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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR A RANDOM LEDGER MODEL

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Abstract

Distributed ledgers – decentralized databases maintained by network consensus – are often modeled as directed acyclic graphs (DAGs) to capture the causal structure of data addition. Although blockchain systems like Bitcoin use linear chains, alternatives such as *tangle* in IOTA employ random DAGs. In such mechanisms each new transaction approves multiple predecessors selected through a randomized process. Prior work has established a fluid-limit approximation of the tangle's growth, governed by a delay differential equation. In this paper we go beyond the fluid limit by analyzing the next-order behavior. We show that the fluctuations around the deterministic limit converge to a Gaussian process and derive a stochastic delay differential equation (SDDE) that describes this next-order approximation.

Keywords: Functional central limit theorem; delay differential equations; stochastic processes; directed acyclic graph; distributed ledger; tangle; IOTA

2020 Mathematics Subject Classification: Primary 60F17
Secondary 60H20; 34K50; 68M10

1. Introduction

Digital ledger technology (DLT) encompasses an array of algorithms, databases, and protocols used in applications such as crypto-currencies, smart contracts, and data marketplaces. One notable mathematical aspect of DLT is the graphical structure of the databases. In the case of the blockchain [13], which is used as the foundation for Bitcoin and other cryptocurrencies, the graphical structure is a linear directed graph where a vertex represents a block, a container of data including a set of transactions, a timestamp, and a cryptographic hash linking it to its predecessor, while the edges represent the cryptographic dependences that enforce this sequential structure. The graph grows by iteratively adding new blocks, approximately one every ten minutes. To overcome the shortcomings of Bitcoin, many DLT models have been proposed employing different mechanisms. In this paper we will focus on a stochastic directed acyclic graph (DAG) model known as the tangle, which was devised as the basis for the IOTA cryptocurrency [19]. Unlike in blockchain, each transaction in the tangle is linked to several other transactions which are chosen using a randomized protocol. In addition, this protocol

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prevents directed cycles forming, so the resulting structure is a random DAG. As new transactions arrive, the model generates an increasing sequence of random DAGs, and the growth of this structure is described by a stochastic process. Due to its origins in IOTA, we will use the term 'tangle' to describe our DAG-based DLT. The basic model of the tangle which we use in this paper is described in detail in [12] and [17].

Here we provide a high-level view of the four stages through which a transaction passes before it is securely stored in the ledger. The tangle has a natural topological ordering, arising from the sequence of times at which new transactions arrive at the graph (we assume that these times are distinct). There is a crucial proof of work (POW, [21]) step occurring in the time between a new transaction's arrival and the time when a new vertex representing this transaction is inserted in the tangle. This POW results in the creation of a unique numerical ID for the new vertex. This ID encodes information about the data content of the new transaction, and also encodes the IDs of up to two 'older' vertices in the tangle whose inclusion serves as the mechanism for cryptographic security.

First stage. A newly arrived transaction (hereafter referred to as an arrival) undergoes POW, which lasts for a time duration h. We will provide details about this proof of work stage below, but for the moment it is enough to know that it involves the selection of two 'older' vertices in the graph, and the construction of the ID for the new vertex; this ID is the output of a cryptographic hash which entangles the IDs of the older vertices with the data of the newly arrived transaction.

Second stage. It begins when the POW has been completed. Upon completion of POW a new vertex is attached to the graph, whose ID is the output of the cryptographic hash as described above in the first stage. Two new directed edges are also added to the graph at the same time, directed from the new vertex to the two older vertices which were selected for its POW step in the first stage.

Third stage. It begins at some later (random) time when the vertex is first selected by a subsequent 'younger' arrival for its POW step.

Fourth stage. When the POW step in the third stage is complete, the transaction is considered to be fully verified.

Our analysis of the tangle considers vertices which are in their second and third stages. They are called *tips*, and they have the property that they do not lie at the 'head' of any directed edge in the DAG (we follow the convention that a directed edge points from its tail to its head), that is, they are attached only to outgoing edges. It follows that a vertex arrived at time t_n will be added to the set of tips at time t_{n+h}.

The vertices in the second stage are called *free tips* and ones in the third stage are called *pending tips*. Every tip starts out as a free tip, then evolves into a pending tip, and finally becomes a non-tip vertex which is attached to both incoming and outgoing edges. These mentioned types of vertices with corresponding transformations are illustrated in Figure 1. We now give two main assumptions of the model we use within this chapter.

Assumption 1. (*Constant inter-arrival times.*) Consistent with the analysis in [12], we assume here that the times between consecutive arrivals are constant and equal to λ^{-1}, where λ is the arrival rate.

Assumption 2. (*Constant delay times.*) Also following [12], throughout this chapter we will make the simplifying assumption that the duration h of POW (proof of work) is constant and is the same for all transactions, and that $h\,\lambda = m$ is integer.

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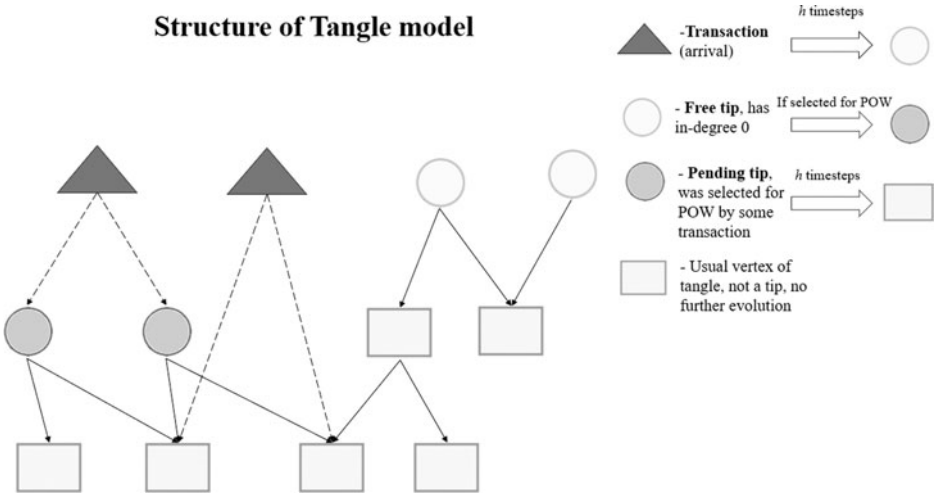


FIGURE 1. Four different types of nodes in a tangle: transaction (or arrival), free tip, pending tip, and usual vertex. At each time we have a new arrival (denoted by a triangle), which is not yet part of the DAG; it selects randomly two tips in the DAG for POW and after h time steps becomes a *free tip*. A free tip might be selected further in the process for POW; then it becomes a *pending tip* (circle). If a tip becomes pending, then after h time steps it ceases to be a tip (square). Outgoing edges from a vertex show its choice of vertices selected by POW.

The random DAG model just described, with the assumptions stated above, was originally proposed and analyzed informally in [17], and was studied rigorously by King in [12], where it was shown that the discrete tangle process is ergodic, and that after rescaling by λ^{-1} the processes $(A^{(\lambda)}, B^{(\lambda)})$ converge in probability, on every finite time interval, to the unique solution $(a(t), b(t))$ of a coupled first-order delay differential equation (DDE). In particular, [12, Theorem 2] establishes a quantitative tail bound of the form

$$\mathbb{P}\big[\sup_{0\le t\le T} |A^{(\lambda)}(t) - a(t)| > \varepsilon\big] \le C\,\lambda^{-1}\varepsilon^{-2}$$

(and the analogous bound for $B^{(\lambda)} - b$), which implies that the fluctuation of the rescaled processes around the fluid limit is of order $\lambda^{-1/2}$ in probability; Theorem 3 of [12] proves global exponential stability of the DDE. The present paper is the natural next step in that program: we identify the law of the $\lambda^{1/2}$-rescaled fluctuations around the fluid limit and show that they converge to a Gaussian stochastic process, characterized as the unique solution of an explicit stochastic delay differential equation (SDDE); the precise statement is Theorem 1.

The stochasticity in the model arises solely from the random selection of two existing tips at each new arrival, and the delay in both the fluid-limit DDE and the limiting SDDE comes from the proof of work duration h: a vertex chosen at time t_n becomes a non-tip only at time $t_n + h$, so the dynamics at time t_n depend on the state of the process at time $t_n - h$. This is the elementary source of the delayed terms appearing both in the fluid-limit equation of [12] and in the SDDE derived here.

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1.1. Relation to previous work

This paper extends and clarifies the limiting behavior of the model introduced and described in [12]. Due to the widespread use of DLT implementations, extensive research using DAG models has been conducted, particularly in the context of blockchain [10, 13]. However, the linear structure of Blockchain's underlying graph introduces limitation in adding new blocks, such as the increasing competition among miners to solve cryptographic puzzles, which in turn slows down the system's growth. More information about blockchain protocols and their comparisons may be found in [20] and [21]. For that reason, various protocols were developed and analyzed as an alternative for Bitcoin [15]; one such modification is the *tangle model*, first proposed in [17]. These DAG-based DLTs have been drawing increasing attention in recent years such as the analysis of the resilience of the tangle model under parasite chain attacks [4], or the stability of unverified transactions [8], but are still less studied compared to Blockchain.

In addition, the stochastic DAG model is normally described as a delay process due to an essential mechanism in DLTs, POW, which has been studied extensively in the context of queueing theory [1, 14] and through the use of stochastic differential equations [16]. These methods have been applied to various domains and are particularly relevant to the analysis of ledger technologies with inherent delays.

The novel contribution of this paper is a second-order, mathematically rigorous refinement of the fluid limit of [12]: we identify the limiting law of the rescaled fluctuations of the free-tip process and characterize it as the unique solution of an explicit linear SDDE driven by Brownian motion. We emphasize that, given the simplifying assumptions of the model (constant inter-arrival times, constant POW duration, uniform tip selection), this result should be read as a structural mathematical refinement of the fluid limit rather than as a quantitative performance guarantee for deployed ledger systems; relaxations of these assumptions are discussed briefly in Section 7. Our approach builds upon the functional central limit theorem framework discussed in [3].

