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DETERMINANTS OF AGGREGATE WEALTH

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This paper analyzes aggregate wealth based on separable risk aversion and intertemporal substitution. In an infinite-period overlapping-generations model with survival uncertainty, economic growth, and permanent and temporary income shocks, our closed-form solution permits us to derive rich behavioral conclusions and to assess the relative importance of the major components of aggregate wealth. In particular, the separation of the attitudes toward risk and substitution allows us to disentangle behaviors relating to the intertemporal substitution motive from behaviors relating to the precautionary motive.

Keywords: Aggregate Wealth, Risk Aversion, Intertemporal Substitution, Precautionary Saving

1. INTRODUCTION

Recently, the role of precautionary saving against uninsurable income uncertainty has drawn great attention, as several leading researchers have claimed that precautionary saving constitutes at least half of aggregate wealth [see Skinner (1988), Zeldes (1989), and Caballero (1990, 1991)]. However, existing theories on aggregate wealth have been based on expected utilities with either constant absolute risk aversion (CARA) or constant relative risk aversion. As we know, these two utilities either do not permit the notion of intertemporal substitution or entangle the latter with risk aversion. As a result, the behaviors relating to intertemporal substitution are mixed with the behaviors relating to risk aversion. As Irvine and Wang (1994) point out, intertemporal substitution is as important as a precautionary motive in determining aggregate wealth. Therefore, the separation of the two factors may be a key to our understanding of aggregate wealth.

We adopt Weil's (1993) non-expected CARA utility, which has two parameters to represent risk aversion and intertemporal substitution separately. Weil (1993) limits his analysis to individual saving. We embed Weil's (1993) model in an infinite-period, overlapping-generations model with survival uncertainty to extend his work to the analysis of aggregate wealth. In addition, we allow economic growth with permanent and temporary output shocks. By separating the intertemporal substitution motive from the precautionary motive, and by allowing income growth, we hope to gain a better understanding of aggregate wealth. Note here that the

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presence of economic growth is important to the behavior of nonprecautionary wealth.

Because of the special features of Weil's utility, our aggregate wealth can be divided into a precautionary component and a nonprecautionary component. Risk aversion is only relevant to precautionary wealth, whereas the elasticity of intertemporal substitution is primarily relevant to nonprecautionary wealth. The role of interest rates, economic growth, and the rate of time preference in aggregate wealth is closely related to the attitude toward intertemporal substitution, whereas the role of survival uncertainty, income risk, and the persistence of shocks is closely related to the attitude toward risk.

The analysis of individual saving is distinctly different from the analysis of aggregate wealth. In an overlapping-generations model, increases in saving by the young generally imply decreases in saving by the old, and vice versa. If a person saves more when she is young, she will spend more when she is old so as to run her wealth down to zero before she dies. The movement of aggregate wealth is determined by the net effect of the two opposing forces.

The paper is organized as follows: Section 2 presents a model of aggregate wealth and its solution. Section 3 analytically and numerically analyzes behaviors of aggregate wealth, individual wealth, and consumption. Section 4 concludes the paper with a few remarks. All derivations are included in the Appendix.

2. MODEL

2.1. Intertemporal Utility

Let us briefly review the utility formulation of Weil (1993). In addition, we introduce a special feature, survival rate, into the formulation. This feature is crucial for the discussion of aggregate wealth in a model with infinite periods.

Without any uncertainty, a popular utility is

$$U_t = \sum_{i=0}^{\infty} \beta^i c_{t+i}^{\rho}, \tag{1}$$

where $0 \leq \rho \leq 1$ and $0 \leq \beta \leq 1$. Let p be the probability of surviving through period $t+1$, given the fact that one is alive in period t:

$$p \equiv \text{Prob (survive through period } t+1 \mid \text{alive in period } t).$$

We call p the *survival rate*. The unconditional probability of reaching age t is p^{t-1}. With this survival uncertainty, (1) should be revised to[1]

$$U_t = \sum_{i=0}^{\infty} (p\beta)^i c_{t+i}^{\rho}. \tag{2}$$

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Equation (2) represents the same preferences as the following utility:

$$U_t = \left[(1 - p\beta)\sum_{i=0}^{\infty}(p\beta)^i c_{t+i}^{\rho}\right]^{1/\rho}. \tag{3}$$

The latter formulation allows ρ to be negative: $\rho \in (-\infty, 1]$. In particular, the case with $\rho = 0$ can be obtained by taking the limit $\rho \to 0$, i.e.,

$$U_t = (1 - p\beta)\sum_{i=0}^{\infty}(p\beta)^i \ln c_{t+i}.$$

The elasticity of intertemporal substitution for the utility in (3) is $\eta \equiv 1/(1 - \rho)$.

The U_t in (3) satisfies the following iterative process:

$$U_t = \left[(1 - p\beta)c_t^{\rho} + p\beta U_{t+1}^{\rho}\right]^{1/\rho}. \tag{4}$$

Conversely, under some minor conditions (restrictions on the consumption space[2]), any $\{U_t\}_{t=0}^{\infty}$ satisfying (4) must be the U_t in (3). The advantage of (4) is that it can be used for the definition of a more general utility.

If future consumption is random, then U_{t+1} is random. In this case, let $\hat{U}_{t+1}$ be the certainty equivalent of U_{t+1} and replace (4) with

$$U_t = \left[(1 - p\beta)c_t^{\rho} + p\beta \hat{U}_{t+1}^{\rho}\right]^{1/\rho}. \tag{5}$$

Here, $\eta \equiv 1/(1 - \rho)$ is now the elasticity of intertemporal substitution between current consumption and the certainty equivalent of future consumption. In Weil's (1993) formulation, the certainty equivalent[3] is defined by constant absolute risk aversion θ:

$$e^{-\theta \hat{U}_{t+1}} = E_t(e^{-\theta U_{t+1}}). \tag{6}$$

Here, $\theta = 0$ corresponds to risk neutrality, that is, $\hat{U}_{t+1} = E_t U_{t+1}$. Combining (5) and (6) gives a recursive intertemporal utility formulation:

$$U_t = \left[(1 - p\beta)c_t^{\rho} + p\beta\left(-\frac{1}{\theta}\ln E_t e^{-\theta U_{t+1}}\right)^{\rho}\right]^{1/\rho}. \tag{7}$$

This formulation also has been axiomatically rationalized by Kreps and Porteus (1978).

2.2. Individual Problem

For each individual, we use Weil's (1993) formulation. For the aggregate economy, we use an overlapping-generations model with a constant population size, infinitely lived individuals with survival risk, and economic growth with permanent and temporary shocks. Instead of dealing with individual consumption demand as Weil (1993) did, we concentrate on aggregate wealth in equilibrium.

