THE REVIEW OF SYMBOLIC LOGIC
Volume 3, Number 3, September 2010

POSITIVE LOGIC WITH ADJOINT MODALITIES: PROOF THEORY, SEMANTICS, AND REASONING ABOUT INFORMATION

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Abstract. We consider a simple modal logic whose nonmodal part has conjunction and disjunction as connectives and whose modalities come in adjoint pairs, but are not in general closure operators. Despite absence of negation and implication, and of axioms corresponding to the characteristic axioms of (e.g.) **T**, **S4**, and **S5**, such logics are useful, as shown in previous work by Baltag, Coecke, and the first author, for encoding and reasoning about information and misinformation in multiagent systems. For the propositional-only fragment of such a dynamic epistemic logic, we present an algebraic semantics, using lattices with agent-indexed families of adjoint pairs of operators, and a cut-free sequent calculus. The calculus exploits operators on sequents, in the style of "nested" or "tree-sequent" calculi; cut-admissibility is shown by constructive syntactic methods. The applicability of the logic is illustrated by reasoning about the muddy children puzzle, for which the calculus is augmented with extra rules to express the facts of the muddy children scenario.

§1. Introduction. Modal logics include various modalities, represented as unary operators, used to formalize and reason about extra modes such as time, provability, belief, and knowledge, applicable in various areas (we have that of security protocols in mind). Like disjunction and conjunction, modalities often come in pairs, for example, $\Diamond$ and $\Box$: one preserves disjunctions and the other conjunctions. According to the intended application, further axioms such as monotonicity and idempotence can be imposed on the modalities.

As well as relational (or Kripke) models, one may consider as models for such logics various ordered structures, such as lattices with operators. The question then arises as to what is the simplest way of obtaining a pair of these operators. If the lattice is a Boolean algebra and thus has negation, any join-preserving operator (such as $\Diamond$) immediately provides us with a meet-preserving one (such as $\Box$) by De Morgan duality. In a Heyting algebra, the lack of De Morgan duality will cause one of these operators to preserve meets only in one direction. What if no negation is present, for example, in a distributive lattice? The categorical notion of adjunction (aka Galois connection) is useful here: any (arbitrary) join-preserving endomorphism on a lattice has a Galois right adjoint, which (by construction) preserves meets. For example, in the category `Sup` of sup-lattices with join-preserving maps, every such map is residuated, that is, has a right adjoint.

Received: May 20, 2009

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doi:10.1017/S1755020310000134

https://doi.org/10.1017/S1755020310000134 Published online by Cambridge University Press

In this paper, we consider a minimal modal logic where the underlying logic has only two binary nonmodal connectives—conjunction and disjunction—and where the modalities are adjoint but have no closure-type properties (such as idempotence). As algebraic semantics one may consider a bounded distributed lattice, the modalities thereof being residuated lattice endomorphisms. Examples are quantales and Heyting algebras when one argument of their residuated multiplication and conjunction (respectively) is fixed. One may also consider a relational semantics. In the proof of relational completeness, the absence of negation prohibits us from following standard canonical model constructions, as we can no more form maximally consistent sets. We overcome this by developing an equivalent Hilbert-style axiomatization for our logic and then using the general Sahlqvist results of Gehrke *et al.* (2005) based on completion of algebras with operators.

We provide a sequent calculus, which contains, in addition to axioms for the logical constants $\top$, $\bot$, only the operational left and right rules for each connective and operator. We prove admissibility of the structural rules of *Contraction*, *Weakening*, and *Cut* by constructive syntactic methods. In the absence of negation and of closure-type properties for the modalities, developing well-behaved sequent calculus rules for the modalities (in particular the left rule for the right adjoint $\Box$) was a challenging task; a calculus not obviously allowing cut-elimination was given in Baltag *et al.* (2007). Our sequents are a generalization of Gentzen's where the contexts (antecedents of sequents), as well as formulae, have a structure and can be nested. For application, we augment our calculus with a rule that allows us to encode assumptions of epistemic scenarios, and show that *Cut* is still admissible.

We interpret our adjoint modalities as *information* and *uncertainty* and use them to encode and prove epistemic properties of the puzzle of muddy children. Owing to the absence of negation, we can only express and prove positive versions of these epistemic properties. But, our proofs are simpler than the proofs of traditional modal logics, for example, those in Huth & Ryan (2000). In a nutshell, in just one proof step the adjunction is unfolded and the information modality is replaced by the *uncertainty* modality; in the next proof step, the assumptions of the scenario are imported into the logic via the assumption rule. At this stage the modalities are eliminated and the proof continues in a propositional setting. Since our information modality is not necessarily truthful, we are able to reason about more challenging versions of epistemic scenarios, for example, when agents are dishonest and their deceitful communications lead to false information. Properties of these more challenging versions have not been proved in traditional modal logic in computer science approaches, like that of Huth & Ryan (2000).

A cut-elimination theorem for intuitionistic linear logic with a modal operator is presented in Restall (2000, p. 122), as corollary of a rather general theorem requiring certain syntactic conditions to hold; it is easily extendible to a distributive lattice setting, and, as Restall remarks, structural rules can be varied to our "heart's content." Our treatment differs: *Weakening* and *Contraction* are built into our rules, rather than (optionally) included as primitive, and, in not using Restall's general theorem about *Cut*-admissibility, we are (we believe) better placed for a proposed extensive development where *Cut*-admissibility is to be shown for a more complex calculus covering two kinds of sequent, with not just propositions but also actions, as in Baltag *et al.* (2007) and Sadrzadeh (2006).

Our approach is similar to deep inference systems, for example, by Brünnler (2006) and Kashima (1994) for full modal logics. Of these two formalisms, the closest to ours is that for the tense logic of Kashima. Other than differences in logic (presence of negation and two-sided sequents), which lead to different modal rules (based on De Morgan duality), our proof theoretic techniques have (we believe) some advantages over those of Kashima: (1)

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we formalize deep substitution in the nested sequents and as a result do not need to develop two different versions of the calculus and prove soundness and completeness separately, and (2) our *Cut*-admissibility proofs are done explicitly via a syntactic construction rather than as a consequence of semantic completeness.

Some proof systems encode modalities by introducing semantic labels to encode accessibility and satisfaction relations; these are better placed to produce cut-free systems for adjoint modalities, for example, the comprehensive work of Negri (2005) and Simpson (1993) for classical and intuitionistic modal logics. The former in particular can be adapted to provide a cut-free labeled sequent calculus for our logic. However, that these systems are strongly based on the relational semantics of modal logic and mix it with the syntax of the logic does not fit well with the spirit of the algebraic motivation behind our own rather sparse logic.

Finally, Bonnette & Goré (1998) present a labeled sequent system for the minimal tense logic $\mathbf{K_t}$, which, with negation, implication and half of the tense operators removed and the remaining operators ('always in the future' and 'some time in the past') cloned, once for every agent, is similar to our logic. Their labeled sequent system is much more geared to efficient implementation (and correspondingly less geared to human use) than ours; moreover, it is not clear how the agent-indexed multitude of pairs of modal operators would be modeled in that particular labeled style or whether the lack, in our negation-free context, of a negation normal form is more than just a notational difficulty. Goré (1998) presents Belnap-style display calculi for a wide range of substructural logics, in some cases with adjoint pairs of modalities, and mentions Wansing (1994) for the specific case of display calculi for tense logics; despite the advantage of a uniform and general cut-admissibility theorem, as in Restall's work, these approaches seem much less suitable than our own for human use, because of the wide range of primitive structural rules and notational conventions required to display principal formulae.

On the application side, adjoint modalities have been originally used to reason about time in the context of tense logics, for example, in Prior (1968) and von Karger (1998). Their epistemic application is novel and was initiated in the dynamic epistemic algebra of Baltag *et al.* (2007) and Sadrzadeh (2006), comprising a quantale of "actions" with an "update" operation on a lattice of "propositions." The logic of the algebra is an abstraction from the Dynamic Epistemic Logic of Baltag & Moss (2004), developed for reasoning about information flow in multiagent systems. A sound and complete sequent calculus was developed in Baltag *et al.* (2007) and in Sadrzadeh (2006), but the eliminability of its cut rules (necessary to prove completeness) is problematic. This paper takes a first step to solve the problem, by solving it for the propositional part of the logic. We shall endeavor to extend the cut-free calculus of this paper with "actions" and "updates" in a sequel.

The present paper is the full version of a conference paper: Sadrzadeh & Dyckhoff (2009).

§2. Sequent calculus for positive logic with adjoint modalities.

2.1. Sequent calculus. We refer to our logic as APML for "adjoint positive modal logic," with the suffix $Tree$ when we consider a tree-style sequent calculus. The set M of *formulae* m of APML is generated over a set $\mathcal{A}$ of *agents* A and a set At of *atoms* p by the following grammar:

$$m ::= \bot \mid \top \mid p \mid m \wedge m \mid m \vee m \mid \Box_A m \mid \blacklozenge_A m$$

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Items I and *contexts* Γ are generated by the following syntax:

$$I ::= m \mid \Gamma^A \qquad \Gamma ::= I\ multiset$$

where Γ^A will be interpreted as $\blacklozenge_A(\bigwedge \Gamma)$, for $\bigwedge \Gamma$ the meet of the interpretations of elements in Γ.