1.2. Structure of the paper

In Section 2 we formalize the tangle model, introducing key definitions and assumptions, including the structure of the directed acyclic graph (DAG), the proof of work (POW) mechanism, and the stages of transaction processing. Section 3 outlines the main result of the paper, the limiting behavior of the fluctuation term. Here we construct the limiting functions and the stochastic delay differential equation (SDDE) that the fluctuation satisfies. Section 4 presents the formulation of the intermediate lemmas required to prove the main result, while Section 5 provides the proof of the main theorem, Theorem 1. Section 6 offers the proofs of the lemmas defined in Section 4. Finally, Section 7 summarizes the obtained results and explores potential extensions of the current model by relaxing some of the assumptions made throughout the paper.

2. Background and formulation of the model

2.1. Overview and definitions

The growth of the tangle is controlled by the arrival rate of new transactions, the delay in the proof of work, and also by the protocol that is used to select the two older vertices for the ID computation in the first stage. In this work we will assume that these two older vertices are chosen randomly and uniformly with replacement from the set of tips at the time of arrival.

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Notice that these tips may be either already *pending* or *free*, which is important for our further study.

We first provide a definition of the variables, whose update rules will be stated in (2) and (3):

$$\begin{aligned} t_n &= n\,\lambda^{-1} = \text{arrival time of the } n\text{th transaction},\\ \mathcal{O}_n &= \text{set of vertices selected for POW by arrival at time } t_n,\\ \mathcal{L}(t) &= \text{set of tips at time } t,\\ L(t) &= |\mathcal{L}(t)| = \text{number of tips at time } t,\\ X(t) &= \text{number of free tips at time } t,\\ W(t) &= \text{number of pending tips at time } t,\\ L(t) &= X(t) + W(t). \end{aligned}$$

Note that every POW starts at an arrival time t_n and by Assumption 1 is completed at time $t_n + h = t_{n+m}$, and therefore the set $\{t_n\}$ contains both the arrival times and the POW completion times. Therefore the processes $\{L(t), X(t), W(t)\}$ are piecewise constant, with jumps only at the times $\{t_n\}$. By convention we choose these processes to be right continuous, and we define

$$L_n = L(t_n), \quad X_n = X(t_n), \quad W_n = W(t_n).$$

Since updates occur only at the times $\{t_n\}$ and $\{t_n + h = t_{n+m}\}$, it follows that $\mathcal{L}(t)$ is constant and equal to $\mathcal{L}(t_{n-1})$ in the interval $[t_{n-1}, t_n)$.

The protocol for selection of vertices at the arrival time t_n is simply this: select randomly and uniformly with replacement twice from the set $\mathcal{L}(t_{n-1})$ to obtain the set $\mathcal{O}_n$. Note that $|\mathcal{O}_n| \in \{1, 2\}$ since the same vertex may be selected twice (though this is unlikely when the arrival rate is large). It will be useful to further distinguish the number of free tips that were selected, so we define

$$U_n = \text{number of tips in } \mathcal{O}_n \text{ that were free tips at time } t_{n-1}.$$

It follows that $U_n \in \{0, 1, 2\}$, and we let $\mathcal{F}(n)$ denote the σ-algebra generated by $\{U_1, \ldots, U_n\}$:

$$\mathcal{F}(n) = \sigma(U_1, \ldots, U_n). \tag{1}$$

We have defined U_n to be the number of vertices in the set $\mathcal{O}_n$ that were free at time t_{n-1}. After selection these free vertices become pending vertices at time t_n, hence they will never contribute to any of the subsequent values $U_{n+1}, U_{n+2}, \ldots$. Because the POW time is h, the total number of pending vertices at any arrival time t_n is the sum of $\{U_n, U_{n-1}, \ldots, U_{n-m+1}\}$. That is,

$$W_n = \sum_{j=n-m+1}^{n} U_j. \tag{2}$$

We also have the following update equations:

$$\begin{aligned} X_{n+1} &= X_n + 1 - U_{n+1},\\ L_{n+1} &= L_n + 1 - U_{n+1-m}, \end{aligned} \tag{3}$$

The term $+1$ appears on the right-hand side of (3) because this is the number of transactions that complete their POW at time t_{n+1}. Also the term $-U_{n+1}$ appears in X_{n+1} because this is

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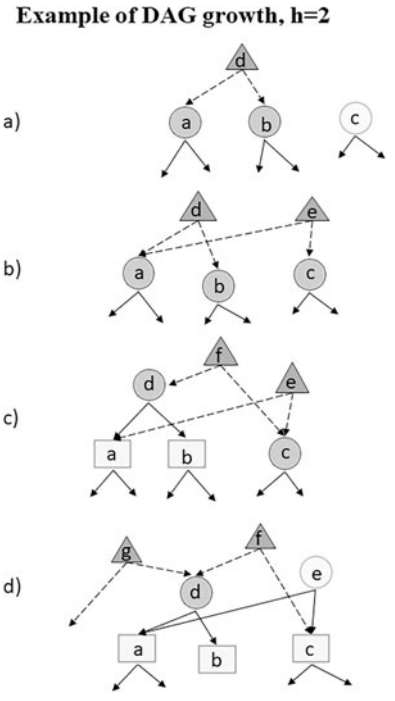


FIGURE 2. Graph representation of tip growth. A particular example of model updates at four consecutive times. Here we chose time for proof of work (POW) $h = 2$. (a) The tips are vertices a, b, and c. Vertices a and b are selected as the parents for the new arrival d, so they become pending tips at this time. (b) Tips: a, b, c. New arrival e selects a, c, so c becomes a pending tip, while a is already pending. (c) Tips: c, d. Vertex d has completed POW by this time, which also means that a, b cease to be the tips. New arrival f selects c, d. (d) Tips: d, e. By this time, e has completed POW so becomes a free tip now and d is still pending. New arrival g selects d and some other tip.

the number of free tips that are selected at the $(n+1)$th step, thereby becoming pending tips. The term $-U_{n+1-m}$ appears in L_{n+1} because this counts the number of vertices that cease to be tips between the nth and $(n+1)$th steps. The recursion (3) requires initial values X_0, L_0, W_0 and past selection counts $(U_{-m+1}, \ldots, U_0)$; these are specified in Section 2.3. An example of the tangle growth process is illustrated in Figure 2.

Since each tip is either free or pending, we have the equation

$$L_n = W_n + X_n.$$

Finally, all tips that were pending at time t_{n-m} will no longer be tips at time t_n, while all the free tips at time t_{n-m} will still be tips at time t_n. In addition, every tip that completed its POW after time $t_n - h$ will still be a tip at time t_n, so we have

$$L_n = X_{n-m} + m. \tag{4}$$

Notice that (4) implies that the sequence $\{L_n\}$ is a function of the sequence $\{X_n\}$. We will henceforth regard $\{X_n\}$ as the fundamental process and regard $\{L_n\}$ as a derived process. Also, $\{L_n, X_n, W_n\}$ are measurable with respect to $\mathcal{F}(n)$.

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2.2. The random tip growth model

For the random tip growth model the conditional distribution of the random variable U_{n+1} is

$$\begin{aligned}\mathbb{P}(U_{n+1}=2 \mid \mathcal{F}(n)) &= \frac{X_n(X_n-1)}{{L_n}^2},\\ \mathbb{P}(U_{n+1}=0 \mid \mathcal{F}(n)) &= \frac{{W_n}^2}{{L_n}^2},\\ \mathbb{P}(U_{n+1}=1 \mid \mathcal{F}(n)) &= \frac{2W_nX_n+X_n}{{L_n}^2}.\end{aligned} \tag{5}$$

Note that

$$\mathbb{E}[U_{n+1} \mid \mathcal{F}(n)] = 2\,\frac{X_n}{L_n} - \frac{X_n}{L_n^2}. \tag{6}$$

It is evident that $\{X_n\}$ is not a Markov process because the distribution of X_{n+1} depends on L_n, which itself depends on U_{n-m} (however, note that in the extended space $\mathbb{R}^m$, the vector $(X_n, X_{n-1}, \ldots, X_{n-m+1})$ forms a Markov chain).

2.3. Initial DAG and initial conditions

Because the dynamics depend on POW completions over the past h time units, the formulation of the processes $X(t)$ and $L(t)$ for $t \geq 0$ requires initial conditions to be provided over the interval $[-h, 0]$. We now describe how these initial conditions correspond to a concrete initial DAG.

For each $j = -m+1, \ldots, 0$ introduce $u_j \in \{0, 1, 2\}$, and let ξ_0 be a positive integer. Consider a DAG observed at time $t_{-m} = -h$, and suppose it contains

$$X_{-m} = \xi_0 + \sum_{j=-m+1}^{0} u_j$$

free tips and $W_{-m} = 0$ pending tips, so that $L_{-m} = X_{-m}$.

Over the interval $(-h, 0]$, no further POW completions occur; the only events are the m arrivals at times $t_{-m+1}, \ldots, t_0$. At each arrival t_j, $j = -m+1, \ldots, 0$, the prescribed u_j free tips are selected and become pending:

$$U_j = u_j, \quad X_j = X_{j-1} - u_j, \quad W_j = W_{j-1} + u_j, \quad L_j = L_{j-1}.$$

Summing these updates yields, at time $t_0 = 0$, the values

$$\begin{aligned}X_0 &= X_{-m} - \sum_{j=-m+1}^{0} u_j = \xi_0,\\ W_0 &= \sum_{j=-m+1}^{0} u_j,\\ L_0 &= X_0 + W_0,\end{aligned}$$

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matching the declared initial conditions of the tangle process. Together with the update rules (3), these initial values are sufficient to generate the process for $n \geq 1$. Telescoping (3) on $1 \leq n \leq m$, with $U_j = u_j$ for the indices $j \in \{-m+1, \ldots, 0\}$, gives the explicit formula

$$L_n = \xi_0 + \sum_{j=n-m+1}^{0} u_j + n, \quad n = 1, \ldots, m.$$

Thus the sequence $(\xi_0, u_{-m+1}, \ldots, u_0)$ encodes the deterministic history of free-tip selections on $(-h, 0]$ that produces the initial DAG observed at $t_0 = 0$.

