcal{O}/\overset{+}{\sim}$ and a half-line $\Gamma^+_{[O]^+} : [0, \infty) \longrightarrow \mathbb{R}^2$ as in Lemma 7.2. Moreover, consider any open neighborhood $U_{[O]^+} \subset \mathbb{R}^2$ of the half-line $\Gamma^+_{[O]^+}$. Therefore, for any indexing $[O]^+ = \{O_1, \ldots, O_r\}$, there exists a family $\{\Gamma_O^+\}_{O \in [O]^+}$ of pairwise disjoint half-lines $\Gamma_O^+ : [0, \infty) \longrightarrow \mathbb{R}^2$, each positively transverse to $\mathcal{F}$, such that:*

- *the half-line Γ_O^+ is contained in $U_{[O]^+}$ for all $O \in [O]^+$;*
- *the endpoint $\Gamma_O^+(0)$ lies on the leaf ϕ for all $O \in [O]^+$;*
- *the forward orbit $\overline{L(\phi)} \cap O$ is contained in Γ_O^+ for all $O \in [O]^+$;*
- *the endpoints $\{\Gamma_O^+(0)\}_{O \in [O]^+}$ are ordered along the leaf ϕ according to $\preceq$.*

Proof. Using the Homma–Schoenflies theorem, we can assume that:

- $\Gamma^+_{[O]^+}([0, \infty)) = [0, \infty) \times \{0\}$, oriented positively in the first coordinate;
- $\phi = \{0\} \times \mathbb{R}$, oriented downwards;
- $[0, \infty) \times [-1, 1] \subset U_{[O]^+}$ is vertically foliated by $\mathcal{F}$.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

Consider an arbitrary enumeration $[O]^+ = \{O_1, \dots, O_r\}$. For any $O_i, O_j \in [O]^+$, we have $\partial_L C_{O_i} = \partial_L C_{O_j}$. We denoted this set by $\partial_L C_{[O]^+}$. Now, since the closed set $[0, \infty) \times [-1, 1]$ is trivially foliated by $\mathcal{F}$ and contains the transverse half-line $\Gamma^+_{[O]^+}$, we get that $[0, \infty) \times [-1, 1]$ is disjoint from $\partial_L C_{[O]^+}$. By Lemma 2.2, there exists a family $\{V_\phi\}_{\phi \in \partial_L C_{[O]^+}}$ of sufficiently small trivial neighborhoods for the leaves in $\partial_L C_{[O]^+}$ such that each V_ϕ is disjoint from $[0, \infty) \times [-1, 1]$.

Since each leaf of $\mathcal{F}$ intersects $\bigcup_{i=1}^r O_i$ at most once, by moving the points of each orbit $O_i \in [O]^+$ vertically, we obtain a homeomorphism $\varphi : [0, \infty) \times [-1, 1] \longrightarrow [0, \infty) \times [-1, 1]$ that fixes each vertical and satisfies $\varphi(O_i \cap \overline{L(\phi)}) \subset [0, \infty) \times \{1/i\}$ for all $i \in \{1, \dots, r\}$. For each $i \in \{1, \dots, n\}$, consider the path $\Gamma_i^+ : [0, \infty) \longrightarrow \mathbb{R}^2$ defined by $\Gamma_i^+(t) = (t, 1/i)$.

Observe that each path $\varphi^{-1} \circ \Gamma_i^+$ is positively transverse to $\mathcal{F}$ and disjoint from all trivial neighborhoods in $\{V_\phi\}_{\phi \in \partial_L C_{[O]^+}}$. As in the proof of Theorem A, this is enough to imply that each $\varphi^{-1} \circ \Gamma_i^+$ is a half-line.

Therefore, we end up with a family $\{\varphi^{-1} \circ \Gamma_i^+\}_{i=1}^r$ of pairwise disjoint half-lines positively transverse to $\mathcal{F}$, contained in $U_{[O]^+}$, with their endpoints $\{\varphi^{-1} \circ \Gamma_i^+(0)\}_{i=1}^r$ ordered increasingly along the leaf ϕ, i.e. $\varphi^{-1} \circ \Gamma_i^+(0) < \varphi^{-1} \circ \Gamma_{i+1}^+(0)$ according to the orientation of ϕ for all $i \in \{1, \dots, r-1\}$.

This concludes the proof of the lemma. □

7.2. *Proof of Theorem B.*

Proof. Before we start, we pay attention to the fact that the enumeration $\mathcal{O} = \{O_1, \dots, O_r\}$ fixed in the statement of the proposition satisfies $O_i \lesssim_L O_{i+1}$ for all $i \in \{1, \dots, r-1\}$. This implies that each class $[O]^+ \in \mathcal{O}/\overset{+}{\sim}$ is associated to an interval of $\{1, \dots, r\}$ given by

$$I_{[O]^+} = \{i \in \{1, \dots, r\} \mid [O_i]^+ = [O]^+\}.$$

Moreover, the intervals in $\{I_{[O]^+}\}_{[O]^+ \in \mathcal{O}/\overset{+}{\sim}}$ are pairwise disjoint and their arrangement inside the set $\{1, \dots, r\}$ is determined by the total order $\lesssim_L$ on $\mathcal{O}/\overset{+}{\sim}$.

