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Application of Vectorial Wave Method in Free Vibration Analysis of Cylindrical Shells

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Abstract. The vectorial form of the Wave Propagation Method (VWM), regarding the dispersion of harmonic plain (elasto-dynamic) waves within certain wave-guides, is developed for the vibration analysis of circular cylindrical shells. To obtain this goal, all plain waves are divided into positive-negative going wave vectors along with the shell axis. Based on the Flügge thin shell theory, the shell continuity as well as boundary conditions are well satisfied by introducing the propagation and reflection matrices. Furthermore, all elements of the reflection matrix are derived for certain classical supports. As an example, for demonstrating the feasibility of VWM in the shell vibration analysis, a circular cylindrical shell with two ended flexible support is adopted. The natural frequencies of the system as well as mode shapes are obtained using VWM. The aquired results are compared with those of the previous works and found in excellent agreement. It is also found that VWM could mathematically provide a reduced dimensional matrix (dominant matrix) to calculate the natural frequencies of the system. Accordingly, the proposed method can provide high computational efficiency and remarkable accuracy, simultaneously.

AMS subject classifications: 70J30, 74K25

Key words: Elasto-dynamic waves, vectorial wave method, reflection matrices, circular cylindrical shell, Flügge theory, free vibration.

1 Introduction

Vibration of cylindrical shells, a critical industrial problem for many years, has long since come to the attention of scientific literature [1]. Presently, research in this area is underway to provide ways to achieve higher accuracy as well as higher computation speeds with minimum geometrical and mechanical limitations.

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For free and forced vibration analysis in cylindrical shells, several ways have been proposed. Based on the cost of the required calculations for a specific accuracy, such methods can be divided into iterative and non-iterative methods. Each one of these methods could provide a special dominant matrix for the calculation of natural frequencies. Two important categories of the iterative methods are: numerical methods and close form solutions. The numerical methods based on the energy, weighted residual, differential quadrature and so on [2–4] give a large size dominant matrix (depending on the number of required frequencies) and hence require the largest amount of calculations. In the close form solutions [5, 6], as another sets of the iterative methods, depending on the complexity of the problem and the required accuracy, a converged solution can be obtained by using several analytical terms, leading to a large size dominant matrix. In a comparison, the non-iterative methods provide smaller size dominant matrices independent of the required accuracy and number of frequencies. Hence these methods are very fast and have a better convergency and accuracy [7].

It should be noted that vibration analysis of the elementary geometries such as beams, plates and more complex geometries such as cylindrical shells is a key for the analysis, control and optimization of more complex systems. Hence, any raise in the accuracy or any reduction in the computational cost of the solution method in vibration analysis of such elementary systems is of great interest, as it could open new approaches for the analysis of higher order or more complex systems, real time analysis, optimizations or inverse problems.

Furthermore, the low computational cost methods are of interest for monitoring, system failure detection as well as inverse problems in real time monitoring systems, in which deal with limited computational resources and system memories. From this point of view, any reduction in the required computational cost or memory of the solution approach is of great engineering and mathematical interest.

As far as shell vibrational problems are concerned, extensible non-iterative solutions generally include transfer function method, TFM (Laplace transform of shell Green's functions) [8] and wave propagation method (wave function expansion) [9–11]. Based on wave propagation method, two general techniques could be addressed for the analysis of shell vibrations in recent studies: state-space technique (SST) [12] and wave based method (WBM) [7].

The vectorial form of wave propagation method (VWM) can provide accurate and low cost non-iterative solutions for the vibration analysis of a system. This methodology has frequently been employed in the vibration analysis of non-shell mechanical elements, such as bars, beams and plates [13–19]. However, perhaps due to lake of relevant vector-matrix relations, the application of VWM in the vibration analysis of shells has not been addressed yet.

In the present study, VWM is developed for vibration analysis of circular cylindrical shells and the relevant matrix relations are introduced. Accordingly, using VWM, the positive-negative going wave vectors are defined and continuity and boundary conditions of the shell are satisfied by determining the propagation and reflection matrices. It

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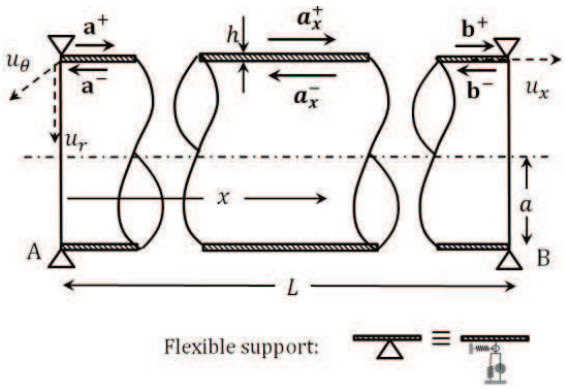


Figure 1: Wave propagation in a circular cylindrical shell with elastic supports.

is highly crucial to determine such matrices as they would enable VWM to solve more complex problems such as non-uniform thickness [13, 14] or issues related to active control [15]. In this regard, all elements of reflection matrix are determined for certain classical supports (clamped (CL), simply supported (SS) and shear diaphragm (SD)).

To evaluate the feasibility of VWM in analyzing the vibrations of shells, a circular cylindrical shell with two ended flexible supports is adopted. The obtained results are well compared with the results of previous studies, for different conditions. Ultimately, by using VWM, relevant mode shapes are obtained and the effects of support characteristics on natural frequencies in an asymmetric combination of the ended supports are investigated. The proposed method provide a reduced dimensional dominant matrix to calculate frequencies, compared with the other non-iterative methods, i.e., TFM, SST and WBM. In fact, using VWM, whole dominant matrix is divided into certain numbers of sub-matrices (propagation and reflection matrices). Accordingly, the proposed method is fully capable of fulfilling high numerical efficiency and well accuracy at the same time (as demonstrated in the given results).

Consider the cylindrical shell illustrated in Fig. 1 as a representative case with certain springs at its ends. The equation of motion of the problem is expressed in Section 2. In Section 3, the relevant vector and matrix relations for using in VWM will be introduced and discussed. In Sections 4 certain relations for calculations of frequencies and mode shapes of Fig. 1 are provided. In Section 6, the main results of the study will be discussed and verified.

