J. K-Theory **12** (2013), 3–14
doi:10.1017/is013005020jkt228
©2013 ISOPP

Remarks on Murre's conjecture on Chow groups

Dedicated to Professor Aderemi Kuku
on the Occasion of his 70th Birthday

by

KEJIAN XU AND ZE XU*

Abstract

For certain product varieties, Murre's conjecture on Chow groups is investigated. More precisely, let k be an algebraically closed field, X be a smooth projective variety over k and C be a smooth projective irreducible curve over k with function field K. Then we prove that if X (resp. X_K) satisfies Murre's conjectures (A) and (B) for a set of Chow-Künneth projectors $\{\pi_i', 0 \leq i \leq 2\dim X\}$ of X (resp. for $\{(\pi_i')_K\}$ of X_K) and if for any j, $\mathrm{CH}^j_{\mathrm{alg}}(X_K;\mathbb{Q}) \subseteq \mathrm{Ker}((\pi'_{2j})_K)$, then the product variety $X \times C$ also satisfies Murre's conjectures (A) and (B). As consequences, it is proved that if C is a curve and X is an elliptic modular threefold over k (an algebraically closed field of characteristic 0) or an abelian variety of dimension 3, then Murre's conjecture (B) is true for the fourfold $X \times C$.

Key Words: motivic decomposition, Chow group, curve, abelian variety, elliptic modular threefold.

Mathematics Subject Classification 2010: 14C25.

1. Introduction

We will work with the category $\mathscr{V}(k)$ of smooth projective varieties over a field k. Let $X \in \mathscr{V}(k)$ be irreducible and of dimension d. Let $H(X) := H^*_{et}(\overline{X}, \mathbb{Q}_\ell)$ be the ℓ-adic cohomology groups over a (fixed) algebraic closure $\overline{k}$ of k, where $\overline{X} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ and $\ell \neq \mathrm{ch}(k)$ is a prime, and let $\mathrm{cl}_X : Z^i(X) \to H^{2i}(X)$ be the cycle map associated to $H(X)$, where $Z^i(X)$ is the group of algebraic cycles of codimension i on X (By definition, an algebraic cycle on X is a finite $\mathbb{Z}$-linear combination of integral subvarieties of X). We have the well-known Künneth formula:

$$H^{2d}(X \times X) \simeq \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X).$$

*This research is supported by National Natural Science Foundation of China (Grant No. 10871106).

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

Let $\Delta_X \subseteq X \times X$ be the diagonal. Then $\mathrm{cl}_{X\times X}(\Delta_X)$ has the Künneth decomposition:

$$\mathrm{cl}_{X\times X}(\Delta_X) = \pi_0^{\mathrm{hom}} + \pi_1^{\mathrm{hom}} + \ldots + \pi_{2d}^{\mathrm{hom}},$$

where $\pi_i^{\mathrm{hom}} \in H^{2d-i}(X) \otimes H^i(X)$ is the i-th Künneth component.

Let $A^j_{\mathrm{num}}(X)$ (resp. $A^j_{\mathrm{rat}}(X) = \mathrm{CH}^j(X)$) be the groups of algebraic cycles of codimension j modulo numerical equivalence (resp. rational equivalence). Grothendieck's Künneth standard conjecture states that the π_i^{hom} are all algebraic (i.e., all of them are in the image of the cycle map). Another classical conjecture asserts that homological equivalence coincides with numerical equivalence ([14]). Assuming these two conjectures, the diagonal (modulo numerical equivalence) has a canonical decomposition into a sum of orthogonal idempotents (also called projectors)

$$\Delta_X = \pi_0^{\mathrm{num}} + \pi_1^{\mathrm{num}} + \ldots + \pi_{2d}^{\mathrm{num}} \tag{1}$$

in the correspondence ring $A^d_{\mathrm{num}}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$, as dreamed by Grothendieck. In this setting, in the category of Grothendieck motives $\mathscr{M}_k^{\mathrm{num}}$ ([14]) (with respect to numerical equivalence), the motive $h(X) \in \mathscr{M}_k^{\mathrm{num}}$ has a canonical decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \ldots \oplus h^{2d}(X), \tag{2}$$

where $h^i(X) := (X, \pi_i^{\mathrm{num}}, 0) \in \mathscr{M}_k^{\mathrm{num}}$ (See [14] for details).