3. Statement of results

Throughout this section we fix an arbitrary terminal time $T > 0$ on which all convergence statements are formulated, and we write $C[s_1, s_2]$ for the space of real-valued continuous functions on $[s_1, s_2]$. We assume that $\lambda h = m$, where m is a positive integer. Our results concern a sequence of models where $\lambda \to \infty$, or equivalently $m \to \infty$. For each λ the tangle process is the sequence of random variables $\{X_n, L_n\}$ indexed by the arrival times. It was shown previously [12] that after scaling by λ^{-1} these processes converge weakly on every finite interval as $\lambda \to \infty$ to continuous functions $a, b\colon [0, T] \to [0, 1]$ with $a(t) \leq b(t)$ for all t, which satisfy the following coupled first-order delay differential equations:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 1 - 2\,\frac{a(t)}{b(t)}, \quad \frac{\mathrm{d}b}{\mathrm{d}t} = 1 - 2\,\frac{a(t-h)}{b(t-h)}. \tag{7}$$

More precisely, for each λ and for each $n \geq 1$, we define

$$A^{(\lambda)}(t_n) = \lambda^{-1} X_n, \quad B^{(\lambda)}(t_n) = \lambda^{-1} L_n. \tag{8}$$

We then define the continuous processes $\{A^{(\lambda)}(t), B^{(\lambda)}(t)\}$ by linear interpolation on each interval $[t_n, t_{n+1}]$ (this is a modification of the definitions used in [12], where the scaled processes were piecewise constant). The statement of weak convergence says that for any $T > 0$ and $\varepsilon > 0$,

$$\mathbb{P}\big(\sup_{0 \leq t \leq T} |A^{(\lambda)}(t) - a(t)| \geq \varepsilon\big) \to 0 \quad \text{as } \lambda \to \infty,$$

with a similar statement for the difference $B^{(\lambda)}(t) - b(t)$. In this paper we are interested in the leading-order correction term as $\lambda \to \infty$. To this end we will prove that there is a Gaussian process $z(t)$ such that

$$\lambda^{1/2}\,(A^{(\lambda)}(t) - a(t)) \to z(t) \quad \text{as } \lambda \to \infty \quad \text{(convergence in law)}.$$

We also derive an explicit expression for the stochastic delay differential equation satisfied by $z(t)$. This kind of result can be described as an FCLT with delay. The result quantifies the probability distribution of the rescaled fluctuations of the tangle process around its fluid limit in the large-λ regime under the simplifying assumptions of the model.

In order to formulate the results, we recall the definition of initial conditions for the equation (7): the pair $\alpha = (a(0), \{u(t)\colon\ -h \leq t \leq 0\})$ is a *DDE initial condition* if $a(0) > 0$, $u(\cdot)$ is integrable, and

$$0 \leq u(t) \leq 2 \quad \text{for all } -h \leq t \leq 0.$$

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For each DDE initial condition α, the fluid equation (7) admits a unique solution (a_α, b_α) on $[0, \infty)$, constructed by the method of steps [6].

A DDE initial condition α does *not*, on its own, single out a tangle process. To obtain a sequence of rescaled tangle processes whose fluctuations can be compared with the fluid solution (a_α, b_α), we must pick, for each λ, a discrete initial condition $(\xi_0, u_{-m+1}, \dots, u_0)$ for the tangle process in the sense of Section 2.3, and we must do so consistently with α. 'Consistently' here means that the rescaled processes $A^{(\lambda)}, B^{(\lambda)}$ generated from this discrete initial condition stay close to (a_α, b_α) on the initial interval $[0, h]$. We quantify this closeness by requiring

$$\sup_{0\le n\le m} |B^{(\lambda)}(t_n) - b_\alpha(t_n)| \le \lambda^{-1}; \tag{9}$$

Lemma 1 below shows that for every α and every λ such a discrete initial condition exists.

For a given DDE initial condition α we also define the map $\phi_\alpha : C[-h, T] \to C[0, T]$ as follows: for all $f \in C[-h, T]$ and $t \in [0, T]$,

$$(\phi_\alpha \circ f)(t) = f(t) + \int_0^t \frac{2}{b_\alpha(u)} f(u)\,\mathrm{d}u - \int_0^t \frac{2a_\alpha(u)}{(b_\alpha(u))^2} f(u-h)\,\mathrm{d}u. \tag{10}$$

Before stating Theorem 1, we briefly indicate the form one should expect for the limiting equation. By (3) and (6), the conditional mean of the increment $X_{n+1} - X_n$ is

$$\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] = 1 - 2\,\frac{X_n}{L_n} + \frac{X_n}{L_n^2}.$$

The first two terms, after rescaling by λ^{-1}, reproduce the right-hand side of (7) along the fluid trajectory; the third term has order λ^{-1} (since $L_n \ge m = \lambda h$) and vanishes in the limit. A first-order Taylor expansion of X_n/L_n around the fluid solution (a_α, b_α), together with the identity $L_n = X_{n-m} + m$ from (4), expresses the leading-order fluctuation of this drift as a linear combination of the fluctuation at time t and the fluctuation at time $t-h$, the delayed term arising precisely from the dependence of L_n on X_{n-m}. This linear combination is exactly the integrated operator ϕ_α defined in (10). Then, after computing quadratic variation for the limiting process, we will apply the martingale representation theorem to identify the limit of the $\lambda^{1/2}$-rescaled fluctuations as a centered Gaussian process satisfying the linear SDDE (12) below.

Theorem 1. (Gaussian limit for rescaled fluctuations of a DDE.) *Fix $T > 0$. Let $\alpha = (a(0), \{u(s) : -h \le s \le 0\})$ be a DDE initial condition with fluid solution (a_α, b_α) of (7). For each λ, let $A^{(\lambda)}, B^{(\lambda)}$ be the rescaled tangle processes (8) arising from a discrete initial condition $(\xi_0, u_{-m+1}, \dots, u_0)$ of Section 2.3 satisfying (9), extended to all $t \in [0, T]$ by linear interpolation between the arrival times $\{t_n\}$. For all $\lambda > 0$ we define*

$$z^{(\lambda)}(t) = \begin{cases} 0, & -h \le t \le 0, \\ \lambda^{1/2}\,(A^{(\lambda)}(t) - a_\alpha(t)), & 0 \le t \le T. \end{cases} \tag{11}$$

Then $\{z^{(\lambda)}(t)\}$ $(-h \le t \le T)$ converges in law as $\lambda \to \infty$. Let $z(t)$ denote the limit process. Then $z(t)$ is a Gaussian process on $C([-h, T])$ that satisfies $z(t) = 0$ for $-h \le t \le 0$, and for $t \in [0, T]$ satisfies the SDDE

$$(\phi_\alpha \circ z)(t) = \int_0^t \left[2\,\frac{a_\alpha(s)}{b_\alpha(s)}\left(1 - \frac{a_\alpha(s)}{b_\alpha(s)}\right)\right]^{1/2} \mathrm{d}B(s), \tag{12}$$

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where B(s) is standard Brownian motion.

The proof will be provided in Section 5.

4. Lemmas and other preliminaries

When proving Theorem 1, we separate out some of the key steps in the following lemmas. These lemmas will be proved in Section 6.

The overall strategy follows the martingale-based approach to functional central limit theorems developed in [3], which fits within the broader framework of Ethier and Kurtz [7] (martingale-problem characterization of weak limits, Chapter 4; FCLT-type weak convergence of discrete processes to continuous limits, Chapter 7). The main feature of the tangle requiring adaptation is the delay: L_n depends on X_{n-m} via (4), so that the limit equation is a stochastic delay differential equation rather than an SDE. The argument proceeds in four steps.

(i) *Initial condition matching* (Lemma 1). Construct, from the DDE initial condition α, a discrete sequence $(u_{-m+1}, \dots, u_0)$ for the tangle process such that the rescaled process $B^{(\lambda)}$ matches the fluid solution b_α on $[0, h]$ up to an error of order λ^{-1}.

(ii) *Martingale decomposition and quadratic variation* (Lemmas 2, 3). Write the increment of the rescaled process as a martingale-difference term plus a deterministic drift, and identify the limit of the predictable quadratic variation.

(iii) *Moment bounds and tightness* (Lemmas 4, 5, 6). Obtain L^4-bounds on $z^{(\lambda)}$ and on its increments uniformly in λ, which together with Kolmogorov's tightness criterion yields tightness of the family $\{z^{(\lambda)}\}$ on $C[-h, T]$.

(iv) *Identification of the limit* (Section 5). Pass to the limit in the martingale identity and apply the martingale representation theorem to identify the limit as the unique solution of the SDDE (12).

The delay enters at every step through the variable X_{n-m}, which controls L_n via (4) and therefore appears in both the drift and the diffusion coefficients.

The first lemma constructs admissible discrete initial conditions for the tangle process from a given continuous DDE initial condition $\alpha = (a(0), u(\cdot))$. The conclusion is that the rescaled $B^{(\lambda)}$ matches b_α on the initial interval $[0, h]$ up to $O(\lambda^{-1})$, which is the regime needed throughout the rest of the proof.

Lemma 1. *For each α, λ there is a sequence $(u_{-m+1}, \dots, u_0)$ with $u_i \in \{0, 1, 2\}$ for all i such that*

$$\sup_{0 \le n \le m} |B^{(\lambda)}(t_n) - b_\alpha(t_n)| \le \lambda^{-1}. \tag{13}$$

Remark 1. The discrete values $u_i \in \{0, 1, 2\}$ in Lemma 1 are obtained by stepwise quantization of the continuous initial profile $u(\cdot)$, in such a way that the cumulative partial sums match the integral of u up to $O(\lambda^{-1})$. Pointwise approximation of u by the sequence (u_i) is neither required nor used; only the integrated/cumulative deviation enters the bound (13).