2 Governing equations

Consider a circular cylindrical shell of length L, thickness h, material density ρ and middle layer radius a. By defining $\eta = h/\sqrt{12a}$ as a dimensionless coefficient and $c_L = (E/\rho(1-\mu^2))^{0.5}$ as dispersion speed of the longitudinal waves, the equation of motion

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of the shell is:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} u_x \\ u_\theta \\ u_r \end{Bmatrix} = \mathbf{0}. \tag{2.1}$$

Where L_{ij} $(i,j=1,\cdots,3)$ are differential operators based on Flügge thin shell theory [7,11] and are expressed in Appendix. Moreover, $u_{(x,\theta,r)}$ are the orthogonal components of the shell displacements in relevant directions (see Fig. 1). The displacement fields in terms of harmonic wave functions can be represented as:

$$u_x(x,\theta,t) = Ce^{i(\lambda x + n\theta + \omega t)}, \quad u_\theta(x,\theta,t) = \alpha u_x, \quad u_r(x,\theta,t) = \beta u_x, \tag{2.2}$$

where, C is a constant and α and β are the displacement ratios, i.e., $\alpha = u_\theta/u_x$ and $\beta = u_r/u_x$. λ is the axial wave number, n is the circumferential mode number and lastly ω is the circular frequency. Substituting Eq. (2.2) into Eq. (2.1), it gives:

$$\begin{bmatrix} \begin{pmatrix} \chi_{11} = \\ A_{11} - \Lambda^2 \end{pmatrix} & \chi_{12} = B_{12}\Lambda & \chi_{13} = B_{13}\Lambda + D_{13}\Lambda^3 \\ \chi_{12} & \chi_{22} = A_{22} + C_{22}\Lambda^2 & \chi_{23} = A_{23} + C_{23}\Lambda^2 \\ -\chi_{13} & -\chi_{23} & \begin{pmatrix} \chi_{33} = A_{33} + C_{33}\Lambda^2 \\ + E_{33}\Lambda^4 \end{pmatrix} \end{bmatrix} \begin{Bmatrix} 1 \\ \alpha \\ \beta \end{Bmatrix} C = \mathbf{0}, \tag{2.3}$$

where, $\Lambda = \lambda a$ is the dimensionless wave number and χ_{ij} $(i,j=1,2,3)$ are the matrix elements. The expressions $(A,B,C,D)_{ij}$ $(i,j=1,2,3)$ are coefficients as shown in Appendix. For a non-trivial solution, the determinant of the matrix $[\chi_{ij}]$ has to be equal to zero. Setting $|\chi_{ij}| = 0$, we left with a 4^{th} order polynomial in terms of Λ^2 as:

$$A_8\Lambda^8 + A_6\Lambda^6 + A_4\Lambda^4 + A_2\Lambda^2 + A_0 = 0, \tag{2.4}$$

where Zhou and Yang [8] as well as Zhou et al. [11] have well discussed the an analytical solution for the eight possible roots of the Eq. (2.4). Five coefficient, A_i $(i=0,2,4,6,8)$, are represented in the Appendix. Now, the displacement fields of the shell middle layer, in the axial, tangential and radial directions can be described as a linear combination of eight harmonic functions (wave function expansion) as follows:

$$U_x'(x,\theta) = e^{in\theta}\left(\sum_{j=1}^{4} a_j^+ e^{i\lambda_j x} + \sum_{j=5}^{8} a_j^- e^{i\lambda_j x}\right), \tag{2.5a}$$

$$U_\theta'(x,\theta) = e^{in\theta}\left(\sum_{j=1}^{4} \alpha_j a_j^+ e^{i\lambda_j x} + \sum_{j=5}^{8} \alpha_j a_j^- e^{i\lambda_j x}\right), \tag{2.5b}$$

$$U_r'(x,\theta) = e^{in\theta}\left(\sum_{j=1}^{4} \beta_j a_j^+ e^{i\lambda_j x} + \sum_{j=5}^{8} \beta_j a_j^- e^{i\lambda_j x}\right), \tag{2.5c}$$

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where the characteristics of the positive and negative going wave amplitudes are respectively denoted by a_j^+ and a_j^- (j=1,$\cdots$,8). Depending on the corresponding wave numbers, such wave amplitudes can be decaying, standing or propagating. Based on the conjugation roots of Eq. (2.4), the wave numbers in the above composition were arranged as $\lambda_m=-\lambda_{m+4}$, $\lambda_m<0$, where $m=1,\cdots,4$, where each wave number corresponds to a wave. Thus, the relations between the ratios of displacements can be written as:

$$\alpha_m=-\alpha_{m+4},\quad \beta_m=-\beta_{m+4},\quad m=1,\cdots,4, \tag{2.6}$$

where the above results are similar with the results reported for non-shell element wave guides [14].

3 Wave vectors and matrices

For free vibration analysis of a cylindrical shell which uses VWM, the vectors of positive and negative going waves had to be defined and hence, the consideration of the following form for wave amplitudes results in:

$$\begin{aligned}\mathbf{a}_x^{+(-)}(x)&=\begin{bmatrix} e^{i\lambda_{1(5)}x} & 0 & 0 & 0\\ 0 & e^{i\lambda_{2(6)}x} & 0 & 0\\ 0 & 0 & e^{i\lambda_{3(7)}x} & 0\\ 0 & 0 & 0 & e^{i\lambda_{4(8)}x}\end{bmatrix}\begin{Bmatrix} a_{1(5)}\\ a_{2(6)}\\ a_{3(7)}\\ a_{4(8)}\end{Bmatrix}\\ &=\underline{F}^{+(-)}(x)\mathbf{a}^{+(-)},\end{aligned} \tag{3.1}$$

where $\underline{F}^{\pm}$ are the propagation matrices, $\mathbf{a}^{\pm}$ are the wave amplitudes and finally $\mathbf{a}_x^{\pm}$ are the wave vectors, all going in the positive and negative directions, respectively. By defining the vectors of amplitude $\overline{\mathbf{w}}$ and $\overline{\mathbf{f}}$, for the displacement vector, $\mathbf{w}$ and force vector, $\mathbf{f}$ we have:

$$\mathbf{w}(x,\theta)=\overline{\mathbf{w}}(x)e^{in\theta}=\begin{bmatrix}U_x' & -\frac{\partial U_r'}{\partial x} & U_\theta' & U_r'\end{bmatrix}^T e^{in\theta}, \tag{3.2a}$$

$$\mathbf{f}(x,\theta)=\overline{\mathbf{f}}(x)e^{in\theta}=\begin{bmatrix}N_{xx}' & M_{xx}' & T_{x\theta}' & V_{xr}'\end{bmatrix}^T e^{in\theta}. \tag{3.2b}$$

It should be noted that "N" and "M" denote force and moment resultants, respectively as shown in Fig. 2. The terms "T" and "V" are the Kirchhoff parameters and the prime symbol ($'$) means the division of the term to ($e^{in\theta}$). All the mentioned parameters are given in the Appendix. By substituting Eq. (2.5) in Eq. (3.2), the following results can be obtained:

$$\overline{\mathbf{w}}(x)=\underline{\psi}^+\mathbf{a}_x^+ +\underline{\psi}^-\mathbf{a}_x^-, \tag{3.3a}$$

$$\overline{\mathbf{f}}(x)=\underline{\phi}^+\mathbf{a}_x^+ +\underline{\phi}^-\mathbf{a}_x^-, \tag{3.3b}$$