Thus, *contexts* are finite multisets of items, whereas *items* are either formulae or agent-annotated contexts. The use of multisets rather than sets makes the role of the *Contraction* rule explicit, with the rules in a form close to the requirements of an implementation. The union of two multisets is indicated by a comma, as in Γ, Γ' or (treating an item I as a one-element multiset) as in Γ, I. *Sequents* are of the form $\Gamma \vdash m$.

If one of the items inside a context is replaced by a "hole" [], we have a *context-with-a-hole*. More precisely, we have the notions of *context-with-a-hole* Δ and *item-with-a-hole* J, defined using mutual recursion as follows:

$$\Delta ::= \Gamma, J \qquad J ::= [] \mid \Delta^A$$

and so a context-with-a-hole is a context (i.e., a multiset of items) together with an *item-with-a-hole*, that is, either a hole or an agent-annotated context-with-a-hole. To emphasize that a context-with-a-hole is not a context, we use Δ for the former and Γ for the latter; similarly for items-with-a-hole J and items I.

Given a context-with-a-hole Δ and a context Γ, the result $\Delta[\Gamma]$ of *applying* the first to the second, that is, replacing the hole [] in Δ by Γ, is a context, defined recursively (together with the application [yielding not an item but a context] of an item-with-a-hole to a context) as follows:

$$(\Gamma', J)[\Gamma] = \Gamma', J[\Gamma] \qquad ([])[\Gamma] = \Gamma \qquad (\Delta^A)[\Gamma] = \Delta[\Gamma]^A$$

Given contexts-with-a-hole Δ', Δ, and an item-with-a-hole J, the *combinations* $\Delta' \bullet \Delta$ and $J \bullet \Delta$ are defined as follows by mutual recursion on the structures of Δ' and J:

$$(\Gamma, J) \bullet \Delta = \Gamma, (J \bullet \Delta) \qquad ([]) \bullet \Delta = \Delta \qquad (\Delta''^A) \bullet \Delta = (\Delta'' \bullet \Delta)^A$$

LEMMA 2.1. *Given contexts-with-a-hole* Δ', Δ, *an item-with-a-hole* J, *and a context* Γ, *the following equations between contexts hold:*

$$(\Delta' \bullet \Delta)[\Gamma] = \Delta'[\Delta[\Gamma]] \qquad (J \bullet \Delta)[\Gamma] = J[\Delta[\Gamma]]$$

Proof. Routine. □

We have the following initial sequents (in which p is restricted to being an atom):

$$\frac{}{\Gamma, p \vdash p}\ Id \qquad \frac{}{\Delta[\bot] \vdash m}\ \bot L \qquad \frac{}{\Gamma \vdash \top}\ \top R$$

The rules for the lattice operations and the modal operators are:

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$$\frac{\Delta[m_1, m_2] \vdash m}{\Delta[m_1 \wedge m_2] \vdash m}\ \wedge L \qquad \frac{\Gamma \vdash m_1 \quad \Gamma \vdash m_2}{\Gamma \vdash m_1 \wedge m_2}\ \wedge R$$

$$\frac{\Delta[m_1] \vdash m \quad \Delta[m_2] \vdash m}{\Delta[m_1 \vee m_2] \vdash m}\ \vee L \qquad \frac{\Gamma \vdash m_1}{\Gamma \vdash m_1 \vee m_2}\ \vee R1 \qquad \frac{\Gamma \vdash m_2}{\Gamma \vdash m_1 \vee m_2}\ \vee R2$$

$$\frac{\Delta[m^A] \vdash m'}{\Delta[\blacklozenge_A(m)] \vdash m'}\ \blacklozenge_A L \qquad \frac{\Gamma \vdash m}{\Gamma', \Gamma^A \vdash \blacklozenge_A(m)}\ \blacklozenge_A R$$

$$\frac{\Delta[(\Box_A m, \Gamma)^A, m] \vdash m'}{\Delta[(\Box_A m, \Gamma)^A] \vdash m'}\ \Box_A L \qquad \frac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R$$

The two indicated occurrences of p in the *Id* rule are *principal*. Each right rule has its conclusion's succedent as its *principal formula*; in addition, the $\blacklozenge_A R$ rule has Γ^A as a *principal item* and Γ' (which is there to ensure admissibility of *Weakening*) as its *parameter*. Each left rule has a *principal item*; these are as usual, except that the $\Box_A L$ rule has the formula $\Box_A m$ *principal* as well as the principal item $(\Box_A m, \Gamma)^A$.

Note that the $\Box_A L$ rule duplicates the principal item in the conclusion into the premiss; in examples, we may omit this duplicated item for simplicity. This duplication allows a proof of the admissibility of *Contraction*, and thus of completeness. To see its necessity, note that the following sequent is (according to the algebraic semantics in Section 3.) valid:

$$\blacklozenge_A(\Box_A(m \vee n)) \quad \vdash \quad (m \wedge \blacklozenge_A(\Box_A(m \vee n))) \vee (n \wedge \blacklozenge_A(\Box_A(m \vee n)))$$

It is, however, not derivable unless the principal item of $\Box_A L$ is duplicated into the rule's premiss.

As a standard check on the rules, we show the following:

LEMMA 2.2. *For every formula m and every context Γ, the sequent $\Gamma, m \vdash m$ is derivable.*

Proof. By induction on the size of m. In case m is an atom, or $\bot$, or $\top$, the sequent $\Gamma, m \vdash m$ is already initial. For compound m, consider the cases. Meet and join are routine. Suppose m is $\blacklozenge_A(m')$; by inductive hypothesis, we can derive $m' \vdash m'$, and by $\blacklozenge_A R$ we can derive $\Gamma, m'^A \vdash \blacklozenge_A(m')$, whence $\Gamma, \blacklozenge_A(m') \vdash \blacklozenge_A(m')$ by $\blacklozenge_A L$. Now suppose m is $\Box_A m'$. By inductive hypothesis, we can derive $(\Gamma, \Box_A m')^A, m' \vdash m'$, and by $\Box_A L$ we get $(\Gamma, \Box_A m')^A \vdash m'$; from this we obtain $\Gamma, m \vdash m$ by $\Box_A R$. □

We refer to instances of this derived sequent also as Id. Since we use multisets (for contexts) rather than sets or lists, the rules of exchange and associativity are inexpressible. As an example of a derivation, we prove the above valid sequent (the second premiss of $\vee L$ is just like the first):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\Box_A(m \vee n))^A, m \vdash m}\ Id \quad \dfrac{\dfrac{}{\Box_A(m \vee n) \vdash \Box_A(m \vee n)}\ Id}{(\Box_A(m \vee n))^A, m \vdash \blacklozenge_A(\Box_A(m \vee n))}\ \blacklozenge_A R}{(\Box_A(m \vee n))^A, m \vdash m \wedge \blacklozenge_A(\Box_A(m \vee n))}\ \wedge R}{(\Box_A(m \vee n))^A, m \vdash (m \wedge \blacklozenge_A(\Box_A(m \vee n))) \vee (n \wedge \blacklozenge_A(\Box_A(m \vee n)))}\ \vee R \quad \dots}{(\Box_A(m \vee n))^A, m \vee n \vdash (m \wedge \blacklozenge_A(\Box_A(m \vee n))) \vee (n \wedge \blacklozenge_A(\Box_A(m \vee n)))}\ \vee L}{(\Box_A(m \vee n))^A \vdash (m \wedge \blacklozenge_A(\Box_A(m \vee n))) \vee (n \wedge \blacklozenge_A(\Box_A(m \vee n)))}\ \Box_A L}{\blacklozenge_A(\Box_A(m \vee n)) \vdash (m \wedge \blacklozenge_A(\Box_A(m \vee n))) \vee (n \wedge \blacklozenge_A(\Box_A(m \vee n)))}\ \blacklozenge_A L$$

https://doi.org/10.1017/S1755020310000134 Published online by Cambridge University Press

To allow induction on the sizes of items, we need a precise definition, with a similar definition for contexts. The *size of a formula* is just the (weighted) number of operator occurrences, counting each operator $\blacklozenge_A$ and $\Box_A$ as having Weight 2; the *size of an item* Γ^A is the size of Γ plus 1, and the *size of a context* is the sum of the sizes of its items. The *size of a sequent* $\Gamma \vdash m$ is just the sum of the sizes of Γ and m. Note that each premiss of a rule instance has lower size than the conclusion, except for the rule $\Box_A L$, whose presence leads to nontermination of a naive implementation of the calculus.

