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PAPER

An $(\infty, 0)$-model structure on Eilenberg–Zilber opetopic sets

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Abstract

We construct a model for weak $(\infty, 0)$-categories based on the theory of opetopes. The combinatorial complexity of opetopes, even just positive ones, presents a significant obstacle to developing model structures on the corresponding presheaf category. We overcome this by defining a full subcategory $\widehat{\mathsf{pOpe}}_{\iota EZ}$ of $\widehat{\mathsf{pOpe}}_{\iota}$, which we prove to be a reflective subcategory of presheaves on pOpe_{ι} and possesses the regularity needed for homotopical arguments.
Our main results are the following:

- The category $\widehat{\mathsf{pOpe}}_{\iota EZ}$ carries a cofibrantly generated model structure for $(\infty, 0)$-categories, constructed via a modification of the Cisinski–Olschok theory.
- This model structure is Quillen equivalent to the Kan–Quillen model structure on sSet.

These results establish $\widehat{\mathsf{pOpe}}_{\iota EZ}$ as a valid opetopic model for spaces, while the underlying combinatorial framework provides a foundation for future development of opetopic weak $(\infty, 1)$-model.

Keywords: Opetopes; higher category theory; model category structure; homotopical algebra; abstract homotopy theory

1. Introduction

Most homotopical models of higher categories are formalized as certain presheaves on a fixed category of shapes, which encode composition and associativity rules. The most common choices for these shapes include simplices, cubes, globes, and related structures. Opetopes, introduced by Baez and Dolan, fit within this framework but are distinguished by the fact that they were explicitly designed for this purpose from the outset – unlike other shape categories, which often emerged from more classical considerations (a similar approach is found in Joyal's theta category).

Initially defined using operads to address homotopical aspects of higher categories, opetopes were later redefined by several authors and linked to the theory of computads and rewriting, establishing their role in a more computational branch of higher category theory.

In this work, we focus on positive opetopes introduced by Zawadowski and later redefined by Leclerc. Two main achievements of this theory motivate our study:

Theorem 1.1 (Zawadowski (2023), Corollary 13.5). *The category of presheaves over the category of positive opetopes* pOpe *is equivalent to the category of many-to-one computads.*

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Theorem 1.2 (Zawadowski (2017), Theorem 4.15). *The category* pOpe_ι *of positive opetopes with* ι*-contractions forms a test category.*

The first theorem indicates that the combinatorics of positive opetopes aligns closely with the theory of strict higher categories: higher categories can be thought of as generated by computads. The second theorem tells us that there exists a cofibrantly generated model structure on $\widehat{\mathsf{pOpe}_\iota}$ representing the homotopy theory of $(\infty, 0)$-categories. Our goal is to go beyond the mere existence of this model structure and construct a different one: with an explicit presentation via Cisinski–Olschok theory, and Quillen equivalent to the Kan–Quillen model structure on sSet. However, the combinatorial complexity of the category of positive opetopes – most importantly, the fact that it is not an Eilenberg–Zilber category – poses a significant obstacle to achieving this goal directly. While Cisinski–Olschok theory does not itself require the Eilenberg–Zilber condition on the base category, the failure of opetopic sets to satisfy the Eilenberg–Zilber lemma gives rise to difficulties of two kinds. First, it is heuristically unappealing: two distinct nondegenerate cells degenerating to the same cell, or a cell degenerating to another in two distinct ways, do not correspond to any meaningful property of identities of higher morphisms, and allowing such phenomena would likely force homotopical workarounds for the resulting pathologies. Second, it leads to significant technical obstacles, most notably the lack of a convenient generating set of cofibrations in the form of boundary inclusions, which is needed for standard constructions such as filtration by skeleta.

In principle, one could try restricting the class of ι-contractions in such a way that the resulting category would satisfy the Eilenberg–Zilber condition, similarly to Chanavat and Hadzihasanovic (2026). We illustrate our reason for not pursuing that route with the following example: in that case, there would be no degeneracy morphism σ from a triangular opetope to a 1-dimensional globular opetope; since the latter cannot be included in the former as a face, σ cannot have a section. Yet in an $(\infty, 0)$-category one would clearly like to have the possibility of inserting an identity on either endpoint of the domain edge of a globular diagram, obtaining a triangular diagram. If such a possibility is not built into the combinatorics of the exponent category, one is again forced to implement some homotopical mitigations, which are likely to obscure the presentation of generators of a model structure.

Therefore, we restrict our attention to a certain full subcategory of well-behaved opetopic sets, called Eilenberg–Zilber opetopic sets ($\widehat{\mathsf{pOpe}_{\iota EZ}}$). In a sense, this continues the study of ideas introduced by Bergner and Rezk (2013) and Berger and Moerdijk (2011), who imposed additional assumptions on the exponent category, making the corresponding presheaves well-behaved. In our approach, we give a different meaning to this idea, formulating similar demands as conditions on presheaves, which select a certain well-behaved subcategory. The decisive argument in favor of this approach is that a certain functor ζ does not preserve monomorphisms between general opetopic sets. However, it does preserve monomorphisms between Eilenberg–Zilber opetopic sets, justifying our restriction to this subclass.

The functor ζ is derived from the nerve functor, which maps each representable to the nerve of the poset of faces ordered by inclusion. The nerve functor alone cannot be used for the comparison with the Kan–Quillen structure: the image of its right adjoint is not contained within $\widehat{\mathsf{pOpe}_{\iota EZ}}$, and furthermore it completely forgets the crucial property of non-invertibility of morphisms (unlike ζ, it destroys completely the information about the direction of edges). The functor ζ addresses these issues by exploiting the rewriting-theoretic nature of opetopes; we refer to Chapter 3 for its precise definition.

Although the category of Eilenberg–Zilber opetopic sets is not a topos, we have successfully generalized the Cisinski–Olschok theory in a way that still applies in this setting. The main results of this article are:

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- Corollary 5.18 and Definition 5.24: The construction of a combinatorial model structure on $\widehat{\mathsf{pOpe}_{\iota EZ}}$ for $(\infty, 0)$-categories.
- Theorem 5.33: The proof that this model structure is Quillen equivalent to the Kan–Quillen model structure on simplicial sets.

These results establish $\widehat{\mathsf{pOpe}_{\iota EZ}}$ as a valid opetopic model for spaces, while the underlying combinatorial framework provides a foundation for future development of directed models.

1.1 Outline

The first chapter introduces the necessary definitions and foundational results concerning positive opetopes.

In the second chapter, we study Eilenberg–Zilber presheaves and the class of cellular monomorphisms. We introduce the notion of a tidy Reedy category (of which pOpe_ι is a principal example), which is an abstract framework for developing our theory of Eilenberg–Zilber presheaves.

The third chapter develops the connection between positive opetopes and simplicial sets through the lens of rewriting theory, including the definition of the functor ζ. It also presents related notions of ideals on positive opetopes and opetopic associahedra. In the final chapter, we construct a model category structure on Eilenberg–Zilber opetopic sets and prove that it is Quillen-equivalent to the Kan–Quillen model structure on simplicial sets.

2. Positive Opetopes and Contraction Morphisms

2.1 Positive-to-one posets and dendritic face complexes

The category pOpe of positive opetopes was introduced by Zawadowski in Zawadowski (2007a) as part of his study of the connection between computads and categories of presheaves, inspired by earlier works by Harnik and Makkai. Positive opetopes are higher-dimensional combinatorial structures used to model composition in higher categories. Later, alternative descriptions of pOpe were proposed by other authors, including a simpler axiomatization based on dendritic face complexes due to Leclerc Leclerc (2024).

Below, we present both definitions, starting with Leclerc's formulation, which is more accessible to those new to the subject. Along the way, we introduce several key notions, such as ι-contractions, which are essential for our goal of fitting opetopes into the framework of model categories. This chapter serves as a foundation for subsequent work, where certain presheaves on the category pOpe_ι will become models of weak higher categories. No original results are presented here; instead, our goal is to provide a clear and comprehensive introduction to the topic.

Definition 2.1. *A **positive-to-one poset** is a quadruple* $P := (|P|, \dim, \prec^+, \prec^-)$, *where*

- $|P|$ *is a finite set,*
- $\dim : |P| \to \mathbb{N}$ *is a function*
- $\prec^+$ *and* $\prec^-$ *are binary relations on* $|P|$.

This data is subject to the following conditions:

- $x \prec^+ y$ *or* $x \prec^- y \implies \dim(x) + 1 = \dim(y)$,
- $\prec^+ \cap \prec^- = \emptyset$,
- $\dim(x) \geq 1 \implies \exists!\, y, y \prec^+ x \wedge \exists\, z, z \prec^- x$,
- $\dim(x) = 1 \implies \exists!\, y, y \prec^- x$.

The transitive-reflexive closure of $\prec := \prec^+ \cup \prec^-$ endows $|P|$ with the structure of a poset, for which dim becomes an order-preserving map.

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Definition 2.2. ***The boundary*** $\partial(x)$ *of* x *is the set* $y \in |P| \mid y \prec x$*). Note that* $\partial(x)$ *is not closed under taking faces of its elements; in particular, it is not a positive-to-one poset.*

The codomain $\gamma(x)$ *of* x *is the unique element* y *such that* $y \prec^{+} x$ *(sometimes identified with the singleton containing that element, by abuse of notation).*

The boundary $\partial(x)$ *of* x *is the set* $\{y \in |P| \mid y \prec x\}$*. Note that* $\partial(x)$ *is not closed under the operation of taking the face of a cell in* $\partial(x)$*, in particular, it is not a positive-to-one poset.*

Finally, we set $P_k = \dim^{-1}(k)$ and $P_{\leq k} = \dim^{-1}(\{0, \ldots, k\})$.