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where, $\underline{\psi}^{\pm}$ and $\underline{\phi}^{\pm}$ are matrices related to the negative and positive going wave vectors, corresponding to $\overline{\mathbf{w}}$ and $\overline{\mathbf{f}}$, respectively. These matrices can be defined as:

$$\underline{\psi}^{+(-)}=\begin{bmatrix}\boldsymbol{\Psi}_{1(5)} & \boldsymbol{\Psi}_{2(6)} & \boldsymbol{\Psi}_{3(7)} & \boldsymbol{\Psi}_{4(8)}\end{bmatrix}, \tag{3.4a}$$

$$\underline{\phi}^{+(-)}=\begin{bmatrix}\boldsymbol{\Phi}_{1(5)} & \boldsymbol{\Phi}_{2(6)} & \boldsymbol{\Phi}_{3(7)} & \boldsymbol{\Phi}_{4(8)}\end{bmatrix}. \tag{3.4b}$$

Each matrix column, i.e., $\underline{\psi}$ and $\underline{\phi}$, for $j=1,\cdots,8$, is as follows:

$$\boldsymbol{\Psi}_j=\begin{Bmatrix}1\\ -\mathrm{i}\beta_j\lambda_j\\ \alpha_j\\ \beta_j\end{Bmatrix}, \tag{3.5a}$$

$$\boldsymbol{\Phi}_j=\begin{Bmatrix}k\left(\mathrm{i}\lambda_j+\dfrac{\mu}{a}(\mathrm{i}n\alpha_j+\beta_j)\right)+\dfrac{D}{a}\beta_j\lambda_j^2\\ D\left(\lambda_j^2\beta_j+\dfrac{\mathrm{i}n\mu}{a^2}(\alpha_j-\mathrm{i}n\beta_j)+\dfrac{1}{a}\mathrm{i}\lambda_j\right)\\ \dfrac{1-\mu}{2}\left(\mathrm{i}k\left(\lambda_j\alpha_j+\frac{n}{a}\right)+\dfrac{3}{a^2}D\lambda_j(\mathrm{i}\alpha_j+n\beta_j)\right)\\ D\left(\dfrac{n}{a^2}\lambda_j(-\mu\alpha_j+(2-\mu)\mathrm{i}n\beta_j)+\lambda_j^2\left(\mathrm{i}\lambda_j\beta_j-\dfrac{1}{a}\right)+\dfrac{n^2(1-\mu)}{2a^3}\right)\end{Bmatrix}, \tag{3.5b}$$

where, $k=Eh/(1-\mu^2)$ and $D=Eh^3/12(1-\mu^2)$ are the membrane and bending stiffness constants of the shell, respectively. The coefficient μ is the Poisson's ratio and E is the Young modulus. The propagation matrices, $\underline{F}^{\pm}$, can be specified through the following equations:

$$\mathbf{b}^{+}=\underline{F}^{+}\mathbf{a}^{+},\quad \mathbf{a}^{-}=\underline{F}^{-}\mathbf{b}^{-}\Rightarrow F_{11}^{+}=\frac{b_{11}^{+}}{a_{11}^{+}},\quad F_{11}^{-}=\frac{a_{11}^{-}}{b_{11}^{-}},\cdots, \tag{3.6}$$

where, as previously defined, $\underline{F}^{\pm}=\underline{F}^{\pm}(L)$ are the propagation matrices in the position $x=L$. Consequently, $\underline{F}^{\pm}(L)$ can be represented in the matrix form as:

$$\underline{F}^{+(-)}=\begin{bmatrix}\mathrm{e}^{\mathrm{i}\lambda_{1(5)}L} & 0 & 0 & 0\\ 0 & \mathrm{e}^{\mathrm{i}\lambda_{2(6)}L} & 0 & 0\\ 0 & 0 & \mathrm{e}^{\mathrm{i}\lambda_{3(7)}L} & 0\\ 0 & 0 & 0 & \mathrm{e}^{\mathrm{i}\lambda_{4(8)}L}\end{bmatrix}. \tag{3.7}$$

In order to examine the influences of flexible support characteristics on the vibrations of a cylindrical shell, it is necessary to define the stiffness matrices. These matrices, in location $x=0$ (denoted by A in Fig. 1) are defined as follows:

$$\underline{K}_A=\begin{bmatrix}K_{xA} & 0 & 0 & 0\\ 0 & K_{r\theta A} & 0 & 0\\ 0 & 0 & K_{\theta A} & 0\\ 0 & 0 & 0 & K_{rA}\end{bmatrix}, \tag{3.8}$$

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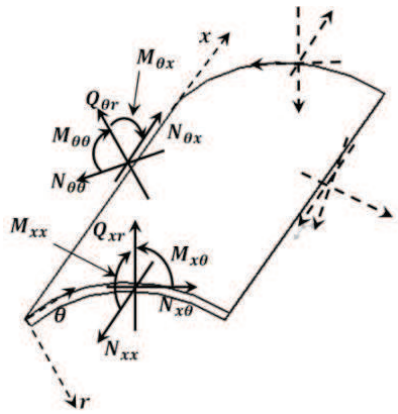


Figure 2: Force and moment resultants in a circular cylindrical shell.

where, the diagonal elements $K_{(x,r,\theta,r\theta)A}$ are the stiffness coefficients of the springs on the left end of the shell. To satisfy the boundary conditions at both ends of the cylinder (see Fig. 1), we should have:

$$\left(\mathbf{f}=\pm\underline{K}\overline{\mathbf{w}}\right)_{A,B(x=0,L)}. \tag{3.9}$$

Substituting Eq. (3.3) into Eq. (3.9), it gives:

$$\underline{\phi}^{+}\mathbf{a}^{+}+\underline{\phi}^{-}\mathbf{a}^{-}=\underline{K}_{A}(\underline{\psi}^{+}\mathbf{a}^{+}+\underline{\psi}^{-}\mathbf{a}^{-}), \tag{3.10a}$$

$$\underline{\phi}^{+}\mathbf{b}^{+}+\underline{\phi}^{-}\mathbf{b}^{-}=-\underline{K}_{A}(\underline{\psi}^{+}\mathbf{b}^{+}+\underline{\psi}^{-}\mathbf{b}^{-}). \tag{3.10b}$$

Factoring Eq. (3.10) by $\mathbf{a}^{\pm}$ and $\mathbf{b}^{\pm}$, it gives:

$$\mathbf{a}^{+}=\underline{R}_{A}\mathbf{a}^{-}, \tag{3.11a}$$

$$\mathbf{b}^{-}=\underline{R}_{B}\mathbf{b}^{+}. \tag{3.11b}$$

Where, $\underline{R}_{A,B}$ are reflection matrices at both ends of the cylinder as:

$$\underline{R}_{A}=(\underline{\phi}^{+}-\underline{K}_{A}\underline{\psi}^{+})^{-1}(\underline{K}_{A}\underline{\psi}^{-}-\underline{\phi}^{-}), \tag{3.12a}$$

$$\underline{R}_{B}=-(\underline{K}_{B}\underline{\psi}^{-}+\underline{\phi}^{-})^{-1}(\underline{\phi}^{+}+\underline{K}_{B}\underline{\psi}^{+}). \tag{3.12b}$$

