Ergod. Th. & Dynam. Sys. (1998), **18**, 717–723
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Analyticity and metric transitivity on the torus

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(*Received* 13 *May* 1996 *and accepted in revised form* 26 *March* 1997)

Abstract. Suppose we are given an analytic divergence free vector field (X, Y) on the standard torus. We can find constants a and b and a function $F(x, y)$ of period one in both x and y such that $(X, Y) = (a - F_y, b + F_x)$. For a given F, let P be the map sending (x, y) into $(F_y(x, y), -F_x(x, y))$. Let A be the image of the torus under this map and let B be the image under this map of the set of points (x, y) at which $F_{xx}F_{yy} - (F_{xy})^2$ vanishes. For any point (a, b) in the complement of the interior of A, the flow on the torus arising from the differential equations $dx/dt = a - F_y(x, y)$, $dy/dt = b + F_x(x, y)$ is metrically transitive if and only if a/b is irrational. For any point in A but not in B the flow is not metrically transitive. Moreover, if a/b is irrational but the flow on the torus is not metrically transitive and we use our differential equations to define a flow in the entire plane (rather than on the torus), this flow has a nonstationary periodic orbit. It is an open question whether a point (a, b) in the interior of A can give rise to a metrically transitive flow.

Suppose we are given differential equations

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y)$$

in the plane, where X and Y are of period one in x and y and are sufficiently smooth. Suppose further that the divergence of the vector field (X, Y) equals zero. We may then regard our differential equations as giving rise to a measure preserving flow on the torus. It is well known that this flow is metrically transitive if and only if the ratio of $\int_0^1 \int_0^1 X(x, y)\, dx\, dy$ to $\int_0^1 \int_0^1 Y(x, y)\, dx\, dy$ is irrational, provided the vector field does not vanish anywhere. We wish to see what we can say if we drop the assumption that the vector field vanishes nowhere. From now on in order to get a result bearing on this question we will assume that $X(x, y)$ and $Y(x, y)$ are real analytic. We can rewrite the vector field (X, Y) as $(a - F_y, b + F_x)$ where F is analytic and of period one in x and y and

$$(a, b) = \left(\int_0^1 \int_0^1 X(x, y)\, dx\, dy, \int_0^1 \int_0^1 Y(x, y)\, dx\, dy \right).$$

We then ask, for a given F, for which pairs (a, b) is the flow metrically transitive?

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Let P be the map of the torus into $\mathbb{R}^2$ sending (x, y) into $(F_y(x, y), -F_x(x, y))$. Thus P sends (x, y) into the pair (a, b) such that the corresponding differential equations have (x, y) as a stationary point.

Let A be the image under P of the torus and let B be the image under P of those points of the torus at which the Hessian $F_{xx}F_{yy} - (F_{xy})^2$ vanishes. By Sard's theorem B has Lebesgue measure zero and is therefore nowhere dense. By the remarks made above, if (a, b) does not belong to A, the flow on the torus is metrically transitive if and only if a/b is irrational. At the end of this paper we will see that the same criterion applies if (a, b) belongs to the boundary of A.

We are first going to prove the following results.

THEOREM 1. *If (a, b) belongs to A but not to B the flow on the torus is not topologically transitive.*

THEOREM 2. *Suppose a/b is irrational but our flow is not metrically transitive. Then if we use our differential equations to define a flow in the entire plane rather than on the torus, this flow has a nonstationary periodic orbit.*

As we will see, for flows on the torus of the type we are considering, metric transitivity and topological transitivity are equivalent.

We start by proving the following.

PROPOSITION 1. *The following statements are equivalent.*

(1) (a) *a and b are both different from zero and the ratio of these two numbers is irrational.*
    (b) *There does not exist a closed topological disc on the torus that is invariant under the flow.*
(2) *The flow on the torus is topologically transitive.*
(3) *The flow on the torus is metrically transitive.*

Proof. (1) $\rightarrow$ (2). Recall that a point p is called quasi-regular provided for every continuous function f on the torus,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(tx)\, dt$$

exists. The set of points that are both quasi-regular and recurrent has measure one [**5**]. For any quasi-regular point (x_0, y_0), $\lim_{t\to\infty}(x(t)/t)$ and $\lim_{t\to\infty}(y(t)/t)$ exist, since these are the same as

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(x(t), y(t))\, dt \quad \text{and} \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T Y(x(t), y(t))\, dt,$$

respectively. (Here $(x(t), y(t))$ is the orbit passing through (x_0, y_0) at time zero.)