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The individuals in our model are the same when they are born, with the same preferences as represented by the intertemporal utility in (7), and the same endowment as defined by an initial nonhuman wealth A_0 and a lifetime income process $\{y_t\}_{t=0}^{\infty}$. There is also a capital market for lending and borrowing at a constant interest rate r.[4] We denote $\hat{r} \equiv 1 + r$.

Given survival rate p and population size 1 for the whole society in any period, the population size of individuals dying in any period is $1 - p$. The population size of the newborns in each period is assumed to be $1 - p$, which maintains the population size at 1.

Saving and economic growth naturally should be related closely to each other. Economic growth thus may be important to aggregate wealth. To incorporate it into our model, we propose an income process of the form

$$y_{t+1} = (\hat{a} - \lambda)\hat{a}^t y_0 + \lambda y_t + \varepsilon_{t+1}, \tag{8}$$

where $a \in \mathbf{R}$, $\hat{a} \equiv 1 + a$, $\lambda \in [0, 1]$, and $\{\varepsilon_t\}$ is white noise. Let $\bar{y}_t \equiv E(y_t)$. In the *steady state* where $y_t = \bar{y}_t$ for all t, (8) implies that $y_t = \hat{a}^t y_0$.[5] That is, output grows at a constant rate a in the steady state. The persistence of income shocks is measured by λ, with $\lambda = 0$ corresponding to a purely transitory shock, and $\lambda = 1$ corresponding to a permanent shock. That is, in our formulation, income has three components: the growth component $x_t \equiv (\hat{a} - \lambda)\hat{a}^t y_0$, the persistent component λy_t, and the random component ε_{t+1}. Each component has a unique role in aggregate wealth.

Each individual faces the following lifetime utility maximization problem:

$$\begin{aligned} V(A_{t-1},\, x_{t-1},\, y_t) = \max \Bigg[& (1 - p\beta) c_t^{\frac{\eta-1}{\eta}} + p\beta \left(-\frac{1}{\theta} \ln E_t e^{-\theta V(A_t,\, x_t,\, y_{t+1})} \right)^{\frac{\eta-1}{\eta}} \Bigg]^{\frac{\eta}{\eta-1}} \\ & \text{s.t. } A_t = \hat{r} A_{t-1} + y_t - c_t \\ & y_{t+1} = x_t + \lambda y_t + \varepsilon_{t+1} \\ & \text{given } A_0,\, y_0, \end{aligned} \tag{9}$$

where E_t is the expectations operator conditional on information Φ_t available at time t, c_t is consumption, A_t is wealth, and β is the discount factor of time preference. Let

$$\delta \equiv (p\beta\hat{r})^{\eta}, \qquad \varepsilon^* \equiv \frac{1}{\varphi} \ln E_t e^{-\varphi \varepsilon_{t+1}}, \qquad \varphi \equiv \frac{\theta}{\hat{r} - \lambda} (\hat{r} - \delta^{1/\eta})^{\frac{\eta}{\eta-1}} (\hat{r} - \delta)^{\frac{1}{1-\eta}}.$$

The Appendix derives the solution in a feedback form, depending on the state[6] (A_{t-1}, x_{t-1}, y_t):

$$c_t = (\hat{r} - \delta) \left[A_{t-1} + \frac{x_t + r y_t - \varepsilon^*}{r(\hat{r} - \lambda)} \right]. \tag{10}$$

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By using (10) and the budget conditions in (9), we can express c_t in a recursive form:

$$c_{t+1} = \delta c_t + \frac{\hat{r} - \delta}{\hat{r} - \lambda}\left(\frac{a}{r}x_t + \varepsilon_{t+1} + \varepsilon^*\right),$$

which indicates the economic role of δ. By using (10) and the budget equations in (9) again, we also can express A_t in a recursive form:

$$A_t = \delta A_{t-1} + \frac{\delta - \lambda}{\hat{r} - \lambda}y_t - \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)}\left(x_t - \varepsilon^*\right),$$

which then can be used recursively to yield

$$A_t = \delta^t A_0 + \frac{\delta - \lambda}{\hat{r} - \lambda}\sum_{i=1}^{t}\delta^{t-i}y_i - \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)}\sum_{i=1}^{t}\delta^{t-i}x_i + \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)}\frac{1 - \delta^t}{1 - \delta}\varepsilon^*. \quad \textbf{(11)}$$

2.3. Aggregate Equilibrium

The population size of the age t group is $(1 - p)p^{t-1}$, and so, aggregate wealth is

$$W = (1 - p)\sum_{t=1}^{\infty}p^{t-1}A_t. \quad \textbf{(12)}$$

At any moment in time, there are $1 - p$ newborns, demanding a total wealth endowment $(1 - p)A_0$. At the same time, $1 - p$ individuals die, leaving a total accidental wealth $(1 - p)W$. Following Caballero (1991), the equilibrium condition is to equalize the total supply of wealth and the total demand for wealth, which implies

$$A_0 = W. \quad \textbf{(13)}$$

Substituting (13) into (11) and then into (12) gives an equation for W, which then can be solved for the equilibrium *aggregate wealth*:

$$W^* = \frac{1 - p\delta}{1 - \delta}\frac{1 - p}{p}\frac{\delta - \lambda}{\hat{r} - \lambda}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}y_i$$
$$- \frac{\hat{a}(1 - p)(\hat{a} - \lambda)(\hat{r} - \delta)y_0}{r(\hat{r} - \lambda)(1 - \delta)(1 - \hat{a}p)} + \frac{(\hat{r} - \delta)\varepsilon^*}{r(\hat{r} - \lambda)(1 - \delta)}. \quad \textbf{(14)}$$

The existence of this equilibrium solution needs the following assumption.[7]

Assumption 1. $p\delta < 1$ and $p\hat{a} < 1$.

It is not surprising that the two conditions in Assumption 1 guarantee the existence of a stationary equilibrium—the two conditions are statements about the curvature of an agent's felicity function and the ways in which the turnover of the population is sufficiently rapid compared to the economy's income growth

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trend. These types of conditions are fairly common in growth models with linear production technologies.