Eqs. (3.12) satisfy the flexible boundary conditions of the shell. The presence of flexible support elements of reflection matrices in Eq. (3.12), containing several parameters, result in large relationships. However for some classical supports, the reduced elements may be represented; an example is the case of shear diaphragm supports ($K_x=K_{r\theta}=0$ and tending both constants K_r and K_θ to infinity), using Eqs. (3.4a), (3.5a) and (3.8) for a stiffness

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matrix of the support, $\underline{K}_{SD}$, is evaluated as:

$$\underline{\phi}^{+}-\underline{K}_{SD}\underline{\psi}^{+}=\begin{bmatrix}\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14}\\ & & \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24}\\ & & \phi_{31}-K_\theta\alpha_1 & \phi_{32}-K_\theta\alpha_2 & \phi_{33}-K_\theta\alpha_3 & \phi_{34}-K_\theta\alpha_4\\ & & \phi_{41}-K_r\beta_1 & \phi_{42}-K_r\beta_2 & \phi_{43}-K_r\beta_3 & \phi_{44}-K_r\beta_4\end{bmatrix}_{K_{r,\theta}\to\infty}$$
$$\simeq\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -K_\theta\alpha_1 & -K_\theta\alpha_2 & -K_\theta\alpha_3 & -K_\theta\alpha_4\\ -K_r\beta_1 & -K_r\beta_2 & -K_r\beta_3 & -K_r\beta_4\end{bmatrix}_{K_{r,\theta}\to\infty}, \tag{3.13}$$

in the same way, with respect to Eqs. (2.6), (3.4a) and (3.8), one can find:

$$\underline{K}_{SD}\underline{\psi}^{-}-\underline{\phi}^{-}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -K_\theta\alpha_1 & -K_\theta\alpha_2 & -K_\theta\alpha_3 & -K_\theta\alpha_4\\ -K_r\beta_1 & -K_r\beta_2 & -K_r\beta_3 & -K_r\beta_4\end{bmatrix}_{K_{r,\theta}\to\infty}. \tag{3.14}$$

Thus from Eqs. (3.12) to (3.14) the reflection matrix of the SD support at the left end of cylinder is an identity matrix, $\underline{I}$, as:

$$\underline{R}_{SD}\,|_A=(\underline{\phi}^{+}-\underline{K}_{SD}\underline{\psi}^{+})^{-1}(K_{SD}\underline{\psi}^{-}-\underline{\phi}^{-})=\underline{I}, \tag{3.15}$$

where the above relation clearly shows that in the presence of SD supports at the shell boundaries, all components of the incident wave vector are reflected completely. With the same algebraic operation, the reflection matrices for SS boundary conditions can be derived as:

$$\underline{R}_{SS}\,|_A$$
$$=\frac{1}{\mathcal{T}}\begin{bmatrix}\mathcal{T}+2\beta_1\lambda_1\mathcal{A} & 2\beta_2\lambda_2\mathcal{A} & 2\beta_3\lambda_3\mathcal{A} & 2\beta_4\lambda_4\mathcal{A}\\ & & -2\beta_1\lambda_1\mathcal{B} & \mathcal{T}-2\beta_2\lambda_2\mathcal{B} & -2\beta_3\lambda_3\mathcal{B} & -2\beta_4\lambda_4\mathcal{B}\\ & & 2\beta_1\lambda_1\mathcal{C} & 2\beta_2\lambda_2\mathcal{C} & \mathcal{T}+2\beta_3\lambda_3\mathcal{C} & 2\beta_4\lambda_4\mathcal{C}\\ & & -2\beta_1\lambda_1\mathcal{D} & -2\beta_2\lambda_2\mathcal{D} & -2\beta_3\lambda_3\mathcal{D} & \mathcal{T}-2\beta_4\lambda_4\mathcal{D}\end{bmatrix}, \tag{3.16}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{T}$ are the functions of displacements ratio and wave numbers as:

$$\mathcal{A}=\alpha_2(\beta_3-\beta_4)+\alpha_3(\beta_4-\beta_2)+\alpha_4(\beta_2-\beta_3), \tag{3.17a}$$
$$\mathcal{B}=\alpha_1(\beta_3-\beta_4)+\alpha_3(\beta_4-\beta_1)+\alpha_4(\beta_1-\beta_3), \tag{3.17b}$$
$$\mathcal{C}=\alpha_1(\beta_2-\beta_4)+\alpha_2(\beta_4-\beta_1)+\alpha_4(\beta_1-\beta_2), \tag{3.17c}$$
$$\mathcal{D}=\alpha_1(\beta_2-\beta_3)+\alpha_2(\beta_3-\beta_1)+\alpha_3(\beta_1-\beta_2), \tag{3.17d}$$
$$\begin{aligned}\mathcal{T}=&\alpha_1\beta_2(\beta_3(\lambda_2-\lambda_3)+\beta_4(\lambda_4-\lambda_2))+\alpha_2\beta_1(\beta_3(\lambda_3-\lambda_1)+\beta_4(\lambda_1-\lambda_4))\\ &+\beta_3\beta_4(\lambda_3-\lambda_4)(\alpha_1-\alpha_2)+\alpha_3\beta_1(\beta_2(\lambda_1-\lambda_2)+\beta_4(\lambda_4-\lambda_1))\\ &+\alpha_4\beta_1(\beta_2(\lambda_2-\lambda_1)+\beta_3(\lambda_1-\lambda_3))+\beta_2(\alpha_3\beta_4(\lambda_2-\lambda_4)+\alpha_4\beta_3(\lambda_3-\lambda_2)).\end{aligned} \tag{3.17e}$$

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Furthermore, the reflection matrix for CL supports can be given as:

$$\underline{R}_{CL}|_A = \begin{bmatrix} 1+\frac{2(\mathcal{A}'+\beta_1\lambda_1+\mathcal{A})}{\mathcal{T}} & \frac{2(\mathcal{A}'+\beta_2\lambda_2+\mathcal{A})}{\mathcal{T}} & \frac{2(\mathcal{A}'+\beta_3\lambda_3+\mathcal{A})}{\mathcal{T}} & \frac{2(\mathcal{A}'+\beta_4\lambda_4+\mathcal{A})}{\mathcal{T}} \\ \frac{2(\mathcal{B}'-\beta_1\lambda_1+\mathcal{B})}{\mathcal{T}} & 1+\frac{2(\mathcal{B}'-\beta_2\lambda_2+\mathcal{B})}{\mathcal{T}} & \frac{2(\mathcal{B}'-\beta_3\lambda_3+\mathcal{B})}{\mathcal{T}} & \frac{2(\mathcal{B}'-\beta_4\lambda_4+\mathcal{B})}{\mathcal{T}} \\ \frac{2(\mathcal{C}'+\beta_1\lambda_1+\mathcal{C})}{\mathcal{T}} & \frac{2(\mathcal{C}'+\beta_2\lambda_2+\mathcal{C})}{\mathcal{T}} & 1+\frac{2(\mathcal{C}'+\beta_3\lambda_3+\mathcal{C})}{\mathcal{T}} & \frac{2(\mathcal{C}'+\beta_4\lambda_4+\mathcal{C})}{\mathcal{T}} \\ \frac{2(\mathcal{D}'-\beta_1\lambda_1+\mathcal{D})}{\mathcal{T}} & \frac{2(\mathcal{D}'-\beta_2\lambda_2+\mathcal{D})}{\mathcal{T}} & \frac{2(\mathcal{D}'-\beta_3\lambda_3+\mathcal{D})}{\mathcal{T}} & 1+\frac{2(\mathcal{D}'-\beta_4\lambda_4+\mathcal{D})}{\mathcal{T}} \end{bmatrix}, \tag{3.18}$$

