CLASSICAL ELECTRODYNAMICS AS A DISTRIBUTION THEORY. II

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ABSTRACT. In a previous paper by the author (3) it was shown how the theory of distributions of L. Schwartz enables a mathematically consistent formalism to be given for a system composed of point charges interacting through their classical electromagnetic field. In the present paper a definition of the energy and momentum of the field lying in a space-like surface is given, and it is shown that from this four-vector it is possible to derive the usual equation of conservation of total energy and momentum.

1. *Introduction.* In a previous paper by the author (3) (referred to hereafter as I), it was shown how the theory of distributions of Schwartz (2) may be used to obtain a mathematically consistent formalism for a system composed of point charges interacting through their classical electromagnetic fields. This formalism may be used to exhibit the precise limitations on physical concepts defined in the formalism. In particular, the limitations on the concepts of field and field energy were discussed in I. Whereas the concept of field and the derivation of the self-field by means of the principle of conservation of energy and momentum were considered in detail in I, the concept of the field energy associated with any space-like surface was only very briefly considered in § 7 of I. It was conjectured in I that the field energy momentum $P_\mu(\sigma)$ (in the notation of I) residing in the space-like surface σ and associated with a single point charge, can be given meaning if and only if the world line L of the point charge cuts σ orthogonally. This would lead to the result that the principle of conservation of total energy and momentum of the system is not directly meaningful, since in particular the total energy or momentum of the system would not be defined at any time as a number, but only as a distribution $P_\mu(t)$ with support contained in the time axis. The general conservation principle would become that $P_\mu(t)$ is a time-independent distribution.

It is the purpose of this paper to show that the above unphysical situation does not occur.† In the particular case that σ is the surface $\sigma_t: x_0 = t$, a relativistically invariant *partie-finie* definition is given for $P_\mu(t) = Pf \int_{\sigma_t} T_{\mu 0}(x, t)\, d^3x$. It is then shown, with the aid of the theorem of bounded convergence, that $P_\mu(t)$ is a differentiable function of t, and satisfies the equation of conservation of total energy and momentum, equation (5). This result is immediately extended to the case of a family (σ_λ) of space-like surfaces filling space-time, and sufficiently differentiable. The corresponding equation of conservation of total energy and momentum is now equation (14). A further extension to a system composed of more than one point charge is again immediate.

Also in this paper is given an extension of the definition of the tensor transformation

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of a function under the Lorentz group to the tensor transformation of a distribution under this group. This definition reduces to that for functions when the tensor distribution is a tensor function, and enables the usual covariance properties of physical quantities in classical electrodynamics to be extended to the correct distribution form of this theory.

2. Partie-finie *definitions*. The notation used throughout this paper will be the same as that used in I. We take $\mathscr{D}$, $\mathscr{D}_t$ as the vector spaces of indefinitely differentiable functions with compact supports in the Euclidean spaces respectively of four dimensions with points $\mathbf{x} = (t, x)$ and of one dimension with points t.

The stress tensor distribution $\mathfrak{T}_{\mu\nu}$ defined in equation (27) of I is equal to the conventional stress tensor $T_{\mu\nu}$ of classical electrodynamics outside the world lines L_i of the particles and is the distribution obtained by extending $T_{\mu\nu}$ over the L_i by means of a Lorentz invariant *partie-finie* definition. Since $T_{\mu\nu}$ is not integrable over the L_i then $\mathfrak{T}_{\mu\nu}$ is a distribution which cannot be reduced to a function of space-time everywhere, and so the concept of localization of field energy cannot be considered as completely correct. This means that the field energy-momentum $P_\mu(\sigma)$ residing in the space-like surface σ cannot be defined in the usual way as the functional of σ, $P_\mu(\sigma) = \int_\sigma \mathfrak{T}_{\mu\nu} d\sigma^\nu$, from the stress tensor $\mathfrak{T}_{\mu\nu}$. To see if it is possible to define the field energy-momentum residing in σ we will take for simplicity one charged particle moving along L. The point of intersection of L with σ will be denoted by P_σ.