LEMMA 2.3. *The following* Weakening *rule is admissible:*

$$\frac{\Delta[\Gamma] \vdash m}{\Delta[\Gamma, \Gamma'] \vdash m}\ Wk$$

Proof. Induction on the depth of the derivation of the premiss and case analysis (on the rule used in the last step). Suppose the last step is by $\blacklozenge_A R$, with $m = \blacklozenge_A m'$, and with premiss $\Gamma^* \vdash m'$ and conclusion $\Gamma'', \Gamma^{*A} \vdash m$, so $\Delta[\Gamma] = \Gamma'', \Gamma^{*A}$. To obtain $\Delta[\Gamma, \Gamma']$ from this there are two possibilities. In the first case, Γ occurs inside Γ^{*A}, and we make a routine use of the inductive hypothesis and reapply $\blacklozenge_A R$ with the same parameter. In the second case, we just use the $\blacklozenge_A R$ rule with a different parameter. Other cases are straightforward. □

LEMMA 2.4. *The $\blacklozenge_A L$ and $\Box_A R$ rules are invertible, that is, the following are admissible:*

$$\frac{\Delta[\blacklozenge_A(m)] \vdash m'}{\Delta[m^A] \vdash m'}\ \blacklozenge_A Inv \qquad \frac{\Gamma \vdash \Box_A m}{\Gamma^A \vdash m}\ \Box_A Inv$$

Proof. Induction on the height of the derivation of the premiss. □

LEMMA 2.5. *The $\wedge L$, $\vee L$, and $\wedge R$ rules are invertible.*

Proof. Induction on the height of the derivation of the premiss. □

LEMMA 2.6. *The following* Item Contraction *rule is admissible*

$$\frac{\Delta[I, I] \vdash m}{\Delta[I] \vdash m}\ IContr$$

Proof. Strong induction on the size of the item I, with a subsidiary induction on the height of the derivation of the premiss, together with case analysis and the above inversion lemmas. Consider the cases of the last step; first, when I is nonprincipal, we permute the contraction up and (keeping I fixed) apply the subsidiary induction hypothesis; when the premiss is an initial sequent, so is the conclusion; when the step is by $\blacklozenge_A R$ with I principal (and thus of the form Γ'^A) the premiss of that step has antecedent Γ' from which the copy of I is absent, allowing reuse of the $\blacklozenge_A R$ rule to yield $\Delta[I] \vdash m$; and, when I is otherwise principal, the last step is one of the four one-premiss left rules. The $\Box_A L$ case is handled by the subsidiary inductive hypothesis (for the two cases, where I is an item $(\Box_A m', \Gamma)^A$ and where it is a formula $\Box_A m'$ inside such an item), and the other cases ($\wedge L$, $\vee L$, $\blacklozenge_A L$) are handled by the invertibility lemmas and the main inductive hypothesis. □

COROLLARY 2.7. *The following* Contraction *rule is admissible*

$$\frac{\Delta[\Gamma, \Gamma] \vdash m}{\Delta[\Gamma] \vdash m}\ Contr$$

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Proof. Induction on the size of the context Γ, by Lemma 2.6. □

LEMMA 2.8. *The rule* $\top L^-$ *is admissible:*

$$\frac{\Delta[\top] \vdash m}{\Delta[\Gamma] \vdash m}\ \top L^-$$

Proof. Induction on the depth of the derivation of the premiss and case analysis. □

THEOREM 2.9. *The* Cut *rule is admissible*

$$\frac{\Gamma \vdash m \quad \Delta'[m] \vdash m'}{\Delta'[\Gamma] \vdash m'}\ Cut$$

Proof. Strong induction on the rank of the cut, where the *rank* is given by the pair (size of cut formula m, sum of heights of derivations of premisses).

To clarify the different reductions used (and to show how all cases are covered), we present the different cases in tabular form: in the top row are the different cases for the last step of the first premiss of the cut and in the left column are the different cases for the last step of the derivation of the second premiss of the cut. The letters refer to the case in the treatment below. The attributes like "nonprincipal" refer to the status of the cut formula w.r.t. the rule.

	Id	$\bot L$	$\top R$	$\wedge L$	$\vee L$	$\blacklozenge_A L$	$\Box_A L$	$\wedge R$	$\vee R$	$\blacklozenge_A R$	$\Box_A R$
Id (Principal)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11) (a)
$\bot L$ (Nonprincipal)	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	(11) (b)
$\top R$											(11) (c)
$\wedge L$ (Nonprincipal)											(11) (d)
$\vee L$ (Nonprincipal)											(11) (e)
$\blacklozenge_B L$ (Nonprincipal)											(11) (f)
$\Box_B L$ (Nonprincipal)											(11) (g)
$\Box_A L$ (Principal)											(11) (h)
$\wedge R$											(11) (i)
$\vee R$											(11) (j)
$\blacklozenge_B R$											(11) (k)
$\Box_B R$											(11) (l)

1. The first premiss is an instance of Id.

$$\frac{\dfrac{}{\Gamma', p \vdash p}\ Id \qquad \Delta'[p] \vdash m'}{\Delta'[\Gamma', p] \vdash m'}\ Cut$$

is transformed to

$$\frac{\Delta'[p] \vdash m'}{\Delta'[\Gamma', p] \vdash m'}\ Wk$$

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2. The first premiss is an instance of $\bot L$.

$$\dfrac{\dfrac{}{\Delta[\bot] \vdash m}\ \bot L \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[\bot]] \vdash m'}\ Cut$$

is transformed, using Lemma 2.1. to identify $\Delta'[\Delta[\bot]]$ and $(\Delta' \bullet \Delta)[\bot]$, to

$$\dfrac{}{\Delta'[\Delta[\bot]] \vdash m'}\ \bot L$$

3. The first premiss is an instance of $\top R$.

$$\dfrac{\dfrac{}{\Gamma \vdash \top}\ \top R \qquad \Delta'[\top] \vdash m'}{\Delta'[\Gamma] \vdash m'}\ Cut$$

transforms to the following using Lemma 2.8.

$$\dfrac{\Delta'[\top] \vdash m'}{\Delta'[\Gamma] \vdash m'}\ \top L^{-}$$

4. The first premiss is an instance of $\wedge L$. Straightforward.
5. The first premiss is an instance of $\vee L$. Straightforward.
6. The first premiss is an instance of $\blacklozenge_A L$.

$$\dfrac{\dfrac{\Delta[m^A] \vdash m'}{\Delta[\blacklozenge_A(m)] \vdash m'}\ \blacklozenge_A L \qquad \Delta'[m'] \vdash m''}{\Delta'[\Delta[\blacklozenge_A(m)]] \vdash m''}\ Cut$$

transforms (using Lemma 2.1.) to

$$\dfrac{\Delta[m^A] \vdash m' \qquad \dfrac{\Delta'[m'] \vdash m'' \qquad}{\Delta'[\Delta[m^A]] \vdash m''}\ Cut}{\Delta'[\Delta[\blacklozenge_A(m)]] \vdash m''}\ \blacklozenge_A L$$

7. The first premiss is an instance of $\Box_A L$.

$$\dfrac{\dfrac{\Delta[(\Box_A m, \Gamma)^A, m] \vdash m'}{\Delta[(\Box_A m, \Gamma)^A] \vdash m'}\ \Box_A L \qquad \Delta'[m'] \vdash m''}{\Delta'[\Delta[(\Box_A m, \Gamma)^A]] \vdash m''}\ Cut$$

transforms to

$$\dfrac{\Delta[(\Box_A m, \Gamma)^A, m] \vdash m' \qquad \Delta'[m'] \vdash m''}{\dfrac{\Delta'[\Delta[(\Box_A m, \Gamma)^A, m]] \vdash m''}{\Delta'[\Delta[(\Box_A m, \Gamma)^A]] \vdash m''}\ \Box_A L}\ Cut$$

8. The first premiss is an instance of $\wedge R$.

$$\dfrac{\dfrac{\Gamma \vdash m_1 \qquad \Gamma \vdash m_2}{\Gamma \vdash m_1 \wedge m_2}\ \wedge R \qquad \Delta[m_1 \wedge m_2] \vdash m'}{\Delta[\Gamma] \vdash m'}\ Cut$$

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transforms to

$$\dfrac{\Gamma \vdash m_2 \qquad \dfrac{\Gamma \vdash m_1 \qquad \dfrac{\Delta[m_1 \wedge m_2] \vdash m'}{\Delta[m_1, m_2] \vdash m'}\, Inv \wedge L}{\Delta[\Gamma, m_2] \vdash m'}\, Cut}{\dfrac{\Delta[\Gamma, \Gamma] \vdash m'}{\Delta[\Gamma] \vdash m'}\, Contr}\, Cut$$