For convenience we will henceforth omit the subscript α on the functions $a(t)$, $b(t)$. For $n \ge 1$ we define

$$G_{n+1} = X_{n+1} - X_n - \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n], \tag{14}$$

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where $\mathcal{F}_n$ was defined in (1). It follows that $\{G_n\}$ is a martingale difference sequence with respect to the filtration $\{\mathcal{F}_n\}$.

The next lemma identifies the predictable conditional variance of the martingale difference G_{n+1} along the fluid solution. This quantity will be the source of the diffusion coefficient in the limiting SDDE.

Lemma 2. *For every $n \geq 0$,*

$$\mathbb{E}\big[G_{n+1}^2 \mid \mathcal{F}_n\big] = 2\,\frac{x_n}{x_{n-m}+h}\left(1-\frac{x_n}{x_{n-m}+h}\right) + R_n, \tag{15}$$

where R_n is an $\mathcal{F}_n$-measurable random variable satisfying

$$|R_n| \leq \frac{C(h)}{\lambda} \quad a.s., \tag{16}$$

for a constant $C(h) < \infty$ depending only on h.

The next result is an L^4-bound on partial sums of G_n, obtained from the discrete Burkholder inequality applied to the martingale $\sum_{j=1}^{n} G_j$. It will be used both to prove tightness and to control the remainder term in the martingale decomposition.

Lemma 3. *There is a universal constant $c_{\mathrm{B}} < \infty$ such that for all $n \geq 1$,*

$$\left\| \sum_{j=1}^{n} G_j \right\|_4 \leq 2\, c_{\mathrm{B}}\, n^{1/2}. \tag{17}$$

Recall that $z^{(\lambda)}(t) = \sqrt{\lambda}(A^{(\lambda)}(t) - a_\alpha(t))$ for $t \in [0, T]$ and $z^{(\lambda)}(t) = 0$ for $t \in [-h, 0]$, as defined in (11). Our next result is a uniform L^4-bound on $z^{(\lambda)}$, valid for all $t \in [0, T]$ and all $\lambda > 0$ under the regularity hypotheses on the initial conditions stated above. This bound enters the tightness argument and the identification of the limit.

Lemma 4. *For any $T > 0$ there exists $C_5 < \infty$ such that*

$$\|z^{(\lambda)}(t)\|_4 \leq C_5 \quad \textit{for all } \lambda > 0.$$

We also define for $n \geq 1$ the deterministic coefficients

$$H_1(n) = -\frac{2}{a(t_n - h) + h},$$
$$H_2(n) = \frac{2\,a(t_n)}{(a(t_n - h) + h)^2},$$

which are obtained by linearizing the conditional drift of $X_{n+1} - X_n$ around the fluid solution $(a(t), b(t))$, using (4) to write b in terms of a. The next lemma is the central decomposition of the rescaled fluctuation: the increment of $z^{(\lambda)}$ splits into a linear-in-$z^{(\lambda)}$ deterministic part with delayed coefficients H_1, H_2, a martingale-difference term $\lambda^{-1/2} G_{n+1}$, and a higher-order remainder $\lambda^{-3/2} E_n$ controlled in L^4.

Lemma 5. *For all $n \geq 1$,*

$$z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_n) = \lambda^{-1}\,(z^{(\lambda)}(t_n)\,H_1(n) + z^{(\lambda)}(t_{n-m})\,H_2(n))) + \lambda^{-1/2}\,G_{n+1} + \lambda^{-3/2}\,E_n,$$

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where the remainder term E_n satisfies the uniform bound

$$\|E_n\|_4 \le c \tag{18}$$

for some constant $c \ge 0$.

Finally we will need the following fourth-moment estimate on the increments of $z^{(\lambda)}$. Together with the boundary value $z^{(\lambda)}(0) = 0$, the estimate (19) is exactly the hypothesis of Kolmogorov's tightness criterion [2, Theorem 7.3], and therefore yields tightness of the family $\{z^{(\lambda)}\}$ on $C[-h, T]$.

Lemma 6. *There is a constant C (depending on T) such that for all $0 \le s, t \le T$, for all $\lambda > 0$,*

$$\mathbb{E}\big[|z^{(\lambda)}(t) - z^{(\lambda)}(s)|^4\big] \le C\,(t-s)^2. \tag{19}$$

5. Proof of Theorem 1

For all $\lambda > 0$, the process $z^{(\lambda)}(t)$ defines a probability measure on $C([-h, T])$. Tightness of the family $\{z^{(\lambda)}\}$ on $C([-h, T])$ is implied by Lemma 6 together with $z^{(\lambda)}(0) = 0$ (which makes the at-zero condition trivial), via Kolmogorov's tightness criterion [2, Theorem 7.3].

Tightness implies that the family $\{z^{(\lambda)}\}$ is relatively compact in the space of Borel measures on $C([0, T])$. Hence there is a subsequence $z^{(i)} = z^{(\lambda_i)}$, where $\{h\lambda_i\}$ is an increasing sequence of integers, which converges to a process $z(t)$ on $C([0, T])$. Skorokhod's theorem implies the existence of a version such that $z^{(i)}$ converges almost surely to $z(t)$:

$$z^{(i)} = z^{(\lambda_i)} \to z \quad \text{a.s. as } i \to \infty.$$

We next establish some properties of the limit process $z(t)$. Recall the definition (10).

Lemma 7. *Let $0 \le s < t \le T$ and let $g \in C_b(C[0, s])$ be any bounded continuous map on $C[0, s]$. Then*

$$\mathbb{E}[((\phi \circ z)(t) - (\phi \circ z)(s))\, g(\{z(u) \colon 0 \le u \le s\})] = 0.$$

Proof. As shown in [12], there exists $c_1 < \infty$ such that $|a'(t)| \le c_1$. Therefore we have the uniform bounds

$$|a(u) - a(t_n)|,\ |b(u) - b(t_n)| \le c_1\, \lambda^{-1} \quad \text{for all } u \in [t_n, t_{n+1}], \tag{20}$$

and we also have the bound

$$b(u) \ge h \quad \text{for all } u \in [0, T]. \tag{21}$$

Since $z^{(\lambda)}(0) = 0$ and $|X_{n+1} - X_n| \le 1$, we deduce

$$|z^{(\lambda)}(t)| \le \lambda^{1/2}\, T(1 + c_1) \quad \text{for all } t \in [0, T] \tag{22}$$

and

$$|z^{(\lambda)}(t) - z^{(\lambda)}(t_n)| \le \lambda^{-1/2}\,(1 + c_1) \quad \text{for all } t \in [t_n, t_{n+1}]. \tag{23}$$

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Lemma 4 implies that $\mathbb{E}\big[z^{(\lambda)}(t)^2\big]$ is bounded uniformly for $t \in [0, T]$, and hence there exists $C_3 < \infty$ such that

$$\big\|\big(\phi \circ z^{(\lambda)}\big)(t)\big\|_2 \le C_3 \quad \text{for all } t \in [0, T]. \tag{24}$$

Furthermore, Fatou's lemma implies that

$$\mathbb{E}\big[(z(t))^2\big] \le \liminf_{n\to\infty} \mathbb{E}\big[(z^{\lambda_n}(t))^2\big]$$

and so $(\phi \circ z)(t)$ also satisfies the uniform bound (24).

Using Lemma 5, we deduce

$$z^{(\lambda)}(t_n) = \lambda^{-1} \sum_{j=0}^{n-1} \big(z^{(\lambda)}(t_j)\, H_1(j) + (z^{(\lambda)}(t_{j-m})\, H_2(j)\big) + \sum_{j=0}^{n-1} \big(\lambda^{-1/2}\, G_{j+1} + E_j\big),$$

and therefore

$$\begin{aligned}\big(\phi \circ z^{(\lambda)}\big)(t_n) = &\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\frac{2}{b(u)} z^{(\lambda)}(u) - \frac{2}{b(t_j)} z^{(\lambda)}(t_j)\right] \mathrm{d}u \\ &- \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\frac{2a(u)}{b(u)^2} z^{(\lambda)}(u-h) - \frac{2a(t_j)}{b(t_j)^2} z^{(\lambda)}(t_{j-m})\right] \mathrm{d}u \\ &+ \sum_{j=0}^{n-1} \big(\lambda^{-1/2}\, G_{j+1} + E_j\big).\end{aligned}$$

Using (20), (21), (23), and (18), we deduce that there exists $C_1 < \infty$ such that

$$\left|\big(\phi \circ z^{(\lambda)}\big)(t_n) - \sum_{j=0}^{n-1} \lambda^{-1/2}\, G_{j+1}\right| \le C_1\, \lambda^{-1/2} \quad \text{for all } t_n \in [0, T]. \tag{25}$$

By the same reasoning, we deduce that for any $k < n$,

$$\left|\big(\phi \circ z^{(\lambda)}\big)(t_n) - \big(\phi \circ z^{(\lambda)}\big)(t_k) - \sum_{j=k}^{n-1} \lambda^{-1/2}\, G_{j+1}\right| \le C_1\, \lambda^{-1/2} \quad \text{for all } t_n \in [0, T]. \tag{26}$$