It follows that γ is a function $P_{\geq 1} \to |P|$ of the degree -1, where $P_{\geq 1} = \dim^{-1}(\{1, 2, 3, \ldots\})$.

Repeated application of γ resulting in an element of dimension k is denoted by $\gamma^{(k)}$, that is $\gamma^{(k)}(x) = \gamma^{\dim(x)-k}(x)$.

We interpret elements of $|P|$ as generalized higher-dimensional morphisms.

Additionally, we can interpret dim topologically. Given a positive-to-one poset, we can associate with it a regular CW-complex. Cells in this complex correspond to the elements of the positive-to-one poset. For more details, see Hadzihasanovic (2024).

Definition 2.3. *A* ***dendritic face complex*** *is a positive-to-one poset* $(|P|, \dim, \prec^{+}, \prec^{-})$ *satisfying the following additional conditions:*

- ***(Greatest element)****: there exists a greatest element in the partial order generated by* $\prec$
- ***(Oriented Thinness)****: for any triple* $x \prec^{\alpha} y \prec^{\beta} z$ *(where* $\alpha, \beta \in \{+, -\}$*), there exists a unique* $y' \neq y$ *such that* $x \prec^{\alpha'} y' \prec^{\beta'} z$ *for some* $\alpha', \beta' \in \{+, -\}$*. Furthermore, the equality* $\alpha\beta = -\alpha'\beta'$ *holds*
- ***(Acyclicity)****: if* $\dim(x) > 1$*, and we have* $y_1, \ldots, y_n \prec^{-} x$ *and* $z_1, \ldots, z_{n-1}$ *satisfying* $z_i \prec^{-} y_i$ *and* $z_i \prec^{+} y_{i+1}$*, then* $y_1 \neq y_n$

Definition 2.4. ***Morphism of positive-to-one posets*** $f : P \to Q$ *is a map between underlying sets* $f : |P| \to |Q|$ *such that*

- $f(\dim(x)) = \dim(f(x))$*, that is* f *preserves the dimension*
- *if* $x, y \in |P|$ *satisfy* $x \prec^{\alpha} y$*, then* $f(x) \prec^{\alpha} f(y)$ *for* $\alpha \in \{+, -\}$
- *for every* $x \in |P|$*,* f *induces a bijection between* $\delta(x)$ *and* $\delta(f(x))$*.*

Notation 2.5. *Let* P *be a positive-to-one-poset. We introduce the following notation:*

- *we write* $|P|$ *for the set of all cells of* P*, to emphasize the process of forgetting the poset structure and distinguish it from* P *when viewed as an object in the category of positive-to-one-posets. Formally,* $|P| = \bigcup_{i=0}^{\dim(P)} P_i$*, where* $\dim(P)$ *is a maximum of a* $\dim$ *function*
- *if* $x \in |P|$*, then* $\langle x \rangle$ *denotes the smallest subobject of* P *generated by* x*; that is, the smallest subobject of* P *containing* x *and closed under the operations* δ *and* γ
- *if* P *is a dendritic face complex,* α_P *denotes the* ***top cell*** *of* P*, that is the cell of maximal dimension*

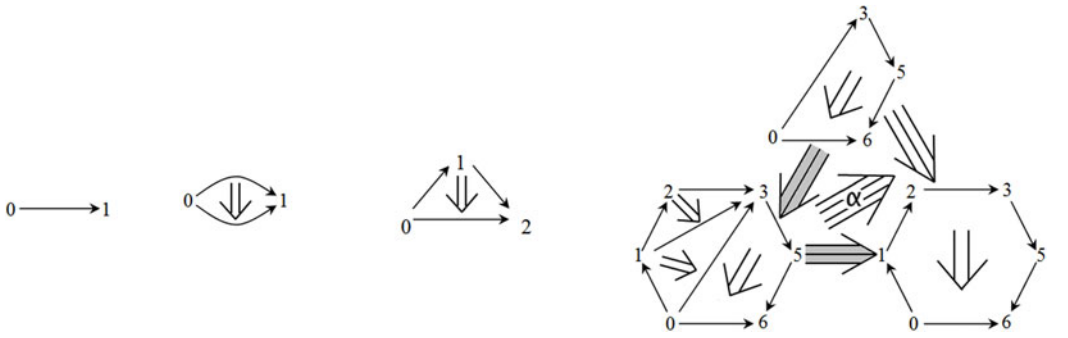
In particular, $\langle \alpha_P \rangle = P$*.*

Example 2.6. *On the picture below, we show example figures of some positive opetopes. Each element of dimension* n *is represented by an* n*-fold arrow, indicating a mapping from the domain to the codomain.*

Note that in the case of the last (4-dimensional) opetope, we have repeated some edges in order to account for the complexity of the combinatorics involved. To clarify the meaning of the diagram, consider the cell labeled α *in the rightmost opetope: the domain of that arrow consists of the two gray*

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3-fold cells and their boundaries. This serves as an illustration of how the domain and codomain structures are encoded in the diagram.



Definition 2.7. *A* ***contraction morphism*** *of positive-to-one posets $f : P \to Q$ is a map between underlying sets $f : |P| \to |Q|$ such that:*

(1) ***Monotonicity of the dimension:***

$$\dim(f(p)) \leq \dim(p), \text{ i.e., } f \text{ does not increase the dimension.}$$

(2) ***Preservation of codomains:***

$$f(\gamma^k(p)) = \gamma^k(f(p)), \quad \text{for } k \geq 0 \text{ and } p \in P_{k+1}.$$

The ***kernel*** *of f is defined as:*

$$\ker(f) = \{p \in |P| \mid \dim(p) > \dim(f(p))\}.$$

(3) ***Preservation of domains:***

- *If* $\dim(f(p)) = \dim(p)$, *then f restricts to a bijection:*

$$\delta(p) \setminus \ker(f) \to \delta(f(p))$$

- *If* $\dim(f(p)) = \dim(p) - 1$, *then f restricts to a bijection:*

$$\delta(p) \setminus \ker(f) \to f(p)$$

- *If* $\dim(f(p)) < \dim(p) - 1$, *then:*

$$\delta(p) \subseteq \ker(f)$$

Definition 2.8. *The category* DFC *is a category with dendritic face complexes as objects, where morphisms are morphisms of positive-to-one-posets.*

The category DFC_ι *has the same objects, but its morphisms are contraction morphisms as described above.*

When viewed as morphisms between computads, contractions are exactly those maps that send generators either to generators or to (possibly iterated) identities on generators of smaller dimensions, Zawadowski (2017) Thm 2.13.

The main result of Leclerc (2024) states that the category of dendritic face complexes is equivalent to the original category pOpe of positive opetopes introduced by Zawadowski. The category pHg of positive hypergraphs of Zawadowski is equivalent to the category of positive-to-one-posets (with the morphisms described above).

The definition of contraction morphisms is taken from Zawadowski (2017) and applied to the formalism of positive-to-one-posets without any change.

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The category DFC_ι can be endowed with a structure of a Reedy category.

Definition 2.9. *Let $\mathcal{A}$ be a small category, equipped with a degree function $d : \mathrm{obj}(\mathcal{A}) \to \Lambda$ on objects, where $(\Lambda, <)$ is a well-ordered set, and suppose that $\mathcal{A}_+$ and $\mathcal{A}_-$ are subcategories of $\mathcal{A}$ which contain all of its objects. Then we say that $(\mathcal{A}, \mathcal{A}_+, \mathcal{A}_-)$ is a* ***Reedy category*** *if and only if these structures satisfy:*

- *if $\alpha : x \to y$ is a non-identity arrow in $\mathcal{A}_+$ (respectively, $\mathcal{A}_-$) then $d(x) < d(y)$ (respectively, $d(x) > d(y)$), and*
- *every arrow α of $\mathcal{A}$ has a unique factorization $\alpha = \overrightarrow{\alpha} \circ \overleftarrow{\alpha}$ where $\overrightarrow{\alpha} \in \mathcal{A}_+$ and $\overleftarrow{\alpha} \in \mathcal{A}_-$.*

We call the morphisms of $\mathcal{A}_-$ ***codegeneracy maps*** *and the morphisms of $\mathcal{A}_+$* ***coface maps****.*

For more exhaustive treatment of the theory of Reedy categories, we refer the reader to Riehl and Verity (2014). To ground these notions, consider $\mathcal{A} = \Delta$, the simplex category, where d corresponds to the dimension of simplices.

Definition 2.10. *We define the degree function $d : \mathrm{Ob}(\mathsf{DFC}) \to \mathbb{N}$ by the formula $P \mapsto \#|P|$.*

The category $(\mathsf{DFC}_\iota)_+$ is equal to DFC, while the category $(\mathsf{DFC}_\iota)_-$ is a category with morphisms being those contraction maps $f : P \to Q$ for which the underlying function is surjective.