Furthermore, Murre ([16]) expects that the conjectural decomposition (1) exists even in the Chow group $\mathrm{CH}^d(X \times X; \mathbb{Q}) := \mathrm{CH}^d(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q} := A^d_{\mathrm{rat}}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and hence in the category of Chow motives $\mathscr{M}_k^{\mathrm{rat}}$ (with respect to rational equivalence), $h(X) \in \mathscr{M}_k^{\mathrm{rat}}$ has a similar decomposition as in (2). In this new setting, the decomposition is not canonical any more. However, from this conjectural decomposition, Murre ([16]) conjectured a very interesting filtration on rational Chow groups which relates rational equivalence to homological equivalence in finite steps as done by the conjectural Bloch-Beilinson filtration.

More precisely, as in [16], we will say that X has a *Chow-Künneth decomposition* over k if there exist $\pi_i \in \mathrm{CH}^d(X \times X; \mathbb{Q}), 0 \le i \le 2d$, satisfying

(i) π_i are mutually orthogonal projectors;

(ii) $\sum_i \pi_i = \Delta_X$;

(iii) $\mathrm{cl}_{X\times X}(\pi_i) = \pi_i^{\mathrm{hom}}$ (the i-th Künneth component).

Equivalently, the Chow motive of X has a (Chow-Künneth) decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \ldots \oplus h^{2d}(X),$$

where $h^i(X) := (X, \pi_i, 0) \in \mathscr{M}_k^{\mathrm{rat}}$.

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

Note that any correspondence $\alpha \in \mathrm{CH}^d(X \times X;\mathbb{Q})$ acts on the Chow group $\mathrm{CH}^p(X;\mathbb{Q})$ by $\alpha(Z) = p_{2*}(\alpha \cdot p_1^* Z) \in \mathrm{CH}^p(X;\mathbb{Q})$ for any $Z \in \mathrm{CH}^p(X;\mathbb{Q})$, where p_i are the projections.

Then, Murre proposed in [16] the following famous conjecture.

Murre's Conjecture

(A): There exists a Chow-Künneth decomposition for every irreducible variety $X \in \mathscr{V}(k)$ of dimension d.

(B): $\pi_0,\ldots,\pi_{j-1}$ and $\pi_{2j+1},\ldots,\pi_{2d}$ act as zero on $\mathrm{CH}^j(X;\mathbb{Q})$.

(C): Let $F^v\mathrm{CH}^j(X;\mathbb{Q}) = \mathrm{Ker}\pi_{2j} \bigcap \mathrm{Ker}\pi_{2j-1} \bigcap \ldots \bigcap \mathrm{Ker}\pi_{2j-v+1}$. Then the filtration $F^\bullet$ is independent of choice of the π_i.

(D): $F^1\mathrm{CH}^j(X;\mathbb{Q}) = \mathrm{CH}^j_{\mathrm{hom}}(X;\mathbb{Q}) := \mathrm{Ker}(\mathrm{cl}_X)$.

It was shown by Jannsen ([10]) that Murre's conjecture is equivalent to the Bloch-Beilinson conjecture on rational Chow groups and that the two conjectural filtrations proposed respectively by Murre, and Bloch and Beilinson coincide. The main advantage of Murre's conjecture over Bloch-Beilinson's is that one can check the statements for specific varieties as we will do in this paper.

Until now, Murre's conjecture is verified for only a few special varieties. It is known that (A) is true for curves, surfaces ([15]), abelian varieties ([5][18]), some threefolds ([3][4]), some special fourfolds ([12][20]), certain modular varieties ([7][8]) and varieties with finite-dimensional associated Chow motives such that the Künneth components of the diagonal are algebraic ([10][11]). As for the other parts of Murre's conjecture, it is known that (B) and (D) are true for the product of a curve and a surface ([16]), that (B) is true for the product of two surfaces ([12]) and that some part of (D) is true for the product of two surfaces ([12][13]). Jannsen ([11]) proved that (A), (B), (C) and (D) are true for some very special higher dimensional varieties over some special ground fields; in particular, he proved that if k is a rational or elliptic function field (in one variable) over a finite field $\mathbb{F}$ and X_0 is an arbitrary product of rational and elliptic curves over $\mathbb{F}$, then (A)-(D) hold for $X_0 \times_{\mathbb{F}} k$. Gordon and Murre ([9]) proved that (A), (B) and (D) are true for elliptic modular threefolds over a field of characteristic 0.