2.4. Steady-State Equilibrium

Notice that the incomes in (14) are realized incomes. If the sample of incomes is in the steady state where $y_t = \bar{y}_t$ for all t, then

$$y_t = y_0 \hat{a}^t, \tag{15}$$

and

$$W^* = \frac{\hat{a}(1-p)[r(\delta-\lambda)-(\hat{r}-\delta)(\hat{a}-\lambda)]}{r(1-\delta)(\hat{r}-\lambda)(1-\hat{a}p)} y_0 + \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)(1-\delta)}\varepsilon^*. \tag{16}$$

If ε_{t+1} is conditionally normally distributed: $\varepsilon_{t+1} \mid \Phi_t \sim N(0, \sigma^2)$, then

$$\varepsilon^* = \frac{\theta\sigma^2}{2(\hat{r}-\lambda)} (\hat{r} - \delta^{1/\eta})^{\frac{\eta}{\eta-1}} (\hat{r}-\delta)^{\frac{1}{1-\eta}}.$$

Substituting (13) and (16) into (11) then gives

$$A_t^* = \frac{r(\delta-\lambda)-(\hat{r}-\delta)(\hat{a}-\lambda)}{r(\hat{r}-\lambda)(\hat{a}-\delta)} \left[1 + \frac{a(1-\delta p)}{(1-\delta)(1-\hat{a}p)} \left(\frac{\delta}{\hat{a}}\right)^t\right] \hat{a} y_t + \frac{(\hat{r}-\delta)\varepsilon^*}{r(\hat{r}-\lambda)(1-\delta)}. \tag{17}$$

Further substituting (17) into (10) gives

$$c_t^* = \frac{\hat{r}-\delta}{\hat{r}-\lambda} \left\{ \left[\hat{a} - \lambda + \frac{r(\delta-\lambda)-(\hat{r}-\delta)(\hat{a}-\lambda)}{(1-\delta)(1-\hat{a}p)} (1-\delta p) \left(\frac{\delta}{\hat{a}}\right)^{t-1}\right] \times \frac{a y_t}{r(\hat{a}-\delta)} + \frac{\varepsilon^*}{1-\delta} \right\}. \tag{18}$$

Several remarks are in order. First, even though adjustments in wealth can be passed to future generations in this model, there is no voluntary bequest motive. Consequently, inheritance comes solely from involuntary bequests.

Second, the absence of income insurance in our model is realistic and standard in these models because of the moral hazard problems that would be associated with its introduction. In other words, we deal with uninsurable income risks.

Third, the absence of an insurance market for lifetime uncertainty in our model is based on Yaari's (1964) result that the conditions governing the evolution of the equations of motion for saving and consumption are the same in the no-lifetime-uncertainty case as in the case in which lifetime is uncertain but with an annuity market where the rate of interest is "fair." Thus, if the terminal conditions of the

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problem were the same in each of these cases, the actual path of wealth accumulation would be the same, and the inclusion of a market for life insurance would be equivalent to the path under a certain lifetime.[8]

Finally, because our model does not include physical capital, we are unable to discuss many issues in growth theory. However, we think that physical capital plays a crucial role only in the transition state of an economy. Because our interest is the aggregate wealth in a steady state, it should not make a major difference in our conclusions if physical capital were included. Besides that, the tractability of a model without physical capital allows us to analyze some important issues in the literature of wealth accumulation that a growth model cannot handle easily, such as the separation of precautionary and nonprecautionary wealth. These are the main two reasons that researchers in the area of aggregate wealth, such as Modigliani and Brumberg (1954), Skinner (1988), Zeldes (1989), and Caballero (1990, 1991), have used exogenous output processes. Here, the constant output growth rate in our model can be treated as the steady-state output growth rate, which should be determined mainly by exogenous factors such as technology and individual preferences.

3. ANALYSIS

We now analyze the steady-state equilibrium. In our discussions, we restrict $r, \delta, a \in [0, 1]$. Also, Assumption 1 is satisfied in our benchmark.

3.1. Aggregate Wealth

Aggregate wealth in (16) can be divided into a nonprecautionary component W_N^* and a precautionary component W_P^*:

$$W_N^* \equiv \frac{\hat{a}(1-p)[r(\delta-\lambda)-(\hat{r}-\delta)(\hat{a}-\lambda)]}{r(1-\delta)(\hat{r}-\lambda)(1-\hat{a}p)}y_0, \tag{19}$$

$$W_P^* \equiv \frac{(\hat{r}-\delta)\varepsilon^*}{r(\hat{r}-\lambda)(1-\delta)} = \frac{\theta\sigma^2(\hat{r}-\delta^{1/\eta})^{\frac{\eta}{\eta-1}}(\hat{r}-\delta)^{\frac{2-\eta}{1-\eta}}}{2r(\hat{r}-\lambda)^2(1-\delta)}. \tag{20}$$

whereas W_N^* is unrelated to income uncertainty, W_P^* results from income uncertainty. Intertemporal substitution has a role in both precautionary and nonprecautionary wealth, but risk aversion is only relevant to precautionary wealth, and economic growth is only relevant to nonprecautionary wealth. This separation would be impossible with a traditional expected utility, for which risk aversion and elasticity of intertemporal substitution are entangled and are represented by a single parameter.

Higher income risk raises aggregate wealth by raising precautionary wealth:

$$\frac{\partial W^*}{\partial \sigma} = \frac{\partial W_P^*}{\partial \sigma} = \frac{2(\hat{r}-\delta)\varepsilon^*}{r(\hat{r}-\lambda)(1-\delta)\sigma} > 0.$$

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This effect has attracted considerable attention recently,[9] as people generally find that income uncertainty is of major importance in aggregate wealth.

As expected, higher risk aversion also raises aggregate wealth by raising precautionary wealth:

$$\frac{\partial W^*}{\partial \theta} = \frac{\partial W_P^*}{\partial \theta} = \frac{(\hat{r} - \delta)\varepsilon^*}{r(\hat{r} - \lambda)(1 - \delta)\theta} > 0. \tag{21}$$

Note that attitude toward risk, as measured by θ, has nothing to do with nonprecautionary wealth. However, expected utilities, which mix attitude toward risk and attitude toward intertemporal substitution, inevitably entangle precautionary and nonprecautionary wealth. The result in (21) thus will not be as clear-cut under expected utility.

Precautionary wealth is independent of economic growth, and, if $\delta \leq \lambda$,

$$\frac{\partial W^*}{\partial a} = \frac{\partial W_N^*}{\partial a} < 0.$$

This results from the so-called consumption smoothing motive.[10]

Aggregate income in the steady state is

$$Y \equiv (1 - p) \sum_{t=1}^{\infty} p^{t-1} y_t = \frac{\hat{a}(1 - p)y_0}{1 - \hat{a}p}.$$

If we use this expression to replace y_0 with Y, then (16) becomes

$$\frac{W^*}{Y} = \frac{r(\delta - \lambda) - (\hat{r} - \delta)(\hat{a} - \lambda)}{r(1 - \delta)(\hat{r} - \lambda)} + \frac{(\hat{r} - \delta^{1/\eta})^{\frac{\eta}{\eta - 1}} (\hat{r} - \delta)^{\frac{2 - \eta}{1 - \eta}}}{2r(\hat{r} - \lambda)^2(1 - \delta)} (\theta Y) \left(\frac{\sigma}{Y}\right)^2.$$