One can endow dendritic face complexes with a significantly different degree function, taking values in the set of eventually zero sequences of natural numbers, endowed with a colexicographic order:

$$\mathrm{Ob}(\mathsf{DFC}) \ni P \mapsto (\#|P_0|, \#|P_1|, \dots) \in \mathbb{N}^{\mathbb{N}}$$

While the first degree function is certainly easier to introduce and understand, often the second turns out to be better suited for inductive arguments. In this example, the codomain is no longer equal to the set of natural numbers, in particular it does not satisfy the definition of a Reedy category stated in Riehl and Verity (2014). However, the theory resulting from replacing $\mathbb{N}$ with an arbitrary well-ordered set is a straightforward generalization of the most common case and also covered in the literature (see Hovey (1999), Definition 5.2.1)

On the other hand, one cannot use the dimension of the greatest cell as a degree function for the above choice of $\mathsf{DFC}_{\iota -}$. This is evident from the examples presented below.

Example 2.11. *The only (isomorphism class of) dendritic face complex P with the greatest cell of the dimension* 0*, is a point. It has a single* 0*-cell and no cells of higher dimensions.*

The only (isomorphism class of) dendritic face complex with the greatest cell of the dimension 1*, is an edge. It has two distinct* 0*-cells, a single* 1*-cell, and no cells of higher dimension.*

There exists an infinite countable set of pairwise non-isomorphic dendritic face complexes with the greatest cell of the dimension 2*.*

More precisely, let $n \in \mathbb{N}_{\geq 1}$ $P^n_0 = \{0, \dots, n\}$, $P^n_1 = \{(i, i+1) \mid i \in \{0, \dots, n-1\}\} \sqcup \{a\}$ and $P^n_2 = \{b\}$.

Set $\delta((i, i+1)) = \{i\}$, $\gamma((i, i+1)) = \{i+1\}$, $\delta(a) = \{0\}$, $\gamma(a) = \{n\}$, $\delta(b) = \{(i, i+1) \mid i \in \{0, \dots, n-1\}\}$, $\gamma(b) = \{a\}$.

This endows P^n with a structure of a dendritic face complex with $2n + 3$ distinct cells.

There are exactly two surjective contractions morphisms $P^2 \to P^1$, one with kernel equal $\{(0, 1)\}$ and the second with kernel given by $\{(1, 2)\}$

Definition 2.12. *We say that a dendritic cell complex is* ***unary****, if $\#\delta(\alpha_P) = 1$.*

Example 2.13. *An edge and P_1 are both unary dendritic cell complexes.*

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2.2 Positive hypergraphs and positive opetopes

In this section, we present the original definition of the category of positive opetopes due to Zawadowski Zawadowski (2007a).

Definition 2.14. *A **positive hypergraph** is a triple* $S := (\{S_k\}_{k\in\mathbb{N}}, \{\delta_k\}_{k\in\mathbb{N}}, \{\gamma_k\}_{k\in\mathbb{N}})$*, where each* S_k *is a finite set,* $\delta_k : S_{k+1} \to S_k$ *is a total relation and* $\gamma_k : S_{k+1} \to S_k$ *is a function.*

Furthermore, we assume that only finitely many sets S_k *are nonempty and that the relation* $\delta_0 : S_1 \to S_0$ *is a function.*

*A **morphism of positive hypergraphs*** $f : S \to T$ *is a family of functions* $f_k : S_k \to T_k$*, for* $k \in \mathbb{N}$*, such that, for* $k > 0$ *and* $a \in S_k$*, we have* $\gamma(f(a)) = f(\gamma(a))$ *and* f_{k-1} *restricts to a bijection* $f_a : \delta(a) \to \delta(f(a))$*.*

*The **category of positive hypergraphs** is denoted by* pHg*.*

Set S_k is called a set of k-cells of S. We identify $\gamma(x)$ with $\{\gamma(x)\}$ (as we did in the case of dendritic face complexes) and as a consequence, we introduce the following notation: $\delta\delta(x) = \bigcup_{y\in\delta(x)} \delta(y)$, $\gamma\delta(x) = \{\gamma(y) \mid y \in \delta(x)\}$.

Definition 2.15. *We define a binary relation of **lower order*** $<^{S_k,-}$ *on for* S_k *for* $k > 0$ *as the transitive closure of the relation* $\lhd^{S_k,-}$ *on* S_k *such that, for* $a, b \in S_k$*,* $a \lhd^{S_k,-} b$ *iff* $\gamma(a) \in \delta(b)$*.*

On S_k*, we write* $a <^- b$ *instead of* $a <^{S_k,-} b$*, and similarly for* $\leq^-, \bowtie^-$*.*

We write $a \bowtie^- b$ iff either $a <^- b$ or $b <^- a$, and we write $a \leq^- b$ iff either $a = b$ or $a <^- b$.

Definition 2.16. *We also define a binary relation of **upper order*** $<^{S_k,+}$ *on* S_k *for* $k \geq 0$ *as the transitive closure of the relation* $\lhd^{S_k,+}$ *on* S_k *such that, for* $a, b \in S_k$*,* $a \lhd^{S_k,+} b$ *iff there is* $\alpha \in S_{k+1}$ *such that* $a \in \delta(\alpha)$ *and* $\gamma(\alpha) = b$*.*

We write $a \bowtie^+ b$ iff either $a <^+ b$ or $b <^+ a$, and we write $a \leq^+ b$ iff either $a = b$ or $a <^+ b$.

Definition 2.17. *A positive hypergraph S is a **positive opetopic cardinal** if it is non-empty, that is* $S_0 \neq \emptyset$*, and it satisfies the following four conditions:*

(1) ***(Globularity)****: For* $a \in S_{\geq 2}$*:*

$$\gamma\gamma(a) = \gamma\delta(a) \setminus \delta\delta(a), \quad \delta\gamma(a) = \delta\delta(a) \setminus \gamma\delta(a);$$

(2) ***(Strictness)****: For* $k \in \mathbb{N}$*, the relation* $<^{S_k,+}$ *is a strict order;* $<^{S_0,+}$ *is linear;*

(3) ***(Disjointness)****: For* $k > 0$*,*

$$\bowtie^{S_k,-} \cap \bowtie^{S_k,+} = \emptyset;$$

(4) ***(Pencil linearity)****: For any* $k > 0$ *and* $x \in S_{k-1}$*, the sets*

$$\{a \in S_k \mid x = \gamma(a)\} \quad \text{and} \quad \{a \in S_k \mid x \in \delta(a)\}$$

are linearly ordered by $<^{S_k,+}$*.*

Definition 2.18. *We define **the size** of a positive opetopic cardinal S to be the sequence of natural numbers* $\mathrm{size}(S) = \#\big(S_n \setminus \delta(S_{n+1})\big)_{n\in\mathbb{N}}$*, with almost all being equal to* 0*.*

*We say that S is a **positive opetope** iff* $\mathrm{size}(S)_l = 1$*, for* $l \in \mathbb{N}$*.*

*The **category of positive opetopes*** pOpe *is the full subcategory of* pHg *on positive opetopes.*

For a given positive opetope P, we set $|P| := \bigcup_{n\in\mathbb{N}} P_n$. Moreover, for a given cell x of P, we denote by $\dim(x)$ the unique $k \in \mathbb{N}$ such that $x \in P_k$.

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Finally, we set $\gamma^{(n)}(x) = \gamma^{\dim(x)-n}(x)$ if $n \geq \dim(x)$ and $\gamma^{(n)}(x) = x$ otherwise.

Theorem 2.19 (Leclerc (2024)). *The categories* DFC *and* pOpe *are equivalent.*

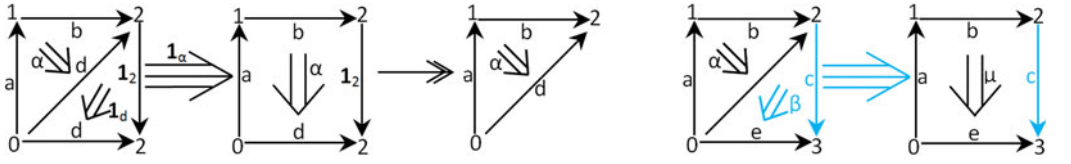
On objects, the pair of functors exhibiting that equivalence is given as follows:

- given a positive-to-one poset
$$(P, \dim, \prec^+, \prec^-)$$
we assign to it a positive hypergraph
$$(\{P_k\}_{k\in\mathbb{N}}, \{\delta_k\}_{k\in\mathbb{N}}, \{\gamma_k\}_{k\in\mathbb{N}})$$
with $P_k = \dim^{-1}(k)$, $\gamma_k(x) = \gamma(x)$ and $\delta_k(x) = \delta(x)$
- given a positive hypergraph
$$(\{P_k\}_{k\in\mathbb{N}}, \{\delta_k\}_{k\in\mathbb{N}}, \{\gamma_k\}_{k\in\mathbb{N}})$$
we assign to it a positive-to-one poset
$$(\coprod_{k\in\mathbb{N}} P_k, \dim, \prec^+, \prec^-)$$
with dim on each summand of the coproduct being equal to the index of that summand, $(x \prec^+ y) \Leftrightarrow (x = \gamma_{\dim(y)+1}(y))$ and $(x \prec^- y) \Leftrightarrow (x \in \delta_{\dim(y)+1}(y))$

Via this equivalence, contraction morphisms for positive opetopes are defined as those morphisms that correspond under the equivalence to contraction morphisms of dendritic face complexes (Definition 3.7). Hence, we do not repeat the definition here.