where $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ and $\mathcal{D}'$ are as follows:

$$\mathcal{A}' = \alpha_2\beta_3\beta_4(\lambda_3-\lambda_4)+\alpha_3\beta_2\beta_4(\lambda_4-\lambda_2)+\alpha_4\beta_2\beta_3(\lambda_2-\lambda_3), \tag{3.19}$$
$$\mathcal{B}' = \alpha_1\beta_3\beta_4(\lambda_4-\lambda_3)+\alpha_3\beta_1\beta_4(\lambda_1-\lambda_4)+\alpha_4\beta_1\beta_3(\lambda_3-\lambda_1), \tag{3.20}$$
$$\mathcal{C}' = \alpha_1\beta_2\beta_4(\lambda_2-\lambda_4)+\alpha_2\beta_1\beta_4(\lambda_4-\lambda_1)+\alpha_4\beta_1\beta_2(\lambda_1-\lambda_2), \tag{3.21}$$
$$\mathcal{D}' = \alpha_1\beta_2\beta_3(\lambda_3-\lambda_2)+\alpha_2\beta_1\beta_3(\lambda_1-\lambda_3)+\alpha_3\beta_1\beta_2(\lambda_2-\lambda_1). \tag{3.22}$$

4 Frequencies and mode shapes

To examine the feasibility of the proposed method for shell vibrations, a circular cylindrical shell with two ended supports, as mentioned before, is investigated using the required vector-matrix relations, obtained in the previous section. Fig. 1, as a case study, demonstrates the wave motion within the shell and in the vicinity of its boundaries. In this regard, Eqs. (3.6) and (3.11) could be expressed by the amplitude wave vector a^+ as follows:

$$(\underline{R}_A\underline{F}^-\underline{R}_B\underline{F}^+-\underline{I})\mathbf{a}^+=\mathbf{0}. \tag{4.1}$$

The above equation provides a 4×4 coefficient matrix (dominant matrix) to calculate the natural frequencies. The dominant matrix consists of certain sub-matrices, multiplied by each other, i.e., the reflection and propagation matrices. It should be noted that the size of the dominant matrix is independent of certain geometrical and mechanical complexities; hence, the effect of changes in geometry can be reflected in the component of the dominant matrix [13]. This is the main advantage of the VWM method. However, finding appropriate components regarding the geometry of the system is a crucial issue. Once, the components of the dominant matrix are defined (as represented in Eqs. (3.7) and (3.12)), the dominant matrix can be easily utilized for the analysis of the problem with arbitrary support conditions. Here, for non-trivial results in Eq. (4.1), i.e., $\mathbf{a}^+\neq\mathbf{0}$, the determinant of the dominant matrix must be zero as:

$$|\underline{R}_A\underline{F}^-\underline{R}_B\underline{F}^+-\underline{I}|=\mathbf{0}. \tag{4.2}$$

By using the above results and invoking Eqs. (3.6) and (3.12), all natural frequencies of the circular cylindrical shell can be determined. For some special cases of classical boundary

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conditions it is possible to derive analytical expressions for axial wave numbers. For example, in the case of SD-SD supports, both reflection matrices become identity matrices, as mentioned in Eq. (3.15) and then from Eq. (4.2), the following relation for exact wave numbers can be found:

$$|\underline{F}^{+}(2L)-\underline{I}|=(e^{i2\lambda_1 L}-1)(e^{i2\lambda_2 L}-1)(e^{i2\lambda_3 L}-1)(e^{i2\lambda_4 L}-1)=0, \tag{4.3}$$

where at least one of the above wave numbers should be equal to $m\pi/L$ (calculations revealed that $\lambda_1=m\pi/L$). In general and for arbitrary flexible supports, however, it is impossible to provide such expressions. Hence, using Eq. (4.2), the error parameter ($C(\Omega)$) is introduced here as:

$$C(\Omega)=|\underline{R}_A\underline{F}^{-}\underline{R}_B\underline{F}^{+}-\underline{I}|. \tag{4.4}$$

Thus, the natural frequencies could be directly obtained from $C(\Omega)=0$. In order to extract the natural frequencies of the system, a method for extracting the roots of the equation $C(\Omega)=0$ is required. One of the well-known methods capable of extracting natural frequencies from the equation $C(\Omega)=0$ is "root searching methods" or the Wittrick-Williams algorithm [17].

In order to determine the mode shapes of the problem, the normalized vector of a^{+} in Eq. (4.1) should be solved. By setting a_i^{+} ($i=2,3,4$) for $a_1^{+}=1$, it is possible to get the mode shapes under different support conditions. Using Eqs. (3.3), (3.6) and (3.11), the following relation is achieved for the displacements vectors as:

$$\overline{\mathbf{w}}(x)=(\underline{\psi}^{+}\underline{F}(x)+\underline{\psi}^{-}\underline{F}(L-x)R_B\underline{F}(L))\mathbf{a}^{+}. \tag{4.5}$$

Regarding the homogeneity of the constructing materials of the shell, the wave propagation matrices is simplified as $\underline{F}^{+}=\underline{F}^{-}=\underline{F}$.

5 Numerical results

Using the obtained relations in the previous sections, certain numerical examples are presented here. For the case studies, the following common geometrical and mechanical properties are adopted: $a=1$m and $E=210$GPa. Other characteristics considered in the examples where given separately in the text. Moreover, we have written a code to perform the calculations using the MATLAB language program. All of the results and charts are determined applying the written code in MATLAB.

To examine the proposed method in the presence of arbitrary non-rigid supports, certain flexible supports are considered for the shell. The considered flexible supports can have arbitrary stiffness in the x, r, θ and $r\theta$ directions, i.e., 0 to ∞. However, four different flexible supports are noted in this study where expressed by E^{I}, E^{II}, E^{III} and E^{IV}. These supports are flexible only in one direction, i.e., radial (E^{I}), circumferential (E^{II}), axial (E^{III}) or both radial and circumferential (E^{IV}) directions and are rigid in other directions. The flexible supports was represented by the stiffness matrix in Eq. (3.8).