Let e_1 and e_2 be the elements of the one-dimensional homology group of the torus determined by the closed curves on the torus coming from the straight line segments in the plane from the origin to $(1, 0)$ and $(0, 1)$, respectively. If p is a recurrent quasi-regular point on the torus coming from the point (x_0, y_0) in the plane, choose a straight

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line segment through (x_0, y_0) and everywhere transverse to the orbits of the flow and a sequence of positive numbers t_n going to infinity such that for each n the point $t_n p$ lies on the straight line segment. Let C_n be the closed curve obtained by closing up the orbit from p to $t_n p$ by a straight line segment, and let $\bar{C}_n$ be the element of the one-dimensional homology group of the torus (with real coefficients) determined by C_n. Then $\lim_{n\to\infty}(\bar{C}_n/t_n)$ exists and equals $\alpha e_1 + \beta e_2$ where $\alpha = \lim_{t\to\infty}(x(t)/t)$ and $\beta = \lim_{t\to\infty}(y(t)/t)$. We call this limit the asymptotic cycle determined by p and denote it by A_p [7]. If μ is the measure on the torus determined by $dx \wedge dy$, then

$$\begin{aligned}
&\int A_p \, d\mu(p) \\
&= \int \lim_{T\to\infty} \left(\frac{x(Tp) - x(p)}{T} e_1 + \frac{y(Tp) - y(p)}{T} e_2 \right) d\mu(p) \\
&= \int \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T X(tp)\, dt \right) d\mu(p) e_1 + \int \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T Y(tp)\, dt \right) d\mu(p) e_2 \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int X(tp)\, d\mu(p) \right) dt e_1 + \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int Y(tp)\, d\mu(p) \right) dt e_2 \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int X(p)\, d\mu(p) \right) dt e_1 + \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int Y(p)\, d\mu(p) \right) dt e_2 \\
&= a e_1 + b e_2.
\end{aligned}$$

If p and p' are quasi-regular points on the same orbit, $A_p = A_{p'}$. If p and p' are non-stationary points on different orbits, C_n and C'_n can intersect only in points that lie either on the line segment joining $t_n p$ to p or on the line segment joining $t'_n p'$ to p'.

However, since the time it takes a point on our original line segment to return to that segment (if it does so at all) is bounded from below by a positive number, there exists a $T_0 > 0$ such that the number of intersections of C'_n with the line segment joining $t_n p$ to p is less than t_n/T_0 and the number of intersections of C_n with the line segment joining $t'_n p'$ to p' is less than t'_n/T_0. It follows that the number of intersections of C_n with C'_n is less than $(t_n/T_0) + (t'_n/T_0)$. Therefore, the intersection number of the homology classes $(1/t_n)\bar{C}_n$ and $(1/t'_n)\bar{C}'_n$ is less than $(1/t_n T_0) + (1/t'_n T_0)$. Taking the limit as n goes to infinity, we see that the intersection class of A_p and $A_{p'}$ is zero in the homology ring of the torus. Hence there is a one-dimensional subspace of the one-dimensional homology ring of the torus that contains A_p for every recurrent quasi-regular point p.

From now on assume that statements (1a) and (1b) of our proposition hold. Since $\int A_p \, d\mu(p) = a e_1 + b e_2$ it follows that the set of points p for which A_p is a non-zero multiple of $a e_1 + b e_2$ is of positive measure. It is known that for almost all quasi-regular points p and any continuous function f on the torus,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(tp)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^0 f(tp)\, dt.$$

From this we can conclude the following.

LEMMA 1. *There exists an orbit $(x(t), y(t))$ in the plane such that*

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \lim_{t\to-\infty} \frac{x(t)}{y(t)} = \frac{a}{b}$$

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and a/b is therefore irrational. Moreover, either $\lim_{t\to\infty} x(t) = \infty$ *and* $\lim_{t\to-\infty} x(t) = -\infty$ *or* $\lim_{t\to\infty} x(t) = -\infty$ *and* $\lim_{t\to-\infty} x(t) = \infty$*, and similarly for* $y(t)$.

Next let K denote the set of points on the torus where the vector field vanishes.

LEMMA 2. *K does not contain any simple closed curve.*

Proof. Suppose C_0 is a simple closed curve contained in K. Since we are assuming statement (1b) of our proposition holds, C_0 cannot be the boundary of a closed topological disc on the torus. Therefore, C_0 is not homotopic to zero. Let q_0 be a point on C_0 corresponding to the point (u_0, v_0) in the plane.