Example 2.20. *In the picture below, on the left we have depicted an example of a δ-contraction (in fact, an epimorphic one). On the right, kernel of this epimorphism is marked with blue color.*

To clarify the meaning of this transformation, note that the cells on the left diagram are marked with their images under the morphism. For instance, the cell labeled a is mapped to the corresponding cell a on the right, while the 1*-cell labeled* $\mathbf{1}_2$ *is mapped to the* 0*-cell labeled* 2*. This illustrates how individual components are preserved or collapsed under the contraction.*



3. Eilenberg–Zilber Presheaves

We begin this chapter by revisiting basic definitions and properties concerning presheaves over Reedy categories (as introduced in Section 1.3 of Cisinski (2019)), presenting them in a form more conducive to the development of the theory that follows.

After recalling the basic theory of Reedy categories, we introduce two key distinctions: first, between general presheaves and EZ-presheaves (the latter being those satisfying a certain double uniqueness condition on nondegenerate cells), and second, between cellular monomorphisms (those built inductively via the Reedy structure) and general monomorphisms. In the usual setting, where $\mathcal{A}$ is an elegant Reedy category, these distinctions become irrelevant. However, this is not the case for many categories of interest, such as the category pOpe_ι (consult Cavallo and

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Sattler (2022) for a different example). We introduce the degeneracy-forking condition on the Reedy category – monomorphisms between EZ-presheaves over such category to a large extent behave as monomorphisms in sSet (for instance, we have well-behaved constructions analogous to skeleta in sSet), ruling out many pathologies occurring in a general case.

In the second section, we introduce the notion of a **tidy Reedy category** – a Reedy category over which EZ-presheaves form a category that shares with sSet even more desired properties from the homotopical point of view, such as local presentability. The majority of this chapter serves the purpose of establishing these properties.

3.1 Basic theory of EZ-presheaves

In this section, we fix a Reedy category $\mathcal{A}$ with degree function $d : \mathrm{ob}(\mathcal{A}) \to \Lambda$, where Λ is an arbitrary well-ordered set. For a presheaf $X \in \widehat{\mathcal{A}}$, we extend d to sections by setting $d(e) := d(P)$ whenever $e \in X_P$ (i.e., e is a section over P). To emphasize the topological interpretation, sections of a presheaf are called **cells**.

Definition 3.1. *Let $X \in \widehat{\mathcal{A}}$ be a presheaf and $\alpha \in X_P$ for some $P \in \mathcal{A}$. We say that the cell α is **degenerate** if $\alpha = \sigma^* \beta$ for some non-identity $\sigma \in \mathrm{Mor}(\mathcal{A}_-)$ and some $\beta \in X_Q$. Otherwise, we say that α is **nondegenerate**. The set of nondegenerate cells of X is denoted by X^{nd}.*

*For a pair of cells $a', a \in X$, if there exists a non-identity morphism $\sigma \in \mathrm{Mor}(\mathcal{A}_-)$ such that $a = \sigma^* a'$, then we say that a' **degenerates to** a. Moreover, if $a' \in X^{nd}$, we call a' a **predegenerate cell** of a.*

Definition 3.2. *The α-**skeleton** $\mathrm{Sk}_\alpha(X)$ is the largest subpresheaf of X such that every nondegenerate section e of $\mathrm{Sk}_\alpha(X)$ satisfies $d(e) \leq \alpha$.*

Definition 3.3. *We define **the boundary** $\partial \text{ㅗ}_P$ of a representable presheaf ㅗ_P as the presheaf given by the formula $\partial \text{ㅗ}_P = \mathrm{colim}_{\beta < d(P)} Sk_\beta \text{ㅗ}_P$.*

Definition 3.4. *A presheaf $X \in \widehat{\mathcal{A}}$ is called an **EZ-presheaf** (Eilenberg–Zilber presheaf) if every section of X is the image of a unique nondegenerate section under a unique degeneracy map.*

Remark 3.5. If every $X \in \widehat{\mathcal{A}}$ is an Eilenberg–Zilber presheaf, then $\mathcal{A}$ is by definition an elegant Reedy category Bergner and Rezk (2013).

Example 3.6. *If X is a presheaf on Δ (or more generally, over any Eilenberg–Zilber category), then it is an EZ-presheaf. This is the statement of the Eilenberg–Zilber Lemma (Cisinski (2019), Lemma 1.3.6).*

A presheaf X can fail to satisfy the Eilenberg–Zilber condition for two different reasons:

(1) There exists a nondegenerate $\alpha \in X$ and two distinct codegeneracy maps σ_1, σ_2 acting on α such that $\sigma_1^* \alpha = \sigma_2^* \alpha$.
(2) There exist a pair of distinct nondegenerate cells $\alpha, \beta \in X$ and a pair of codegeneracy maps σ_1, σ_2 acting on them such that $\sigma_1^* \alpha = \sigma_2^* \beta$.

Definition 3.7. *Let $i : \mathcal{A}_- \hookrightarrow \mathcal{A}$ be the inclusion functor and let $X \in \widehat{\mathcal{A}}$ be a presheaf. The **category of cells** of X, denoted by $\int_{\mathcal{A}} X$, is the category of elements of the restricted presheaf $i^* X$ on $\mathcal{A}_-$. In symbols, $\int_{\mathcal{A}} X := \int_{\mathcal{A}_-} i^* X$.*

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More explicitly, $\oint_{\mathcal{A}} X$ has the same objects as $\int_{\mathcal{A}} X$ (i.e., pairs (P, x) with $P \in \mathcal{A}$ and $x \in X_P$), but a morphism $(f, \alpha) : (P, x) \to (Q, y)$ in $\oint_{\mathcal{A}} X$ is a morphism in $\int_{\mathcal{A}} X$ such that the underlying arrow $f : P \to Q$ belongs to the subcategory $\mathcal{A}_-$.

By abuse of notation, we identify the sets $\bigsqcup_{P \in \mathcal{A}} X_P \cong \mathrm{Ob}(\int_{\mathcal{A}} X) = \mathrm{Ob}(\oint_{\mathcal{A}} X)$.

Lemma 3.8. *A presheaf $X \in \widehat{\mathcal{A}}$ is an EZ-presheaf if and only if every connected component of its category of cells has a terminal object.*

Proof. The category $\oint_{\mathcal{A}} X$ is isomorphic to the coproduct of its maximal connected subcategories:

$$\oint_{\mathcal{A}} X \cong \coprod_{s \in \mathcal{S}} \mathcal{C}_s$$

Note that the uniqueness of the predegenerate cell of $a \in X_P$ in the definition of an EZ-presheaf corresponds to the existence of a weakly terminal object in the category $\mathcal{C}_s$ containing a. The uniqueness of the degenerating map means precisely that this weakly final object is, in fact, terminal in a strict sense. □

Observation 3.9. *A skeleton of an EZ-presheaf is an EZ-presheaf itself. More generally, if X is an EZ-presheaf and U is some set of cells of X which satisfies the following condition:*

- *if $x \in U$ and $y \in X_P$ is a predegenerate cell of x, then $y \in U$*

then the presheaf Y obtained from X by deleting images by codegeneracy maps of cells contained in U is also an EZ-presheaf.

Definition 3.10. *Inclusion morphisms $\partial \mathbb{L}_P \overset{\partial_P}{\hookrightarrow} \mathbb{L}_P$ are called **boundary inclusions**. The class of transfinite compositions of pushouts of boundary inclusions is called the class of **cellular monomorphisms**.*

Definition 3.11. *If $\mathcal{A}$ is a Reedy category such that for every codegeneracy map $\sigma : P \to Q$ there exist a pair of distinct codegeneracy maps $\sigma', \sigma'' : R \to P$ such that $\sigma \circ \sigma' = \sigma \circ \sigma''$, then we say that $\mathcal{A}$ is a **degeneracy-forking** Reedy category.*

The idea behind that notion can be motivated as follows – if $f : X \to Y$ is a monomorphism of simplicial sets, then it both reflects (which is easy to prove) and preserves (which is somewhat harder) nondegenerate cells, that is, α is a nondegenerate cell of X if and only if $f(\alpha)$ is a nondegenerate cell of Y. This is very tightly related to the fact that Y can be obtained from X by attaching some new nondegenerate cells, one by one, expressing f as a transfinite composition of pushouts of boundary inclusions.

Usually, this is proved by the Eilenberg–Zilber lemma, in particular relying on the fact that every degeneracy has a section (some face map).

More precisely, if $f(\alpha) \in Y_n$ is degenerate, then it is a degeneracy of some of its faces: $f(\alpha) = \sigma^* \delta^* f(\alpha)$ (δ is a section of the degeneracy σ by which the cell α was degenerate), so by functoriality, $\alpha = \sigma^* \delta^* \alpha$, meaning that α must be a degeneracy of some of its faces as well.

Unfortunately, this argument does not apply to many categories of interest, most notably $\widehat{\mathsf{pOpe}_{\iota EZ}}$, because not every degeneracy has a section. The concept of a degeneracy-forking Reedy category allows us to derive the same conclusion as above, but from different premises.