In this paper, we consider Murre's conjecture for certain product varieties. Concretely, we consider such a problem: if the conjecture is true for X, when is it also true for the product of X with a curve? We generalize Murre's discussion given in [17], and in particular, for an algebraically closed field k, $X \in \mathscr{V}(k)$ and an irreducible curve $C \in \mathscr{V}(k)$ with function field $K = k(C)$, we prove as the main result (Theorem 2.4) of this paper that if X has a set of Chow-Künneth projectors $\{\pi'_i, 0 \leq i \leq 2\dim X\}$ that satisfies (B) and such that X_K, endowed with the induced Chow-Künneth decomposition, also satisfies (B) and that for any j, $\mathrm{CH}^j_{\mathrm{alg}}(X_K;\mathbb{Q}) \subseteq \mathrm{Ker}((\pi'_{2j})_K)$, then induced Chow-Künneth decomposition of

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

$X \times C$ satisfies (B). As consequences, we prove that if C is a (smooth projective connected) curve, then Murre's conjecture (B) is true for $X \times C$, where X is an elliptic modular threefold over k (an algebraically closed field of characteristic 0) or X is an abelian variety of dimension 3. This implies particularly that (B) is true for two new kinds of fourfolds other than products of two surfaces considered in [12] and [13]. Vial [20] has proved independently some new results using a quite different method.

2. The Main Theorem and Consequences

First, for product varieties, we have the following easy lemma.

Lemma 2.1 *([16]) Assume that Y_i $(i = 1,2)$ are smooth projective irreducible varieties over k. Let $Y = Y_1 \times Y_2$. If Y_i $(i = 1,2)$ have a Chow-Künneth decomposition, then Y has also a Chow-Künneth decomposition.*

Indeed, if $\{\pi_{pY_i}\}_{0\le p\le 2dimY_i}$ is a set of Chow-Künneth projectors for Y_i, then the naturally induced Chow-Künneth projectors of Y are

$$\pi_{kY} = \sum_{p+q=k} \pi_{pY_1} \otimes \pi_{qY_2} \ (0 \le k \le 2(dimY_1 + dimY_2)),$$

where $\pi_{pY_1} \otimes \pi_{qY_2} := \tau_*(\pi_{pY_1} \times \pi_{qY_2})$ with $\tau : Y_1 \times Y_1 \times Y_2 \times Y_2 \to Y_1 \times Y_2 \times Y_1 \times Y_2$ switching the second and the third factor.

Now let C be a smooth projective curve with a rational point e defined over a field k. It is well-known (see [19] for details) that C has a Chow-Künneth decomposition

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C),$$

where $h^i(C) = (C, \pi_i'')$ with

$$\pi_0'' = [e \times C], \ \ \pi_2'' = [C \times e], \ \ \pi_1'' = \Delta_C - \pi_0'' - \pi_2''.$$

Assume that the irreducible variety $X \in \mathscr{V}(k)$ has the Chow-Künneth decomposition

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X), \ \ h^i(X) = (X, \pi_i').$$

Then, the product variety $X \times C$ has the Chow-Künneth decomposition

$$h(X \times C) = \bigoplus_{m=0}^{2(d+1)} (X \times C, \pi_m), \tag{3}$$

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

where, explicitly,

$\pi_0 = \pi'_0 \otimes \pi''_0$,
$\pi_1 = \pi'_1 \otimes \pi''_0 + \pi'_0 \otimes \pi''_1$,
$\pi_m = \pi'_m \otimes \pi''_0 + \pi'_{m-1} \otimes \pi''_1 + \pi'_{m-2} \otimes \pi''_2$ $(2 \le m \le 2d)$,
$\pi_{2d+1} = \pi'_{2d-1} \otimes \pi''_2 + \pi'_{2d} \otimes \pi''_1$,
$\pi_{2d+2} = \pi'_{2d} \otimes \pi''_2$.

Let

$$\mathrm{CH}^j_{\mathrm{alg}}(X;\mathbb{Q}) := \{Z \in \mathrm{CH}^j(X;\mathbb{Q}) : Z \sim_{\mathrm{alg}} 0\},$$

where $Z \sim_{\mathrm{alg}} 0$ means that Z is algebraically equivalent to 0.