This formula can be used conveniently for numerical analysis because we need not assign a value to Y. The benchmark parameter values are chosen as

$$\begin{aligned} r &= 5\%, \qquad a = 2\%, \qquad \lambda = 98\%, \qquad \sigma/Y = 20\%, \\ \theta Y &= 3, \qquad \eta = 1.8, \qquad p = 96\%, \qquad \beta = 98\%. \end{aligned} \tag{22}$$

All parameter values except η are chosen on the basis of stylized facts, representing the annual data for the United States. The elasticity $\eta = 1.8$ is chosen so that W^*/Y is close to 3, which is also a stylized fact. Because the existing literature provides little indication of the range of η, we borrow the well-known stylized fact on W^*/Y in the determination of η. The interest rate $r = 5\%$ and the output growth rate $a = 2\%$ are used widely in the literature [see, e.g., Caballero (1990) and Irvine and Wang (1994) for the value of the interest rate]. The discount factor $\beta = 98\%$ means that the rate of time preference is about 2%, which is used by Irvine and Wang (1994). The rate of income uncertainty $\sigma/Y = 20\%$ comes from Caballero (1991), who concludes that $\sigma/Y > 10\%$. A 96% survival rate implies 16% population size of retirees (age 65 or older), which is realistic.[11] The variable θY corresponds to

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the relative risk aversion, which often has been assigned a value between 2 and 4 [see, e.g., Skinner (1988), Zeldes (1989) and Caballero (1990, 1991)].[12] Empirical studies suggest a λ value that is fairly close to 1; we choose $\lambda = 0.98$ [see, e.g., Prescott (1986) and McCallum (1989, p. 194)].

We have

$$\frac{\partial W_P^*}{\partial \lambda} = \frac{(\hat{r} - \delta)\varepsilon^*}{r(\hat{r} - \lambda)^2(1 - \delta)} > 0, \qquad \frac{\partial W_N^*}{\partial \lambda} = -\frac{a\hat{a}(1 - p)(\hat{r} - \delta)y_0}{r(1 - \delta)(1 - \hat{a}p)(\hat{r} - \lambda)^2} < 0.$$

Because higher persistence allows shocks to last longer and thus increases intertemporal uncertainty, it boosts precautionary wealth, just as the variability of shocks does. Interestingly, higher persistence also means a more intertemporally stable expected income. With higher persistence, expected future income is related more closely to current income. This works against wealth accumulation. That is, people accumulate less nonprecautionary wealth if the expected component of their income is more intertemporally stable. The overall effect of persistence on aggregate wealth is generally positive. In the benchmark (22), aggregate wealth increases in λ iff $\lambda > 80\%$. Note that if there were no survival risk, the persistence of shocks would not affect nonprecautionary wealth.[13]

In the benchmark, we have

$$\frac{\partial W_N^*}{\partial \eta} > 0, \qquad \frac{\partial W_P^*}{\partial \eta} < 0, \qquad \frac{\partial W^*}{\partial \eta} < 0. \tag{23}$$

The concept of risk aversion deals with *random* fluctuations within a period, whereas the concept of intertemporal substitution deals with *expected* fluctuations across periods. Stronger preference toward intertemporal substitution means that the individuals are more tolerant toward expected fluctuations in consumption over their lifetime, whereas stronger risk aversion means that the individuals are less tolerant toward random fluctuations in consumption within a period.

In a growing economy, nonprecautionary motives are encouraged by higher tolerance toward expected consumption fluctuations; as a result, nonprecautionary wealth increases.[14] As nonprecautionary saving becomes relatively more important, some saving switches from precautionary saving to nonprecautionary saving on the margin, implying a smaller precautionary wealth.

The effect of the interest rate on wealth accumulation is always of interest. In the benchmark, we have

$$\frac{\partial W_N^*}{\partial r} < 0, \qquad \frac{\partial W_P^*}{\partial r} > 0, \qquad \frac{\partial W^*}{\partial r} > 0. \tag{24}$$

In our OG model with a survival rate, the direction of saving's movement by the old often is not the same as the movement of aggregate wealth. A higher interest rate may induce more nonprecautionary saving and result in a higher individual wealth late in one's life [the substitution effect; see the empirical findings of Summers (1981)]. However, the individuals die out over time, which means more

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wealth endowment to newborns. Newborns with a higher wealth endowment may save less (the wealth effect). The net effect of the intergenerational wealth and substitution effects on nonprecautionary wealth is negative. Similarly, a higher interest rate may induce less precautionary saving [see the empirical findings of Carroll (1992)], which results in less individual wealth late in one's life. This means less endowment to the newborns, which induces more saving from younger generations. The net effect on precautionary wealth is positive.

In the benchmark, we have

$$\frac{\partial W_N^*}{\partial p} < 0, \qquad \frac{\partial W_P^*}{\partial p} > 0, \qquad \frac{\partial W^*}{\partial p} < 0. \tag{25}$$

A higher survival rate means fewer bequests and thus lower nonprecautionary wealth. As each individual lives longer, the same degree of income risk in each period will have more effect on one's lifetime consumption plans. Thus, a higher survival rate boosts precautionary wealth.

The discount factor β enters the utility in the same way as p does, and their effects on the two wealth components are the same. However, their net effect on aggregate wealth is the opposite:

$$\frac{\partial W_N^*}{\partial \beta} < 0, \qquad \frac{\partial W_P^*}{\partial \beta} > 0, \qquad \frac{\partial W^*}{\partial \beta} > 0.$$

As each individual increases her preference toward future consumption, more wealth is accumulated against the same degree of income risk in order to protect future consumption. Thus, a higher discount factor boosts precautionary wealth.

Individuals with more patience may save more (the substitution effect) and have more wealth late in life, but these individuals die out over time, which means more wealth endowment to newborns. Newborns with a higher wealth endowment may save less (the wealth effect). The net effect of the intergenerational wealth and substitution effects on nonprecautionary wealth is negative. Note that, when there is no economic growth, the discount factor has no effect on nonprecautionary wealth.

An interesting issue is the relative importance of economic growth, interest rate, and income uncertainty in aggregate wealth. We use the elasticities of aggregate wealth as a measure of their importance. Using (16), in the benchmark we have[15]

$$\begin{aligned}
e_r &\equiv \frac{r}{W^*}\frac{\partial W^*}{\partial r} = 18.7, \qquad \frac{\partial e_r}{\partial \eta} > 0, \qquad \frac{\partial e_r}{\partial \theta} < 0; \\
e_\sigma &\equiv \frac{\sigma/Y}{W^*}\frac{\partial W^*}{\partial (\sigma/Y)} = 27.2, \qquad \frac{\partial e_\sigma}{\partial \eta} > 0, \qquad \frac{\partial e_\sigma}{\partial \theta} < 0; \\
e_a &\equiv \frac{a}{W^*}\frac{\partial W^*}{\partial a} = -18.0, \qquad \frac{\partial e_a}{\partial \eta} < 0, \qquad \frac{\partial e_a}{\partial \theta} > 0.
\end{aligned} \tag{26}$$

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Many recent theoretical and empirical findings suggest that income uncertainty is about twice as important as interest rates.[16] Our calculated e_r and e_σ confirm this observation. However, it is surprising to find that economic growth is as important as interest rate.