Outside P_σ in σ the distribution $\mathfrak{T}_{\mu\nu}$ reduces to the function $T_{\mu\nu}$ of equation (26) of I, which is not summable over the point P_σ in σ. What we wish to determine here is if it is possible to extend $T_{\mu\nu}$ over P_σ by a suitable *partie-finie* definition. In general, this definition would be of the following form. A two-dimensional closed and sufficiently smooth surface $S(\epsilon)$, depending on the positive number ϵ, will be drawn in σ to surround P_σ and to contract to P_σ as $\epsilon \to 0$. We denote by $OS(\epsilon)$ the set of points in σ on the opposite side of $S(\epsilon)$ to P_σ. The infinite part $I_\mu(\sigma, S(\epsilon))$ of the integral $\int_{OS(\epsilon)} T_{\mu\nu}(\mathbf{x})\,\phi(\mathbf{x})\,d\sigma^\nu$ may be obtained uniquely, for any $\phi \in \mathscr{D}$, as a linear combination of a finite number of terms proportional to $\epsilon^\lambda(\log\epsilon)^\mu$ with $\lambda \leqslant 0$, $\mu \geqslant 0$, the term $\lambda = \mu = 0$ being excluded, and the coefficients being distributions with supports P_σ. The finite limiting quantity

$$\lim_{\epsilon\to 0}\left[\int_{OS(\epsilon)} T_{\mu\nu}\phi\, d\sigma^\nu + I_\mu(\sigma, S(\epsilon))\,.\,\phi\right]$$

is then a distribution $\epsilon\mathscr{D}'$, by (2) (vol. I, Theorem XIII), and will be denoted by

$$Pf\int_\sigma T_{\mu\nu} d\sigma^\nu\,.\,\phi \quad \text{or} \quad P_\mu(\sigma)\,.\,\phi.$$

3. *Tensor distributions*. It is useful to extend the concept of a tensor function of space-time to that of a tensor distribution with its support in four-dimensional space-time. To achieve this extension let λ be a proper Lorentz transformation $\mathbf{x}' = \lambda\mathbf{x}$ or $x'_\mu = a^\nu_\mu x_\nu$. This induces a transformation λ^* from $\mathscr{D}$ to $\mathscr{D}$ defined by

$$\lambda^*\phi(\mathbf{x}') = \phi(\lambda\mathbf{x}),$$

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and further induces the transformation λ on $\mathscr{D}'$ to $\mathscr{D}'$ defined by

$$(\lambda T)_{\mathbf{x}'} . \phi(\mathbf{x}') = T_{\mathbf{x}} . \phi(\lambda \mathbf{x}), \tag{1}$$

or

$$\lambda T . \phi = T . \lambda^* \phi.$$

Equation (1) can be given meaning in general only if the support of T cuts $\lambda^{-1}K$ in a compact set, for any compact K. Then we may define an indexed family of distributions $T^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_n}$ with $0 \leqslant \nu_i, \mu_j \leqslant 3$ to be a tensor with contravariant rank n and covariant m, if in the dashed frame of reference the distributions evaluated as $T'^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_n}$ are related to those in the undashed frame by

$$T'^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_n} = a^{\sigma_1}_{\mu_1} \dots a^{\sigma_n}_{\mu_n} a^{\nu_1}_{\tau_1} \dots a^{\nu_m}_{\tau_m} (\lambda T^{\tau_1 \dots \tau_m}_{\sigma_1 \dots \sigma_n}). \tag{2}$$

In the particular case $m = n = 0$ our definition reduces to that of the invariant distributions considered by Methe ((1), p. 228). If T is a function, then (2) reduces to the usual transformation for a tensor function of space-time, as can be readily verified.

In the particular case of the distribution j_μ of equation (5) of I,

$$\begin{aligned}(j'_\mu)_{\mathbf{x}'} . \phi(\mathbf{x}') &= \sum_{i=1}^{n} e_i \int_{L_i} \phi(z'_\mu(s))\, dz'_\mu/ds \, . \, ds = a^\nu_\mu \sum_{i=1}^{n} e_i \int_{L_i} \phi(\lambda \mathbf{z}(s))\, dz_\nu/ds \, . \, ds \\ &= a^\nu_\mu j_\nu . \lambda^* \phi = a^\nu_\mu (\lambda j_\nu)_{\mathbf{x}'} . \phi(\mathbf{x}').\end{aligned}$$

Hence the four distributions j_μ are the contravariant components of a vector-distribution, so that by equation (1) of I the potentials A_μ are also.

The Lorentz invariance of equation (2) of I and the contravariant rank 2 of $F_{\mu\nu}$ are now evident. The invariant definition of $\mathfrak{T}_{\mu\nu}$ ensures the contravariant rank 2 of this distribution. Thus the definition of equation (2) is seen to give exactly the same Lorentz transformation properties to the quantities and equations occurring in the distribution form of classical electrodynamics as in the function form of this theory.