By applying Lemma 7.3 to the leaf ϕ and the set of orbits $\mathcal{O}$, we obtain a family of pairwise disjoint half-lines $\{\Gamma^+_{[O]^+}\}_{[O]^+ \in \mathcal{O}/\overset{+}{\sim}}$, each positively transverse to $\mathcal{F}$ and satisfying:

- the endpoints $\Gamma^+_{[O]^+}(0)$ lie on the leaf ϕ;
- the forward orbit $\overline{L(\phi)} \cap O_i$ is contained in $\Gamma^+_{[O]^+}$ for all $i \in I_{[O]^+}$;
- for any two distinct classes $[O]^+, [O']^+ \in \mathcal{O}/\overset{+}{\sim}$ satisfying $[O]^+ \lesssim_L [O']^+$, we have

$$\Gamma^+_{[O]^+}(0) < \Gamma^+_{[O']^+}(0) \quad \text{along the leaf } \phi.$$

Next, consider a family $\{U_{[O]^+}\}_{[O]^+ \in \mathcal{O}/\overset{+}{\sim}}$ of pairwise disjoint open sets satisfying

$$\Gamma^+_{[O]^+} \subset U_{[O]^+} \quad \text{for all } [O]^+ \in \mathcal{O}/\overset{+}{\sim}.$$

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

Finally, for each class $[O]^+ \in \mathcal{O}/\overset{+}{\sim}$, we apply Lemma 7.4 to the half-line $\Gamma^+_{[O]^+}$ and the open neighborhood $U_{[O]^+}$, with respect to the enumeration $[O]^+ = \{O_i\}_{i\in I_{[O]^+}}$. This yields a family of pairwise disjoint half-lines $\{\Gamma^+_i\}_{i\in I_{[O]^+}}$, each positively transverse to $\mathcal{F}$ and satisfying:

- the half-line Γ^+_i is contained in the open neighborhood $U_{[O]^+}$;
- the forward orbit $\overline{L(\phi)} \cap O_i$ is contained in Γ^+_i;
- the endpoint $\Gamma^+_i(0)$ lies on the leaf ϕ;
- The points $\{\Gamma^+_i(0)\}_{i\in I_{[O]^+}}$ are ordered $\Gamma^+_i(0) < \Gamma^+_{i+1}(0)$ along the leaf ϕ.

By repeating this process for each class $[O]^+ \in \mathcal{O}/\overset{+}{\sim}$, we obtain a family of pairwise disjoint half-lines $\{\Gamma^+_i\}_{1\le i\le r}$ that satisfies the conditions stated in the theorem. □

By combining Remark 4.6 with Theorem B, we obtain directly the following corollary, which provides an alternative definition of $\lesssim_L$ and $\lesssim_R$ in terms of proper transverse trajectories (under generic conditions on $\mathcal{F}$). This result is stated below.

COROLLARY 7.5. *Let $\mathcal{O} \subset \mathrm{Orb}(f)$ be a finite set of orbits and assume that $\mathcal{F}$ is generic for $\mathcal{O}$. Let $\phi \in \mathcal{F}$ be a leaf and let $O, O' \in \mathcal{O}(\phi)$ be two distinct orbits. Then, we have that $O \lesssim_L O'$ if, and only if, there exists proper transverse trajectories Γ_O and $\Gamma_{O'}$ of the orbits O and O', respectively, such that the following conditions hold:*

(i) *the trajectories Γ_O and $\Gamma_{O'}$ are disjoint on the left side of ϕ, i.e. $\Gamma_O \cap \Gamma_{O'} \cap \overline{L(\phi)} = \varnothing$;*

(ii) *the intersection points $p_O \in \Gamma_O \cap \phi$ and $p_{O'} \in \Gamma_{O'} \cap \phi$ are ordered as $p_O < p_{O'}$ according to the orientation along the leaf ϕ.*

We obtain an analogous result for $\lesssim_R$ by replacing each L with R in the above statement.

Moreover, under generic conditions on $\mathcal{F}$, we also have the following corollary.