Let $g \in C_b(C[0, t_k])$, and define

$$\|g\| = \sup_{f \in C[0,t_k]} |g(f\{u\colon 0 \le u \le t_k\})|.$$

Since $\mathbb{E}[G_j \mid \mathcal{F}_k] = 0$, for any $j \ge k$ we have

$$\mathbb{E}\left[\sum_{j=k}^{n-1} \lambda^{-1/2}\, G_{j+1}\, g\big(z^{(\lambda)}(\{u\colon 0 \le u \le t_k\})\big)\right] = 0,$$

and so it follows from (26) that

$$\left|\mathbb{E}\Big[\Big(\big(\phi \circ z^{(\lambda)}\big)(t_n) - \big(\phi \circ z^{(\lambda)}\big)(t_k)\Big)\, g\big(z^{(\lambda)}(\{u\colon 0 \le u \le t_k\})\big)\Big]\right| \le C_1\, \|g\|\, \lambda^{-1/2}. \tag{27}$$

We will now derive a regularized version of the bound (27). Let $\Gamma > 0$. For any $f \in C[0, T]$, we define $f_\Gamma \in C[0, T]$ by

$$f_\Gamma(t) = \mathrm{mid}\{-\Gamma, \Gamma, f(t)\}.$$

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It follows that

$$f_\Gamma(t) = f(t) - (f(t) - \Gamma)\,\mathcal{I}_{\{f(t)>\Gamma\}} - (f(t) + \Gamma)\,\mathcal{I}_{\{f(t)<-\Gamma\}},$$

where $\mathcal{I}_A$ is the indicator function of the event A. Note that

$$\left|\mathbb{E}\left[(f(t) - \Gamma)\,\mathcal{I}_{\{f(t)>\Gamma\}}\right]\right| \le \left|\mathbb{E}\left[f(t)\,\mathcal{I}_{\{f(t)>\Gamma\}}\right]\right| \le \|f(t)\|_2\,(\mathbb{P}(f(t) > \Gamma))^{1/2}.$$

We will apply this for the function $f(t) = \left(\phi \circ z^{(\lambda)}\right)(t)$, so we have $\|f(t)\|_2 \le C_3$ and

$$\mathbb{P}(f(t) > \Gamma) \le \Gamma^{-1}\left\|\left(\phi \circ z^{(\lambda)}\right)(t)\right\|_2^2 \le C_3^2\,\Gamma^{-1}.$$

We define

$$\mathcal{E}_1(t) = \left|\mathbb{E}\left[\left(\left(\phi \circ z^{(\lambda)}\right)(t) - \Gamma\right)\mathcal{I}_{\{(\phi\circ z^{(\lambda)})(t)>\Gamma\}}\, g\left(z^{(\lambda)}(\{u\colon 0 \le u \le t_k\})\right)\right]\right|,$$
$$\mathcal{E}_2(t) = \left|\mathbb{E}\left[\left(\left(\phi \circ z^{(\lambda)}\right)(t) + \Gamma\right)\mathcal{I}_{\{(\phi\circ z^{(\lambda)})(t)<-\Gamma\}}\, g\left(z^{(\lambda)}(\{u\colon 0 \le u \le t_k\})\right)\right]\right|.$$

Therefore we deduce

$$\mathcal{E}_1(t),\ \mathcal{E}_2(t) \le \|g\|\,C_3^2\,\Gamma^{-1/2}.$$

Substituting into the regularized version of (27), we get

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\left(\phi \circ z^{(\lambda)}\right)_\Gamma(t_n) - \left(\phi \circ z^{(\lambda)}\right)_\Gamma(t_k)\right) g\left(z^{(\lambda)}(\{u\colon 0 \le u \le t_k\})\right)\right]\right| \\
&\quad \le \left|\mathbb{E}\left[\left(\left(\phi \circ z^{(\lambda)}\right)(t_n) - \left(\phi \circ z^{(\lambda)}\right)(t_k)\right) g\left(z^{(\lambda)}(\{u\colon 0 \le u \le t_k\})\right)\right]\right| \\
&\qquad + \mathcal{E}_1(t_n) + \mathcal{E}_1(t_k) + \mathcal{E}_2(t_n) + \mathcal{E}_2(t_k) \\
&\quad \le C_1\,\|g\|\,\lambda^{-1/2} + 4\,\|g\|\,C_3^2\,\Gamma^{-1/2}.
\end{aligned}$$

This extends to all $s, t \in [0, T]$. Since both functions $\left(\phi \circ z^{(\lambda)}\right)_\Gamma$ and g are continuous and uniformly bounded on $C[0, T]$, we use weak convergence along the sequence $\{\lambda_n\}$ to conclude that for $0 \le s < t \le T$,

$$|\mathbb{E}[((\phi \circ z)_\Gamma(t) - (\phi \circ z)_\Gamma(s))\, g(z(\{u\colon 0 \le u \le s\}))]| \le 4\,\|g\|\,C_3^2\,\Gamma^{-1/2}.$$

Furthermore, $|(\phi \circ z)_\Gamma(t)| \le |(\phi \circ z)(t)|$ and $(\phi \circ z)(t)$ is integrable, so by dominated convergence we deduce that

$$\begin{aligned}
&\mathbb{E}[((\phi \circ z)(t) - (\phi \circ z)(s))\, g(z(\{u\colon 0 \le u \le s\}))] \\
&\quad = \lim_{\Gamma\to\infty} \mathbb{E}[((\phi \circ z)_\Gamma(t) - (\phi \circ z)_\Gamma(s))\, g(z(\{u\colon 0 \le u \le s\}))] \\
&\quad = 0.
\end{aligned}$$

□

The continuous bounded functions are dense in the space of bounded functions on $C[0, s]$, and therefore for any event A in the σ-algebra $\mathcal{G}(s)$ generated by $\{z(u)\colon 0 \le u \le s\}$ we have

$$\mathbb{E}[((\phi \circ z)(t) - (\phi \circ z)(s))\,\mathcal{I}_A] = 0.$$

Therefore

$$\mathbb{E}[((\phi \circ z)(t) - (\phi \circ z)(s)) \mid \mathcal{G}(s)] = 0 \quad \text{a.s.},$$

and hence we conclude that $(\phi \circ z)$ is a martingale adapted to the filtration $\{\mathcal{G}(t)\}$. Now using (25), we can express $(\phi \circ z)(t_n)$ as

$$\left(\phi \circ z^{(\lambda)}\right)(t) = \sum_{k=0}^{j} \frac{1}{\sqrt{\lambda}} G_{k+1} + \frac{R_1^{(j)}}{\sqrt{\lambda}}, \quad t \in [t_j, t_{j+1}],\ |R_1^{(j)}| < C_1 \quad \text{for all } 0 \le j < N.$$

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From Lemma 2 it can be derived that

$$\mathbb{E}\left[\left(\phi \circ z^{(\lambda)}\right)^2(t) - \left(\phi \circ z^{(\lambda)}\right)^2(s)\right] = \sum_{k=l}^{j} \frac{1}{\lambda}\mathbb{E}[Q(x_k, x_{k-m})] + \frac{R_2^{(l,j)}}{\lambda},$$

$$t \in (t_j, t_{j+1}], \quad s \in (t_l, t_{l+1}], \quad \left|R_2^{(l,j)}\right| \le C_4 < \infty \quad \text{for all } 0 \le l \le j \le N,$$

where

$$Q(x, y) \stackrel{\text{def}}{=} 2\frac{x}{y+h}\left(1 - \frac{x}{y+h}\right).$$

We now record a Lipschitz bound for Q on the relevant domain

$$D = \{(x, y) \in \mathbb{R}^2 : y \ge 0,\ 0 \le x \le y + h\},$$

which contains both (x_k, x_{k-m}) (since $X_k \le L_k = X_{k-m} + m$) and $(a(t_k), a(t_{k-m}))$ for all k. Setting $u = x/(y+h) \in [0, 1]$ on D, we compute

$$\frac{\partial Q}{\partial x} = \frac{2}{y+h}(1-2u), \quad \frac{\partial Q}{\partial y} = -\frac{2u}{y+h}(1-2u),$$

so that on D

$$\left|\frac{\partial Q}{\partial x}\right|, \left|\frac{\partial Q}{\partial y}\right| \le \frac{2}{y+h} \le \frac{2}{h}.$$

Consequently Q is Lipschitz on D with constant $K = 2/h$ depending only on the proof of work duration: for any $(x_1, y_1), (x_2, y_2) \in D$,

$$|Q(x_1, y_1) - Q(x_2, y_2)| \le \frac{2}{h}(|x_1 - x_2| + |y_1 - y_2|). \tag{28}$$

We now estimate the difference between the sum above with exact values x_k in Q and the corresponding sum with the fluid solutions $a(t_k)$. Applying (28) with $K = 2/h$,

$$\begin{aligned}
&\left|\frac{1}{\lambda}\sum_{k=l}^{j}\mathbb{E}[Q(x_k, x_{k-m}) - Q(a(t_k), a(t_{k-m}))]\right| \\
&\quad \le K\frac{1}{\lambda}\left(\sum_{k=l}^{j}\mathbb{E}|x_k - a(t_k)| + \sum_{k=l}^{j}\mathbb{E}|x_{k-m} - a(t_{k-m})|\right) \\
&\quad = K\frac{1}{\lambda^{3/2}}\left(\sum_{k=l}^{j}\mathbb{E}|z(t_k)| + \sum_{k=l}^{j}\mathbb{E}|z(t_{k-m})|\right) \\
&\quad \le 2C_1\frac{1}{\lambda^{3/2}}(j - l + 1) \\
&\quad \le 2C_1 T\frac{1}{\sqrt{\lambda}}.
\end{aligned}$$

Let

$$S(t_{j+1}) = \sum_{k=0}^{j}\frac{1}{\lambda}Q(a(t_k), a(t_{k-m})) = \sum_{k=0}^{j}\frac{1}{\lambda}Q(a(t_k), a(t_k - h)),$$