In the proof of Theorem 2.4, we need the following computations.

Lemma 2.2 *For any $Z \in CH^j(X \times C;\mathbb{Q})$, we have*

(i) $(\pi'_m \otimes [e \times C])(Z) = \pi'_m(Z(e)) \times [C]$, where $Z(e) := p_{1*}(Z \cdot (X \times [e]))$ with $p_1 : X \times C \to X$ the first projection;

(ii) $(\pi'_m \otimes [C \times e])(Z) = \pi'_m(p_{1*}Z) \times [e]$.

Proof: (i) We have

$$\begin{aligned}(\pi'_m \otimes [e \times C])(Z) &= p_{34*}(\tau_*(\pi'_m \times [e \times C]) \cdot (Z \times [X \times C])) \\ &= p_{34*}((\tau_*(\pi'_m \times [e]) \cdot (Z \times [X])) \times [C]) \\ &= p_{3*}(\tau_*(\pi'_m \times [e]) \cdot (Z \times [X])) \times [C] \\ &= \pi'_m(Z(e)) \times [C],\end{aligned}$$

where p_i (resp. p_{ij}) denotes the projection to the i-th (resp. the (i,j)-th) factor of a product variety and τ is the map that switches the second and third factor. (Note that in the following, we keep similar notations.) In the second equality, commutativity of intersection product and exterior product is used. Note that in the last equality, we have used the following computation.

$$\begin{aligned}p_{3*}(\tau_*(\pi'_m \times [e]) \cdot (Z \times [X]) &= p_{3*}(\tau_*((\pi'_m \times [C]) \cdot [X \times X \times e]) \cdot (Z \times [X])) \\ &= p_{3*}((p_{13}^*\pi'_m \cdot [X \times e \times X]) \cdot (Z \times [X])) \\ &= p_{3*}(p_{13}^*\pi'_m \cdot ((Z \cdot [X \times e]) \times [X])) \\ &= p_{2*}p_{13*}(p_{13}^*\pi'_m \cdot ((Z \cdot [X \times e]) \times [X])) \\ &= p_{2*}(\pi'_m \cdot p_{13*}((Z \cdot [X \times e]) \times [X])) \\ &= p_{2*}(\pi'_m \cdot (p_{1*}(Z \cdot [X \times e]) \times [X])) \\ &= \pi'_m(Z(e)).\end{aligned}$$

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

The fiber square formula is used in the second equality, commutativity of intersection product and exterior product is used in the third equality and the projection formula is used in the fifth equality.

(ii) Similar to (i), we have

$$\begin{aligned}(\pi'_m \otimes [C \times e])(Z) &= p_{34*}(\tau_*(\pi'_m \times [C \times e]) \cdot (Z \times [X \times C])) \\ &= p_{34*}(\tau_*(\pi'_m \times [C]) \cdot (Z \times [X]) \times [e]) \\ &= p_{3*}(\tau_*(\pi'_m \times [C]) \cdot (Z \times [X])) \times [e] \\ &= \pi'_m(p_{1*}Z) \times [e].\end{aligned}$$

□

Lemma 2.3 *Let $Z \in CH^j(X \times C; \mathbb{Q})$. Then*
(i) $(\pi'_m \otimes \Delta_C)(Z) = p_{23*}(p_{13}^* Z \cdot (\pi'_m \times [C]))$;
(ii) $(id_X \times f)^*((\pi'_m \otimes \Delta_C)(Z)) = (\pi'_m)_K((id_X \times f)^* Z)$, where $K = k(C)$ is the function field of C, $f : Spec(K) \longrightarrow C$ is the natural morphism and $(\pi'_m)_K = \pi'_m \times \Delta_{Spec(K)}$.

Proof: (i) Let $\delta_C : C \longrightarrow C \times C$ be the diagonal morphism of C. Then, we have

$$\begin{aligned}(\pi'_m \otimes \Delta_C)(Z) &= p_{34*}(\tau_*(\pi'_m \times \Delta_C) \cdot (Z \times [X \times C])) \\ &= p_{34*}(\tau_*(\mathrm{id}_{X\times X} \times \delta_C)_*(\pi'_m \times \Delta_C) \cdot (Z \times [X \times C])) \\ &= p_{23*}((\pi'_m \times \Delta_C) \cdot (\mathrm{id}_{X\times X} \times \delta_C)^* \tau^* Z)) \\ &= p_{23*}(p_{13}^* Z \cdot (\pi'_m \times [C])).\end{aligned}$$

Note that the projection formula is used in the third and fourth equality.