COROLLARY 7.6. *Distinct orbits $O, O' \in \mathcal{O}$ have a weak $\mathcal{F}$-transverse intersection if, and only if, every pair of proper transverse trajectories Γ_O and $\Gamma_{O'}$ of O and O' intersect.*

Proof. It suffices to show that any $O, O' \in \mathcal{O}$ not having a weak $\mathcal{F}$-transverse intersection must admit disjoint proper transverse trajectories. If $C_O \cap C_{O'} = \varnothing$, this result is trivial. Assume that $C_O \cap C_{O'} \neq \varnothing$ and consider a leaf $\phi \in C_O \cap C_{O'}$ that is disjoint from O and O'. Now, consider $\phi_L, \phi_R \in C_O \cap C_{O'}$ with $L(\phi_L) \subset L(\phi_R)$ sufficiently close to ϕ so that the set $\overline{L(\phi_R) \cap R(\phi_L)}$ is disjoint from O and O'. By Corollary 7.5, we get proper transverse trajectories Γ_O and $\Gamma_{O'}$ for O and O' that are disjoint on $L(\phi_L) \cup R(\phi_R)$. Since O and O' do not have a weak $\mathcal{F}$-transverse intersection, Γ_O and $\Gamma_{O'}$ can be taken to intersect the leaves ϕ_L and ϕ_R in the same order. Thus, we can easily modify Γ_O and $\Gamma_{O'}$ inside $\overline{L(\phi_R) \cap R(\phi_L)}$ so that they become disjoint, while preserving the properties of being proper. □

8. *Adaptive orders across maximal leaf domains*

We begin this section with an illustrative scenario that motivates Theorem C (Figure 43).

Let $\mathcal{O} \subset \mathrm{Orb}(f)$ be a finite collection of orbits and let $\mathcal{D}_\mathcal{O}$ be the decomposition by maximal leaf domains induced by $\mathcal{O}$. Consider a maximal leaf domain $D \in \mathcal{D}_O$ and a

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

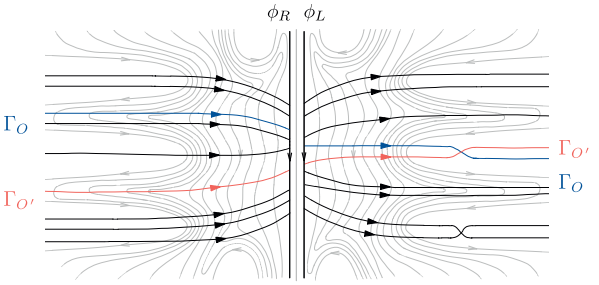


FIGURE 43. Illustrative scenario that explains the idea behind the proof of Theorem C.

pair of sufficiently close leaves $\phi_L, \phi_R \in D$ satisfying $L(\phi_L) \subset L(\phi_R)$. Now, assume that there exists a family of proper transverse trajectories $\{\Gamma_O\}_{O \in \mathcal{O}(D)}$ for $\mathcal{O}(D)$ such that

$$\bigcap_{O \in \mathcal{O}(D)} \Gamma_O \cap \overline{L(\phi_L) \cap R(\phi_R) \cap D} = \varnothing.$$

We want such a family $\{\Gamma_O\}_{O \in \mathcal{O}(D)}$ to follow the following principles.

- If two orbits $O, O' \in \mathcal{O}(D)$ have a weak $\mathcal{F}$-transverse intersection, we ask ourselves

  'Does there exist some $D' \in \mathcal{D}_{\mathcal{O}}$ satisfying $D \rightsquigarrow D'$ for which $O, O' \in \mathcal{O}(D')$?'

  If the answer is 'YES', then we want the trajectories Γ_O and $\Gamma_{O'}$ to be disjoint inside D, leaving their obligatory intersection to happen in D', or later if possible.

  If the answer is 'NO', then we want the obligatory intersection between Γ_O and $\Gamma_{O'}$ to happen in a single point inside D, specifically, in between the leaves ϕ_L and ϕ_R;
- If two orbits $O, O' \in \mathcal{O}(D)$ do not have a weak $\mathcal{F}$-transverse intersection, then we simply want the trajectories Γ_O and $\Gamma_{O'}$ to be disjoint.

Observe that any family $\{\Gamma_O\}_{O \in \mathcal{O}(D)}$ as described above would prove Theorem C in the particular case where $\mathcal{O} = \mathcal{O}(D)$. Therefore, to prove Theorem C in the general case, we not only need to construct such a family $\{\Gamma_O\}_{O \in \mathcal{O}(D)}$ for each $D \in \mathcal{D}_{\mathcal{O}}$, but also to do so in a coherent way, so that the resulting family $\{\Gamma_O\}_{O \in \mathcal{O}}$ solves the theorem.

This problem is solved by defining a 'standard' total order $\leq_\phi$ on the set $\mathcal{O}(\phi)$ of orbits crossing a leaf $\phi \in \mathcal{F}$, and an 'adaptive' total order $\leq_D$ on the set $\mathcal{O}(D)$ of orbits crossing a maximal leaf domain $D \in \mathcal{D}_{\mathcal{O}}$.