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Table 1: Comparison of natural frequency parameters Ω in different n and two h/a, $L=20$m, $\rho=7850$kg/m^3.

SD-SD					SD-SD				
$\frac{h}{a}$	n	Ref. [12]	Present	%	$\frac{h}{a}$	n	Ref. [12]	Present	%
0.05	1	0.0161065	0.0161065	0.000	0.002	1	0.0161011	0.0161011	0.000
	2	0.0393038	0.0393038	0.000		2	0.0054532	0.0054532	0.000
	3	0.1098527	0.1098527	0.000		3	0.0050418	0.0050418	0.000
	4	0.2103446	0.2103446	0.000		4	0.0085340	0.0085340	0.000

Table 2: Comparison the frequency parameters, Ω for two different ended supports ($L=20$m, $h=0.01$m, $\rho=7850$kg/m^3).

	SD-SD			CL-CL		
n	Ref. [4]	Present	Ref. [11]	Ref. [4]	Present	Ref. [11]
1	0.016101	0.016101	0.016101	0.032885	0.032792	0.032781
2	0.009382	0.009382	0.009381	0.013932	0.013901	0.013898
3	0.022105	0.022105	0.022105	0.022672	0.022667	0.022666
4	0.042095	0.042095	0.041967	0.042208	0.042208	0.041998
5	0.068008	0.068008	0.067936	0.068046	0.068046	0.067936

Table 3: Comparison the frequency parameters, Ω for different flexible supports ($K_{r,\theta,r\theta}$=10), L=20m, h=0.01m, $\rho=7850$kg/m^3.

	$E^{II}-E^{II}$		$E^{I}-E^{I}$		$E^{IV}-E^{IV}$	
m, n	Ref. [11]	Present	Ref. [11]	Present	Ref. [11]	Present
1,1	0.030069	0.030081	0.032741	0.032752	0.032758	0.032769
1,2	0.013871	0.013875	0.013884	0.013888	0.013889	0.013894

Each table in this section, Tables 1 to 4, is provided for a comparison with the similar papers [4, 11, 12] and in different conditions. Here, the rigid springs are utilized to simulate the fixed directions of the mentioned elastic supports. For instance, E^I support was provided by setting all stiffness coefficients of the springs in the x, θ and $r\theta$ directions to a large amount, e.g., 10^{30} and so on.

Zhang and Xiang [12] have utilized the state-space technique (SST) to study the free vibration of cylindrical shells. They have examined the case of SD-SD supports for the thickness of two cylindrical shells, $h/a=0.05$ and 0.002. Table 1 compares the accuracy of the results acquired using VWM with those reported by Zhang and Xiang [12] and acquired employing the state-space technique. Table 1 shows that the obtained frequencies in the present study are in excellent agreement with those addressed by Zhang and Xiang [12]. Table 2 compares the frequency parameters calculated in this paper with those in two other investigations (see [4, 11]) for SD-SD and CL-CL ended supports. In this comparison our obtained frequency parameters where compared with two different shell theories and two different solution methods. This table indicates that most of the results gained via VWM are completely in accordance with those mentioned in [4].

Table 3 provides a comparison of results in the presence of elastic supports with those illustrated in the paper [11]. In this table, a symmetric composition of E^I, E^{II} or E^{IV} is in-

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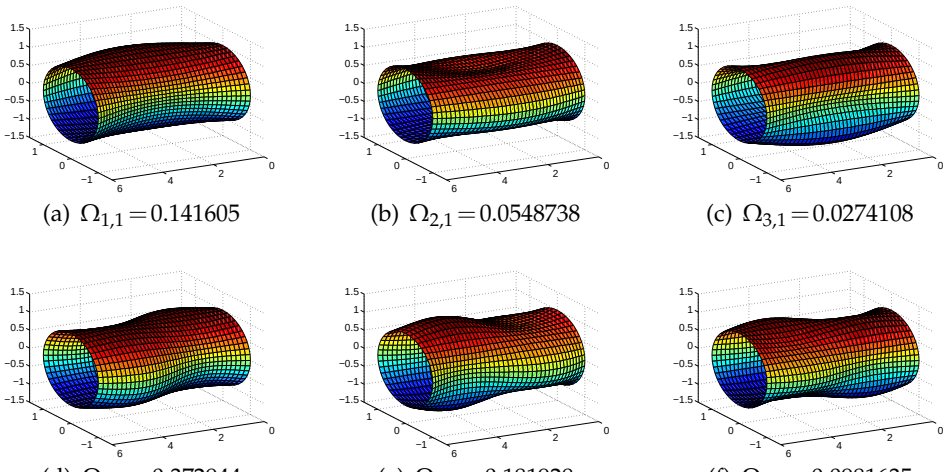
Table 4: Comparison the fundamental frequency parameters, Ω_f for different L/a ratios ($K_r=10$), $h=0.01$m, $\rho=7850$kg/m^3.

		E^I-E^I		
L/a	m,n	Ref. [11]	Present	Diff.%
10	1,3	0.030603	0.030615	0.040
20	1,2	0.013884	0.013888	0.030
30	1,2	0.009356	0.009357	0.011

vestigated for both ends of the cylinder. Based on this table, there exist minor differences among the symmetric combinations of the elastic supports. In order to evaluate VWM for different axial lengths of a circular cylindrical shell, certain comparisons were drawn (Table 4). In this table the symmetric supports, i.e., E^I-E^I, are presented. Once again a good proximity can be observed between the two results.

In accordance with Eq. (4.5), Fig. 3 illustrate the resulting mode shapes for six combinations of mode numbers ($n=1,\cdots,3$, $m=1,2$). The considered characteristics for these figures where, $h=0.002$m, $L=6$m and $\rho=7800$kg/m^3. The natural frequencies presented in Fig. 3 are comparable with those belonging to [3].

Most of the investigations provide examples through symmetric elastic supports for the issue [3,11]. In this regard, some examples of asymmetric flexible ended supports are presented here. Figs. 4(a) to (d) represent natural frequency variations of a circular shell with $CL-E^{I,IIorIII}$ ended supports under different conditions. Moreover, Figs. 4(a) and (b) expressed the variations of frequency parameter for two different circumferential and axial mode numbers ($n=1,2$ and $m=1,2$). Furthermore, Figs. 4(c) and (d) evaluate the



(a) $\Omega_{1,1}=0.141605$ (b) $\Omega_{2,1}=0.0548738$ (c) $\Omega_{3,1}=0.0274108$

(d) $\Omega_{1,2}=0.372944$ (e) $\Omega_{2,2}=0.181928$ (f) $\Omega_{3,2}=0.0981635$

Figure 3: Mode shapes of a circular cylindrical shell with E^I-E^I supports ($\Omega_{n,m}$).