(ii) From (i) and the following diagram

$$\begin{array}{ccc} X \times X \times \mathrm{Spec}(K) \simeq X_K \times_K X_K & \xrightarrow{p_{X_K}} & X_K \\ \downarrow{\scriptstyle \mathrm{id}_X \times \mathrm{id}_X \times f} & & \downarrow{\scriptstyle \mathrm{id}_X \times f} \\ X \times X \times C & \xrightarrow{p_{23}} & X \times C \end{array}$$

we have

$$\begin{aligned}(\mathrm{id}_X \times f)^*((\pi'_m \otimes \Delta_C)(Z)) &= (\mathrm{id}_X \times f)^* p_{23*}(p_{13}^* Z \cdot \pi'_m \times [C]) \\ &= p_{X_K*}(\mathrm{id}_X \times \mathrm{id}_X \times f)^*(p_{13}^* Z \cdot \pi'_m \times [C]) \\ &= p_{X_K*}((\mathrm{id}_X \times \mathrm{id}_X \times f)^* p_{13}^* Z \cdot (\mathrm{id}_X \times \mathrm{id}_X \times f)^*(\pi'_m \times [C]))\end{aligned}$$

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

$$= p_{X_K *}(((\mathrm{id}_X \times f)^* Z \times_K X_K) \cdot \pi'_m \times \Delta_{\mathrm{spec}K})$$
$$= (\pi'_m \otimes \Delta_{\mathrm{spec}K})((\mathrm{id}_X \times f)^* Z)$$
$$= (\pi'_m)_K((\mathrm{id}_X \times f)^* Z).$$

In the second equality, a slight generalization of the formula in [6, Prop.1.7, p.18] is used. In the third equality functoriality is used; while in the fourth equality, we use the isomorphism $X \times X \times \mathrm{Spec}(K) \simeq X_K \times_K X_K$. □

Now we can give our main theorem as follows.

Theorem 2.4 *Let k be an algebraically closed field, $X \in \mathscr{V}(k)$ and $C \in \mathscr{V}(k)$ an irreducible curve with function field $K = k(C)$. Assume that X has a Chow-Künneth decomposition $\{\pi'_i, 0 \le i \le 2dimX\}$ that satisfies Murre's conjecture (B), X_K endowed with the Chow-Künneth decomposition $\{(\pi'_i)_K, 0 \le i \le 2dimX\}$ also satisfies Murre's conjecture (B), and for any j,*

$$CH^j_{alg}(X_K; \mathbb{Q}) \subseteq Ker((\pi'_{2j})_K).$$

Then the Chow-Künneth decomposition of $X \times C$ given as in (3) satisfies Murre's conjecture (B).

Proof: Let $Z \in \mathrm{CH}^j(X \times C; \mathbb{Q})$. Easy computations show that (B) is true if Z is of the form $Z' \times [C]$ with $Z' \in \mathrm{CH}^j(X; \mathbb{Q})$. So, we can assume $Z(e) = 0$, since we have

$$Z = (Z - Z(e) \times [C]) + Z(e) \times [C],$$
$$(Z - Z(e) \times [C])(e) = Z(e) - Z(e) = 0.$$

Assume that

$$0 \le m \le j - 1 \quad \text{or} \quad 2j + 1 \le m \le 2(d + 1).$$

Then, for $m \ge 1$, we have

$$0 \le m - 1 \le (j - 1) - 1 \quad \text{or} \quad 2(j - 1) + 1 \le m - 1,$$

and for $m \ge 2$, we have

$$0 \le m - 2 \le (j - 1) - 1 \quad \text{or} \quad 2(j - 1) + 1 \le m - 1.$$

From Lemma 2.2 and the assumptions on X, we have (note that $p_{1*}Z \in \mathrm{CH}^{j-1}(X; \mathbb{Q})$)

$$(\pi'_m \otimes [e \times C])(Z) = \pi'_m(Z(e)) \times [C] = 0,$$

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

$$(\pi'_{m-1}\otimes[e\times C])(Z)=\pi'_{m-1}(Z(e))\times[C]=0,$$
$$(\pi'_{m-1}\otimes[C\times e])(Z)=\pi'_{m-1}(p_{1*}Z)\times[e]=0,$$
$$(\pi'_{m-2}\otimes[C\times e])(Z)=\pi'_{m-2}(p_{1*}Z)\times[e]=0.$$