Now, we come back to the general setting. Let $\mathcal{O} \subset \mathrm{Orb}(f)$ be a finite collection of orbits and let $\mathcal{D}_{\mathcal{O}}$ be the decomposition by maximal leaf domains induced by $\mathcal{O}$. Before we proceed with the definitions of the standard and adaptive orders, we chose to denote the equivalence relation of backward and forward $\mathcal{F}$-asymptotic orbits in $\mathcal{O}$ by $\sim$, that is,

$$O \sim O' \iff O \overset{+}{\sim} O' \text{ and } O \overset{-}{\sim} O'.$$

Then, we fix an arbitrary total order $\leq_{[O]}$ on each equivalence class $[O] \in \mathcal{O}/\sim$. This choice of total orders $\{\leq_{[O]}\}_{[O] \in \mathcal{O}/\sim}$ determine the standard and adaptive orders defined below.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

8.0.1. *Standard ordering over leaves.* For each leaf $\phi \in \mathcal{F}$, we consider the relation $\preceq_\phi$ on the set $\mathcal{O}(\phi)$, defined as follows:

$$O \preceq_\phi O' \iff \begin{cases} O \lesssim_R O' & \text{if } O \,\bar{\not\sim}\, O', \\ O \lesssim_L O' & \text{if } O \,\bar{\sim}\, O' \text{ and } O \,\overset{+}{\not\sim}\, O', \\ O \preceq_{[O]} O' & \text{if } O \sim O'. \end{cases}$$

Roughly speaking, the relation $\preceq_\phi$ primarily compares orbits in $\mathcal{O}(\phi)$ using the relation $\lesssim_R$. If the compared orbits are backward $\mathcal{F}$-asymptotic, then it tries to compare them using the relation $\lesssim_L$. At last, if the orbits are both forward and backward $\mathcal{F}$-asymptotic, then it uses the total order $\preceq_{[O]}$ defined on the equivalence class $[O] \in \mathcal{O}/\!\sim$ of these orbits.

LEMMA 8.1. *For any leaf $\phi \in \mathcal{F}$, the relation $\preceq_\phi$ is a total order on the set $\mathcal{O}(\phi)$.*

Proof. Let $\phi \in \mathcal{F}$ be a leaf and recall that $\lesssim_R$ induces a total order on the set $\mathcal{O}(\phi)/\bar{\sim}$. Thus, we can enumerate the set $\mathcal{O}(\phi)/\bar{\sim}$ increasingly with respect to $\lesssim_R$, that is,

$$\mathcal{O}(\phi)/\bar{\sim} = \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\}, \quad \text{with } \mathcal{O}_i \lesssim_R \mathcal{O}_{i+1} \text{ for all } 1 \le i < r.$$

Then, for each $i \in \{1, \ldots, r\}$, since $\lesssim_L$ induces a total order on the set $\mathcal{O}_i/\overset{+}{\sim}$, we can enumerate it increasingly with respect to $\lesssim_L$, that is,

$$\mathcal{O}_i/\overset{+}{\sim} = \{\mathcal{O}_{i,1}, \mathcal{O}_{i,2}, \ldots, \mathcal{O}_{i,m(i)}\}, \quad \text{with } \mathcal{O}_{i,j} \lesssim_L \mathcal{O}_{i,j+1} \text{ for all } 1 \le j < m(i).$$

At last, note that, for any $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, m(i)\}$, it holds that $\mathcal{O}_{i,j} \in \mathcal{O}(\phi)/\!\sim$. Thus, there exists an equivalence class $[O]_{i,j} \in \mathcal{O}/\!\sim$ such that $\mathcal{O}_{i,j} \subset [O]_{i,j}$. We can enumerate $\mathcal{O}_{i,j}$ increasingly with respect to the initially fixed total order $\preceq_{[O]_{i,j}}$, that is,

$$\mathcal{O}_{i,j} = \{O_{i,j,1}, \ldots, O_{i,j,n(i,j)}\}, \quad \text{with } O_{i,j,k} \preceq_{[O]_{i,j}} O_{i,j,k+1} \text{ for all } 1 \le k < n(i,j).$$

At the end, we obtained a triple index enumeration of the set $\mathcal{O}(\phi) = \{O_{i,j,k}\}_{i,j,k}$, where the indices run through $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, m(i)\}$ and $k \in \{1, \ldots, n(i,j)\}$.