Regarding the second column in (26), a higher elasticity of intertemporal substitution means higher tolerance toward intertemporal fluctuations in consumption, which naturally increases the importance of factors that favor saving, such as interest rates and income uncertainty. Likewise, it decreases the importance of those factors that favor consumption smoothing, such as economic growth. More tolerant agents react more to interest rate and income uncertainty. Likewise, because e_a measures the importance of the consumption-smoothing motive, higher tolerance of consumption fluctuations reduces it.

Regarding the third column in (26), risk aversion measures intolerance toward consumption fluctuations, which pushes the importance in the other direction, resulting in the opposite signs in the second and third columns.[17]

3.2. Individual Wealth

Individual wealth can also be divided into a nonprecautionary component and a precautionary component. Note that precautionary wealth is independent of age. The reason is as follows: Our model is stationary—tomorrow's shock is like today's shock. The uncertain lifetime framework prevents an agent from anticipating his or her demise for sure, and so, the agent always behaves as if there is another tomorrow. Death is always a surprise. Hence, an individual's age should not influence his or her decision making on precautionary saving.

The differences between the equilibrium individual wealth in (17) and the individual wealth demand in (11) are reflected by the difference in the wealth endowment. Because of the discount factor δ that applies to the wealth endowment in the wealth demand, the wealth demand by the old behaves similarly to the equilibrium wealth, whereas the wealth demand by the young behaves differently. Here, we distinguish the young from the old by the average age in the population.

In the benchmark, the individual wealth profile A_t^* is plotted in Figure 1. It has a stylized hump, achieving the maximum at $t = 26$ or actual age 46. The individual finishes off her wealth at $t = 60$ or actual age 80.[18]

3.3. Consumption

Weil (1993) has provided a detailed discussion of consumption demand. Here, we discuss only the equilibrium consumption and its comparison with the consumption demand of Weil (1993).

The equilibrium consumption in (18) is of the Keynesian type, that is, a linear function of current income, whereas the consumption demand in (10) obeys the permanent-income hypothesis. Without risk, the equilibrium consumption also obeys the permanent-income hypothesis.

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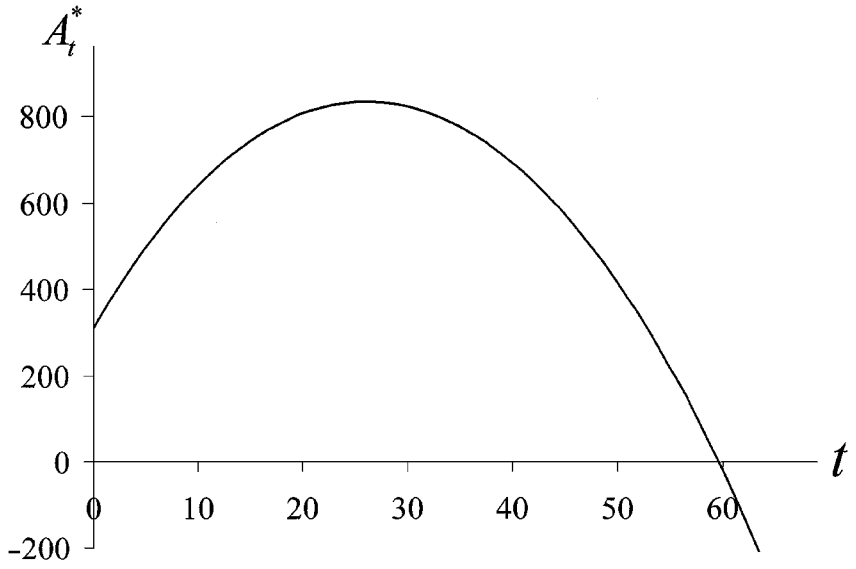


FIGURE 1. Individual wealth profile.

Let μ be the marginal propensity to consume out of current income from the consumption demand, and μ^* be the marginal propensity to consume out of current income from the equilibrium consumption. As Weil (1993) points out, μ is increasing in λ. Hence, the consumption demand reacts more to permanent than to transitory shocks in income. This is consistent with the well-known prediction by Milton Friedman (Wood and Woods 1990). However, this conclusion is based on a partial equilibrium. By inspecting (18), this conclusion is reversed for μ^* if $a \leq r$. Everyone may demand more consumption, but in equilibrium, the relative price will adjust in favor of saving; in our particular case, as fueled by intergenerational wealth transfers in equilibrium, when the interest rate is high, such a price effect is strong enough to reverse the direction.

Although μ does not depend on age, μ^* increases with age. This is expected because, as one gets older, a smaller proportion of income needs to be saved.

An individual who has stronger preference toward intertemporal substitution, that is, a larger η, will have a larger μ. In the benchmark, such a conclusion also holds for μ^*. As we know, the elasticity of intertemporal substitution is a measure of tolerance toward intertemporal variation in consumption. With the possibility of early death, there is a chance for individual wealth not to be consumed. Because of that, individuals with stronger preference toward intertemporal substitution choose to consume more when young.

Although μ is not affected by economic growth, in the benchmark, higher economic growth leads to a lower μ^*. Because the equilibrium precautionary saving is not affected by economic growth, higher economic growth reduces the saving

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rate. See Carroll and Weil (1993) for some empirical findings.[10] This is due to the so-called consumption-smoothing motive.

Weil (1993) shows that an increase in the interest rate increases the propensity to consume from wealth if the elasticity of intertemporal substitution is greater than one. What will happen in equilibrium where intergenerational transfer of wealth is an important factor? In our benchmark, we have

$$\frac{\partial \mu^*}{\partial r} < 0, \qquad \frac{\partial c_t^*}{\partial r} > 0,$$

for all t. A higher interest rate means a higher price for current consumption (the substitution effect), which encourages nonprecautionary saving and results in a lower propensity. The precautionary saving, however, works against such a tendency: A higher interest rate implies a bigger wealth (the wealth effect), which enhances the effects of income uncertainty and lowers the precautionary saving. The precautionary motive dominates, driven by the wealth effect and fueled by intergenerational wealth transfers.

Finally, in the benchmark, we have

$$\frac{\partial \mu^*}{\partial p} < 0, \qquad \frac{\partial c_t^*}{\partial p} < 0$$

for all t. A reduction in the survival uncertainty suggests a longer life expectancy, which naturally reduces the consumption propensity and encourages saving. Notice here that aggregate wealth is not increased by an increase in saving from this source; see (25). Intergenerational wealth transfers and the distribution of the population are the key underlying factors that reverse the direction of aggregate wealth.