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with linear interpolation on $[t_j; t_{j+1}]$. Now we perform a procedure similar to that above:

$$\begin{aligned}&\left|\mathbb{E}\left[\left(\left(\phi \circ z^{(\lambda)}\right)^2(t) - \left(\phi \circ z^{(\lambda)}\right)^2(s) - (S(t) - S(s))\right) * g(z^{\lambda}([0; s]))\right]\right| \\ &\quad = \left|\mathbb{E}\left[\left(\left(\phi \circ z^{\lambda}\right)^2(t) - \left(\phi \circ z^{\lambda}\right)^2(s) - \int_s^t Q(a(r), a(r-h))\,\mathrm{d}r\right) * g(z^{(\lambda)}([0; s]))\right]\right| \\ &\quad \leq C_5 \|g\| \lambda^{-1/2}.\end{aligned}$$

Similar reasoning leads to the conclusion that

$$(\phi \circ z)^2(t) - \int_0^t Q(a(s), a(s-h))\,\mathrm{d}s$$

is also a continuous square-integrable martingale adapted to the filtration $\{\mathcal{G}(t)\}$. The process $\phi \circ z$ is therefore a continuous local martingale with predictable quadratic variation

$$\langle \phi \circ z \rangle_t = \int_0^t Q(a(s), a(s-h))\,\mathrm{d}s.$$

Since $\langle \phi \circ z \rangle_t$ is absolutely continuous in t, we may apply the representation theorem for continuous local martingales with absolutely continuous quadratic variation ([11, Theorem 3.4.2]; the one-dimensional case is due to Doob [5]) to obtain a standard Brownian motion B such that

$$\begin{aligned}(\phi \circ z)(t) &= \int_0^t Q(a(s), a(s-h))^{1/2}\,\mathrm{d}B(s) \\ &= \int_0^t \left(2\frac{a(s)}{a(s-h)+h}\left(1 - \frac{a(s)}{a(s-h)+h}\right)\right)^{1/2} \mathrm{d}B(s), \qquad (29)\end{aligned}$$

where B is a standard Brownian motion. Equation (29), viewed through ϕ, is a linear delay equation in z with deterministic coefficients, driven by a Wiener integral with deterministic integrand. Differentiating (10) gives the differential form

$$\mathrm{d}z(t) = \left[-\frac{2}{b(t)}\,z(t) + \frac{2a(t)}{b(t)^2}\,z(t-h)\right]\mathrm{d}t + Q(a(t), a(t-h))^{1/2}\,\mathrm{d}B(t),$$

with $z(t) = 0$ for $-h \leq t \leq 0$. For each fixed $s \in [0, T]$, we define the *fundamental solution* $K(\cdot\,, s)\colon [s, T] \to \mathbb{R}$ to be the unique continuous function on $[s, T]$ satisfying $K(s, s) = 1$ and, for $t \in (s, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t, s) = -\frac{2}{b(t)}\,K(t, s) + \frac{2a(t)}{b(t)^2}\,K(t-h, s), \qquad (30)$$

where we adopt the convention $K(t-h, s) = 0$ whenever $t - h < s$ (so that the delayed term is absent until the first interval has been completed). For use in stochastic integration we then extend K to the whole square $[0, T]^2$ by setting $K(t, s) = 0$ for $t < s$; the resulting function is continuous in s for each fixed t but jumps from 0 to 1 at $t = s$, as is standard for the impulse-response interpretation of fundamental solutions.

The function $K(t, s)$ is constructed by the method of steps, exactly as in the construction of (a_α, b_α) in [12, Section 3.1], working outward from $t = s$ on the half-open intervals $[s, s+h)$, $[s+h, s+2h), \ldots,\ [s+kh, s+(k+1)h),\ \ldots$.

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- On $[s, s+h)$, the delayed argument $t-h$ lies in $[s-h, s)$, so by the convention $K(t-h, s)=0$ and (30) collapses to the ordinary linear ODE $\mathrm{d}/\mathrm{d}tK = -(2/b(t))K$. Its unique solution with $K(s, s)=1$ is

$$K(t, s) = \exp\left(-\int_s^t \frac{2}{b(u)}\,\mathrm{d}u\right), \quad t \in [s, s+h).$$

- On $[s+kh, s+(k+1)h)$ for $k \geq 1$, the delayed argument $t-h$ lies in $[s+(k-1)h, s+kh)$, which was constructed in the preceding step, so $K(t-h, s)$ is a known continuous function on this interval. Equation (30) becomes the inhomogeneous linear ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t, s) + \frac{2}{b(t)}\,K(t, s) = \frac{2a(t)}{b(t)^2}\,K(t-h, s),$$

  solved by the integrating factor

$$\exp\left(\int_{s+kh}^t 2/b(u)\,\mathrm{d}u\right)$$

  with initial value given by the left-limit of K at $t = s+kh$ from the preceding interval (the limit exists because K is continuous in t across interval boundaries by construction).

This recursion terminates after $\lceil (T-s)/h \rceil$ steps and produces a continuous function $K(\cdot\,, s)$ on $[s, T]$. Equivalently, for each fixed $s \in [0, T]$, integrating (30) in the first argument from s to t and using $K(s, s)=1$ gives the integral form

$$K(t, s) = 1 + \int_s^t \left[-\frac{2}{b(u)}K(u, s) + \frac{2a(u)}{b(u)^2}K(u-h, s)\right]\mathrm{d}u, \quad 0 \leq s \leq t \leq T. \tag{31}$$

Define the process

$$\widetilde{z}(t) = \int_0^t K(t, s)\, Q(a(s), a(s-h))^{1/2}\,\mathrm{d}B(s), \quad 0 \leq t \leq T, \tag{32}$$

and extend it to $[-h, 0]$ by setting $\widetilde{z}(t)=0$ there, matching the initial condition imposed on z. This extension is consistent with (32) under the convention $\int_0^t = 0$ for $t < 0$, since $K(t, s)=0$ for $t < s$ forces the Wiener integral in (32) to vanish identically for $t \leq 0$. With this extension the composition $\phi \circ \widetilde{z}$ in (10) is well-defined.

We claim that $\widetilde{z}$ satisfies

$$\left(\phi \circ \widetilde{z}\,\right)(t) = \int_0^t Q(a(s), a(s-h))^{1/2}\,\mathrm{d}B(s),$$

the same integral equation (29) satisfied by z. The verification uses the stochastic Fubini theorem [18, Chapter IV, Theorem 64] to interchange the Lebesgue $\mathrm{d}u$ and Itô $\mathrm{d}B(s)$ integrals; the deterministic-integrand case used here is the simplest setting of that theorem. Throughout the calculation below we use the extension $K(t, s)=0$ for $t < s$ established above, so every inner integral may be written uniformly with limits 0 to t, the actual support being enforced by K

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itself. Substituting the definition of $\widetilde{z}$ into (10), writing $Q_s := Q(a(s), a(s-h))$, we get

$$\begin{aligned}(\phi \circ \widetilde{z})(t) &= \int_0^t K(t,s)\sqrt{Q_s}\,\mathrm{d}B(s) + \int_0^t \frac{2}{b(u)} \int_0^t K(u,s)\sqrt{Q_s}\,\mathrm{d}B(s)\,\mathrm{d}u \\ &\quad - \int_0^t \frac{2a(u)}{b(u)^2} \int_0^t K(u-h,s)\sqrt{Q_s}\,\mathrm{d}B(s)\,\mathrm{d}u \\ &= \int_0^t \left\{ K(t,s) + \int_s^t \left[\frac{2}{b(u)} K(u,s) - \frac{2a(u)}{b(u)^2} K(u-h,s) \right] \mathrm{d}u \right\} \sqrt{Q_s}\,\mathrm{d}B(s).\end{aligned}$$

In the second equality the order of $\mathrm{d}u$ and $\mathrm{d}B(s)$ has been swapped by stochastic Fubini, and the inner $\mathrm{d}u$-integral has had its lower limit shrunk from 0 to s using $K(u,s) = 0$ for $u < s$ (and $K(u-h,s) = 0$ for $u - h < s$, i.e. $u < s + h$, which is automatic on $[s,t]$). By (31) the bracketed expression equals 1 for every $s \in [0,t]$, so

$$(\phi \circ \widetilde{z})(t) = \int_0^t \sqrt{Q(a(s), a(s-h))}\,\mathrm{d}B(s) = (\phi \circ z)(t).$$

The operator ϕ is injective on continuous functions vanishing on $[-h, 0]$: this follows from the method of steps applied to (10), since on each interval $[kh, (k+1)h] \cap [0,T]$ the equation $(\phi \circ f)(t) = 0$ reduces, after differentiation, to a linear ODE with zero initial value whose only solution is $f \equiv 0$. Hence $\widetilde{z} = z$ a.s., and z admits the Wiener-integral representation

$$z(t) = \int_0^t K(t,s)\, Q(a(s), a(s-h))^{1/2}\,\mathrm{d}B(s). \tag{33}$$

The representation (33) identifies the limit z uniquely. Therefore all weak limit points of the family $\{z^{(\lambda)}\}$ coincide with z, and we conclude that $z^{(\lambda)}$ converges in law to z as $\lambda \to \infty$. □

We close this section by recording two structural properties of the limit process z that follow directly from the representation (33). Since z is the Itô integral of a deterministic integrand, it is a centred Gaussian process. To compute its covariance kernel for arbitrary times $t, t' \in [0,T]$, observe that by the support property $K(t,s) = 0$ for $s > t$, both $z(t)$ and $z(t')$ may be written as Itô integrals over the common interval $[0,T]$:

$$z(t) = \int_0^T K(t,s)\, Q_s^{1/2}\,\mathrm{d}B(s), \qquad z(t') = \int_0^T K(t',s)\, Q_s^{1/2}\,\mathrm{d}B(s),$$

the integrands being identically zero on $(t,T]$ and $(t',T]$ respectively. The Itô isometry [11, Proposition 3.2.10], applies with this common upper limit, and yields

$$\mathbb{E}[z(t)z(t')] = \int_0^{t\wedge t'} K(t,s)K(t',s)\, Q(a(s), a(s-h))\,\mathrm{d}s.$$

6. Proofs of lemmas

6.1. Proof of Lemma 1

We have

$$b_\alpha(t) = a(0) + t + \int_{t-h}^0 u(s)\,\mathrm{d}s \quad \text{for } 0 \le t \le h$$