By construction, $\preceq_\phi$ coincides with the lexicographic order on $\mathcal{O}(\phi) = \{O_{i,j,k}\}_{i,j,k}$, i.e.

$$O_{i,j,k} \preceq_\phi O_{i',j',k'} \iff \begin{cases} i < i' & \text{or} \\ i = i' \text{ and } j < j' & \text{or} \\ i = i' \text{ and } j = j' \text{ and } k < k'. \end{cases}$$

This proves that $\preceq_\phi$ is a total order on the set $\mathcal{O}(\phi)$ and concludes the proof. □

8.1. *Adaptive ordering over maximal leaf domains.* For each $D \in \mathcal{D}_\mathcal{O}$, we consider an equivalence relation $\overset{D}{\sim}$ on the set $\mathcal{O}(D)$ that is slightly weaker than the relation $\overset{+}{\sim}$.

For any two orbits $O, O' \in \mathcal{O}(D)$, we say that $O \overset{D}{\sim} O'$ if:

- either $O, O' \in \mathcal{O}_\omega(D)$ and they satisfy $O \overset{+}{\sim} O'$;
- or $O, O' \in \mathcal{O}_{\text{out}}(D)$ and there exists a leaf $\phi_L \in \partial_L D$ such that $O, O' \in \mathcal{O}(\phi_L)$.

We observe that if the orbits O and O' satisfy $O \overset{+}{\sim} O'$, then they must also satisfy $O \overset{D}{\sim} O'$, but the converse does not hold in general, only if the orbits O and O' are both in

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

$\mathcal{O}_\omega(D)$. Moreover, note that the relation $\lesssim_L$ induces a total order on the set $\mathcal{O}(D)/\overset{D}{\sim}$. This follows from the fact that if the orbits $O, O', O'' \in \mathcal{O}_{\text{out}}(D)$ satisfy $O \lesssim_L O' \lesssim_L O''$ and $O \overset{D}{\sim} O''$, then they must also satisfy $O \overset{D}{\sim} O''$.

With this equivalence relation, we can define the relation $\leq_D$ on the set $\mathcal{O}(D)$ as follows:

$$O \leq_D O' \iff \begin{cases} O \lesssim_L O' & \text{if } O \overset{D}{\not\sim} O', \\ O \lesssim_R O' & \text{if } O \overset{D}{\sim} O' \text{ and } O \bar{\not\sim} O', \\ O \leq_{[O]} O' & \text{if } O \sim O'. \end{cases}$$

Roughly speaking, $\leq_D$ primarily compares orbits in $\mathcal{O}(D)$ using the relation $\lesssim_L$. If the compared orbits exit the domain D through the same leaf $\phi_L \in \partial_L D$, then it tries to compare them using $\lesssim_R$. At last, if these orbits are both forward and backward $\mathcal{F}$-asymptotic, then it uses the total order $\leq_{[O]}$ on their equivalence class $[O] \in \mathcal{O}/\sim$ to compare them.

Note that $\leq_D$ may be viewed as an analog of $\leq_\phi$, with the roles of $\lesssim_L$ and $\lesssim_R$ essentially reversed, except in the case where the compared orbits exit D through the same leaf in $\partial_L D$, in which case, the relation $\lesssim_R$ continues to be applied.

LEMMA 8.2. *For any $D \in \mathcal{D}_\mathcal{O}$, the relation $\leq_D$ is a total order on the set $\mathcal{O}(D)$.*

Proof. The proof is completely analogous to the proof of Lemma 8.1. □

9. *Proof of Theorem C*

This section is dedicated to the proof of Theorem C. The version of the theorem stated here is slightly different from the one presented in the introduction (see Remark 9.1).

THEOREM C. *Let $\mathcal{O} \subset \text{Orb}(f)$ be a finite collection of orbits. Assume $\mathcal{F}$ is generic for $\mathcal{O}$. Then, there exists a family $\{\Gamma_O\}_{O \in \mathcal{O}}$ of proper transverse trajectories of $\mathcal{O}$ that satisfies:*

(i) $\#(\Gamma_O \cap \Gamma_{O'}) \leq 1$ *for any pair of distinct orbits $O, O' \in \mathcal{O}$;*

(ii) $\#(\Gamma_O \cap \Gamma_{O'}) = 1$ *if, and only if, O and O' have a weak $\mathcal{F}$-transverse intersection.*

Remark 9.1. The version of Theorem C enunciated in the introduction is a direct consequence of the statement above combined with the following argument. We enumerate $\mathcal{O} = \{O_1, \ldots, O_r\}$ and consider the family of proper transverse trajectories $\{\Gamma_i\}_{1 \leq i \leq r}$ given by Theorem C. Then, consider a collection of planar foliations $\{\mathcal{F}_j\}_{1 \leq j \leq n}$ which are pairwise transverse to each other, and such that $\mathcal{F}_0 = \mathcal{F}$ and for each $1 \leq i \leq r$, there exists $1 \leq j \leq n$ such that $\Gamma_i \in \mathcal{F}_j$. The homeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{D}^2$ in the statement of Theorem C in the introduction is given by the circle compactification of $\mathbb{R}^2$ via ideal boundaries of $\{\mathcal{F}_j\}_{1 \leq j \leq n}$, as described in [**Bon24**].