In the definition of $Pf \int_\sigma T_{\mu\nu} d\sigma^\nu$ given in the previous section, there is a wide freedom of choice for the surface $S(\epsilon)$. It may be seen from the remarks in (2) (vol. I, p. 45), that the value of $Pf \int_\sigma T_{\mu\nu} d\sigma^\nu$ depends on the surface $S(\epsilon)$ chosen. To remove any possible ambiguity it would be satisfactory if the requirement that $Pf \int_\sigma T_{\mu\nu} d\sigma^\nu$ transform as a vector distribution under the Lorentz group restricts this choice of $S(\epsilon)$. This is not the case. For

$$\int_{OS(\epsilon)} T'_{\mu\nu}(\mathbf{x}')\, \phi(\mathbf{x}')\, d\sigma'^\nu = a^\tau_\mu \int_{OS(\epsilon)} T_{\tau\nu}(\mathbf{x})\, \phi(\lambda \mathbf{x})\, d\sigma^\nu \tag{3}$$

for any $\epsilon > 0$ and any sufficiently smooth $S(\epsilon)$, so that $I'_\mu(\sigma, S(\epsilon)) = a^\tau_\mu I_\tau(\sigma, S(\epsilon))$ for any ϵ, and

$$Pf \int_\sigma T'_{\mu\nu} d\sigma'^\nu . \phi(\mathbf{x}') = a^\tau_\mu Pf \int_\sigma T_{\tau\nu} d\sigma^\nu . \phi(\lambda \mathbf{x}). \tag{4}$$

It is, however, necessary to restrict the choice of $S(\epsilon)$ for the following reason. The requirement of special relativity, that the laws of physics be independent of the particular coordinate system used to formulate them in, must be applied here to the

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Pf definition. Since the definition of the Pseudo-function $Pf\int_\sigma T_{\mu\nu}d\sigma^\nu$ must be regarded as one of the laws of physics, we thus require that this definition be independent of the choice of the coordinate frame which is used. Further it is only the concept of a space-like surface which is invariant, so that we require the definition of $Pf\int_\sigma T_{\mu\nu}d\sigma^\nu$ to be independent of the shape of σ, except for its space-like character. This definition can only be achieved if $S(\epsilon)$ is the intersection of σ with some three-dimensional surface $D(\epsilon)$, defined say, by the equation $f(x) = \epsilon$. Hence $f(x)$ must be invariant under any homogeneous proper Lorentz transformation and also under any linear translation in space-time. The only choice of $f(x)$ would seem to be an invariant function of $\gamma_\mu = x_\mu - z_\mu(s)$ (as defined in equation (17) of I) and thus a function f of $\gamma_\mu\gamma^\mu$. Assuming that $f(\gamma_\mu\gamma^\mu)$ is suitably well-behaved then $D(\epsilon)$ is required to be identical with the surface $T(\epsilon)$ defined by $\gamma_\mu\gamma^\mu = -\epsilon$. We have now removed any ambiguity in the possible choice of $S(\epsilon)$, the unique choice being $S(\epsilon) = T(\epsilon) \cap \sigma$.

A similar argument shows that the *Pf* definition of equation (27) of I is the only one which satisfies the condition of invariance under Lorentz transformation and any linear translation in space-time.

4. *Identification of field energy and momentum.* Let us now take σ as the hyperplane $\sigma_t : x_0 = t$. Since $T_{\mu 0}$ tends to zero rapidly enough we may form $P_\mu(t) = Pf\int T_{\mu 0}(x, t)\, d^3x$, where *Pf* will now refer to the invariant definition given above. Then we wish to identify $P_\mu(t)$ as the field energy-momentum residing in σ_t. To achieve this identification it is necessary to verify that the following conditions are satisfied:

(i) $P_\mu(t)$ is a differentiable function of t and satisfies the equation of conservation of energy and momentum

$$dP_\mu/dt = -F_\mu(s_t)\, ds_t/dt, \tag{5}$$

where $z_0(s_t) = t$ and it is assumed that the $z_\mu(s)$ are differentiable functions of s at least to the third order. We will assume from now on that $z_\mu(s)$ are differentiable in s to a high enough order for the continuity and differentiability properties of functions considered below to be immediate.