Since C_0 can be parametrized by a periodic function from the real line into the torus, we can lift this to a parametric curve $(u(t), v(t))$ in the plane such that $u(0) = u_0$, $v(0) = v_0$. If we parametrize C_0 properly, either we will have that $\lim_{|t|\to\infty}(v(t)/u(t))$ exists and is rational while $\lim_{t\to\infty} u(t) = \infty$ and $\lim_{t\to-\infty} u(t) = -\infty$ or we will have that $u(t)$ is bounded while $\lim_{t\to\infty} v(t) = \infty$ and $\lim_{t\to-\infty} v(t) = -\infty$. By Lemma 1 there is a solution curve $(x(t), y(t))$ to our differential equations in the plane such that $\lim_{|t|\to\infty}(x(t)/y(t))$ exists and is irrational, and as t goes from $-\infty$ to $+\infty$ either $x(t)$ goes from $-\infty$ to ∞ or from ∞ to $-\infty$ and similarly for $y(t)$. Let G be the function that sends a point in the plane with polar coordinates (ρ, θ) into the point with polar coordinates $(\arctan\rho, \theta)$. Then G sends the plane homeomorphically onto the interior of the circle about the origin of radius $\pi/2$. The curve $(x(t), y(t))$ is sent by G onto a simple arc with endpoints on the boundary of the circle which are diametrically opposite. Similarly $(u(t), v(t))$ is sent onto a simple arc with a different pair of diametrically opposite endpoints. It is intuitively clear that these two arcs must intersect, and a rigorous proof can be given using basic facts about plane topology. This would imply that the orbit $(x(t), y(t))$ contains a point on the curve $(u(t), v(t))$. However, any point $(u(t), v(t))$ is stationary, so we have a contradiction and our lemma is proved. □

Now let K_0 be any component of the set of points K on the torus where our vector field vanishes. If K_0 consists of more than one point there is a proof in [**6**] of the well known fact that since K is the set of zeros of a real analytic function, it (and therefore K_0) is a one-dimensional complex with an even number of branches emanating from any point. By Lemma 2, K_0 does not contain any simple closed curve. Hence K_0 must be a dendrite [**9**]. However, any dendrite contains at least two endpoints and in this case there would have to be a point with exactly one branch emanating from it. Thus K_0 can only consist of a single point. Since K is a compact complex, it must be finite.

Since our vector field is analytic, a point where the vector field vanishes must either be a focus, be encircled by arbitrarily small periodic orbits, or have a neighbourhood consisting of a finite number of sectors, each of which is either parabolic, elliptic, or hyperbolic [**4**]. Because our flow is measure preserving, the recurrent points are dense, which implies the only possibility is that our point has a neighbourhood consisting of hyperbolic sectors. The number of hyperbolic sectors at a singular point must be even. If the number is $2n$ then our singular point must have index $-(n-1)$ [**3**]. There are only finitely many singular points and because we are on the torus the sum of the indices

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must be zero. Thus each singular point has a neighbourhood consisting of two hyperbolic sectors. By altering the flow (and perhaps the differentiable structure on the torus) in a small neighbourhood of each singular point we arrive at a smooth flow on the torus whose orbits are the same as those we had originally, except that we no longer have any stationary points. (At each singular point there is, for the original flow, exactly one orbit approaching the singular point as $t \to +\infty$ and one approaching the point as $t \to -\infty$.)

To prove that our original flow was topologically transitive we need only show that this modified flow has no closed orbits [**8**]. However, the proof of this goes along exactly the same lines as our proof that the set K of stationary points of the flow contains no simple closed curve. This concludes our proof that statement (1) implies statement (2).

(2) $\to$ (3). In [**6**] it is shown that if we have a topologically transitive flow on the torus given by analytic differential equations there are only finitely many stationary points and the flow is orbitally equivalent to a Kronecker straight line flow, modified so as to have a finite number of stationary points. It is also remarked that if in this case there is an invariant probability measure that is zero for points, then this measure is unique and the flow is metrically transitive with respect to it. These assumptions hold in the present case and therefore we get metric transitivity.

(3) $\to$ (1). Since we are assuming metric transitivity we must have topological transitivity. This obviously implies statement (1b). To prove statement (1a) we need only demonstrate the following.

LEMMA 3. *If either $b = 0$ or a/b is rational, the union of the set of nonstationary periodic orbits is an open set whose complement is a $C - W$ complex of dimension at most one.*

Proof. By multiplying our vector field by a suitable positive constant we get a situation where a and b are both integers, if the assumptions of our lemma hold. Thus in proving Lemma 3 we can assume without loss of generality that a and b are both integers.

Now let L be the set of points in the plane where our vector field $(X, Y) = (a - F_y, b + F_x)$ vanishes. As noted above, L is a $C - W$ complex of dimension at most one such that there are an even number of analytic branches emanating from any point that is not isolated. If we let $H(x, y) = F(x, y) + bx - ay$ then since the gradient of H vanishes on L, H must be constant on each component of L.