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Specifically, if $f(\alpha) = \sigma^* \beta$, then $\sigma_1^* f(\alpha) = \sigma_2^* f(\alpha)$, and also $\sigma_1^* \alpha = \sigma_2^* \alpha$ because f is mono. If X is an EZ-presheaf, α must be degenerate in this case (i.e., some cell is a degeneracy of α in two distinct ways).

In summary, in informal terms, this condition ensures that a cell of an EZ-presheaf that "appears" degenerate is indeed degenerate.

Lemma 3.12. *If よ_P is a representable presheaf, $a \in \text{よ}_P \setminus \partial\, \text{よ}_P$, then a is a degeneracy of the top cell (i.e., the cell corresponding via the Yoneda embedding to the identity map of P).*

Proof. Take the morphism $R \xrightarrow{\mu} P$ corresponding to the cell a via Yoneda embedding. Using the assumption that $\mathcal{A}$ is a Reedy category, we factor it as $R \to Q \xrightarrow{\phi} P$, where ϕ is a coface map. Since a is not in $\partial\, \text{よ}_P$, $\phi = 1_P$ and hence μ must be a codegeneracy map. □

Lemma 3.13. *Every representable presheaf is an EZ-presheaf.*

Proof. Let よ_P be a representable presheaf and $x \in (\text{よ}_P)_Q$. By the Yoneda lemma, we can equivalently view x as a morphism $x : Q \to P$. We can factor $x = \delta \circ \sigma$ uniquely, where δ is a coface map and σ is a codegeneracy map. The cell of よ_P corresponding to the morphism δ is the only nondegenerate cell with x as its image by some degeneracy map. This degeneracy map can only be σ^*. □

Theorem 3.14. *Let X be an EZ-presheaf and $\phi \in Hom(\partial\, \text{よ}_P, X)$. Then the pushout displayed on the diagram below is also an EZ-presheaf.*

$$\begin{array}{ccc} \partial\, \text{よ}_P & \xrightarrow{\phi} & X \\ \downarrow & \ulcorner & \downarrow \\ \text{よ}_P & \xrightarrow{\hat{\phi}} & \widetilde{X} \end{array}$$

Proof. Since colimits are pointwise in presheaves, for every Q, each cell in $\widetilde{X}_Q \setminus X_Q$ is a unique degeneracy of the image of the main cell in よ_P via $\hat{\phi}$ and conversely, if some cell $a \in \widetilde{X}_Q$ is a degeneracy of the image of the main cell in よ_P, then $a \in \widetilde{X}_Q \setminus X_Q$.

Let $a \in \widetilde{X}_Q$ for some Q. If $a \in X_Q$, then there is a unique cell $a' \in X_Q^{nd}$ which degenerates to a, since X is an EZ-presheaf. By what was written above, it follows that a' is also nondegenerate in $\widetilde{X}$ and no other nondegenerate cell of $\widetilde{X}$ degenerates to a.

Similarly, if $a \in \widetilde{X} \setminus X$, then a is a degeneracy of $\hat{\phi}$. By the same reasoning as in Lemma 3.13, a is not a degeneracy of any nondegenerate cell distinct from $\hat{\phi}$.

More compactly, we have shown that $\oint_{\mathcal{A}} \widetilde{X}$ is obtained from $\oint_{\mathcal{A}} X$ just by adding one new connected component, corresponding to degeneracies of the cell $\hat{\phi}$. □

Lemma 3.15. *Let $Z \xhookrightarrow{g} W$ be a monomorphism of presheaves on a degeneracy-forking Reedy category. Then the following are equivalent:*

- *morphism g satisfies the following conditions:*
  - (1) *for every nondegenerate cell $e \in W_P \setminus Z_P$ and for every pair σ_1, σ_2 of distinct degeneracy operators acting on e, the following holds: $\sigma_1^* e \neq \sigma_2^* e$.*
  - (2) *for every pair of cells $e \in W_P$, $e' \in W_Q$ such that e' is a degeneracy of e, $e' \in Z_Q$ implies $e \in Z_P$*

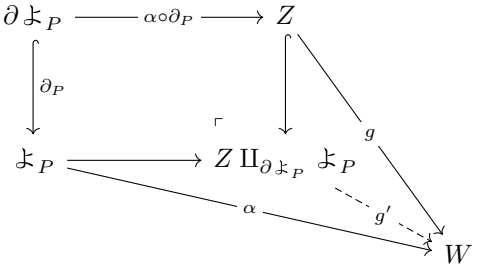
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(3) *for all pairs of distinct nondegenerate cells $e \in W_P$, $e' \in W_Q \setminus Z_Q$ and all pairs of degeneracy operators σ_1, σ_2 such that σ_1 acts on e and σ_2 acts on e', $\sigma_1^* e \neq \sigma_2^* e'$*

- *g is a cellular monomorphism.*

Proof. We will only show that the three conditions listed above imply that g is a cellular monomorphism (the other implication, not using the degeneracy-forking condition, is left to the reader).

Using transfinite induction, it suffices to show that if α is a nondegenerate cell in $W_P \setminus Z_P$ with boundary contained in Z, then the map g' shown on the diagram is a monomorphism that satisfies the same conditions *(1)-(3)* as the map g.



First, we will show that g' is a monomorphism.

Suppose that there are two distinct cells $e, e' \in \left(Z \amalg_{\partial よ_P} よ_P\right)_Q$ such that $g'(e) = g'(e')$. Without loss of generality, $e \notin Z_Q$, so the previous lemma implies that it must be a degeneracy of a top cell of $よ_P$.

If $e' \notin Z_Q$, then e' is also a degeneracy of a top cell of $よ_P$. The top cell of $よ_P$ is sent to α, which is assumed to be nondegenerate in W, hence by *(1)* distinct degeneracy operators acting on α give different cells, so e and e' must be mapped by g to different degeneracies of α.

If $e' \in Z_Q$, then the pair of cells e', α violates the condition $it(2)$.

Conditions *(1)* and *(3)* for g' are obvious, since the image of g' contains the image of g.

To verify the condition *(2)*, we use the assumption that $\mathcal{A}$ is degeneracy-forking. Suppose that there exists $d \in \left(Z \amalg_{\partial よ_P} よ_P\right)_Q$ such that one can find $e \in \left(W \setminus Z \amalg_{\partial よ_P} よ_P\right)_R$ which degenerates to d. Since the condition *(2)* was satisfied for g, we note that d must be a degeneracy of a top cell of $よ_P$.

We can assume that e is nondegenerate in W and that d is nondegenerate in $Z \amalg_{\partial よ_P} よ_P$. Explicitly, let e' be the predegenerate cell of e (unique by condition *(3)*) in a presheaf W and let d' be some predegenerate cell of d in a presheaf Z (obviously e' is not an image of any cell in Z, since otherwise, e would be an image of a cell in Z). We note that e' degenerates to e which degenerates to d and also d' degenerates to d. Let d'' be the predegenerate cell of d'' in Y. Both e', d'' are nondegenerate cells of an EZ-presheaf W which degenerate to the same cell d, and since $e' \notin Z_Q$, they must be equal by *(3)*. In particular, we can assume that d is a top cell of P.

Let $\sigma : R \to Q \in \mathrm{Mor}(\mathcal{A})$ be the codegeneracy map corresponding to the degeneracy operator sending e to d. Let the pair $\sigma', \sigma'' : S \to R$ of distinct codegeneracy morphisms satisfy $\sigma \circ \sigma' = \sigma \circ \sigma''$. Then $\sigma'^*(d) \neq \sigma''^*(d)$ since there are no identifications between degeneracies of the top cell of P in the pushout $Z \amalg_{\partial よ_P} よ_P$. On the other hand, they must be equal, since $d = \sigma^*(e)$. A contradiction. □

Observation 3.16. *One can think of cellular monomorphisms $X \hookrightarrow Y$ as those monomomorphisms, which correspond to morphisms $\oint_{\mathcal{A}} X \hookrightarrow \oint_{\mathcal{A}} Y$ which only add new connected components, each of which has a terminal object.*

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Lemma 3.17. *If $\mathcal{A}$ is a degeneracy-forking Reedy category, then every monomorphism between EZ-presheaves is a cellular monomorphism.*

Proof. Suppose $X \overset{f}{\hookrightarrow} Y$ is a monomorphism between EZ-presheaves, which is not cellular. This means that some cell $e \in \big(Y \setminus \mathrm{Im}(f)\big)_Q$ has a degeneracy (by the operator σ^*) which is an image of a cell $d \in X_R$ (this is the negation of the condition *(2)* from the previous lemma, conditions *(1)* and *(3)* are trivially satisfied, since X and Y are EZ-presheaves).

Since X and Y are EZ-presheaves, we can assume that e is nondegenerate in Y and d is nondegenerate in X. Explicitly, let e' be the predegenerate cell of e in a presheaf Y and let d' be the unique predegenerate cell of d in a presheaf X (obviously e' is not an image of any cell in X, since otherwise, e would be an image of a cell in X). It suffices to observe that e' degenerates to $f(d')$. Indeed, e' degenerates to e which degenerates to $f(d)$ and also $f(d')$ degenerates to $f(d)$. Let d' be the predegenerate cell of $f(d')$ in Y. Both e', d' are nondegenerate cells of an EZ-presheaf Y which degenerate to the same cell $f(d)$, hence they must be equal. In other words, e' is a predegenerate cell of $f(d')$ in a presheaf Y.