9. The first premiss is an instance of $\vee R$.

$$\dfrac{\dfrac{\Gamma \vdash m_i}{\Gamma \vdash m_1 \vee m_2}\, \vee R_i \qquad \Delta[m_1 \vee m_2] \vdash m'}{\Delta[\Gamma] \vdash m'}\, Cut$$

transforms to

$$\dfrac{\Gamma \vdash m_i \qquad \dfrac{\Delta[m_1 \vee m_2] \vdash m'}{\Delta[m_i] \vdash m'}\, Inv \vee L}{\Delta[\Gamma] \vdash m'}\, Cut$$

10. The first premiss is an instance of $\blacklozenge_A R$.

$$\dfrac{\dfrac{\Gamma \vdash m}{\Gamma', \Gamma^A \vdash \blacklozenge_A(m)}\, \blacklozenge_A R \qquad \Delta'[\blacklozenge_A(m)] \vdash m'}{\Delta'[\Gamma', \Gamma^A] \vdash m'}\, Cut$$

is transformed to

$$\dfrac{\Gamma \vdash m \qquad \dfrac{\Delta'[\blacklozenge_A(m)] \vdash m'}{\Delta'[m^A] \vdash m'}\, Inv \blacklozenge_A L}{\dfrac{\Delta'[\Gamma^A] \vdash m'}{\Delta'[\Gamma', \Gamma^A] \vdash m'}\, Wk}\, Cut$$

11. The first premiss is an instance of $\Box_A R$. This now depends on the form of the second premiss.

(a) Id

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\, \Box_A R \qquad \dfrac{}{\Delta[\Box_A m], p \vdash p}\, Id}{\Delta[\Gamma], p \vdash p}\, Cut$$

transforms to

$$\dfrac{}{\Delta[\Gamma], p \vdash p}\, Id$$

(b) $\perp L$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\, \Box_A R \qquad \dfrac{}{\Delta[\Box_A m][\perp] \vdash m'}\, \perp L}{\Delta[\Gamma][\perp] \vdash m'}\, Cut$$

transforms to

$$\dfrac{}{\Delta[\Gamma][\perp] \vdash m'}\, \perp L$$

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(c) $\top R$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{}{\Delta[\Box_A m] \vdash \top}\,\top R}{\Delta[\Gamma] \vdash \top}\,Cut$$

transforms to

$$\dfrac{}{\Delta[\Gamma] \vdash \top}\,\top R$$

(d) $\wedge L$, nonprincipal

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m][m_1, m_2] \vdash m'}{\Delta[\Box_A m][m_1 \wedge m_2] \vdash m'}\,\wedge L}{\Delta[\Gamma][m_1 \wedge m_2] \vdash m'}\,Cut$$

transforms to

$$\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][m_1, m_2] \vdash m'}{\dfrac{\Delta[\Gamma][m_1, m_2] \vdash m'}{\Delta[\Gamma][m_1 \wedge m_2] \vdash m'}\,\wedge L}\,Cut$$

(e) $\vee L$, nonprincipal

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m][m_1] \vdash m' \quad \Delta[\Box_A m][m_2] \vdash m'}{\Delta[\Box_A m][m_1 \vee m_2] \vdash m'}\,\vee L}{\Delta[\Gamma][m_1 \vee m_2] \vdash m'}\,Cut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][m_1] \vdash m'}{\Delta[\Gamma][m_1] \vdash m'}\,Cut \qquad \dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][m_2] \vdash m'}{\Delta[\Gamma][m_2] \vdash m'}\,Cut}{\Delta[\Gamma][m_1 \vee m_2] \vdash m'}\,\vee L$$

(f) $\blacklozenge_B L$, nonprincipal

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m][[m'']^B] \vdash m'}{\Delta[\Box_A m][\blacklozenge_B(m'')] \vdash m'}\,\blacklozenge_B L}{\Delta[\Gamma][\blacklozenge_B(m'')] \vdash m'}\,Cut$$

transforms to

$$\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][[m'']^B] \vdash m'}{\dfrac{\Delta[\Gamma][[m'']^B] \vdash m'}{\Delta[\Gamma][\blacklozenge_B(m'')] \vdash m'}\,\blacklozenge_B L}\,Cut$$

(g) $\Box_B L$, nonprincipal

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m][(\Box_B m'', \Gamma')^B, m''] \vdash m'}{\Delta[\Box_A m][(\Box_B m'', \Gamma')^B] \vdash m'}\,\Box_B L}{\Delta[\Gamma][(\Box_B m'', \Gamma')^B] \vdash m'}\,Cut$$

transforms to

$$\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m][(\Box_B m'', \Gamma')^B, m''] \vdash m'}{\dfrac{\Delta[\Gamma][(\Box_B m'', \Gamma')^B, m''] \vdash m'}{\Delta[\Gamma][(\Box_B m'', \Gamma')^B] \vdash m'}\,\Box_B L}\,Cut$$

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(h) $\Box_A L$, principal

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta'[(\Box_A m, \Gamma')^A, m] \vdash m'}{\Delta'[(\Box_A m, \Gamma')^A] \vdash m'}\,\Box_A L}{\Delta'[(\Gamma, \Gamma')^A] \vdash m'}\,Cut$$

transforms to

$$\dfrac{\Gamma^A \vdash m \qquad \dfrac{\Gamma \vdash \Box_A m \qquad \Delta'[(\Box_A m, \Gamma')^A, m] \vdash m'}{\Delta'[(\Gamma, \Gamma')^A, m] \vdash m'}\,Cut}{\dfrac{\dfrac{\Delta'[(\Gamma, \Gamma')^A, \Gamma^A] \vdash m'}{\Delta'[(\Gamma, \Gamma')^A, (\Gamma, \Gamma')^A] \vdash m'}\,Wk}{\Delta'[(\Gamma, \Gamma')^A] \vdash m'}\,Contr}\,Cut$$

(i) $\wedge R$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m] \vdash m_1 \qquad \Delta[\Box_A m] \vdash m_2}{\Delta[\Box_A m] \vdash m_1 \wedge m_2}\,\wedge R}{\Delta[\Gamma] \vdash m_1 \wedge m_2}\,Cut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m] \vdash m_1}{\Delta[\Gamma] \vdash m_1}\,Cut \qquad \dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m] \vdash m_2}{\Delta[\Gamma] \vdash m_2}\,Cut}{\Delta[\Gamma] \vdash m_1 \wedge m_2}\,\wedge R$$

(j) $\vee R$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m] \vdash m_i}{\Delta[\Box_A m] \vdash m_1 \vee m_2}\,\vee R_i}{\Delta[\Gamma] \vdash m_1 \vee m_2}\,Cut$$

transforms to

$$\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m] \vdash m_i}{\dfrac{\Delta[\Gamma] \vdash m_i}{\Delta[\Gamma] \vdash m_1 \vee m_2}\,\vee R_i}\,Cut$$

(k) $\blacklozenge_B R$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m] \vdash m'}{\Gamma', \Delta[\Box_A m]^B \vdash \blacklozenge_B(m')}\,\blacklozenge_B R}{\Gamma', \Delta[\Gamma]^B \vdash \blacklozenge_B(m')}\,Cut$$

transforms to

$$\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m] \vdash m'}{\dfrac{\Delta[\Gamma] \vdash m'}{\Gamma', \Delta[\Gamma]^B \vdash \blacklozenge_B(m')}\,\blacklozenge_B R}\,Cut$$

(l) $\Box_B R$

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\,\Box_A R \qquad \dfrac{\Delta[\Box_A m]^B \vdash m'}{\Delta[\Box_A m] \vdash \Box_B m'}\,\Box_B R}{\Delta[\Gamma] \vdash \Box_B m'}\,Cut$$

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transforms to

$$\dfrac{\dfrac{\Gamma \vdash \Box_A m \qquad \Delta[\Box_A m]^B \vdash m'}{\Delta[\Gamma]^B \vdash m'}\ Cut}{\Delta[\Gamma] \vdash \Box_B m'}\ \Box_B m'$$

□

LEMMA 2.10. *The following rule (the name* K *is roughly from Moortgat, 1995) is admissible:*

$$\dfrac{\Delta[\Gamma^A, \Gamma'^A, (\Gamma, \Gamma')^A] \vdash m}{\Delta[(\Gamma, \Gamma')^A] \vdash m}\ K$$

Proof. Let $\gamma = \bigwedge \Gamma$ and $\gamma' = \bigwedge \Gamma'$. The proof uses Cut and is as follows, where a superfix $*$ indicates several instances of a rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots}{\Gamma, \Gamma' \vdash \gamma}\ \wedge R^* \quad \dfrac{\dots}{\Gamma, \Gamma' \vdash \gamma'}\ \wedge R^* \quad \dfrac{\Delta[\Gamma^A, \Gamma'^A, (\Gamma, \Gamma')^A] \vdash m}{\Delta[\gamma^A, \gamma'^A, (\Gamma, \Gamma')^A] \vdash m}\ \wedge L^*}{\Delta[(\Gamma, \Gamma')^A, (\Gamma, \Gamma')^A, (\Gamma, \Gamma')^A] \vdash m}\ Cut^*}{\Delta[(\Gamma, \Gamma')^A] \vdash m}\ Contr^*$$

□

§3. Semantics.