So, the problem is reduced to prove

$$(\pi'_{m-1}\otimes\Delta_C)(Z)=0,\ \text{if}\ \ 1\le m\le j-1\ \text{or}\ \ 2j+1\le m\le 2(d+1).$$

At first, we show that $(\mathrm{id}\times f)^*Z$ is algebraically equivalent to 0 on X_K. In fact, let η be the generic point of C, that is, $K=k(\eta)$, and let $f_\eta:\mathrm{Spec}(K)\longrightarrow C_K$ be the K-point defined by η. Denote $\eta_K=f_\eta(\mathrm{Spec}(K))$. Then we have

$$Z_K(\eta_K)=p_{X_K*}(Z_K\cdot X\times\eta_K)=p_{X_K*}(Z_K\cdot(\mathrm{id}_X\times f_\eta)_*(X\times\mathrm{Spec}(K)))$$
$$=p_{X_K*}(\mathrm{id}_X\times f_\eta)_*((\mathrm{id}_X\times f_\eta)^*Z_K)=(\mathrm{id}_X\times f_\eta)^*Z_K$$
$$=(\mathrm{id}_X\times f_\eta)^*p_{XC}^{XCK*}(Z)=(\mathrm{id}_X\times f)^*(Z).$$

Similarly, let $g_e:\mathrm{Spec}(K)\longrightarrow C_K$ and $g:\mathrm{Spec}(k)\longrightarrow C$ be the morphisms both defined by e. Denote $e_K=g_e(\mathrm{Spec}(K))$. Then we have $Z_K(e_K)=((\mathrm{id}_X\times g)^*(Z))_K=Z(e)_K=0$. Thus $(\mathrm{id}\times f)^*Z=Z_K(\eta_K)$, which is algebraically equivalent to $Z_K(e_K)$, is algebraically equivalent to 0.

We claim that

$$(\pi'_{m-1})_K((\mathrm{id}\times f)^*Z)=0,\ \ \text{if}\ \ 1\le m\le j-1\ \ \text{or}\ \ 2j+1\le m\le 2(d+1).$$

In fact, if $m=2j+1$, since $(\mathrm{id}\times f)^*Z\in\mathrm{CH}^j(X_K;\mathbb{Q})$ is algebraically equivalent to 0, from the assumption $\mathrm{CH}^j_{\mathrm{alg}}(X_K;\mathbb{Q})\subseteq\mathrm{Ker}((\pi'_{2j})_K)$ we have

$$(\pi'_{m-1})_K((\mathrm{id}\times f)^*Z)=(\pi'_{2j})_K((\mathrm{id}\times f)^*Z)=0;$$

if $1\le m\le j-1$ or $2j+2\le m\le 2(d+1)$, we have $1\le m-1\le j-2$ or $2j+1\le m-1\le 2d+1$, so from the assumptions on X we get $(\pi'_{m-1})_K((\mathrm{id}\times f)^*Z)=0$ since $(\mathrm{id}\times f)^*Z\in\mathrm{CH}^j(X_K;\mathbb{Q})$.

On the other hand, we have the following well known diagram ([2, Lem. (1A.1), p.21]):

$$\begin{array}{ccccccc}
\mathrm{CH}^{j-1}(X\times(C-U);\mathbb{Q}) & \to & \mathrm{CH}^j(X\times C;\mathbb{Q}) & \longrightarrow & \mathrm{CH}^j(X\times U;\mathbb{Q}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{CH}^j(X_K;\mathbb{Q}) & = & \varinjlim_{U\subseteq C}\mathrm{CH}^j(X\times U;\mathbb{Q}) & &
\end{array}$$