Let $\sigma : R \to Q \in \mathrm{Mor}(\mathcal{A})$ be the codegeneracy map corresponding to the degeneracy operator sending e to $f(d)$. Let the pair $\sigma', \sigma'' : l \to R$ of distinct codegeneracy morphisms satisfy $\sigma \circ \sigma' = \sigma \circ \sigma''$. Then $\sigma'^*(f(d)) \neq \sigma''^*(f(d))$ since d is a nondegenerate cell in EZ-presheaf and f is a monomorphism. On the other hand, they need to be equal, since $f(d) = \sigma^*(e)$. A contradiction. □

Corollary 3.18. *If $\mathcal{A}$ is a degeneracy-forking Reedy category and X is an EZ-presheaf, then $\emptyset \to X$ is a cellular monomorphism.*

Proof. Initial presheaf is both an EZ-presheaf and a monomorphism $\emptyset \to \emptyset$ is cellular. Since $\emptyset \hookrightarrow X$ is mono, the previous result implies that it is a cellular monomorphism. As a result, $\emptyset \to X$ is a cellular monomorphism. □

Lemma 3.19. *If $\mathcal{A}$ is a degeneracy-forking Reedy category, then cellular monomorphisms are stable under taking retracts.*

Proof. Let $X \overset{f}{\to} Y$ be a cellular monomorphism and consider the following diagram in which compositions in both rows are identity morphisms:

$$\begin{array}{ccccc} Z & \overset{i}{\hookrightarrow} & X & \overset{r}{\twoheadrightarrow} & Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ W & \overset{i'}{\hookrightarrow} & Y & \overset{r'}{\twoheadrightarrow} & W \end{array}$$

Morphism g is a monomorphism, being a retract of a monomorphism. Suppose that g is not a cellular monomorphism, so at least one of the conditions specified in Lemma 3.15 does not hold. We show that each of those possibilities leads to a contradiction:

(1) there is a nondegenerate cell $e \in W_P \setminus Z_P$ such that $\sigma^*(e) = \sigma'^*(e)$ for some pair of distinct codegeneracy maps σ, σ'. Since this is not the case for X and Y, this means that $i'(e) \in Y$ is degenerate. We pick its predegenerate cell e' (we note that it is of a different shape than e). Then $r'(e')$ is a predegenerate cell of e in W, a contradiction (more generally, this reasoning shows that the image of a nondegenerate cell in W is nondegenerate in Y, which we will use again in the point (3)).

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(2) there is a pair of cells $e \in W_P$, $e' \in W_Q$ such that e degenerates to e' and $e' \in Z_Q$, $e \notin Z_P$. Since f is a cellular monomorphism, $i'(e) \in X_P$ (otherwise f would violate a condition (2)). Then $e = r' \circ i'(e) = r' \circ f \circ i'(e) = g \circ r \circ i'(e) \in Z$, a contradiction.

(3) There are distinct nondegenerate cells $e \in W_P$, $e' \in W_Q \setminus Z_Q$ which degenerate to the same cell. The cell $i'(e')$ is not in the image of f: if $f(d) = i'(e')$, then $g(r(d)) = r'(f(d)) = r'(i'(e')) = e'$. It follows that $i'(e)$ and $i'(e')$ violate condition (2) for the cellular monomorphism f. □

Notation 3.20. *If M is a class of morphisms in a fixed category $\mathcal{C}$, then $\mathrm{l}(M)$ ($\mathrm{r}(M)$) denotes the* ***left orthogonal complement*** *of the class M (****right orthogonal complement****, respectively), (see Riehl (2014), p. 134, where those notions are introduced as ${}^{\boxslash}\mathcal{R}$ and $\mathcal{L}^{\boxslash}$, respectively).*

Corollary 3.21. *Let $\mathcal{A}$ be a degeneracy-forking category. The class of cellular monomorphisms is equal to the class $\mathrm{l}(\mathrm{r}(\{\partial_P\}_{P \in \mathcal{A}}))$.*

If $\emptyset \to X$ is a cellular monomorphism, then X is an EZ-presheaf. More generally, the class of presheaves X such that $\emptyset \to X$ is cellular, is equal to the class of EZ-presheaves.

Proof. The first claim follows from the small object argument, see Riehl (2014).

For the second one, note that if X is a colimit of a sequence of monomorphisms between EZ-presheaves, then X is an EZ-presheaf itself. Combine that observation with the Theorem 3.14 and the fact that $\emptyset$ is an EZ-presheaf.

The last is a combination of the second claim with Corollary 3.18. □

Corollary 3.22. *A retract of an EZ-presheaf over a degeneracy-forking category is also an EZ-presheaf.*

Proof. By Corollary 3.21, X is an EZ-presheaf if and only if $\emptyset \to X$ is a cellular monomorphism, we can apply Lemma 3.19. Alternatively, one can prove that directly from the definition of an EZ-presheaf. □

3.2 Tidy Reedy categories

Now we impose further conditions on $\mathcal{A}$ (which so far was assumed only to be a degeneracy-forking Reedy category) which will ensure that the notion of EZ-presheaf is compatible with the formation of limits. The assumptions posed here will allow us to prove that the full subcategory of EZ-presheaves is a reflective and locally presentable subcategory of the category of presheaves.

Definition 3.23. *A* ***tidy Reedy category*** *is a Reedy category $\mathcal{A}$ satisfying the following conditions:*

- *it is degeneracy-forking*
- *the category $\mathcal{A}_-$ has binary pushouts*

Remark 3.24. The category satisfying only the second condition above is called a **pre-elegant Reedy category**. This notion was introduced by Cavallo and Sattler in Cavallo and Sattler (2022).

Lemma 3.25. *In a pre-elegant Reedy category, every codegeneracy map is an epimorphism.*

Proof. Cavallo and Sattler (2022), Lemma 5.29. □

One can show that the assumption about the existence of pushouts is sufficient to prove that *EZ*-presheaves are closed under taking products, while to ensure the closedness under forming

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equalizers, one also needs the assumption about epimorphisms. Both of those results are easy consequences of the Corollary 3.32, but we decided to include the independent proof of the first of them below. The degeneracy-forking condition is not connected to the question of completeness of the category of Eilenberg–Zilber presheaves; it is assumed because of the results presented earlier. From now on, we will always assume that $\mathcal{A}$ is a tidy Reedy category.

Example 3.26. *Category* Δ *is both a tidy Reedy category and an Eilenberg–Zilber category.*

Category pOpe_ι *is a tidy Reedy category, in which not every codegeneracy map is a split epimorphism. In Section 4.2 of the Chapter 4, we will give a detailed proof of that fact.*

A subcategory Δ' *of the category* Δ*, with the same set of objects, but morphisms being only those order-preserving maps which have a subsegment of the codomain as an image, is a tidy Reedy category, in which also not every codegeneracy map is a split epimorphism.*

The full subcategory of Δ *spanned by the objects* [0], [1], [2] *is elegant (even an Eilenberg–Zilber category), but not tidy (there are no nonidentity surjections with the codomain* [2]*)*

Theorem 3.27. *Suppose that X, Y are EZ-presheaves. Then* $X \times Y$ *is also an EZ-presheaf.*

Proof. Let $(a, b) \in \big(X \times Y\big)_P$ and $(a', b') \in \big(X \times Y\big)_Q, (a'', b'') \in \big(X \times Y\big)_R$ be two nondegenerate cells of $X \times Y$ which degenerate to (a, b):

$$\mu^*(a', b') = (a, b) = \rho^*(a'', b'')$$

Both cells a' and a'' degenerate to a. Since X is an EZ-presheaf, there is $a''' \in X_S$ which degenerates to a' and also to a''. This must factor through the cell which has the shape of the pushout of ρ and μ, call it $\hat{a} \in X_{Q \amalg_P R}$. Cell $\hat{a}$ degenerates to a via an induced pushout map, which we denote by η.

Similarly, we obtain $\hat{b} \in Y_{Q \amalg_P R}$ which degenerates to b via η. But then $(\hat{a}, \hat{b}) \in \big(X \times Y\big)_{Q \amalg_P R}$ degenerates to both (a', b') and (a, b) and since those are nondegenerate, must be equal to both of them.

Now suppose that $(a, b) \in \big(X \times Y\big)_P^{nd}$ and μ, ρ are codegeneracy maps s. t. $\mu^*(a, b) = \rho^*(a, b) = (a', b') \in \big(X \times Y\big)_Q$. It follows that $\mu^* a = \rho^* a = a'$ and since X is an EZ-presheaf, a must be a degeneracy of some cell a'' via κ s. t. $\mu \circ \kappa = \rho \circ \mu$. Cell a'' must degenerate to some cell $\hat{a} \in X_S$, where S is a pushout of μ and ρ. It degenerates to a via the canonical pushout morphism η. Note that $\hat{a}$ must be distinct from a since the pushout of two distinct morphisms is not an identity map (hence it is a proper codegeneracy map and therefore domain is not equal to the codomain).