4. CONCLUDING REMARKS

Using Weil's (1993) utility that separates intertemporal substitution from risk aversion, we are in a better position to understand the roles of interest rates, income risk, economic growth, survival uncertainty, the rate of time preference, and the persistence of shocks in aggregate wealth. The division of precautionary and nonprecautionary wealth makes it possible to analyze how intertemporal substitution and risk aversion affect wealth accumulation. Risk aversion is only relevant to precautionary wealth, whereas intertemporal substitution is more relevant to nonprecautionary wealth.

The importance of the two most important factors in aggregate wealth—interest rates and income risk—is determined separately by intertemporal substitution and risk aversion. More restrictive expected utilities that mix intertemporal substitution with risk aversion cannot properly disentangle the role of interest rates from the role of income risk in aggregate wealth.[19]

As our analysis indicates, individual saving and aggregate wealth are distinctly different matters. One is a demand concept, whereas the other involves an

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equilibrium process. The saving's movement in some age groups, especially the old, can be the opposite of the movement of aggregate wealth. Even consumption demand and equilibrium consumption can move in opposite directions.

NOTES

1. As, for example, in Caballero (1991) and Irvine and Wang (1994).
2. See, for example, Epstein and Zin (1989).
3. In Epstein and Zin's (1989) formulation, the certainty equivalent is defined by constant relative risk aversion θ:

$$\hat{U}_{t+1}^{1-\theta} = E_t\left(U_{t+1}^{1-\theta}\right).$$

This formulation, however, does not permit a closed-form solution for our problem.

4. The interest rate r here can be considered as the real interest rate for an annuity or a long-term indexed bond.
5. We also can consider a more general form:

$$y_{t+1} = (\hat{a} - \lambda)\hat{a}^t\hat{y} + \lambda y_t + \varepsilon_{t+1}.$$

Then, in the steady state, $y_t = \hat{a}^t\hat{y} + \lambda^t(y_0 - \hat{y})$, which has a steady growth rate a and a diminishing deviation term $\lambda^t(y_0 - \hat{y})$. When there is no growth, the general form becomes Weil's (1993) formula of income formulation.

6. The difference between our income process and Weil's (1993) forces us to solve the consumption demand independently. When the growth rate is zero, our solution is the same as Weil's (1993) solution (with $\hat{y} = y_0$).
7. For the positivity of consumption, we need to impose additional conditions on the parameters. For our purposes, however, there is no need to spell out the conditions. For realistic parameter values, as chosen in Section 3, consumption is positive for all age groups.
8. Kotlikoff (1988), however, has argued that fair annuities are difficult to supply in practice because of agent heterogeneity and self-selection.
9. See, for example, Skinner (1988), Zeldes (1989), Caballero (1990, 1991), Carroll (1992), and Irvine and Wang (1994).
10. Four figures from Carroll and Weil (1993) suggest a long-term negative relation between saving rate and economic growth for Japan, South Korea, Singapore, and Hong Kong.
11. Here t is the number of working years and the actual age is $t + 20$.
12. Because existing empirical estimates of the elasticity of intertemporal substitution are done in models that mix risk aversion with intertemporal substitution, we have to reestimate the elasticity for our model and we have done so using a stylized fact. However, we have accepted the empirical estimate of risk aversion made in models with conventional expected utility functions because the risk aversion often is estimated from one-period models for which intertemporal substitution is absent. In other words, existing estimates of the elasticity of intertemporal substitution are unreliable, but the existing estimates of the risk aversion should be reliable.
13. The intertemporal stability here is a different notion of intertemporal stability from one involving transition paths from one steady state to another. Our economy is an endowment economy at a steady state, and our notion of intertemporal stability applies to the economy at a steady state rather than to a transition path from a steady state to another.
14. If there were no economic growth, then consumption would be constant rather than growing, and our argument and the results in (23) would no longer hold.
15. Caution should be taken here. The elasticity values are marginal. This means that for a 5% increase in σ, for example, we should not conclude that the aggregate wealth will increase by $5 \times 27.2 = 136\%$. When σ increases from 20 to 21, with other parameter values being the same, the elasticity drops to 12.5!

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16. See, for example, Modigliani and Brumberg (1954), Skinner (1988), Caballero (1990, 1991), Carroll (1992), and Irvine and Wang (1994).

17. Are our conclusions in this section robust? For that, we try a few other combinations of parameter values. We find the same conclusions. We are thus confident that our conclusions are robust.

18. Individuals are allowed to have negative wealth. Because they receive an income y_t every period, they can afford to borrow to a certain extent. The question is whether the transversality condition should be used, which will put a lower bound on wealth. This turns out to be unnecessary in our model. Because the population size of a cohort quickly goes to zero as it ages, our conclusions on aggregate wealth will be the same whether or not we put a lower bound on wealth. The lower bound will affect only a negligible segment of the population.

19. Interest rates and income risk are important because they can be influenced easily by government policies. Economic growth also may be an important factor considering business cycles during which an economy typically moves from a trough to a peak over about five years.

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APPENDIX

This Appendix solves problem (9) and derives the formulas in Section 2.

A.1. SOLVING THE CONSUMER PROBLEM

Problem (9) is

$$V^{\frac{\eta-1}{\eta}}(A_{t-1}, x_{t-1}, y_t) = \text{optimize}(1-p\beta)c_t^{\frac{\eta-1}{\eta}} + p\beta\left[-\frac{1}{\theta}\ln E_t e^{-\theta V(\hat{r}A_{t-1}+y_t-c_t, x_t, y_{t+1})}\right]^{\frac{\eta-1}{\eta}}. \tag{A.1}$$

Assume

$$V(A_{t-1}, x_{t-1}, y_t) \equiv \phi_1 A_{t-1} + \phi_2 y_t + \phi_3 x_{t-1} + \phi_4, \tag{A.2}$$

with constants ϕ_i. With (A.2), the right side of (A.1) becomes

$$(1-p\beta)c_t^{\frac{\eta-1}{\eta}} + p\beta\left[\phi_1(\hat{r}A_{t-1}+y_t-c_t) + \phi_2(x_t+\lambda y_t) + \phi_3 x_t + \phi_4 - \frac{1}{\theta}\ln E_t e^{-\theta\phi_2\varepsilon_{t+1}}\right]^{\frac{\eta-1}{\eta}}.$$