(ii) For any function $v(t) \in \mathscr{D}_t$,

$$\int P_\mu(t)\, v(t)\, dt = \mathfrak{T}_{\mu 0}.ev, \tag{6}$$

where $e(x) = 1$ everywhere.

If condition (ii) is satisfied then from equation (33) of I,

$$\int P_\mu(t)\,(dv/dt)\, dt = -\partial\mathfrak{T}_{\mu\nu}/\partial x_\nu . ev = \int F_\mu(s_t)\, v(t)\,(ds_t/dt)\, dt.$$

Then by (2) (vol. II, Theorem IX, p. 128), the solution of the distribution differential equation

$$dP_\mu/dt = -F_\mu(s_t)\,(ds_t/dt) \tag{5'}$$

can only be a differentiable function $P_\mu(t)$ which satisfies equation (5), so that condition (i) is satisfied. We thus wish to prove that condition (ii) is satisfied.

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To achieve this let us take coordinates ϵ, θ_1, θ_2 in σ_t, where $\epsilon = -\gamma_\mu \gamma^\mu$, and θ_1, θ_2 are polar coordinates in σ_t. For small ϵ and suitably well-behaved L the function

$$f(\epsilon, \theta_1, \theta_2, t) = \partial(x)/\partial(\epsilon, \theta_1, \theta_2)$$

may be expanded in powers of ϵ as

$$f(\epsilon, \theta_1, \theta_2, t) = \sum_b \epsilon^b f_b(\theta_1, \theta_2, t) + O(\epsilon^c), \tag{7}$$

where $f_b(\theta_1, \theta_2, t)$ are differentiable functions of θ_1, θ_2, t. The summation will involve a finite number of inverse powers of ϵ, and c is some positive number, which will be chosen later. A similar expansion in powers of ϵ may also be given for $T_{\mu 0}(x, t)$, so that we may write

$$T_{\mu 0}(x, t)\, f(\epsilon, \theta_1, \theta_2, t) = \sum_d \epsilon^d g_d(\theta_1, \theta_2, t) + G(\epsilon, \theta_1, \theta_2, t). \tag{8}$$

The functions g_d, G are differentiable in θ_1, θ_2, t, while c in equation (7) and the similar quantity in the expansion of $T_{\mu 0}(x, t)$ in powers of ϵ are chosen so that G is an integrable function of ϵ over the point $\epsilon = 0$. Evaluating the integral of $T_{\mu 0}$ in the region $R(n_1, n)$ cut off in σ_t between $T(1/n)$ and $T(1/n_1)$, where $n_1 \leqslant n$ and n_1 is large enough so that the expansions of equations (7) and (8) are valid, we have

$$\int_{R(n_1, n)} T_{\mu 0}(x, b)\, d^3x = \sum_{d \neq -1} (1+d)^{-1} (n_1^{-1-d} - n^{-1-d}) \int f_d(\theta_1 \theta_2 t)\, d\theta_1 d\theta_2$$
$$+ \int f_{-1}(\theta_1 \theta_2 t)\, d\theta_1 d\theta_2 (\log n_1 - \log n) + \int d\theta_1 d\theta_2 \int_{1/n}^{1/n_1} d\epsilon\, G(\epsilon, \theta_1, \theta_2, t).$$

Now

$$Pf \int_{\sigma_t} T_{\mu 0}(x, t)\, d^3x = \lim_n \left[\int_{OK(1/n_1)} T_{\mu 0}(x, t)\, d^3x + \int_{R(n_1, n)} T_{\mu 0}(x, t)\, d^3x + f_\mu(n, t) \right]$$
$$= \lim_n P_{\mu, n}(t) = P_\mu(t), \tag{9}$$

where $f_\mu(n, t)$ is chosen so that it exactly cancels the singularities arising from the second term in the bracket and $K(1/n)$ is the intersection of σ_t and $T(1/n)$. To achieve this we must take

$$f_\mu(n, t) = \sum_{d < -1} (1+d)^{-1} n^{-1-d} \int f_d(\theta_1 \theta_2 t)\, d\theta_1 d\theta_2 + \log n \int f_{-1}(\theta_1 \theta_2 t)\, d\theta_1 d\theta_2, \tag{10}$$

and so

$$P_{\mu, n}(t) = \sum_{d \neq -1} (1+d)^{-1} n_1^{-1-d} \int f_d(\theta_1 \theta_2 t)\, d\theta_1 d\theta_2 + \log n_1 \int f_{-1}(\theta_1 \theta_2 t)\, d\theta_1 d\theta_2$$
$$+ \int d\theta_1 d\theta_2 \int_{1/n}^{1/n_1} d\epsilon\, G(\epsilon, \theta_1 \theta_2 t) + \int_{OK(1/n_1)} T_{\mu 0}(x, t)\, d^3x$$
$$- \sum_{d \geqslant -\frac{1}{2}} (1+d)^{-1} n^{-1-d} \int f_d(\theta_1 \theta_2 t)\, d\theta_1 d\theta_2. \tag{11}$$