Similarly, we obtain a cell $\hat{b} \in Y_S$ which degenerates to b via η, but then $\eta^*(\hat{a}, \hat{b}) = (a, b)$. Since (a, b) is a nondegenerate cell, it follows that $(\hat{a}, \hat{b}) = (a, b)$ and $\eta = \mathrm{Id}_S$. Since the pushout of two degeneracy maps is an identity only if those maps are equal (since otherwise it would be a morphism strictly lowering the degree), it follows that $\mu = \rho$. □

Observation 3.28. *Similar reasoning shows that arbitrary products of EZ-presheaves are EZ-presheaves.*

Lemma 3.29. *Let X be an EZ-presheaf,* $a, b \in X$ *satisfy* $\sigma^* a = \sigma^* b$ *for some degeneracy operator* σ*. Then* $a = b$*.*

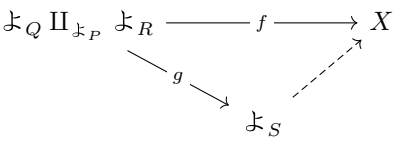
Proof. Let c be the unique nondegenerate cell of X which degenerates to a and b, that is $\sigma_a^* c = a$, $\sigma_b^* c = b$. Then $\sigma^* \sigma_a^* c = \sigma^* \sigma_b^* c$, so by the uniqueness of degeneracy operator $\sigma_a \circ \sigma = \sigma_b \circ \sigma$. Since σ is epi, $\sigma_a = \sigma_b$ and consequently $a = \sigma_a^* c = \sigma_b^* c = b$. □

Definition 3.30. *Let* $P \xrightarrow{\sigma} Q$, $P \xrightarrow{\tau} R$ *be a pair of maps in* $\mathcal{A}_-$ *and let* $S = Q \amalg_P R$*. We will call the canonical map* $\lambda\!\!\!\downarrow_Q \amalg_{\lambda\!\!\!\downarrow_P} \lambda\!\!\!\downarrow_R \to \lambda\!\!\!\downarrow_S$ *a* ***comparing map****.*

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Theorem 3.31. *Let $X \in \widehat{\mathcal{A}}$. Then $X \in \widehat{\mathcal{A}}_{EZ} \iff X$ is orthogonal to all comparing maps.*

Proof. ($\Longrightarrow$) Fix a comparing map $g : よ_Q \amalg_{よ_P} よ_R \to よ_S$.



First, we will show the existence of a dashed arrow. Let $c \in X_T$ be a common predegenerate cell of $f(\alpha_Q)$ and $f(\alpha_R)$. By the universal property of a pushout, there exists a unique codegeneracy map $\kappa : S \to T$. Then $\kappa^* c : よ_S \to X$ is the desired map.

It remains to justify the uniqueness of such a map. Let $\sigma' : R \to S$ be a pushout of σ along τ and suppose that $f_1, f_2 : よ_S \to X$ satisfy $f = f_1 \circ g = f_2 \circ g$. Then $\sigma'^* f_1 = f(\alpha_r) = \sigma'^* f_2$. Now it suffices to apply Lemma 3.29.

($\Longleftarrow$) Suppose that $X \notin \widehat{\mathcal{A}}_{EZ}$. This means that at least one of the following conditions holds:

(1) there exists a pair of distinct cells $a \in X_Q^{nd}$, $b \in X_R^{nd}$ and a pair of codegeneracy maps σ, τ such that $\sigma^* a = \tau^* b \in X_P$

(2) there exists a cell $a \in X_Q^{nd}$ and a pair of distinct codegeneracy maps σ, τ such that $\sigma^* a = \tau^* b \in X_P$

In the first of those cases, X is not orthogonal to a comparing map $よ_Q \amalg_{よ_P} よ_R \to よ_S$: if it was, then the image of α_S in X would be a common predegenerate cell of a and b (which does not exist for a pair of distinct nondegenerate cells).

In the second case, we reason as follows: since σ and τ are distinct, their pushouts are not identities, in particular $d(S) < d(Q)$. If X is orthogonal to a comparing map $よ_Q \amalg_{よ_P} よ_Q \to よ_S$, then the image of α_S in X is a predegenerate cell of a which has a strictly smaller dimension than a, contradicting the fact that a is a nondegenerate cell. □

Recall that a category $\mathcal{C}$ is called a **locally presentable category**, if it satisfies the following conditions:

- $\mathcal{C}$ is locally small
- $\mathcal{C}$ is cocomplete
- there exists a small set S of κ-compact objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is a κ-filtered colimit of a diagram consisting only of objects in S for some regular cardinal κ.

Corollary 3.32. *The category $\widehat{\mathcal{A}}_{EZ}$ is a reflective and locally presentable subcategory of $\widehat{\mathcal{A}}$. In particular, it is bicomplete, and limits in $\widehat{\mathcal{A}}_{EZ}$ can be computed pointwise.*

Proof. Adámek and Rosický (1994), Theorem 1.39. □

Remark 3.33. Local presentability of $\widehat{\mathcal{A}}_{EZ}$ can be also proved differently: by Cavallo and Sattler (2022), Theorem 5.32, EZ-presheaves are characterized as those presheaves, which send pushouts of degeneracy maps to pullbacks of sets. In particular, the category $\widehat{\mathcal{A}}_{EZ}$ is a category of models of limit sketches, thus by Adámek and Rosický (1994) Corollary 1.52 it is locally presentable.

Some colimits of EZ-presheaves can also be computed in the category of all presheaves.

Lemma 3.34. *Let $f : X \hookrightarrow Y$, $g : X \to Z$ be a pair of morphisms between EZ-presheaves such that the first one is a monomorphism. Then the pushout $Y \amalg_X Z$ is an EZ-presheaf.*

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Proof. The morphism f is a cellular monomorphism, being a monomorphism between EZ-presheaves. Since cellular monomorphisms are closed under taking pushouts, $\tilde{f} : Z \hookrightarrow Y \amalg_X Z$ is also a cellular monomorphism. Since Z is an EZ-presheaf, $\emptyset \hookrightarrow Z$ is a cellular monomorphism. It follows that $\emptyset \hookrightarrow Y \amalg_X Z$ is cellular, so $Y \amalg_X Z$ is an EZ-presheaf. □

Lemma 3.35. *The coproduct of a family of EZ-presheaves is an EZ-presheaf.*

Proof. Direct check. □

Theorem 3.36. *A filtered colimit of EZ-presheaves (taken in $\widehat{\mathcal{A}}$) is an EZ-presheaf, that is it is also a colimit in the category $\widehat{\mathcal{A}}_{EZ}$.*

Proof. This follows from Adámek and Rosický (1994), p. 31, and the Theorem 3.31 by observing that both domains and codomains of comparing maps are finitely presentable. □

One can also characterize epimorphisms in the category of EZ-presheaves.

Lemma 3.37. *Let $f : X \to Y$ be a morphism of EZ-presheaves. Then it is an epimorphism if and only if the induced functor $\oint_{\mathcal{A}} f : \oint_{\mathcal{A}} X \to \oint_{\mathcal{A}} Y$ gives a surjection between sets of connected components of the categories involved.*

Proof. Suppose that $\pi_0 \circ \oint_{\mathcal{A}} f : \pi_0 \circ \oint_{\mathcal{A}} X \to \pi_0 \circ \oint_{\mathcal{A}} Y$ is not a surjection.

The functor $\pi_0 \circ \oint_{\mathcal{A}}$ is a left adjoint (being a composition of three left adjoints), in particular, it preserves epimorphisms. It follows that f is not an epimorphism.

Conversely, assume that $\pi_0 \circ \oint_{\mathcal{A}} f : \pi_0 \circ \oint_{\mathcal{A}} X \to \pi_0 \circ \oint_{\mathcal{A}} Y$ is a surjection and let $k, h : Y \to Z$ be a pair of morphisms between EZ-presheaves such that $k \circ f = h \circ f$.

It is enough to check that k and h agree on each cell of Y. Fix $\alpha \in Y$. By the assumption, there exists a codegeneracy map σ and a cell $\beta \in X$ such that $f(\beta) = \sigma^* \alpha$, so $\sigma^* k(\alpha) = k \circ f(\beta) = h \circ f(\beta) = \sigma^* h(\alpha)$. Applying Lemma 3.29, we finish the proof. □

It is also true that if $X \xrightarrow{f} Y$ is a monomorphism (isomorphism), then $\pi_0 \circ \oint_{\mathcal{A}} f : \pi_0 \circ \oint_{\mathcal{A}} X \to \pi_0 \circ \oint_{\mathcal{A}} Y$ is injective (bijective). The converse implications are not true in general.

Lemma 3.38. *If $f : X \to Y$ is a morphism of EZ-presheaves, which is both a monomorphism and an epimorphism, then it is an isomorphism. That is, the category of Eilenberg–Zilber presheaves is balanced.*

Proof. From the assumptions follows that f is a cellular monomorphism such that for every nondegenerate $\alpha \in Y$ there exists $\beta \in X$ and a codegeneracy map σ such that $f(\beta) = \sigma^* \alpha$.

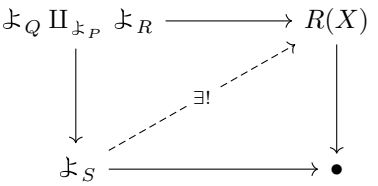
It is enough to check that every nondegenerate cell of Y is an image of some cell of X by f, so suppose that $\alpha \in Y_P \setminus X_P$. This contradicts the point (3) of the Lemma 3.15. □

Let $F : \widehat{\mathcal{A}} \to \mathcal{D}$ be a left adjoint functor. We formulate a sufficient condition for $F|_{\widehat{\mathcal{A}}_{EZ}} \to \mathcal{D}$ to be a left adjoint.