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

where U is non-empty open in C and the left vertical map is $z \mapsto (\mathrm{id} \times f)^* z$. So from Lemma 2.3 (ii), we have

$$(\mathrm{id}_X \times f)^*((\pi'_{m-1} \otimes \Delta_C)(Z)) = (\pi'_{m-1})_K((\mathrm{id} \times f)^* Z) = 0,$$

hence

$$(\pi'_{m-1} \otimes \Delta_C)(Z) = \sum_i Z'_i \times a_i, \text{ with } Z'_i \in \mathrm{CH}^{j-1}(X;\mathbb{Q}) \text{ and } a_i \in \mathrm{CH}^1(C;\mathbb{Q}).$$

In view of $(\pi'_{m-1} \otimes \Delta_C)^2 = \pi'_{m-1} \otimes \Delta_C$, we conclude that

$$\begin{aligned}(\pi'_{m-1} \otimes \Delta_C)(Z) &= \sum_i (\pi'_{m-1} \otimes \Delta_C)(Z'_i \times a_i) \\ &= \sum_i p_{23*}[p_{13}^*(Z'_i \times a_i) \cdot (\pi'_{m-1} \times [C])] \\ &= \sum_i \pi'_{m-1}(Z'_i) \times a_i = 0.\end{aligned}$$

This completes the proof of the theorem. □

Although the theorem above is restricted, we can deduce several interesting consequences.

Corollary 2.5 *If k is an algebraically closed field of characteristic 0 and X is an elliptic modular threefold over k, then (A) and (B) are true for $X \times C$.*

Proof: It was shown in [9] that Murre's conjecture holds for an elliptic modular threefold over a field of characteristic 0. Obviously, conjecture (D) for X_K implies the assumption of Theorem 2.4. So, the result is an immediate consequence of Theorem 2.4. □

Remark 2.6 A more general result has been proved independently by Vial ([20]) using a quite different method. In fact, he shows that Murre's conjecture (A) and (B) hold for any smooth projective 3-fold over a $k \subseteq \mathbb{C}$ which has a Chow-Künneth decomposition such that π_4 factors through a surface. See [20, Lem.4.16] for more details.

By [1] (see also [5] and [16]), for an abelian variety X of dimension g over any field k, we have the following decomposition

$$\mathrm{CH}^j(X;\mathbb{Q}) = \bigoplus_{s=j-g}^{j} \mathrm{CH}^j_s(X),$$

where

$$\mathrm{CH}^j_s(X) := \{\alpha \in \mathrm{CH}^j(X;\mathbb{Q}) | n^*\alpha = n^{2j-s}\alpha, \forall\, n \in \mathbb{Z}\}.$$

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

Corollary 2.7 *Let X be an abelian variety of dimension at most 4 over an algebraically closed field k. Assume that for any j, $CH_0^j(X_K)\bigcap CH_{alg}^j(X_K;\mathbb{Q}) = 0$. Then (A) and (B) are true for $X \times C$.*

In particular, if X is an abelian variety of dimension 3 over an algebraically closed field k, then (A) and (B) are true for $X \times C$.

Proof: It follows from [1] that conjecture (B) is true for an abelian variety of dimension at most 4, equivalently, Beauville's vanishing conjecture holds: $\mathrm{CH}_s^j(X) = 0$ if $s < 0$. By assumption and the fact that algebraic equivalence is adequate, we have

$$\mathrm{CH}^{\,j}_{\mathrm{alg}}(X_K;\mathbb{Q}) = \bigoplus_{s=1}^{j}(\mathrm{CH}_s^j(X_K) \cap \mathrm{CH}^j_{\mathrm{alg}}(X_K;\mathbb{Q})).$$

On the other hand, by Lemma 2.5.1 of [16], we see that for any j,

$$\mathrm{Ker}((\pi'_{2j})_K) = \bigoplus_{s=1}^{j}\mathrm{CH}_s^j(X_K).$$

Then we can apply Theorem 2.4 to end the proof of the first part.