The first-order condition and envelope equation are thus

$$\frac{\eta-1}{\eta}(1-p\beta)c_t^{-\frac{1}{\eta}} - \frac{\eta-1}{\eta}\beta p Q^{-\frac{1}{\eta}}\phi_1 = 0, \tag{A.3a}$$

$$\frac{\eta-1}{\eta}V^{-\frac{1}{\eta}}(A_{t-1}, x_{t-1}, y_t)\phi_1 = \frac{\eta-1}{\eta}\beta p Q^{-\frac{1}{\eta}}\phi_1\hat{r}, \tag{A.3b}$$

where

$$Q \equiv \phi_1(\hat{r}A_{t-1} + y_t - c_t) + \phi_2(x_t + \lambda y_t) + \phi_3 x_t + \phi_4 - \frac{1}{\theta}\ln E_t e^{-\theta\phi_2\varepsilon_{t+1}}.$$

Equations (A.3a) and (A.3b) then imply, respectively,

$$c_t = (\phi_1\beta p)^{-\eta}(1-p\beta)^{\eta}Q, \tag{A.4a}$$

$$V(A_{t-1}, x_{t-1}, y_t) = (\hat{r}\beta p)^{-\eta}Q. \tag{A.4b}$$

Hence,

$$V(A_{t-1}, x_{t-1}, y_t) = \left(\frac{\phi_1}{\hat{r}}\right)^{\eta}(1-p\beta)^{-\eta}c_t. \tag{A.5}$$

Also, (A.4b) gives

$$\phi_1(\hat{r}A_{t-1} + y_t - c_t) + \phi_2(x_t + \lambda y_t) + \phi_3 x_t + \phi_4 - \frac{1}{\theta}\ln E_t e^{-\theta\phi_2\varepsilon_{t+1}}$$
$$= (\hat{r}\beta p)^{\eta}V(A_{t-1}, x_{t-1}, y_t).$$

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Using (A.5), we obtain

$$\phi_1\hat{r}A_{t-1} + (\phi_1 + \lambda\phi_2)y_t + (\phi_2 + \phi_3)x_t + \phi_4 - \frac{1}{\theta}\ln E_t e^{-\theta\phi_2\varepsilon_{t+1}}$$
$$= \left[(\hat{r}\beta p)^\eta + (1 - p\beta)^\eta \phi_1^{1-\eta}\hat{r}^\eta\right] V(A_{t-1}, x_{t-1}, y_t). \quad \textbf{(A.6)}$$

Using (A.2), we find that (A.6) implies

$$\phi_1\hat{r} = \phi_1\hat{r}^\eta\left[(\beta p)^\eta + (1 - p\beta)^\eta\phi_1^{1-\eta}\right], \quad \textbf{(A.7a)}$$
$$\phi_1 + \lambda\phi_2 = \phi_2\hat{r}^\eta\left[(\beta p)^\eta + (1 - p\beta)^\eta\phi_1^{1-\eta}\right], \quad \textbf{(A.7b)}$$
$$\phi_2 + \phi_3 = \phi_3\hat{r}^\eta\left[(\beta p)^\eta + (1 - p\beta)^\eta\phi_1^{1-\eta}\right], \quad \textbf{(A.7c)}$$
$$\phi_4 - \frac{1}{\theta}\ln E_t e^{-\theta\phi_2\varepsilon_{t+1}} = \phi_4\hat{r}^\eta\left[(\beta p)^\eta + (1 - p\beta)^\eta\phi_1^{1-\eta}\right]. \quad \textbf{(A.7d)}$$

Equation (A.7a) gives

$$\hat{r} = \hat{r}^\eta\left[(\beta p)^\eta + (1 - p\beta)^\eta\phi_1^{1-\eta}\right]. \quad \textbf{(A.8)}$$

Then, (A.7b), (A.7c), and (A.7d), respectively, give

$$\phi_1 + \lambda\phi_2 = \phi_2\hat{r}, \qquad \phi_2 + \phi_3 = \phi_3\hat{r}, \qquad \phi_4 - \frac{1}{\theta}\ln E_t e^{-\theta\phi_2\varepsilon_{t+1}} = \phi_4\hat{r}. \quad \textbf{(A.9)}$$

Then, (A.8) and (A.9) imply

$$\phi_1 = (\hat{r} - \delta)^{\frac{1}{1-\eta}}(\hat{r} - \delta^{1/\eta})^{\frac{\eta}{\eta-1}},$$
$$\phi_2 = \tfrac{\phi_1}{\hat{r}-\lambda},$$
$$\phi_3 = \tfrac{\phi_1}{r(\hat{r}-\lambda)},$$
$$\phi_4 = -\tfrac{\phi_1}{r(\hat{r}-\lambda)}\varepsilon^*.$$

Equation (A.5) implies

$$c_t = \left(\frac{\hat{r}}{\phi_1}\right)^\eta (1 - p\beta)^\eta(\phi_1 A_{t-1} + \phi_2 y_t + \phi_3 x_t + \phi_4)$$
$$= \hat{r}^\eta\phi_1^{1-\eta}(1 - p\beta)^\eta\left(A_{t-1} + \frac{y_t}{\hat{r} - \lambda} + \frac{x_t - \varepsilon^*}{r(\hat{r} - \lambda)}\right) = (\hat{r} - \delta)\left(A_{t-1} + \frac{x_t + ry_t - \varepsilon^*}{r(\hat{r} - \lambda)}\right).$$

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A.2. AGGREGATE WEALTH

Substituting (11) and (13) into (12) gives

$$W^* = (1-p)\sum_{t=1}^{\infty} p^{t-1}\left[\delta^t W^* + \frac{\delta-\lambda}{\hat{r}-\lambda}\sum_{i=1}^{t}\delta^{t-i}y_i - \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\sum_{i=1}^{t}\delta^{t-i}x_i + \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\frac{1-\delta^t}{1-\delta}\varepsilon^*\right] = \frac{1-p}{p}\left[\frac{p\delta}{1-p\delta}W + \frac{\delta-\lambda}{\hat{r}-\lambda}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}y_i - \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}x_i + \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\varepsilon^*\sum_{t=1}^{\infty}p^t\frac{1-\delta^t}{1-\delta}\right],$$

if $p\delta < 1$. Then,

$$\frac{p}{1-p}\left[1-\frac{(1-p)\delta}{1-p\delta}\right]W^* = \frac{\delta-\lambda}{\hat{r}-\lambda}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}y_i - \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}x_i + \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\varepsilon^*\sum_{t=1}^{\infty}p^t\frac{1-\delta^t}{1-\delta},$$

implying

$$\frac{p}{1-p}\frac{1-\delta}{1-p\delta}W^* = \frac{\delta-\lambda}{\hat{r}-\lambda}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}y_i - \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}x_i + \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\frac{p\varepsilon^*}{(1-p)(1-p\delta)}. \quad \textbf{(A.10)}$$

We have

$$\sum_{i=1}^{t}\delta^{t-i}x_i = \frac{\hat{a}-\lambda}{\hat{a}-\delta}[\hat{a}^t-\delta^t]\hat{a}y_0.$$

Thus, if $p\hat{a} < 1$,

$$\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}x_i = \frac{\hat{a}-\lambda}{\hat{a}-\delta}\hat{a}y_0\sum_{t=1}^{\infty}p^t(\hat{a}^t-\delta^t) = \frac{p(\hat{a}-\lambda)\hat{a}y_0}{(1-p-ap)(1-p\delta)}.$$

Substituting this into (A.10) then gives

$$\frac{1-\delta}{1-p}W^* = \frac{1-p\delta}{p}\frac{\delta-\lambda}{\hat{r}-\lambda}\sum_{t=1}^{\infty}\sum_{i=1}^{t}p^t\delta^{t-i}y_i - \frac{(\hat{r}-\delta)(\hat{a}-\lambda)\hat{a}y_0}{r(\hat{r}-\lambda)(1-p-ap)} + \frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\frac{\varepsilon^*}{(1-p)},$$

which immediately implies (14).