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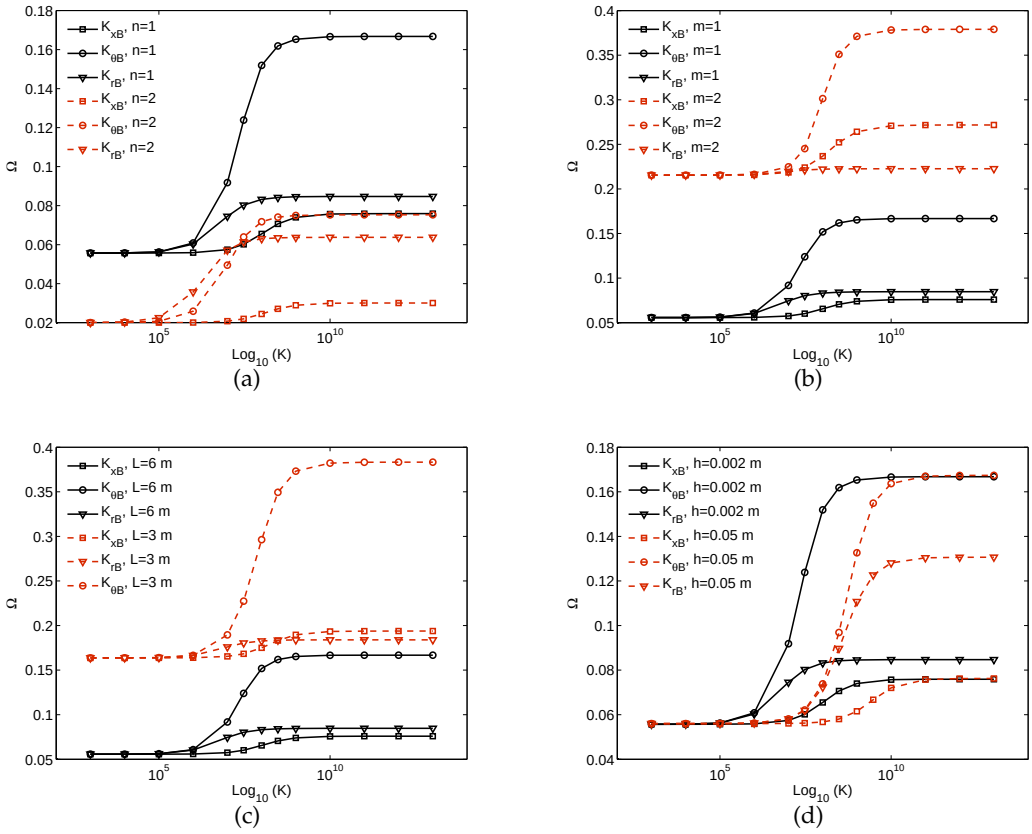


Figure 4: Frequency variation vs. Stiffness of flexible support for a cylindrical shell with clamped-elastic supports.

effects of such geometrical characteristics as length and thickness on the free vibration of the cylinder. For all solid lines plotted in Fig. 4 we have: $n,m=1$, $\rho=7800\text{kg/m}^3$, $L=6\text{m}$, $h=0.002\text{m}$.

In general, the analysis of Figs. 4(a) to (d) leads to the following results:

1. All figures reveal an increase in the natural frequencies in different situations; along with an increase in the stiffness of the flexible support in each direction (this is inevitable due to the increased rigidity of the whole vibration system).

2. Stiffness of the circumferential springs has the maximum variation effect on the natural frequencies in all figures.

3. Depending on the shell thickness and circumferential mode number, n, radial springs can play crucial roles as far as frequency variations are concerned (Fig. 4(d)).

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6 Conclusions

In this paper, vectorial form of wave propagation method (VWM) was developed for vibration analysis of cylindrical shells, based on Flügge classical thin shell theory. In this vein, positive-negative wave vectors going along the cylinder axis were defined. The reflection matrix for an arbitrary flexible support was obtained; furthermore all its elements under such classical boundary conditions as clamped, shear-diaphragm and simply-supported boundaries are derived.

Additionally the non-iterative methods have been applied for shell vibration analysis. Advantages of the proposed method can be summarized as follows:

Using TFM, WBM or SST for the issue, for q numbers of shell segments, an $8q\times 8q$ dominant matrix will be obtained. Accordingly, for a uniform shell, i.e., $q=1$, the dominant matrices provided by the non-iterative methods was, at least, an 8×8 matrix; while the proposed VWM gives a half dimensional, i.e., 4×4, dominant matrix to calculate the natural frequencies (regardless of the numbers of shell segments). Indeed, by using VWM, the total dominant matrix, is divided into certain sub-matrices, i.e., the reflection and propagation matrices. As the size of the dominant matrix is significantly reduced, we believe that the required system memory as well as the computation cost also reduce, too. In addition, the dominant matrix of the system make a direct relationship between system characteristics and system frequencies, as the other non-iterative methods. Hence, the dominant system matrix needs to to be constructed once, without any iteration. Accordingly, the proposed method significantly reduces the required computational resources.

Furthermore, as case studies, free vibrations and the mode shapes of a circular cylindrical shell with two flexible ended supports are analyzed by employing the proposed method. The present results were compared with those of previous studies under different boundary and geometry conditions. Here are the overall conclusions based on previous relations and numerical examples provided in this study:

1. VWM has better numerical performance among the non-iterative methods applied for vibration analysis of the shells.
2. VWM is fully capable of handling the problems including flexible supports.
3. As other non-iterative methods, the proposed method gives very accurate frequency results. In an asymmetric combination of flexible ended supports.
4. Circumferential stiffness of the support has the highest effect on the natural frequency variations in all considered conditions.
5. The radial stiffness depending on the shell thickness and circumferential mode numbers, n can have a remarkable effect on the natural frequencies of the problem.

It is believe that the proposed method may fully be apt to be extended and can tackle future more complex problems in the shell vibration fields, which can be subject of future

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studies. The proposed method may also be of great interest for the analysis of free and forced vibrations of shells with non-uniform thickness or multiple non-rigid intermediate supports in applications, in which low calculation costs, low required memory and high accuracies are simultaneously required.