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and therefore for $0 \le n \le m$

$$b(t_{n+1}) - b(t_n) = \int_{t_n - h}^{t_{n+1} - h} (1 - u(s))\, \mathrm{d}s.$$

Also

$$B^{(\lambda)}(t_{n+1}) - B^{(\lambda)}(t_n) = \lambda^{-1}\,(1 - u_{-m+n}).$$

For the initial value we choose $\xi_0 = \lfloor \lambda a(0) \rfloor$, so it follows that

$$|B^{(\lambda)}(t_0) - b(t_0)| \le \lambda^{-1}.$$

We have

$$B^{(\lambda)}(t_{n+1}) - b(t_{n+1}) = B^{(\lambda)}(t_n) - b(t_n) + \lambda^{-1}\,(1 - u_{-m+n}) - \int_{t_n - h}^{t_{n+1} - h} (1 - u(s))\, \mathrm{d}s.$$

We proceed inductively as follows. Assume that $|B^{(\lambda)}(t_n) - b(t_n)| \le \lambda^{-1}$ and define

$$T_n = B^{(\lambda)}(t_n) - b(t_n) - \int_{t_n - h}^{t_{n+1} - h} (1 - u(s))\, \mathrm{d}s.$$

Note that

$$\left| \int_{t_n - h}^{t_{n+1} - h} (1 - u(s))\, \mathrm{d}s \right| \le \lambda^{-1}$$

and therefore

$$|T_n| \le 2\,\lambda^{-1}.$$

We choose u_{-m+n} according to the following rule:

$$u_{-m+n} = \begin{cases} 0 & \text{if } -2\,\lambda^{-1} \le T_n \le -\lambda^{-1}, \\ 1 & \text{if } -\lambda^{-1} \le T_n \le \lambda^{-1}, \\ 2 & \text{if } \lambda^{-1} \le T_n \le 2\,\lambda^{-1}. \end{cases}$$

Therefore

$$|B^{(\lambda)}(t_{n+1}) - b(t_{n+1})| = |T_n + \lambda^{-1}\,(1 - u_{-m+n})| \le \lambda^{-1},$$

and this proves the induction step. Therefore

$$|B^{(\lambda)}(t_n) - b(t_n)| \le \lambda^{-1} \quad \text{for all } 0 \le n \le \lambda\, T.$$

6.2. Proof of Lemma 2

From the definition (14) and the update equation (3) we have

$$G_{n+1} = 1 - U_{n+1} - \mathbb{E}[1 - U_{n+1} \mid \mathcal{F}_n],$$

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hence

$$\mathbb{E}\big[G_{n+1}^2 \mid \mathcal{F}_n\big] = \mathbb{E}\big[U_{n+1}^2 \mid \mathcal{F}_n\big] - (\mathbb{E}[U_{n+1} \mid \mathcal{F}_n])^2.$$

Using the conditional distribution (5) of U_{n+1} and its conditional expectation (6),

$$\begin{aligned}\mathbb{E}\big[G_{n+1}^2 \mid \mathcal{F}_n\big] &= \frac{4X_n(X_n-1)}{L_n^2} + \frac{2(L_n-X_n)X_n + X_n}{L_n^2} - \left(\frac{2X_nL_n - X_n}{L_n^2}\right)^2 \\ &= \frac{2X_n^2 + 2L_nX_n - 3X_n}{L_n^2} - \frac{X_n^2(4L_n^2 - 4L_n + 1)}{L_n^4} \\ &= 2\,\frac{X_n}{L_n}\left(1 - \frac{X_n}{L_n}\right) + R_n,\end{aligned}$$

where the remainder is

$$R_n = -\frac{3X_n}{L_n^2} + \frac{4X_n^2}{L_n^3} - \frac{X_n^2}{L_n^4}. \tag{34}$$

Using $L_n = X_{n-m} + m$ from (4) together with $X_n/L_n = x_n/(x_{n-m}+h)$ in rescaled variables gives (15). It remains to bound R_n.

Since $0 \le X_n \le L_n$ we have $X_n/L_n \in [0,1]$, and since $L_n \ge m = \lambda h$ pointwise,

$$\left|\frac{X_n}{L_n^2}\right| = \frac{X_n}{L_n}\cdot\frac{1}{L_n} \le \frac{1}{\lambda h}, \qquad \left|\frac{X_n^2}{L_n^3}\right| \le \frac{1}{\lambda h}, \qquad \left|\frac{X_n^2}{L_n^4}\right| \le \frac{1}{(\lambda h)^2}.$$

Combining these in (34),

$$|R_n| \le \frac{3}{\lambda h} + \frac{4}{\lambda h} + \frac{1}{(\lambda h)^2} \le \frac{C(h)}{\lambda} \quad \text{a.s.},$$

with $C(h) := 7/h + 1/h^2$ (taking $\lambda \ge 1$, which is the regime of interest). This proves (16). □

6.3. Proof of Lemma 3

The sequence $\{G_j\}_{j\ge1}$ is a martingale-difference sequence with respect to $\{\mathcal{F}_n\}$ by (14), so the partial sums $\sum_{j=1}^n G_j$ form a discrete-time martingale starting at 0. By the discrete Burkholder inequality [9, Theorem 2.10] applied to this martingale, there is a universal constant $c_{\mathrm{B}} < \infty$ such that

$$\left\|\sum_{j=1}^n G_j\right\|_4 \le c_{\mathrm{B}} \left\|\left(\sum_{j=1}^n G_j^2\right)^{1/2}\right\|_4. \tag{35}$$

Since $|G_j| \le 2$ a.s. we have $\sum_{j=1}^n G_j^2 \le 4\,n$ pointwise, hence

$$\left(\sum_{j=1}^n G_j^2\right)^{1/2} \le 2\,n^{1/2}$$

pointwise, and the right-hand side of (35) is bounded by $2\,c_{\mathrm{B}}\,n^{1/2}$. This proves (17).

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6.4. Proof of Lemma 4

We will first derive a recursion formula for $z^{(\lambda)}(t)$. Since $L_n = X_{n-m} + m$, it follows that $\{X_i, L_{i+m}\}$ are measurable with respect to $\mathcal{F}_n$ for all $i \le n$. The update equation is

$$A^{(\lambda)}(t_{n+1}) = A^{(\lambda)}(t_n) + \lambda^{-1}\left[1 - 2\,\frac{X_n}{L_n} + \frac{X_n}{L_n^2} + G_{n+1}\right]. \tag{36}$$

We define for $x \ge 0$, $y \ge h$

$$H(x, y) = 1 - 2\,\frac{x}{y}.$$

Then from (36) we get

$$z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_n) = \lambda^{-1/2}\,H(X_n, L_n) - \lambda^{1/2}\,[a(t_{n+1}) - a(t_n)] + \lambda^{-1/2}\left[\frac{X_n}{L_n^2} + G_{n+1}\right]. \tag{37}$$

Using the delay differential equation (7), we derive the equation

$$\begin{aligned} a(t_{n+1}) - a(t_n) &= \int_{t_n}^{t_{n+1}} H(a(s), b(s))\,\mathrm{d}s \\ &= \lambda^{-1}\,H(a(t_n), b(t_n)) + \int_0^{\lambda^{-1}} [H(a(t_n + r), b(t_n + r)) - H(a(t_n), b(t_n))]\,\mathrm{d}r. \end{aligned}$$

We have the uniform bounds

$$\left|\frac{\partial H}{\partial x}\right|,\ \left|\frac{\partial H}{\partial y}\right| \le \frac{2}{h} \quad \text{for all } y \ge h,\ 0 \le x \le y, \tag{38}$$

and therefore we deduce that

$$\left|\int_0^{\lambda^{-1}} [H(a(t_n + r), b(t_n + r)) - H(a(t_n), b(t_n))]\,\mathrm{d}r\right| \le 2\,h^{-1}\,\lambda^{-2}.$$

Furthermore, $X_n \le L_n$ and $L_n \ge m$, therefore

$$\frac{X_n}{L_n^2} \le m^{-1} = h^{-1}\,\lambda^{-1}.$$

Substituting into (37), we deduce that

$$z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_n) = \lambda^{-1/2}\,[H(X_n, L_n) - H(a(t_n), b(t_n))] + \lambda^{-1/2}\,G_{n+1} + \lambda^{-3/2}\,E_n', \tag{39}$$

where the remainder term E_n' satisfies the uniform bound

$$|E_n'| \le \frac{2}{h}. \tag{40}$$

It follows that

$$\begin{aligned} &z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(0) \\ &\quad = \lambda^{-1/2} \sum_{j=0}^{n} [H(X_j, L_j) - H(a(t_j), b(t_j))] + \lambda^{-1/2} \sum_{j=0}^{n} G_{j+1} + \lambda^{-3/2} \sum_{j=0}^{n} E_j'. \end{aligned} \tag{41}$$

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We also have

$$H(X_j, L_j) = H\big(A^{(\lambda)}(t_j), B^{(\lambda)}(t_j)\big),$$

and so using (38) we deduce that

$$\begin{aligned} |H(X_j, L_j) - H(a(t_j), b(t_j))| &= \big|H\big(A^{(\lambda)}(t_j), B^{(\lambda)}(t_j)\big) - H(a(t_j), b(t_j))\big| \\ &\le \frac{2}{h}\left(|A^{(\lambda)}(t_j) - a(t_j)| + |B^{(\lambda)}(t_j) - b(t_j)|\right) \\ &= \frac{2}{h}\,\lambda^{-1/2}\left(|z^{(\lambda)}(t_j)| + |z^{(\lambda)}(t_{j-m})|\right). \end{aligned}$$