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A.3. STEADY-STATE CASE

In the steady state, $y_t = x_t + \lambda y_{t-1}$, which can be solved easily:

$$y_t = x_t + \lambda y_{t-1} = x_t + \lambda x_{t-1} + \lambda^2 y_{t-2} + \cdots = \sum_{i=0}^{t-1} \lambda^i x_{t-i} + \lambda^t y_0$$

$$= \hat{a}^t y_0 \left[1 - \left(\frac{\lambda}{\hat{a}}\right)^t\right] + \lambda^t y_0 = \hat{a}^t y_0.$$

Using this, we can write (14) in a simpler form. We have

$$\sum_{i=1}^{t} \delta^{t-i} y_i = \delta^t y_0 \sum_{i=1}^{t} \left(\frac{\hat{a}}{\delta}\right)^i = \frac{\hat{a}}{\delta - 1 - a}(\delta^t - \hat{a}^t) y_0,$$

and

$$\sum_{t=1}^{\infty} \sum_{i=1}^{t} p^t \delta^{t-i} y_i = \frac{\hat{a} y_0}{\delta - 1 - a} \sum_{t=1}^{\infty} p^t (\delta^t - \hat{a}^t) = \frac{p\hat{a} y_0}{(1 - p\delta)(1 - p - ap)}.$$

Hence,

$$W^* = \frac{1-p}{1-\delta} \frac{\delta - \lambda}{\hat{r} - \lambda} \frac{\hat{a}}{1 - p - ap} y_0 - \frac{(1-p)(\hat{r} - \delta)(\hat{a} - \lambda)\hat{a} y_0}{r(\hat{r} - \lambda)(1 - \delta)(1 - p - ap)} + \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)(1 - \delta)} \varepsilon^*,$$

which gives (16).

With the steady-state economic growth in (15), the individual wealth profile in (11) becomes

$$A_t = \delta^t A_0 - \frac{r(\delta - \lambda) - (\hat{r} - \delta)(\hat{a} - \lambda)}{r(\hat{r} - \lambda)(\hat{a} - \delta)} (\delta^t - \hat{a}^t) \hat{a} y_0 + \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)} \frac{1 - \delta^t}{1 - \delta} \varepsilon^*. \quad \textbf{(A.11)}$$

In equilibrium $A_0 = W$, substituting (16) into (A.11) gives the equilibrium individual wealth profile:

$$\begin{aligned} A_t^* &= \delta^t W^* - \frac{r(\delta - \lambda) - (\hat{r} - \delta)(\hat{a} - \lambda)}{r(\hat{r} - \lambda)(\hat{a} - \delta)} (\delta^t - \hat{a}^t) \hat{a} y_0 + \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)} \frac{1 - \delta^t}{1 - \delta} \varepsilon^* \\ &= \frac{\delta^t (1-p) \hat{a} y_0}{r(1-\delta)(\hat{r} - \lambda)(1 - p - ap)} [r(\delta - \lambda) - (\hat{r} - \delta)(\hat{a} - \lambda)] + \frac{(\hat{r} - \delta)\delta^t}{r(\hat{r} - \lambda)(1 - \delta)} \varepsilon^* \\ &\quad - \frac{r(\delta - \lambda) - (\hat{r} - \delta)(\hat{a} - \lambda)}{r(\hat{r} - \lambda)(\hat{a} - \delta)} (\delta^t - \hat{a}^t) \hat{a} y_0 + \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)} \frac{1 - \delta^t}{1 - \delta} \varepsilon^* \\ &= -\frac{r(\delta - \lambda) - (\hat{r} - \delta)(\hat{a} - \lambda)}{r(\hat{r} - \lambda)(1 - p - ap)} \left\{ \frac{1 - p - ap}{\hat{a} - \delta} (\delta^t - \hat{a}^t) - \frac{1 - p}{1 - \delta} \delta^t \right\} \hat{a} y_0 \\ &\quad + \frac{\hat{r} - \delta}{r(\hat{r} - \lambda)} \frac{1}{1 - \delta} \varepsilon^*, \end{aligned}$$

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which immediately implies (17). Substituting (17) into (10) then gives

$$c_t^* = (\hat{r}-\delta)\left\{\frac{\hat{a}[r(\delta-\lambda)-(\hat{r}-\delta)(\hat{a}-\lambda)]}{r(\hat{r}-\lambda)(1-p-ap)(\hat{a}-\delta)}\left[\frac{1-\delta p}{1-\delta}a\delta^{t-1}+(1-p-ap)\hat{a}^{t-1}\right]y_0\right.$$
$$\left.+\frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}\frac{1}{1-\delta}\varepsilon^*+\frac{x_t+ry_t-\varepsilon^*}{r(\hat{r}-\lambda)}\right\}=\frac{(\hat{r}-\delta)\hat{a}[r(\delta-\lambda)-(\hat{r}-\delta)(\hat{a}-\lambda)]}{r(\hat{r}-\lambda)(1-p-ap)(\hat{a}-\delta)}$$
$$\times\left[\frac{1-\delta p}{1-\delta}a\delta^{t-1}+(1-p-ap)\hat{a}^{t-1}\right]y_0+\frac{\hat{r}-\delta}{r(\hat{r}-\lambda)}(x_t+ry_t)+\frac{\hat{r}-\delta}{(\hat{r}-\lambda)(1-\delta)}\varepsilon^*.$$

That is,

$$c_t^*=\frac{\hat{r}-\delta}{\hat{r}-\lambda}\left\{\frac{r(\delta-\lambda)-(\hat{r}-\delta)(\hat{a}-\lambda)}{r(1-p-ap)(\hat{a}-\delta)}\left[\frac{1-\delta p}{1-\delta}a\delta^{t-1}+(1-p-ap)\hat{a}^{t-1}\right]\hat{a}y_0\right.$$
$$\left.+\frac{x_t}{r}+y_t+\frac{1}{1-\delta}\varepsilon^*\right\},$$

which immediately gives (18).

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