Appendix

Differential operators where expressed in Eq. (2.1) are as follows [7,11]:

$$L_{11}=\frac{\partial^2}{\partial x^2}+(1+\eta^2)\frac{1-\mu}{2a^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{c_L^2}\frac{\partial^2}{\partial t^2}, \tag{A.1a}$$

$$L_{12}=L_{21}=\frac{1+\mu}{2a}\frac{\partial^2}{\partial x\partial\theta}, \tag{A.1b}$$

$$L_{13}=-L_{31}=\Big(\frac{\mu}{a}-\eta^2\Big(\frac{1-\mu}{2a}\frac{\partial^2}{\partial\theta^2}-a\frac{\partial^2}{\partial x^2}\Big)\Big)\frac{\partial}{\partial x}, \tag{A.1c}$$

$$L_{22}=(1+3\eta^2)\frac{1-\mu}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{c_L^2}\frac{\partial^2}{\partial t^2}, \tag{A.1d}$$

$$L_{23}=-L_{32}=\Big(-\eta^2\Big(\frac{3-\mu}{2}\Big)\frac{\partial^2}{\partial x^2}+\frac{1}{a^2}\Big)\frac{\partial}{\partial\theta}, \tag{A.1e}$$

$$L_{33}=-\Big(\eta^2\Big(\Big(a\frac{\partial^2}{\partial x^2}+\frac{1}{a}\frac{\partial^2}{\partial\theta^2}\Big)^2+\frac{2}{a^2}\frac{\partial^2}{\partial\theta^2}\Big)+\frac{1+\eta^2}{a^2}+\frac{1}{c_L^2}\frac{\partial^2}{\partial t^2}\Big), \tag{A.1f}$$

where in the above equations, $c_L=\frac{E}{\rho(1-\mu^2)}$ is the speed of dispersion of longitudinal waves and $\eta=\frac{h}{\sqrt{12}a}$ is a dimensionless coefficient.

The coefficients of Eq. (2.3) are as follows:

$$A_{11}=\Omega^2-\frac{1-\mu}{2}n^2(1+\eta^2),\quad B_{12}=-\frac{1+\mu}{2}n,\quad B_{13}=\mathrm{i}\mu-\mathrm{i}n^2\eta^2\frac{1-\mu}{2}, \tag{A.2a}$$

$$D_{13}=\mathrm{i}\eta^2,\quad A_{22}=\Omega^2-n^2,\quad C_{22}=-\frac{1-\mu}{2}(1+3\eta^2),\quad A_{23}=\mathrm{i}n, \tag{A.2b}$$

$$C_{23}=\mathrm{i}n\eta^2\frac{3-\mu}{2},\quad A_{33}=\Omega^2-1-\eta^2(n^2-1)^2,\quad C_{33}=-2n^2\eta^2,\quad E_{33}=-\eta^2, \tag{A.2c}$$

where, Ω is the dimensionless frequency or frequency parameter and is defined as $\Omega=\omega a\sqrt{(1-\mu^2)\rho/E}$.

Coefficients A_i ($i=0,2,4,6,8$) represented in Eq. (2.4) are as follows:

$$A_8=-\frac{1-\mu}{2}(1+3\eta^2)(\eta^2-\eta^4), \tag{A.3a}$$

$$\begin{aligned}A_6=&\Big(\Omega^2-\Big(\frac{1-\mu}{2}(1-\eta^2)-2\Big)n^2+2\mu\Big)\frac{1-\mu}{2}(1+3\eta^2)\eta^2\\&+\Big(\frac{1+\mu}{2}n\eta\Big)^2+(\eta^2-\eta^4)(\Omega^2-n^2)+n^2\eta^4\frac{3-\mu}{4}(1-3\mu),\end{aligned} \tag{A.3b}$$

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$$A_4=\eta^2\Big(-\Omega^2+\Big(2-\mu+\Big(3-3\mu-\Big(\frac{3-\mu}{2}\Big)^2\Big)\eta^2\Big)n^2\Big)\Big(\Omega^2-\frac{1-\mu}{2}n^2(1+\eta^2)\Big)$$
$$-\Big(\Big(\Big(2-n^2\frac{1-\mu}{2}\Big)\eta^2+\mu\Big)(\Omega^2-n^2)+n\eta^2\Big(3-\mu\Big)\frac{1+\mu}{2}n\Big)\Big(\mu-n^2\eta^2\frac{1-\mu}{2}\Big)$$
$$+\Big(\frac{(1+\mu)^2}{2}n^2+2(\Omega^2-n^2-\mu+1)\Big)n^2\eta^2+\frac{1-\mu}{2}(\Omega^2-\eta^2(n^2-1)^2-1)(1+3\eta^2), \tag{A.3c}$$

$$A_2=\Big(-n^2\eta^2(2\Omega^2-2n^2+3-\mu)+(\Omega^2+\eta^2(n^2-1)^2+1)\frac{1-\mu}{2}(1+3\eta^2)\Big)\times$$
$$\Big(\Omega^2-\frac{1-\mu}{2}n^2(1+\eta^2)\Big)-\Big(\Big(\mu-n^2\eta^2\frac{1-\mu}{2}\Big)(\Omega^2-n^2)+2n^2\frac{1+\mu}{2}\Big)\times$$
$$\Big(\mu-n^2\eta^2\frac{1-\mu}{2}\Big)\Big(\Omega^2+\Big(\Big(\frac{1+\mu}{2}\Big)^2-1\Big)n^2\Big)(\Omega^2-\eta^2(n^2-1)^2)n^2, \tag{A.3d}$$

$$A_0=((\Omega^2-n^2)(\Omega^2-\eta^2(n^2-1)^2-1)-n^2)\Big(\Omega^2-\frac{1-\mu}{2}n^2(1+\eta^2)\Big). \tag{A.3e}$$

Force and momentum resultants are defined as follows:

$$T'_{x\theta}=N'_{x\theta}+\frac{1}{a}M'_{x\theta}, \tag{A.4a}$$

$$V'_{xr}=\frac{\partial M'_{xx}}{\partial x}+\frac{1}{a}\frac{\partial}{\partial\theta}(M'_{\theta x}+M'_{x\theta}), \tag{A.4b}$$

$$N'_{xx}=k\Big(\frac{\partial U'_x}{\partial x}+\frac{\mu}{a}\Big(\frac{\partial U'_\theta}{\partial\theta}+U'_r\Big)\Big)-\frac{D}{a}\frac{\partial^2U'_r}{\partial x^2}, \tag{A.4c}$$

$$N'_{x\theta}=\frac{1-\mu}{2}\Big(k\Big(\frac{\partial U'_\theta}{\partial x}+\frac{1}{a}\frac{\partial U'_x}{\partial\theta}\Big)+\frac{D}{a^2}\Big(\frac{\partial U'_\theta}{\partial x}-\frac{\partial^2U'_r}{\partial x\partial\theta}\Big)\Big), \tag{A.4d}$$

$$M'_{xx}=D\Big(-\frac{\partial^2U'_r}{\partial x^2}+\frac{\mu}{a^2}\Big(\frac{\partial U'_\theta}{\partial\theta}-\frac{\partial^2U'_r}{\partial\theta^2}\Big)+\frac{1}{a}\frac{\partial U'_x}{\partial x}\Big), \tag{A.4e}$$

$$M'_{x\theta}=\frac{D(1-\mu)}{a}\Big(\frac{\partial U'_\theta}{\partial x}-\frac{\partial^2U'_r}{\partial x\partial\theta}\Big), \tag{A.4f}$$

$$M'_{\theta x}=-\frac{D(1-\mu)}{2a}\Big(\frac{1}{a}\frac{\partial U'_x}{\partial\theta}-\frac{\partial U'_\theta}{\partial x}+2\frac{\partial^2U'_r}{\partial x\partial\theta}\Big). \tag{A.4g}$$

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