Therefore we get

$$\begin{aligned} |z^{(\lambda)}(t_{n+1})| &\le \frac{2}{h}\,\lambda^{-1}\sum_{j=0}^{n}\left(|z^{(\lambda)}(t_j)| + |z^{(\lambda)}(t_{j-m})|\right) + \lambda^{-1/2}\left|\sum_{j=0}^{n} G_{j+1}\right| + \lambda^{-3/2}\sum_{j=0}^{n}|E_j'| \\ &\le \frac{4}{h}\,\lambda^{-1}\sum_{j=0}^{n}|z^{(\lambda)}(t_j)| + \lambda^{-1/2}\left|\sum_{j=0}^{n} G_{j+1}\right| + \lambda^{-3/2}\sum_{j=0}^{n}|E_j'|. \end{aligned}$$

Using Lemma 3 and the bound (40), we find

$$\|z^{(\lambda)}(t_{n+1})\|_4 \le \frac{4}{h}\,\lambda^{-1}\sum_{j=0}^{n}\|z^{(\lambda)}(t_j)\|_4 + \lambda^{-1/2}\,c\,n^{1/2} + \lambda^{-3/2}\,\frac{2}{h}\,n.$$

Since $n\,\lambda^{-1} \le T$, we have

$$\|z^{(\lambda)}(t_{n+1})\|_4 \le \frac{4}{h}\,\lambda^{-1}\sum_{j=0}^{n}\|z^{(\lambda)}(t_j)\|_4 + c' \quad \text{for all } n \le \lambda T.$$

Then the discrete Grönwall inequality gives

$$\|z^{(\lambda)}(t_{n+1})\|_4 \le c' + c'\,\frac{4}{h}\,T\exp(4T/h) \quad \text{for all } n \le \lambda T.$$

Linear interpolation implies that this bound holds for all $t \in [0, T]$.

6.5. Proof of Lemma 5

We use the equation (41). We define

$$\begin{aligned} H_1(n) &= \frac{\partial H}{\partial x}(a(t_n), b(t_n)) = -\frac{2}{b(t_n)}, \\ H_2(n) &= \frac{\partial H}{\partial y}(a(t_n), b(t_n)) = \frac{2\,a(t_n)}{b(t_n)^2}. \end{aligned}$$

We have the uniform bounds

$$\left|\frac{\partial^2 H}{\partial x^2}\right|, \left|\frac{\partial^2 H}{\partial x\,\partial y}\right|, \left|\frac{\partial^2 H}{\partial y^2}\right| \le \frac{4}{h^2} \quad \text{for all } x, y \ge 0,\ x \le y + h.$$

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Noting that $A^{(\lambda)}(t_n) - a(t_n) = \lambda^{-1/2}\, z^{(\lambda)}(t_n)$ and $B^{(\lambda)}(t_n) - b(t_n) = \lambda^{-1/2}\, z^{(\lambda)}(t_{n-m})$, we conclude that

$$\begin{aligned} H(X_n, X_{n-m}) - H(a(t_n), a(t_{n-m})) &= H\big(A^{(\lambda)}(t_n), A^{(\lambda)}(t_{n-m})\big) - H(a(t_n), a(t_{n-m})) \\ &= \lambda^{-1/2}\, z^{(\lambda)}(t_n)\, H_1(n) + \lambda^{-1/2}\, z^{(\lambda)}(t_{n-m})\, H_2(n) + \lambda^{-1}\, R_n, \end{aligned}$$

where the remainder term satisfies the bound (recall (22))

$$|R_n| \le \frac{2}{h^2}\left(|z^{(\lambda)}(t_n)| + |z^{(\lambda)}(t_{n-m})|\right)^2.$$

It follows from Lemma 4 that

$$\|R_n\|_4 \le c.$$

Substituting into (39), we deduce that

$$z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_n) = \lambda^{-1}\left(z^{(\lambda)}(t_n)\, H_1(n) + z^{(\lambda)}(t_{n-m})\, H_2(n)\right) + \lambda^{-1/2}\, G_{n+1} + \lambda^{-3/2}\, E_n, \tag{42}$$

where

$$\|E_n\|_4 = \|E_n' + R_n\|_4 \le c.$$

6.6. Proof of Lemma 6

We will first prove the inequality (19) for discrete times $\{t_n\}$ and then extend to all $s, t \in [0, T]$. The iterative equation (42) will be the basis for our results. Let $0 \le k \le n \le \lambda^{-1}T$. Then the iterative equation (42) yields

$$\begin{aligned} &|z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_{k+1})| \\ &\quad = \left|\sum_{j=k+1}^{n} z^{(\lambda)}(t_{j+1}) - z^{(\lambda)}(t_j)\right| \\ &\quad \le \sum_{j=k+1}^{n} |z^{(\lambda)}(t_{j+1}) - z^{(\lambda)}(t_j)| \\ &\quad \le \sum_{j=k+1}^{n} \lambda^{-1}\left(|z^{(\lambda)}(t_j)| + |z^{(\lambda)}(t_{j-m})|\right)\left(\frac{2}{h} + \frac{4T}{h^2}\,\lambda^{-1}\right) + \lambda^{-1/2}\left|\sum_{j=k+1}^{n} G_{j+1}\right| + \sum_{j=k+1}^{n} |E_j|. \end{aligned}$$

We take the norm of both sides and use Lemma 4 to derive

$$\begin{aligned} &\|z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_{k+1})\|_4 \\ &\quad \le \sum_{j=k+1}^{n} \lambda^{-1}\,(2\,K_1 \exp(K_2\,T))\left(\frac{2}{h} + \frac{4T}{h^2}\,\lambda^{-1}\right) \\ &\qquad + \lambda^{-1/2}\,\frac{16}{3}\sqrt{\frac{2}{3}}\,2\,(n-k)^{1/2} + \frac{2}{h}\,\lambda^{-3/2}\,(n-k) \\ &\quad = \lambda^{-1/2}\,(n-k)^{1/2}\,C_1 + \lambda^{-1}\,(n-k)\,C_2, \end{aligned}$$

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where

$$C_1 = \frac{16}{3}\sqrt{\frac{2}{3}}\,2,$$
$$C_2 = (2\,K_1 \exp(K_2\,T))\left(\frac{2}{h} + \frac{4T}{h^2}\,\lambda^{-1}\right) + \frac{2}{h}\,\lambda^{-1/2}.$$

Since $\lambda^{-1}\,(n-k) = t_n - t_k$, we deduce

$$\mathbb{E}\Big[\big(z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_{k+1})\big)^4\Big] \le (C_1\,(t_n - t_k)^{1/2} + C_2\,(t_n - t_k))^4 \le C\,(t_n - t_k)^2,$$

where

$$C = (C_1 + C_2\,T^{1/2})^4.$$

This establishes (19) for discrete times.

Let $0 \le s < t \le T$; then there are $k \le n$ and $0 \le \alpha, \beta \le 1$ such that

$$s = \alpha\,t_k + (1-\alpha)\,t_{k+1}, \quad t = \beta\,t_n + (1-\beta)\,t_{n+1}$$

and hence

$$z^{(\lambda)}(s) = \alpha\,z^{(\lambda)}(t_k) + (1-\alpha)\,z^{(\lambda)}(t_{k+1}),$$
$$z^{(\lambda)}(t) = \beta\,z^{(\lambda)}(t_n) + (1-\beta)\,z^{(\lambda)}(t_{n+1}).$$

Define

$$\phi(s,t) = \|z^{(\lambda)}(t) - z^{(\lambda)}(s)\|_4.$$

Then $\phi(s,t)$ is jointly convex in s, t. If $k < n$, then $\sqrt{t-s}$ is jointly concave in s, t for $s \in [t_k, t_{k+1}]$ and $t \in [t_n, t_{n+1}]$. Therefore

$$\begin{aligned}\phi(s,t) &\le \alpha\,\beta\,\phi(t_k,t_n) + \cdots \\ &\le C^{1/4}\,(\alpha\,\beta\,(t_n - t_k)^{1/2} + \cdots) \\ &\le C^{1/4}\,(t-s)^{1/2}.\end{aligned}$$

If $k = n$, then

$$t - s = (\alpha - \beta)\,(t_{n+1} - t_n), \quad \alpha \ge \beta,$$
$$z^{(\lambda)}(t) - z^{(\lambda)}(s) = (\alpha - \beta)\,\big(z^{(\lambda)}(t_{n+1}) - z^{(\lambda)}(t_n)\big)$$

and therefore

$$\begin{aligned}\|z^{(\lambda)}(t) - z^{(\lambda)}(s)\|_4 &\le C^{1/4}\,(\alpha-\beta)\,(t_{n+1} - t_n)^{1/2} \\ &\le C^{1/4}\,(\alpha-\beta)^{1/2}\,(t_{n+1} - t_n)^{1/2} \\ &\le C^{1/4}\,(t-s)^{1/2}.\end{aligned}$$

Therefore the inequality (19) holds for all $s, t \in [0, T]$. □

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7. Discussion and future work

Theorem 1 confirms that the process $z^{(\lambda)}(t)$ converges in law to a Gaussian process as $\lambda \to \infty$. This type of result is often referred to as a *functional central limit theorem*. Indeed, as we established earlier, $A^{(\lambda)}(t) \xrightarrow{d} a(t)$, which is analogous to the weak law of large numbers. The result proved here goes further, showing that

$$z^{(\lambda)}(t) \stackrel{\text{def}}{=} \frac{A^{(\lambda)}(t) - a(t)}{\sqrt{1/\lambda}} \xrightarrow{d} z^*(t),$$

where $z^*(t)$ is a Gaussian process, making this result analogous to the central limit theorem.

The proved result allows us to view the growth process of free tips in this way:

$$X_n \approx \lambda a(t_n) + \sqrt{\lambda} z^*(t_n).$$

In other words, we know the next-order correction term for X_n, in powers of λ.

In this work, we made an assumption that the duration of the proof of work (POW), denoted by h, is constant. This assumption could be relaxed by allowing h to follow a probability distribution, and exploring the dynamics of the tangle model under this scenario could be an interesting direction for future research.

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