Define $G(x, y) = \exp 2\pi i(F(x, y) + bx - ay)$. Since we are assuming that a and b are both integers, $G(x, y)$ arises from a function $\bar{G}$ on the torus. The set K on the torus where our vector field vanishes is a compact $C - W$ complex of dimension at most one and $\bar{G}$ must be constant on each component of K.

It follows that $\bar{G}$ assumes only finitely many distinct values on K. Let these values be $\lambda_1, \ldots, \lambda_n$ and let A_i be the set of points on the torus at which $\bar{G}$ assumes the value λ_i. Since $\bar{G}$ is a function invariant under the flow, UA_i is an invariant set and it contains all points that are stationary under the flow. It follows that the complement of UA_i is composed of nonstationary periodic orbits, because $\bar{G}$ is constant on each orbit in this complement and the gradient of $F(x, y) + bx - ay$ is different from zero at each point in the plane that is outside L. The union of all the nonstationary periodic orbits on the torus is clearly an open set whose complement is a subcomplex of the one-dimensional complex UA_i. □

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We can now prove Theorem 2. In fact if a/b is irrational but our flow is not metrically transitive, there is a closed topological disc on the torus that is invariant under the flow. This can be lifted to a disc D in the plane that is invariant under the flow. Since the flow in D preserves a finite invariant measure that is positive for open sets, the recurrent points are dense in D. However, a recurrent point for a flow in an open disc is either stationary or is a closed orbit. Thus our flow in the plane possesses a nonstationary closed orbit.

Since wherever we have a closed disc on which a flow takes place there must be a stationary point, we get a new proof that if (a, b) is not in the set A described in the statement of Theorem 1 and a/b is irrational we must have metric transitivity. On the other hand, our proposition shows that if a/b is not irrational we do not have metric transitivity.

To prove Theorem 1 we need the following.

LEMMA 4. *If a flow defined by an analytic vector field on the torus has a non-degenerate critical point, it cannot be topologically transitive.*

Proof. It is known [**2**, **6**] that a topologically transitive flow on the torus defined by analytic differential equations has the property that each critical point has a neighbourhood that consists of exactly two hyperbolic sectors. However, a non-degenerate critical point has no such neighbourhood. □

Now we need only observe that if (a, b) belongs to A but not to B and (x_0, y_0) represents a point on the torus that is sent into (a, b) by the map P described in the introduction, the differential equations arising from (a, b) have a non-degenerate critical point at (x_0, y_0).

Next, suppose that (a, b) belongs to the boundary of A. If (x_0, y_0) is sent into (a, b) by P, the Jacobian of P at (x_0, y_0) must be zero. However, this just says that $(a, b) \in B$.

If (a, b) belongs to the boundary of A and a/b is not irrational we know that the flow is not metrically transitive.

THEOREM 3. *If (a, b) belongs to the boundary of A and a/b is irrational, our flow is metrically transitive.*

Proof. Suppose we had a counterexample. Then the corresponding flow in the plane would have a closed orbit C_0. Let (x_0, y_0) be a point in C_0 and let A be a smooth arc transverse to C_0 at (x_0, y_0). If the period of (x_0, y_0) under the flow is T_0, any flow that is a sufficiently close approximation in the C^2 topology to our original flow on the torus will bring (x_0, y_0) back close to itself after time T_0 and therefore there will be a time T_1 close to T_0 that will bring (x_0, y_0) to a point (x_1, y_1) on A.

If (x_1, y_1) were different from (x_0, y_0), it is clear that the perturbed orbit from (x_0, y_0) to (x_1, y_1) together with the subarc of A from (x_0, y_0) to (x_1, y_1) would yield a simple closed curve bounding a closed two cell. If (x_0, y_0) were not equal to (x_1, y_1) (so that the subarc of A was non-degenerate), it is clear that this closed two cell would be sent into a closed two cell contained in it that omitted an open subset, either for positive values of t or for negative values of t. Thus if the approximating flow is measure preserving

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we must have $(x_0, y_0) = (x_1, y_1)$, so the approximating flow is not metrically transitive. However, since (a, b) was assumed to be on the boundary of A we can approximate (a, b) by a point (a', b') in the complement of A such that a'/b' is irrational. This means we can approximate our original flow by a measure-preserving flow that is metrically transitive. This establishes our theorem. □

We have shown that, if (a, b) belongs to the complement of the interior of A, the corresponding flow on the torus is metrically transitive if and only if a/b is irrational. On the other hand, for all points in the interior of A except those on a nowhere dense set of measure zero we know that we do not get metric transitivity. It remains an open problem to determine whether it is possible for a point in the interior of A to produce a metrically transitive flow.

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