3.1. Algebraic semantics.

DEFINITION 3.1. *Let $\mathcal{A}$ be a set, with elements called* agents. *A* DLAM *over $\mathcal{A}$ is a bounded distributive lattice $(L, \top, \bot)$ with two $\mathcal{A}$-indexed families $\{\blacklozenge_A\}_{A \in \mathcal{A}} \colon L \to L$ and $\{\Box_A\}_{A \in \mathcal{A}} \colon L \to L$ of order-preserving maps, with each $\blacklozenge_A$ left adjoint to $\Box_A$. Thus, the following hold, for all $l, l' \in L$:*

$$l \leq l' \quad \textit{implies} \quad \blacklozenge_A(l) \leq \blacklozenge_A(l') \tag{1}$$

$$l \leq l' \quad \textit{implies} \quad \Box_A(l) \leq \Box_A(l') \tag{2}$$

$$\blacklozenge_A(l) \leq l' \quad \textit{iff} \quad l \leq \Box_A(l') \tag{3}$$

PROPOSITION 3.2. *In any* DLAM *the following hold, for all $l, l' \in L$:*

$$\blacklozenge_A(l \vee l') = \blacklozenge_A(l) \vee \blacklozenge_A(l') \tag{4}$$

$$\Box_A(l \wedge l') = \Box_A(l) \wedge \Box_A(l') \tag{5}$$

$$\blacklozenge_A(l \wedge l') \leq \blacklozenge_A(l) \wedge \blacklozenge_A(l') \tag{6}$$

$$\Box_A(l) \vee \Box_A(l') \leq \Box_A(l \vee l') \tag{7}$$

$$\blacklozenge_A(\bot) = \bot \qquad \Box_A(\top) = \top \tag{8}$$

$$\blacklozenge_A(\Box_A(l)) \leq l \tag{9}$$

$$l \leq \Box_A(\blacklozenge_A(l)) \tag{10}$$

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Proof. (4) follows from (1) and (3); similarly (5) follows from (2) and (3). (6) follows routinely from (1); similarly (7) follows from (1). (8) is routine, using (3), $\bot \leq \Box_A(\bot)$ and $\blacklozenge_A(\top) \leq \top$. (9) follows from (3) and $\Box_A(l) \leq \Box_A(l)$; (10) is similar. □

Let L be a DLAM over a set $\mathcal{A}$ of agents A. An *interpretation* of the set M of formulae (over the same set of agents and a given set At of atoms) *in* L is a map $[\![-]\!]\colon At \to L$. The meaning of formulae is obtained by induction on the structure of the formulae:

$$[\![m_1 \vee m_2]\!] = [\![m_1]\!] \vee [\![m_2]\!], \qquad [\![m_1 \wedge m_2]\!] = [\![m_1]\!] \wedge [\![m_2]\!],$$
$$[\![\blacklozenge_A(m)]\!] = \blacklozenge_A([\![m]\!]), \qquad [\![\Box_A m]\!] = \Box_A[\![m]\!],$$
$$[\![\top]\!] = \top, \qquad [\![\bot]\!] = \bot.$$

The meanings of items and of contexts are obtained by mutual induction on their structure:

$$\begin{aligned} [\![m]\!] &= \text{as above} \\ [\![\Gamma^A]\!] &= \blacklozenge_A([\![\Gamma]\!]) \\ [\![I_1, \ldots, I_n]\!] &= [\![I_1]\!] \wedge \cdots \wedge [\![I_n]\!] \\ [\![\emptyset]\!] &= \top \end{aligned}$$

Note that, since $\wedge$ is commutative and associative, the meaning of a context Γ is independent of its presentation as a list of items in a particular order.

A sequent $\Gamma \vdash m$ is *true* in an interpretation $[\![-]\!]$ in L iff $[\![\Gamma]\!] \leq [\![m]\!]$; it is *true* in L iff true in all interpretations in L, and it is *valid* iff true in every DLAM.

LEMMA 3.3. *Let* Γ, Γ' *be contexts with* $[\![\Gamma]\!] \leq [\![\Gamma']\!]$ *and* Δ *a context-with-a-hole. Then*

$$[\![\Delta[\Gamma]]\!] \leq [\![\Delta[\Gamma']]\!].$$

Proof. Routine induction on the structure of Δ (using also a similar result for items-with-a-hole). □

THEOREM 3.4 (SOUNDNESS). *Any derivable sequent is valid, that is, the derivability of* $\Gamma \vdash m$ *implies that* $[\![\Gamma]\!] \leq [\![m]\!]$ *is* true *in any interpretation* $[\![-]\!]$ *in any* DLAM.

Proof. We show that the initial sequents of the sequent calculus are valid and that the rules are truth preserving.

- Initial sequents: these are routine.
- The right rules:
  - $\wedge R$ and $\vee R$ are routine.
  - $\blacklozenge_A R$. We have to show

    $$[\![\Gamma]\!] \leq [\![m]\!] \quad \text{implies} \quad [\![\Gamma', \Gamma^A]\!] \leq [\![\blacklozenge_A(m)]\!]$$

    Assuming $[\![\Gamma]\!] \leq [\![m]\!]$, by definition of $[\![-]\!]$ we have to show $[\![\Gamma']\!] \wedge \blacklozenge_A([\![\Gamma]\!]) \leq \blacklozenge_A([\![m]\!])$, which follows by monotonicity of $\blacklozenge_A$ and definition of meet.
  - $\Box_A R$. We have to show

    $$[\![\Gamma^A]\!] \leq [\![m]\!] \quad \text{implies} \quad [\![\Gamma]\!] \leq [\![\Box_A m]\!]$$

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This follows directly from the definition of $[\![-]\!]$ and property (1) in the definition of a DLAM as follows

$$\blacklozenge_A([\![\Gamma]\!]) \leq [\![m]\!] \quad \text{iff} \quad [\![\Gamma]\!] \leq \Box_A[\![m]\!]$$

- The left rules: these are done by induction on the structure of Δ
  - $\wedge L$ and $\vee L$ are routine.
  - $\blacklozenge_A L$, we have to show

$$[\![\Delta[m^A]]\!] \leq [\![m']\!] \quad \text{implies} \quad [\![\Delta[\blacklozenge_A(m)]]\!] \leq [\![m']\!]$$

    which easily follows from the definition of $[\![\ ]\!]$.
  - $\Box_A L$, we have to show

$$[\![\Delta[(\Box_A m, \Gamma)^A, m]]\!] \leq [\![m']\!] \quad \text{implies} \quad [\![\Delta[(\Box_A m, \Gamma)^A]]\!] \leq [\![m']\!]$$

    for which it is enough to show

$$[\![\Delta[(\Box_A m, \Gamma)^A]]\!] \leq [\![\Delta[(\Box_A m, \Gamma)^A, m]]\!]$$

    By definition of contexts (and items) with holes this breaks down to three cases

    1. $[\![(\Box_A m, \Gamma)^A]\!] \leq [\![(\Box_A m, \Gamma)^A, m]\!]$ which by definition of $[\![\ ]\!]$ is equivalent to the following

$$\blacklozenge_A(\Box_A[\![m]\!] \wedge [\![\Gamma]\!]) \leq \blacklozenge_A(\Box_A[\![m]\!] \wedge [\![\Gamma]\!]) \wedge m$$

       and follows since by Proposition 3.2. and definitions of $\blacklozenge_A$ and $\wedge$ we have

$$\blacklozenge_A(\Box_A[\![m]\!] \wedge [\![\Gamma]\!]) \leq \blacklozenge_A(\Box_A[\![m]\!]) \wedge \blacklozenge_A([\![\Gamma]\!]) \leq \blacklozenge_A(\Box_A[\![m]\!]) \leq m$$

    2. $[\![\Gamma', J[(\Box_A m, \Gamma)^A]]\!] \leq [\![\Gamma', J[(\Box_A m, \Gamma)^A, m]]\!]$ follows from Case 1 by recursively unfolding the definition of an item-with-a-hole.
    3. $[\![\Delta'[(\Box_A m, \Gamma)^A]^B]\!] \leq [\![\Delta'[(\Box_A m, \Gamma)^A, m]^B]\!]$ follows from Case 1 by recursively unfolding the definitions of a context-with-a-hole and an item-with-a-hole.