Proof. Let $\mathcal{D}_\mathcal{O}$ be the decomposition into the maximal leaf domain induced by $\mathcal{O}$ and let $\mathcal{W}_\mathcal{O}$ be the set of critical leaves for $\mathcal{O}$ (see §6). Next, fix an arbitrary total order $\leq_{[O]}$ on each equivalence class $[O] \in \mathcal{O}/\sim$, where $\sim$ consists in the equivalence relation of backward and forward $\mathcal{F}$-asymptotic orbits in $\mathcal{O}$. As explained in §8, this choice of total

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

orders $\{\leq_{[O]}\}_{[O]\in\mathcal{O}/\sim}$ determines the standard order $\leq_\phi$ and the adaptive order $\leq_D$, which are total orders on $\mathcal{O}(\phi)$ and $\mathcal{O}(D)$, respectively, for each $\phi \in \mathcal{F}$ and each $D \in \mathcal{D}_\mathcal{O}$.

For each leaf $\phi \in \mathcal{W}_\mathcal{O}$, we consider a family of pairwise distinct points $\{p_\phi(O)\}_{O\in\mathcal{O}(\phi)}$ lying on the leaf ϕ, such that the points $p_\phi(O)$ are ordered along ϕ according to $\leq_\phi$, that is,

$$O \leq_\phi O' \iff p_\phi(O) \leq p_\phi(O') \quad \text{according to the orientation of } \phi.$$

In the case where the leaf $\phi \in \mathcal{W}_\mathcal{O}$ intersects some orbit $O \in \mathcal{O}(\phi)$, then we can choose the point $p_\phi(O)$ to be exactly the intersection point $\phi \cap O = \{p_\phi(O)\}$.

Fix a leaf domain $D \in \mathcal{D}_\mathcal{O}$ and let $\phi_L, \phi_R \in D$ be two leaves with $L(\phi_L) \subset L(\phi_R)$ that are sufficiently close and sufficiently to the left so that they satisfy

$$\overline{L(\phi_R) \cap R(\phi_L)} \cap \bigcup_{O\in\mathcal{O}(D)} O = \varnothing \quad \text{and} \quad \overline{L(\phi_L)} \cap D \cap \bigcup_{O\in\mathcal{O}_{\text{out}}(D)} O = \varnothing.$$

Now, consider a family of pairwise distinct points $\{q_{R,D}(O)\}_{O\in\mathcal{O}(D)}$ lying on the leaf ϕ_R and ordered according to the standard total order $\leq_{\phi_R}$ on the set $\mathcal{O}(D) = \mathcal{O}(\phi_R)$.

Observe that the order $\leq_{\phi_R}$ allows us to apply the right-analogous version of Theorem B to the leaf ϕ_R and to the set orbits $\mathcal{O}(D)$ indexed increasingly by $\leq_{\phi_R}$; thus, obtaining a family of pairwise disjoint half-lines $\{\Gamma^-_{D,O}\}_{O\in\mathcal{O}(D)}$, each positively transverse to $\mathcal{F}$, that satisfies:

- the endpoint $\Gamma^-_{D,O}(0)$ lies on the leaf ϕ_R for all $O \in \mathcal{O}(D)$;
- the backward orbit $\overline{R(\phi_R)} \cap O$ is contained in $\Gamma^-_{D,O}$ for all $O \in \mathcal{O}(D)$;
- the endpoints $\{\Gamma^-_{D,O}(0)\}_{O\in\mathcal{O}(D)}$ are ordered along the leaf ϕ_R according to $\leq_{\phi_R}$.

We remark that the proof of Theorem B gives us enough flexibility to choose the half-lines in the family $\{\Gamma^-_{D,O}\}_{O\in\mathcal{O}(D)}$ so that they satisfy the following additional properties:

- $\Gamma^-_{D,O}(0) = q_{R,D}(O)$ for any $O \in \mathcal{O}(D)$;
- $p_{\phi'}(O) \in \Gamma^-_{D,O}$ for any $O \in \mathcal{O}_{\text{in}}(D)$, where ϕ' denotes the unique leaf in $\partial_R D \cap C_O$.

At last, for each orbit $O \in \mathcal{O}_{\text{in}}(D)$, we cut $\Gamma^-_{D,O}$ at its intersection point with $\partial_R D$ so that it stays inside D and it joins $\partial_R D$ to ϕ_R. In other words, we redefine $\Gamma^-_{D,O} : [0, 1] \longrightarrow \mathbb{R}^2$ as the sub-arc of the originally defined half-line $\Gamma^-_{D,O}$ that ends at the point $q_{R,D}(O) \in \phi_R$ and starts at the point $p_{\phi'}(O) \in \phi'$, where ϕ' is the unique leaf in $\partial_R D \cap C_O$.