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We may further choose the number of terms in the last summation on the right-hand side of (11) so that $\left|\int d\theta_1 d\theta_2 G(\epsilon\theta_1\theta_2 t)\right|$ is integrable over $\epsilon = 0$. Hence we have that

$$|P_{\mu,n}(t)| \leqslant Q_\mu(t),$$

with

$$Q_\mu(t) = \sum_{d\neq -1} 2(1+d)^{-1} n_1^{-1-d} \left|\int f_d(\theta_1\theta_2 t)\, d\theta_1 d\theta_2\right| + \log n_1 \left|\int f_{-1}(\theta_1\theta_2 t)\, d\theta_1 d\theta_2\right| + \left|\int_{OK(1/n_1)} T_{\mu 0}(x,t)\, d^3x\right| + \int_0^{n_1^{-1}} \left|\int d\theta_1 d\theta_2 G(\epsilon\theta_1\theta_2 t)\right| d\epsilon. \quad (12)$$

From the above discussion it follows that $Q_\mu(t)$ is continuous in t, and so for any $v \in \mathscr{D}_t$ we have that $Q_\mu . |v|$ is integrable. Then by the bounded convergence theorem $P_\mu(t)\, v(t)$ is integrable, and

$$\lim_n \int P_{\mu,n}(t)\, v(t)\, dt = \int P_\mu(t)\, v(t)\, dt. \quad (13)$$

But the left-hand side of equation (13), with the aid of equation (27) of I, is equal to

$$\lim_n \left[\int dt \int_{OK(1/n)} T_{\mu 0}(x,t)\, v(t)\, d^3x + \int dt\, v(t) f_\mu(n,t)\right]$$
$$= \lim_n \left[\int_{OT(1/n)} T_\mu(xt)\, v(t)\, d^4x + \int dt\, v(t) f_\mu(n,t)\right]$$
$$= \mathfrak{T}_{\mu 0} . ev + \lim_n \left[\int dt\, v(t)\, f_\mu(n,t) - I(T(1/n), T_{\mu 0}) . ev\right].$$

This whole expression must be finite, and since the first term is, then so must the second term be. But this term involves a limit in n of a series containing positive powers of n (plus a term involving $\log n$), and this can only be finite provided all the coefficients in this series vanish. Hence we have the required result that

$$\int P_\mu(t)\, v(t)\, dt = \mathfrak{T}_{\mu 0} . ev.$$

The extension of the proof to a system of more than one point charge is immediate. To extend to a general family (σ_λ) of space-like surfaces filling the whole of space time, where $-\infty < \lambda < +\infty$, is also straightforward, provided that the σ_λ are sufficiently differentiable. For then from $Pf\int_{\sigma_\lambda} T_{\mu\nu} d\sigma^\nu$ we may define $P_\mu(\lambda) = Pf\int_{\sigma_\lambda} T_{\mu\nu} d\sigma^\nu . 1$ since $T_{\mu\nu}$ vanishes at infinity in space-like directions fast enough for $P_\mu(\lambda)$ to be defined. We take as new coordinates $\epsilon, \lambda, \theta_1, \theta_2$ (which are polar coordinates on σ_λ). In the case that the transformation $\mathbf{x} \to (\epsilon, \lambda, \theta_1, \theta_2)$ is one to one and sufficiently differentiable both ways, then nearly similar arguments to those used above (with λ replacing t and $\mathbf{x}$ replacing x) show that $P_\mu(\lambda)$ is a differentiable function of λ, and

$$dP_\mu/d\lambda = -F_\mu(s)\, ds/d\lambda. \quad (14)$$

Sufficient conditions have been imposed on the functions $z_\mu(s)$ and on the σ_λ to enable the above results, characterized by equations (13) and (14), to be obtained.

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The minimum conditions of differentiability, etc., which are required for this derivation are not known.

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