□

THEOREM 3.5. Completeness. *Any valid sequent is derivable, that is, if* $[\![\Gamma]\!] \leq [\![m]\!]$ *for every* DLAM *and every interpretation* $[\![-]\!]$ *therein, then* $\Gamma \vdash m$ *is derivable.*

Proof. We follow the Lindenbaum–Tarski proof method of completeness (building the counter model, Blackburn et al., 2001) and show the following

1. The logical equivalence $\cong$ defined as $\vdash\!\dashv$ over the formulae in M is an equivalence relation, that is, it is reflexive, transitive (by the admissibility of *Cut*), and symmetric.
2. The order relation $\leq$ defined as $\vdash$ on the above equivalence classes is a partial order, that is, reflexive, transitive, and antisymmetric.
3. The operations $\wedge$, $\vee$, $\blacklozenge_A$, and $\Box_A$ on the above equivalence classes (defined in a routine fashion) are well defined. To avoid confusion with the brackets of the sequents, that is, $\Delta[\Gamma']$, we occasionally drop the brackets of the equivalence classes and, for example, write $\blacklozenge_A(m)$ for $[\blacklozenge_A(m)]$.

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(a) For $\blacklozenge_A[m] := [\blacklozenge_A(m)]$ we show

$$[m] \cong [m'] \implies [\blacklozenge_A(m)] \cong [\blacklozenge_A(m')]$$

The proof tree of one direction is as follows, the other direction is identically easy

$$\dfrac{\dfrac{m \vdash m'}{m^A \vdash \blacklozenge_A(m')}\ \blacklozenge_A R}{\blacklozenge_A(m) \vdash \blacklozenge_A(m')}\ \blacklozenge_A L$$

(b) For $\Box_A[m] := [\Box_A m]$ we show

$$[m] \cong [m'] \implies [\Box_A m] \cong [\Box_A m']$$

The proof tree of one direction is as follows, the other direction is identically easy

$$\dfrac{\dfrac{m \vdash m'}{(\Box_A m)^A \vdash m'}\ \Box_A L}{\Box_A m \vdash \Box_A m'}\ \Box_A R$$

(c) Similarly for $[m_1] \wedge [m_2] := [m_1 \wedge m_2]$ and $[m_1] \vee [m_2] := [m_1 \vee m_2]$.

4. The above operations satisfy properties of a DLAM, as in Definition 3.1., as follows:

(a) The proofs for properties of $\wedge$ and $\vee$ are routine.

(b) The proof trees for order preservation of $\blacklozenge_A$ and $\Box_A$ are as follows

$$\dfrac{\dfrac{\dfrac{}{m \vdash m'}\ Id}{m^A \vdash \blacklozenge_A m'}\ \blacklozenge_A R}{\blacklozenge_A m \vdash \blacklozenge_A m'}\ \blacklozenge_A L \qquad \dfrac{\dfrac{\dfrac{}{(\Box_A m)^A, m \vdash m}\ Id}{(\Box_A m)^A \vdash m'}\ \Box_A L}{\Box_A m \vdash \Box_A m'}\ \Box_A R$$

(c) The proof trees for the adjunction between $\blacklozenge_A$ and $\Box_A$ are as follows

$$\dfrac{\dfrac{m \vdash \Box_A m'}{m^A \vdash m'}\ \Box_A Inv}{\blacklozenge_A(m) \vdash m'}\ \blacklozenge_A L \qquad \dfrac{\dfrac{\blacklozenge_A(m) \vdash m'}{m^A \vdash m'}\ \blacklozenge_A Inv}{m \vdash \Box_A m'}\ \Box_A R$$

□

3.2. Examples of algebraic semantics. We point out some examples for the algebraic semantics of our calculus.

EXAMPLE 3.6. *The simplest example of a* DLAM *is a Heyting Algebra:*

PROPOSITION 3.7. *A Heyting algebra H is a* DLAM.

To see this let $\blacklozenge_A(-)$ be $h \wedge -$ for some $h \in H$, then, since $\wedge$ is residuated, the Galois right adjoint to $\blacklozenge_A$ exists and is obtained from the implication. For instance we can set

- $\blacklozenge_A(-) = \top \wedge -$ *and we obtain* $\blacklozenge_A = \Box_A = id$,
- $\blacklozenge_A(-) = \bot \wedge - = \bot$ *and we obtain* $\Box_A = \top$,
- $H = \mathcal{A}$ *and we obtain* $\blacklozenge_A(-) = A \wedge -$, *hence* $\Box_A - = A \supset -$ *where* $\supset$ *is the implication.*

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EXAMPLE 3.8. *One can argue that in a Heyting algebra meets are commutative and idempotent but our $\blacklozenge_A$s generally are not. So a closer match would be a residuated lattice monoid:*

PROPOSITION 3.9. *A residuated lattice monoid Q is a* DLAM.

Recall that a residuated lattice monoid Q is a lattice $(Q, \vee, \wedge, \top, \bot)$ with a monoid structure $(Q, \bullet, 1)$ such that the monoid multiplication preserves the joins and has a right adjoint in each argument, that is, $q \bullet - \dashv -/q$ and $- \bullet q \dashv q \setminus -$. Thus if we take $\blacklozenge_A(-)$ to be either $q \bullet -$ or $- \bullet q$ then it will have a right adjoint in each case. For instance, we can set

- *$\blacklozenge_A(-) = 1 \bullet -$ or $- \bullet 1$ and obtain $\blacklozenge_A = \Box_A = id$,*
- *or set $\blacklozenge_A(-) = \bot \bullet -$ or $- \bullet \bot$ and obtain a $\Box_A$ which is a binegation operator, that is, $\neg^l - = -/\bot$ and $\neg^r - = \bot \setminus -$ respectively for each argument.*
- *Alternatively, we can have $L = \mathcal{A}$ and thus obtain $\blacklozenge_A(-) = A \bullet -$, hence $\Box_A - = -/A$, and similarly for the other argument.*

3.3. Relational semantics. In this section, we develop a Hilbert-style calculus APML_{Hilb} for APML and show that this calculus provides an axiomatization equivalent to APML_{Tree}. We show that this logic is sound and complete with regard to *ordered Kripke Frames*, by applying the general Sahlqvist theorem for distributive modal logics, developed in Gehrke *et al.* (2005). The set of formulae M is the same as that of APML_{Tree}. Since the language does not include implication, the sequents are, following Dunn (2005), of the form $m \vdash m'$ for $m, m' \in M$. The axioms and rules are:

Axioms.

$$m \vdash m, \qquad \bot \vdash m, \qquad m \vdash \top$$

$$m \wedge (m' \vee m'') \vdash (m \wedge m') \vee (m \wedge m'')$$

$$m \vdash m \vee m', \qquad m' \vdash m \vee m', \qquad m \wedge m' \vdash m, \qquad m \wedge m' \vdash m'$$

$$\blacklozenge_A(m \vee m') \vdash \blacklozenge_A(m) \vee \blacklozenge_A(m'), \qquad \blacklozenge_A(\bot) \vdash \bot$$

$$\Box_A m \wedge \Box_A m' \vdash \Box_A(m \wedge m'), \qquad \top \vdash \Box_A \top$$

$$\blacklozenge_A(\Box_A m) \vdash m, \qquad m \vdash \Box_A \blacklozenge_A(m)$$

Rules.

$$\frac{m \vdash m' \quad m' \vdash m''}{m \vdash m''}\ cut$$

$$\frac{m \vdash m'' \quad m' \vdash m''}{m \vee m' \vdash m''}\ \vee \qquad \frac{m \vdash m' \quad m \vdash m''}{m \vdash m' \wedge m''}\ \wedge$$

$$\frac{m \vdash m'}{\blacklozenge_A(m) \vdash \blacklozenge_A(m')}\ \blacklozenge_A \qquad \frac{m \vdash m'}{\Box_A m \vdash \Box_A m'}\ \Box_A$$

PROPOSITION 3.10. *APML_{Hilb} is sound and complete with respect to DLAMs.*

Proof. Soundness is easy. Completeness follows from a routine Lindenbaum–Tarski construction. □

PROPOSITION 3.11. *A sequent of the form $m \vdash m'$ is derivable in APML_{Tree} if and only if it is derivable in APML_{Hilb}.*

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Proof. Follows from Proposition 3.10. □

A Hilbert-style modal logic is *Sahlqvist* whenever its modal axioms correspond to first-order conditions of a Kripke frame. According to Sahlqvist's Theorem, these modal logics are sound and complete with regard to their corresponding canonical Kripke models (Blackburn *et al.*, 2001).