Now, consider a family of pairwise distinct points $\{q_{L,D}(O)\}_{O\in\mathcal{O}(D)}$ lying on ϕ_L and ordered increasingly along ϕ_L according to the adaptive total order $\leq_D$.

Observe that the order $\leq_D$ only allows us to apply Theorem B to the leaf ϕ_L and the subset $\mathcal{O}_\omega(D) \subset \mathcal{O}(D)$ indexed increasingly by $\leq_D$, as the order $\leq_D$ is not necessarily compatible with $\lesssim_L$ on orbits leaving D through the same leaf in $\partial_L D$. However, this is enough to obtain (see Figure 44) a family of pairwise disjoint half-lines $\{\Gamma^+_{D,O}\}_{O\in\mathcal{O}_\omega(D)}$ positively transverse to $\mathcal{F}$ such that:

- the endpoint $\Gamma^+_{D,O}(0)$ coincides with the point $q_{L,D}(O) \in \phi_L$ for all $O \in \mathcal{O}_\omega(D)$;
- the forward orbit $\overline{L(\phi_L)} \cap O$ is contained in $\Gamma^+_{D,O}$ for all $O \in \mathcal{O}_\omega(D)$.

Observe that for each orbit $O \in \mathcal{O}_{\text{out}}(D)$, we have a leaf $\phi' \in \partial_L D$ such that $O \in \mathcal{O}(\phi')$ and we can construct a path $\Gamma^+_{D,O} : [-1, 0] \longrightarrow \mathbb{R}^2$ positively transverse to $\mathcal{F}$ that

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

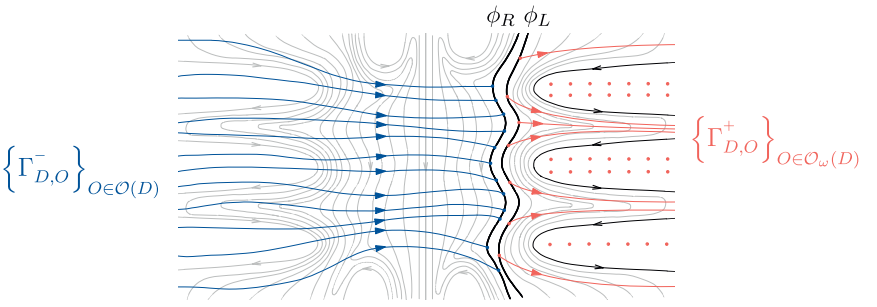


FIGURE 44. Families $\{\Gamma^-_{D,O}\}_{O\in\mathcal{O}(D)}$ and $\{\Gamma^+_{D,O}\}_{O\in\mathcal{O}_\omega(D)}$ constructed in the proof of Theorem C.

connects the point $q_{L,D}(O) \in \phi_L$ to the point $p_{\phi'}(O) \in \phi'$. By constructing these paths one at a time, we can construct them so that the family $\{\Gamma^+_{D,O}\}_{O\in\mathcal{O}_{\text{out}}(D)}$ is pairwise disjoint and, moreover, so that they are each disjoint from the already constructed $\{\Gamma^-_{D,O}\}_{O\in\mathcal{O}_\omega(D)}$.

To summarize, we have two families $\{\Gamma^-_{D,O}\}_{O\in\mathcal{O}(D)}$ and $\{\Gamma^+_{D,O}\}_{O\in\mathcal{O}(D)}$ formed by pairwise disjoint half-lines positively transverse to $\mathcal{F}$ such that, for each $O \in \mathcal{O}(D)$, the following properties hold:

$$\Gamma^-_{D,O}(0) = q_{R,D}(O) \in \phi_R \quad \text{and} \quad \overline{R(\phi_R) \cap D} \cap O \subset \Gamma^-_{D,O},$$
$$\Gamma^+_{D,O}(0) = q_{L,D}(O) \in \phi_L \quad \text{and} \quad \overline{L(\phi_L) \cap D} \cap O \subset \Gamma^+_{D,O}.$$

Now, using the Homma–Schoenflies theorem, we can assume up to conjugacy that:

- the leaves ϕ_R and ϕ_L are vertical, with $\phi_R = \{0\} \times \mathbb{R}$ and $\phi_L = \{1\} \times \mathbb{R}$;
- the square $[0, 1] \times [0, 1]$ is foliated vertically by $\mathcal{F}$;
- the endpoints $\{q_{R,D}(O)\}_{O\in\mathcal{O}(D)}$ lie on the side of the square $\{0\} \times [0, 1]$;
- the endpoints $\{q_{L,D}(O)\}_{O\in\mathcal{O}(D)}$ lie on the side of the square $\{1\} \times [0, 1]$.