Assume that X is an abelian variety of dimension 3 over an algebraically closed field k. By [1], we know that the restricted cycle map

$$c_0 : \mathrm{CH}_0^j(X_K) \longrightarrow H^{2j}(X_K)$$

is injective for $j = 0,1,g-1,g$ if X is an abelian variety of dimension g. So for any j,

$$\mathrm{CH}_0^j(X_K)\bigcap\mathrm{CH}^j_{\mathrm{alg}}(X_K;\mathbb{Q}) \subseteq \mathrm{CH}_0^j(X_K)\bigcap\mathrm{CH}^j_{\mathrm{hom}}(X_K;\mathbb{Q}) = 0.$$

Hence the result follows from the first part. □

Remark 2.8 (i) The assumption in the first part of Corollary 2.7 is needed only when $\dim X = 4$ and $j = 2$, because it always holds if $\dim X \leq 4$ ([1]). This assumption is a consequence of a conjecture of Beauville: the restricted cycle map $c_0 : \mathrm{CH}_0^j(X) \longrightarrow H^{2j}(X)$ is injective for any j.

(ii) The second part of this corollary can also be obtained from a result of Vial [20, Th.4.5].

REFERENCES

1. A. Beauville. Sur l'anneau de Chow d'une variété abélienne. *Math. Ann.* **273**(1986), 647–651

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

2. S. Bloch, Lectures on algebraic cycles, 2ed, Cambridge University Press, New York 2010

3. P. L. del Angel and S. Müller-Stach. Motives of uniruled 3-folds, *Compositio Math.* **112**(1998), 1–16

4. P. L. del Angel, S. Müller-Stach. On Chow motives of 3-folds. *Trans. Amer. Math. Soc.* **352**(2000), 1623–1633

5. C. Deninger and J. P. Murre. Motivic decomposition of abelian schemes and the Fourier transform. *J. Reine Angew. Math.* **422**(1991), 201–219

6. W. Fulton. Intersection theory. *Ergeb. Math. Grenzgeb.* Springer-Verlag, Berlin, 1984

7. B. Gordon, M. Hanamura and J. P. Murre. Relative Chow-Künneth projectors for modular varieties. *J. reine angew. Math.* **558**(2003), 1–14

8. B. Gordon, M. Hanamura and J. P. Murre. Absolute Chow-Künneth projectors for modular varieties. *J. reine angew. Math.* **580**(2005), 139–155

9. B. Gordon and J. P. Murre. Chow motive of elliptic modular surfaces and threefolds. *Math. Inst.,* Univ. of Leiden, Report W 96–16 (1996)

10. U. Jannsen. Motivic sheaves and filtrations on Chow groups. In *Proc. sympos. Pure Math.* **55**(1994), Part 1 , 245–302

11. U. Jannsen. On finite-dimensional motive and Murre's conjecture. In *Algebraic cycles and motives* **2**, 112–142, London Math. Soc. Lecture Note Ser. **344**, Cambridge Univ. Press, Cambridge, 2007

12. B. Kahn, J.P. Murre and C. Pedrini, On the transcendental part of the motive of a surface. *Algebraic cycles and motives* **2**, 143–202, London Math. Soc. Lecture Note Ser. **344**, Cambridge Univ. Press, Cambridge, 2007.

13. K. Kimura. Murre's conjectures for certain product varieties. *J. Math. Kyoto. Univ.* **47**(3)(2007), 621–629

14. J. P. Murre. Lectures on motives. In *Transcendental Aspects of Algebraic Cycles,* Proceedings of Grenoble Summer School (2001), pages 123–170, Cambridge University Press, 2004

15. J. P. Murre. On the motive of an algebraic surface. *J. Reine und angew. Math.* **409**(1990), 190–204

16. J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety, I. The general conjectures and some examples. *Indag. Math. N. S.* **4**(2)(1993), 189–201

17. J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety, II. Verification of the conjectures for threefolds which are the product on a surface and a curve. *Indag. Math. N. S.* **4**(2)(1993), 189–201

18. A. M. Shermenev. The motive of an abelian variety. *Funct. Anal.* **8**(1974), 47–53

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press

19. A. J. Scholl. Classical motives, In *Proceedings of Symposia in Pure Mathematics,* **55**(1994), Part I, 163–187

20. C. Vial. Niveau and coniveau filtrations on cohomology groups and Chow groups, *Proc. LMS,* **105**(5)(2012), 1–35

KEJIAN XU kejianxu@amss.ac.cn

College of Mathematics
Qingdao University
Qingdao 266071
China

ZE XU xuze@amss.ac.cn

Institute of Mathematics
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190
China

Received: August 13, 2011

https://doi.org/10.1017/is013005020jkt228 Published online by Cambridge University Press