PROPOSITION 3.12. APML_{Hilb} *is Sahlqvist.*

Proof. It suffices to show that the two axioms $m \vdash \Box_A \blacklozenge_A(m)$ and $\blacklozenge_A(\Box_A m) \vdash m$ are Sahlqvist. According to the method developed in Gehrke *et al.* (2005), the former sequent is Sahlqvist if and only if m is left Sahlqvist and $\Box_A \blacklozenge_A(m)$ is right Sahlqvist. The first is obvious; the negative generation tree of the latter is as follows

$$- \longrightarrow (\Box_A, -) \longrightarrow (\blacklozenge_A, -) \longrightarrow (m, -)$$

This is right Sahlqvist since the only choice node $\Box_A$ does not occur in the scope of the only universal node $\blacklozenge_A$. The proof of $\blacklozenge_A(\Box_A m) \vdash m$ being Sahlqvist is similar. □

For a Kripke semantics, we consider a simplification of that in Gehrke *et al.* (2005):

DEFINITION 3.13. *A* multimodal Kripke frame *for* APML_{Hilb} *is a tuple*

$$(W, \leq, \{R_A\}_{A \in \mathcal{A}}, \{R_A^{-1}\}_{A \in \mathcal{A}}),$$

where W is a set of worlds, each R_A is a binary relation on W and R_A^{-1} is its converse, and $\leq$ is a partial order on W satisfying

$$\leq \circ R_A^{-1} \subseteq R_A^{-1} \circ \leq \qquad \textit{and} \qquad \geq \circ R_A \subseteq R_A \circ \geq$$

A *Kripke structure* for APML_{Hilb} is a pair $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a multimodal Kripke frame for APML_{Hilb} and $V \subseteq W \times P$ is a binary relation, called a *valuation*. Given such a Kripke structure, the *satisfaction* relation $\models$ is defined on W and formulae of APML_{Hilb} in a routine fashion. The clauses for the modalities are as follows:

- $\mathbb{M}, w \models \blacklozenge_A(m)$ iff $\exists v \in W, \quad w R_A^{-1} v$ and $\mathbb{M}, v \models m$
- $\mathbb{M}, w \models \Box_A m$ iff $\forall v \in W, \quad w R_A v$ implies $\mathbb{M}, v \models m$

From the general Sahlqvist theorem of Gehrke *et al.* (2005) for distributive modal logics and our Propositions 3.12. and 3.11. it follows that

THEOREM 3.14. APML_{Tree} *is sound and complete with respect to Kripke structures for* APML_{Hilb}.

3.4. Representation theorem. We end this section by stating some definitions and results about a concrete construction for DLAMs and a representation theorem for *perfect* DLAMs. They follow from our previous results together with the general definitions and results of Gehrke *et al.* (2005) about representation theorems for distributive modal logics.

DEFINITION 3.15. *The* complex *or* dual *algebra of a multimodal Kripke frame for* APML_{Hilb} *is the set of subsets of W that are downward closed with respect to $\leq$.*

DEFINITION 3.16. *A distributive lattice is called* perfect *whenever it is complete, completely distributive, and join-generated by (i.e., each element of it is equal to the join of) the set of all of its completely join-irreducible elements.*

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LEMMA 3.17. *The complex algebra of a multimodal Kripke frame for* APML_{Hilb} *is closed under intersection, union, and the modal operators (for* $Z \subseteq W$*)*

$$\Box_A Z := \{w \mid \forall v \in W, wRv \implies v \in Z\}$$

and

$$\Diamond_A Z := \{w \mid \exists v \in W, wR^{-1}v, v \in Z\}.$$

PROPOSITION 3.18. *The complex algebra of a multimodal Kripke frame for* APML_{Hilb} *is a perfect DLAM.*

THEOREM 3.19. *Given a perfect DLAM* $\mathcal{L}$*, there is a frame whose complex algebra is isomorphic to* $\mathcal{L}$*.*

Proof. By the above Proposition 3.18., it suffices to construct a frame from $\mathcal{L}$ in a way that the complex algebra of the frame is isomorphic to $\mathcal{L}$. As shown in Lemma 2.26. and Proposition 2.25. of Gehrke *et al.* (2005), the *atom structure* of a perfect DLAM is a such a frame. □

§4. Epistemic applications. Following previous work of Baltag *et al.* (2007) and Sadrzadeh (2006, 2009), we interpret $\blacklozenge_A(m)$ as "agent A's uncertainty about m," that is, in effect, the conjunction of all the propositions that A considers as possible when in reality m holds. Accordingly, $\Box_A m$ will be interpreted as "agent A has information that m." We could use the terminology of *belief*, but wish to avoid this as too suggestive about mental states. Agents can cheat and lie, so "knowledge" is inappropriate.

The intended application of our calculus is to scenarios where extra information is available about the uncertainty of agents. This will always be of the form of one or more assumptions of the form $\blacklozenge_A(p) \supset m''$ where p is an atom and m'' is a disjunction of atoms, for example, $p_1 \vee p_2$. Such assumptions express ideas that would, in the relational semantics, be encoded in the accessibility relation, for example, that such and such a world can access certain other worlds. Such implications are not even formulae of our language; we can however add them as follows, by adding (for each such given assumption) the following evidently sound rule

$$\frac{\Delta[(\Gamma, p)^A, m''] \vdash m}{\Delta[(\Gamma, p)^A] \vdash m}\ Assn$$

It is routine to note that the proofs of admissibility of *Weakening* (2.3) and of *Contraction* (2.6) still work when these extra rules are considered; it is important, for example, that the principal item of $Assn$ be of the form $(\Gamma, p)^A$ rather than $\blacklozenge_A(p)$. The same applies to the invertibility lemmas. Let $\mathrm{APML}^{Assn}_{Tree}$ be the name of the extended calculus.

PROPOSITION 4.1. *The* Cut *rule is admissible in* $\mathrm{APML}^{Assn}_{Tree}$.

Proof. There are three extra cases:

(xii) The first premiss is an instance of $Assn$:

$$\frac{\dfrac{\Delta[(\Gamma, p)^A, m''] \vdash m}{\Delta[(\Gamma, p)^A] \vdash m}\ Assn \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[(\Gamma, p)^A]] \vdash m'}\ Cut$$

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is transformed to

$$\dfrac{\dfrac{\Delta[(\Gamma, p)^A, m''] \vdash m \qquad \Delta'[m] \vdash m'}{\Delta'[\Delta[(\Gamma, p)^A, m'']] \vdash m'}\ Cut}{\Delta'[\Delta[(\Gamma, p)^A]] \vdash m'}\ Assn\ .$$

(xi)(m) The first premiss is an instance of $\Box_A R$ and the second premiss is an instance of $Assn$, with the cut formula $\Box_A m$ nonprincipal, that is, not occurring as an element in the principal item of $Assn$.

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[\Box_A m][(\Gamma', p)^B, m''] \vdash m'}{\Delta[\Box_A m][(\Gamma', p)^B] \vdash m'}\ Assn}{\Delta[\Gamma][(\Gamma', p)^B] \vdash m'}\ Cut$$

is transformed to

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \Delta[\Box_A m][(\Gamma', p)^B, m''] \vdash m'}{\dfrac{\Delta[\Gamma][(\Gamma', p)^B, m''] \vdash m'}{\Delta[\Gamma][(\Gamma', p)^B] \vdash m'}\ Assn}\ Cut$$

(xi)(n) The first premiss is an instance of $\Box_A R$ and the second premiss is an instance of $Assn$, with the cut formula $\Box_A m$ principal, that is, occurring in the principal item of $Assn$.

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \dfrac{\Delta[(\Gamma', \Box_A m, p)^B, m''] \vdash m'}{\Delta[(\Gamma', \Box_A m, p)^B] \vdash m'}\ Assn}{\Delta[(\Gamma', \Gamma, p)^B] \vdash m'}\ Cut$$

is transformed to

$$\dfrac{\dfrac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R \qquad \Delta[(\Gamma', \Box_A m, p)^B, m''] \vdash m'}{\dfrac{\Delta[(\Gamma', \Gamma, p)^B, m''] \vdash m'}{\Delta[(\Gamma', \Gamma, p)^B] \vdash m'}\ Assn}\ Cut$$

□

As an example consider the muddy children puzzle. It goes as follows: n children are playing in the mud and k of them have muddy foreheads. Each child can see the other children's foreheads, but cannot see his own. Their father announces to them 'At least one of you has a muddy forehead' and then asks them "Do you know it is you who has a muddy forehead?". After $k-1$ rounds of 'no' answers by all the children, the muddy ones know that they are muddy. After they announce it in a round of 'yes' answers, the clean children know that they are not muddy.