For each $O \in \mathcal{O}(D)$, let $\gamma_{D,O} : [0, 1] \longrightarrow \mathbb{R}^2$ be the line segment joining $q_{R,D}(O)$ to $q_{L,D}(O)$. Observe that, from the definition of the standard order $\leq_{\phi_R}$ and the adaptive order $\leq_D$, the segments in the family $\{\gamma_{D,O}\}_{O\in\mathcal{O}(D)}$ satisfy the following properties:

- $\#(\gamma_{D,O} \cap \gamma_{D,O'}) = 1$ if and only if the orbits O and O' have a weak $\mathcal{F}$-transverse intersection, and there exists no $D' \in \mathcal{D}_{\mathcal{O}}$ such that $D \rightsquigarrow D'$ and $O, O' \in \mathcal{O}(D')$;
- $\#(\gamma_{D,O} \cap \gamma_{D,O'}) = 0$ otherwise.

For each orbit $O \in \mathcal{O}(D)$, we define the concatenated path $\Gamma_{D,O} := \Gamma^-_{D,O} \cup \gamma_{D,O} \cup \Gamma^+_{D,O}$.

To conclude, we repeat the construction for each maximal leaf domain $D \in \mathcal{D}_{\mathcal{O}}$. Then, we recall that for any orbit $O \in \mathcal{O}(D)$, we have a sequence of maximal leaf domains

$$D_0 \rightsquigarrow D_1 \rightsquigarrow \cdots \rightsquigarrow D_{m_O}$$

such that O belongs to the sets $\mathcal{O}_\alpha(D_0)$ and $\mathcal{O}_\omega(D_{m_O})$. Then, this allows us to construct a proper transverse trajectory of the orbit O as the concatenation of the paths

$$\Gamma_O := \prod_{i=0}^{m_O} \Gamma_{D_i,O}.$$

The resulting family of proper transverse trajectories $\{\Gamma_O\}_{O\in\mathcal{O}}$ proves the theorem. □

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press

Acknowledgements. This project has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No. 945332 .

References

[Bav17] J. Bavard. Conjugacy invariants for Brouwer mapping classes. *Ergod. Th. & Dynam. Sys.* **37**(6) (2017), 1765–1814.

[Bon24] C. Bonatti. Action on the circle at infinity of foliations of $\mathbb{R}^2$. *Enseign. Math.* **72** (2024), 85–146.

[Bro12] L. E. J. Brouwer. Beweis des ebenen Translationssatzes. *Math. Ann.* **72**(1) (1912), 37–54.

[Cai51] S. S. Cairns. An elementary proof of the Jordan–Schoenflies theorem. *Proc. Amer. Math. Soc.* **2**(6) (1951), 860–867.

[Gui94] L. Guillou. Théoréme de translation plane de Brouwer et généralisations du théoréme de Poincaré-Birkhoff. *Topology* **33**(2) (1994), 331–351.

[Han99] M. Handel. A fixed-point theorem for planar homeomorphisms. *Topology* **38**(2) (1999), 235–264.

[Hom53] T. Homma. An extension of the Jordan curve theorem. *Yokohama Math. J.* **1** (1953), 125–129.

[HR57] A. Haefliger and G. Reeb. Variétés (non séparées) à Une dimension et structures feuilletées du plan. *Enseign. Math.* **3**(1) (1957), 107.

[Jor87] C. Jordan. *Cours D'analyse de L'école Polytechnique*. Vol. 3. Gauthier-Villars, Paris, 1887.

[LC04] P. Le Calvez. A foliated version of the Brouwer translation theorem. *Comment. Math. Helv.* **79**(2) (2004), 229–259 (in French).

[LC05] P. Le Calvez. An equivariant foliated version of Brouwer's translation theorem. *Publ. Math. Inst. Hautes Étud. Sci.* **102** (2005), 1–98 (in French).

[LCT18] P. Le Calvez and F. A. Tal. Forcing theory for transverse trajectories of surface homeomorphisms. *Invent. Math.* **212**(2) (2018), 619–729.

[LCT22] P. Le Calvez and F. A. Tal. Topological horseshoes for surface homeomorphisms. *Duke Math. J.* **171**(12) (2022), 2519–2626.

[LR06] F. Le Roux. Points fixes des homéomorphismes de surfaces, 2006. Available at https://www-fourier.ujf-grenoble.fr/sites/ifmaquette.ujf-grenoble.fr/files/LeRoux.pdf.

[LR17] F. Le Roux. An index for Brouwer homeomorphisms and homotopy Brouwer theory. *Ergod. Th. & Dynam. Sys.* **37**(2) (2017), 572–605.

[Sch25] N. Schuback. A foliated viewpoint on homotopy Brouwer theory. *Preprint*, 2025, arXiv:2512.07510.

https://doi.org/10.1017/etds.2026.10321 Published online by Cambridge University Press