To formalize this scenario, assume the children are enumerated and the first k ones are muddy. Consider the propositional atoms s_β for $\beta \subseteq \{1, \ldots, n\}$ where s_β stands for the proposition that exactly the children in β are muddy and $s_\emptyset$ stands for 'no child is muddy'. The formula $\blacklozenge_i(s_\beta)$ stands for the uncertainty of child i about each of these atoms before father's announcement. Since child i can only see the other children's foreheads and not his own, he is uncertain about himself being muddy or not. Let $\overline{k}$ = the set $\{1, \ldots, k\}$ (we write $s_{1,\ldots,k}$ rather than $s_{\{1,\ldots,k\}}$), then the assumption for the uncertainty of the muddy child i is

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$\blacklozenge_i(s_{\overline{k}}) \supset s_{\overline{k}} \vee s_{\overline{k}\setminus i}$, captured in the calculus by the following instance of the assumption rule

$$\frac{\Delta[(\Gamma, s_{\overline{k}})^i, s_{\overline{k}} \vee s_{\overline{k}\setminus i}] \vdash m}{\Delta[(\Gamma, s_{\overline{k}})^i] \vdash m}\ Assn$$

The assumption for the uncertainty of the clean child w is $\blacklozenge_w(s_{\overline{k}}) \supset s_{\overline{k}} \vee s_{\overline{k\cup w}}$ and its $Assn$ rule is similar. For $1 \leq i, j \leq k$, and $k+1 \leq w$ we have that, before the $k-1$'th announcement, a muddy child i is uncertain about having a muddy forehead: $s_{\overline{k}} \vdash \Box_i(s_{\overline{k}} \vee s_{\overline{k}\setminus i})$ (i.e., $\blacklozenge_i(s_{\overline{k}}) \vdash s_{\overline{k}} \vee s_{\overline{k}\setminus i}$). The proof tree of this property is as follows

$$\dfrac{\dfrac{\dfrac{}{(s_{\overline{k}})^i, s_{\overline{k}} \vee s_{\overline{k}\setminus i} \vdash s_{\overline{k}} \vee s_{\overline{k}\setminus i}}\ Id}{(s_{\overline{k}})^i \vdash s_{\overline{k}} \vee s_{\overline{k}\setminus i}}\ Assn}{s_{\overline{k}} \vdash \Box_i(s_{\overline{k}} \vee s_{\overline{k}\setminus i})}\ \Box_i R$$

The uncertainties of children change after each announcement as follows[1]: the k's announcements eliminates the s_γ disjunct from the uncertainty before the announcement when $\gamma \subseteq \{1, \ldots, n\}$ is of size k ; father's announcement eliminates the $s_\emptyset$ disjunct. For example, after the series of 1 to $k-1$'th announcements all the disjuncts except for $s_{\overline{k}}$ will be eliminated from muddy child i's uncertainty; hence his previous uncertainty assumption rule changes to

$$\frac{\Delta[(\Gamma, s_{\overline{k}})^i, s_{\overline{k}}] \vdash m}{\Delta[(\Gamma, s_{\overline{k}})^i] \vdash m}\ Assn$$

The assumption for the uncertainty of the clean child w changes in a similar way. We have that, after the $k-1$'th announcement, a muddy child i obtains information (a) that he is muddy and (b) that other muddy children also obtain information that he is muddy: $s_{\overline{k}} \vdash (\Box_i s_{\overline{k}}) \wedge (\Box_i \Box_j s_{\overline{k}})$. However, a clean child w will be uncertain about being muddy before and after the $k-1$'th announcement: $s_{\overline{k}} \vdash \Box_w(s_{\overline{k}} \vee s_{\overline{k\cup w}})$. The proof tree of the property for a muddy child i (where child j is also muddy) is as follows:

$$\dfrac{\dfrac{\dfrac{}{(s_{\overline{k}})^i, s_{\overline{k}} \vdash s_{\overline{k}}}\ Id}{(s_{\overline{k}})^i \vdash s_{\overline{k}}}\ Assn}{s_{\overline{k}} \vdash \Box_i s_{\overline{k}}}\ \Box_i R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{((s_{\overline{k}})^i, s_{\overline{k}})^j, s_{\overline{k}} \vdash s_{\overline{k}}}\ Id}{((s_{\overline{k}})^i, s_{\overline{k}})^j \vdash s_{\overline{k}}}\ Assn}{(s_{\overline{k}})^i, s_{\overline{k}} \vdash \Box_j s_{\overline{k}}}\ \Box_j R}{(s_{\overline{k}})^i \vdash \Box_j s_{\overline{k}}}\ Assn}{s_{\overline{k}} \vdash \Box_i \Box_j s_{\overline{k}}}\ \Box_i R$$

$$\dfrac{s_{\overline{k}} \vdash \Box_i s_{\overline{k}} \qquad s_{\overline{k}} \vdash \Box_i \Box_j s_{\overline{k}}}{s_{\overline{k}} \vdash (\Box_i s_{\overline{k}}) \wedge (\Box_i \Box_j s_{\overline{k}})}\ \wedge R$$

Consider a twist to the above scenario. Suppose that none of the children are muddy but that the father is a liar (or he cannot see properly) and the children do not suspect this (thus their uncertainties change in the same way as above). After father's false announcement,

[1] The way the uncertainties change after each announcement is formalized in the sequent calculus of previous work of Baltag *et al.* (2007) and Sadrzadeh (2006) via adding a dynamic logic for actions and extra rules for epistemic update; that calculus was not cut-free. Here, we change the assumptions of these uncertainties by hand and defer a full formalization to future work; for more details see next section.

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any child i will (by reasoning) obtain false information that he is the only muddy child: $s_\emptyset \vdash \Box_i s_i$. The proof tree is as follows:

$$\dfrac{\dfrac{\dfrac{}{(s_\emptyset)^i, s_i \vdash s_i}\, Id}{(s_\emptyset)^i \vdash s_i}\, Assn}{s_\emptyset \vdash \Box_i s_i}\, \Box_i R$$

§5. Conclusion and future work. We have developed a tree-style (aka nested) sequent calculus for a positive modal logic where the modalities are adjoints rather than De Morgan duals. We have shown that the structural rules of *Weakening*, *Contraction*, and *Cut* are admissible in our calculus. We have also shown that our calculus is sound and complete with regard to bounded distributive lattices with agent-indexed adjoint pairs of operators. Examples of these are complete Heyting algebras and residuated lattice monoids. Using general results of Gehrke *et al.* (2005), we have shown that our calculus is sound and complete with respect to ordered Kripke frames, by developing a Hilbert-style calculus with the same deductive power. We have motivated the applicability of our modal logic by encoding in it partial assumptions of real-life scenarios and proving epistemic properties of agents in the milestone puzzle of muddy children, but also with a newer misinformation version of it where father's announcement is not necessarily truthful. Since our box modality is not necessarily truthful, we can as well reason about the settings where agents obtain false information as a result of dishonest announcements. Our proof method of unfolding the adjunction and then renaming the left adjoint to its assumed values has made our proofs considerably simpler than the usual proof method of epistemic logics for the muddy children puzzle which uses the fixed point operator of the box. See Kriener *et al.* (2009) for an implementation and the (routine) decidability proof.

Our logic may be seen as a positive version of $\mathbf{K_t}$, that is, tense logic. Thus one can deduce that a proof theory thereof can be obtained by restricting any proof system for tense logic to rules for conjunction and disjunction, the existential past and the universal future that only satisfy the K axiom. We are unaware of a tree-style proof theory for this kind of modal logic in the literature, noting that the presence of $T, 4, 5$ axioms make the proof theory far easier than their absence in the logic. Thus we believe that our tree-style deep inference proof theory and its automated decision procedure are novel and so is its application to epistemic scenarios.

Future directions of our work include:

- A tree-style cut-free sequent calculus that is sound and complete with regard to residuated monoids with adjoint modalities has been developed in Moortgat (1995); it may be extended to quantales. We believe that pairing this extension with what we have in this paper, that is, adding to it the rules for the action of the quantale on its right module such that it remains cut-free, will provide a cut-free sequent calculus for a distributive version of the *Epistemic Systems* of Baltag *et al.* (2007) and thus a negation-free version of the Dynamic Epistemic Logic of Baltag & Moss (2004). This calculus will be an improvement on the algebraic decision procedure of Richards & Sadrzadeh (2009), which only implements a subalgebra of the algebra of Epistemic Systems (namely one that allows $\blacklozenge_A$ only on the right- and $\Box_A$ only on the left-hand side of the sequent).
- A representation theorem for *perfect* DLAMs follows from general results of Gehrke *et al.* (2005). But DLAMs need not be complete and completion involves

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introduction of, in principle, infinitary lattice operations. In Celani & Jansana (1999) similar results are obtained for positive modal logics where $\Box$ and $\Diamond$ come from the same relation; it might be possible to alter their duality theorem and make it suitable for our adjoint modal logic. However, those results, like those of Gehrke *et al.* (2005), are with respect to the less intuitive ordered frames. We are more inclined toward work with the usual nonordered frames, along the lines of Dunn (2005), that is, by using theory and counter theory pairs to build our canonical frames.

- As shown in Propositions 3.7. and 3.9., Heyting algebras and residuated lattice monoids are examples of DLAMs. So in principle our nested tree sequents might be adapted to provide a new sound and complete cut-free proof system for the logics based on these algebras, that is, for intuitionistic and linear logics where the conjunction and tensor (respectively) are treated as adjoint operators. In the former case, we will need extra rules to take care of the commutativity of conjunction, but in the latter case we hope to obtain a new cut-free proof theory for noncommutative intuitionistic multiplicative linear logic. It is also worth investigating how logics with classical negation and thus De Morgan dual connectives can be formulated in this context.

§6. Acknowledgments. Support by Engineering and Physical Sciences Research Council (Grant EP/F042728/1) is gratefully acknowledged, as are helpful comments by Kai Brünnler, Rajeev Goré, Greg Restall, Alex Simpson, and an anonymous referee (who pointed out connections with existing work).

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