Theorem 3.39. *Suppose that $F\big|_{\mathcal{A}_-}$ preserves pushouts. Then the image of the right adjoint R is contained in $\widehat{\mathcal{A}}_{EZ}$. In particular, $F\big|_{\widehat{\mathcal{A}}_{EZ}}$ and R constitute a pair of adjoint functors.*

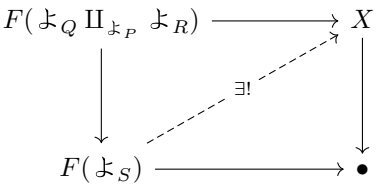
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Proof. The diagram



where the morphism on the left is a generic comparing map, expressing the fact that $R(X)$ is an EZ-presheaf.

By the standard correspondence of lifting problems under adjunction, the existence of the unique filler in this diagram is equivalent to



which is true by the assumption on F (the arrow on the left is an isomorphism). □

4. Functor ζ and Ideals on Opetopes

This chapter has two main objectives. The first is to introduce functors that facilitate comparisons between opetopic and simplicial sets and to systematically study their key properties. The second is to prove that the category pOpe_ι is a tidy Reedy category. Although these two goals may appear unrelated at first glance, they are deeply connected through the concept of ideals in opetopes – a combinatorial analog of the kernel of a map between ω-categories. This notion plays a central technical role and is developed in detail in the middle section of the chapter.

4.1 Adjunction $\zeta \dashv \rho$

In this section, we define the functor $\zeta : \widehat{\mathsf{pOpe}_\iota} \to \mathsf{sSet}$, which will serve as a key tool for comparing the opetopic model structure with the Kan–Quillen model structure on sSet. This functor is closely related to the usual nerve functor $\mathcal{N}$, but unlike $\mathcal{N}$, ζ retains information about the orientations of edges. This feature makes ζ particularly well-suited for comparing models of non-invertible higher categories.

To construct ζ, we begin with the nerve functor $\mathcal{N}$ and apply identifications arising from an abstract rewriting system structure. These identifications align naturally with the rewriting-theoretic properties of positive opetopes, ensuring that ζ captures the desired combinatorial data. For example, on subcategories of graphs inside $\widehat{\mathsf{pOpe}_\iota}$ and sSet, ζ restricts to an isomorphism, while $\mathcal{N}$ maps each edge to a pair of edges with a common codomain.

Definition 4.1. *We define the* ***opetopic nerve functor*** $\mathcal{N}_{\mathsf{pOpe}_\iota} : \mathsf{pOpe}_\iota \to \mathsf{sSet}$ *as follows:* $\mathcal{N}_{\mathsf{pOpe}_\iota}(P)$ *is the set of all non-empty finite ascending sequences* $(x_0, \ldots, x_n)$ *of cells in the opetope* P*, where an ascending sequence satisfies* $x_i \preceq x_j$ *for all* $i \leq j$*. Face operators act on those sequences by deleting cells, while degeneracy operators insert copies of existing cells. This construction is functorial because morphisms in* pOpe_ι *preserve the inclusion order on cells.*

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*A (P, n)-**chain** is the n-simplex in $\mathcal{N}_{\mathsf{pOpe}_\iota}(P)$. In other words, it is a sequence $(x_0, \ldots, x_n)$ of cells of P such that for every $i \in \{0, \ldots, n-1\}$ cell x_i is a (not necessarily proper) face of x_{i+1}.*

Remark 4.2. $\mathcal{N}_{\mathsf{pOpe}_\iota}$ can equivalently be described as the composition $\mathcal{N} \circ \mathcal{P} \colon \mathsf{pOpe}_\iota \to \mathsf{sSet}$, where $\mathcal{P} \colon \mathsf{pOpe}_\iota \to \mathsf{Poset}$ assigns to an opetope P its poset of faces ordered by the relation $\preceq$, and $\mathcal{N} \colon \mathsf{Poset} \to \mathsf{sSet}$ is the standard nerve functor.

Definition 4.3. *Let $\rightsquigarrow$ be the binary relation on $\mathcal{N}_{\mathsf{pOpe}_\iota}(P)$ consisting of pairs of the form:*

$$\Big((x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n),\ (x_0, \ldots, x_{i-1}, \gamma^k(x_i), x_{i+1}, \ldots, x_n)\Big),$$

where $k \in \mathbb{N}_{>0}$ and $x_{i-1} \preceq \gamma^k(x_i)$.

Observation 4.4. *The reflexive closure $\rightsquigarrow_{ref}$ of $\rightsquigarrow$ is closed under simplicial operators: if τ is a simplicial operator and*

$$(x_0, \ldots, x_n) \rightsquigarrow_{ref} (x_0, \ldots, \gamma^k(x_i), \ldots, x_n)$$

then

$$\tau(x_0, \ldots, x_n) \rightsquigarrow_{ref} \tau(x_0, \ldots, \gamma^k(x_i), \ldots, x_n)$$

Definition 4.5. *An **abstract rewriting system** (abbreviated ARS) is a pair (X, R), where X is a set and $R \subset X \times X$ is a binary relation.*

Remark 4.6. Elements of X are usually some expressions, with $(x_1, x_2) \in R$ meaning that x_2 is a reduced (simplified) form of x_2.

This simple idea can be encoded in many ways, for example in Ara et al. (2025) abstract rewriting systems are described as 1-polygraphs, that is graphs.

Definition 4.7. *An ARS is called **terminating** if there is no infinite sequence $\big((x_n, x_{n+1})\big)_{n \in \mathbb{N}}$ of pairs belonging to R.*

Let

$$R^\infty := \bigcup_{n \in \mathbb{N}_{>0}} \overbrace{R \circ \ldots \circ R}^{\text{n-fold composition}}$$

*An ARS is called **locally confluent** if for every $(x, y_1), (x, y_2) \in R$ there exists $z \in X$ such that $(y_1, z), (y_2, z) \in R^\infty$.*

*An ARS is called **confluent** if for every $(x, y_1), (x, y_2) \in R^\infty$ there exists $z \in X$ such that $(y_1, z), (y_2, z) \in R^\infty$.*

*An ARS is said to have **unique normal form property** if every equivalence class of the equivalence relation generated by R contains exactly one normal form.*

Lemma 4.8. *Every confluent ARS has a unique normal form property.*

Proof. Ara et al. (2025), Lemma 1.3.19. □

Lemma 4.9 (Newman's lemma). *A terminating ARS is confluent if and only if it is locally confluent.*

Proof. Ara et al. (2025), Lemma 1.3.21. □

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Lemma 4.10. *The pair* $(\mathcal{N}_{\mathsf{pOpe}_\iota}(P), \rightsquigarrow)$ *forms a confluent abstract rewriting system.*

Proof. We will use Newman's lemma.

Step 1: Reduce the number of generating reductions.

We observe that every relation in $\rightsquigarrow$ decomposes into elementary steps:

$$(x_0, \ldots, x_{i-1}, x_i, \ldots, x_n) \simeq (x_0, \ldots, x_{i-1}, \gamma(x_i), \ldots, x_n),$$

where $x_{i-1} \preceq \gamma(x_i)$.

Step 2: Termination.

This system terminates because reductions strictly decrease the sum of cell dimensions in a sequence.

Step 3: Local confluence.

Consider sequences with reductions at positions i and j. If $x_{i-1} \preceq \gamma(x_i)$ and $x_{j-1} \preceq \gamma(x_j)$, then the sequence $(x'_0, x'_1, \ldots, x'_{n-1}, x'_n)$ with $x'_l = x_l$ for $l \neq i, j$, $x'_i = \gamma(x_i)$, $x'_j = \gamma(x_j)$ remains valid in $\mathcal{N}_{\mathsf{pOpe}_\iota}(P)_n$.

For the sake of transparency, we spell out explicitly the only non-obvious case $j = i + 1$:

$$(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \simeq (x_0, \ldots, x_{i-1}, \gamma(x_i), x_{i+1}, \ldots, x_n),$$
$$(x_0, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots, x_n) \simeq (x_0, \ldots, x_i, \gamma(x_{i+1}), x_{i+2}, \ldots, x_n).$$

We need to show that $(x_0, \ldots, x_{i-1}, \gamma(x_i), \gamma(x_{i+1}), x_{i+2}, \ldots, x_n)$ is an element of $\mathcal{N}_{\mathsf{pOpe}_\iota}(P)_n$. For this, we note that $x_i \preceq \gamma(x_{i+1})$ implies $\gamma(x_i) \preceq \gamma(x_{i+1})$ □

Remark 4.11. Note that our initial ARS can be expressed as a disjoint union of finite abstract rewriting systems indexed by $\mathbb{N}$, since reduction does not change the dimension of a simplex:

$$(\mathcal{N}_{\mathsf{pOpe}_\iota}(P), \rightsquigarrow) = \bigcup_{n\in\mathbb{N}} \Big(\mathcal{N}_{\mathsf{pOpe}_\iota}(P)_n,\ \rightsquigarrow \cap \big(\mathcal{N}_{\mathsf{pOpe}_\iota}(P)_n \times \mathcal{N}_{\mathsf{pOpe}_\iota}(P)_n\big)\Big)$$

Definition 4.12. *Define* $\zeta_{\mathsf{pOpe}_\iota}(P)$ *as the quotient* $\mathcal{N}_{\mathsf{pOpe}_\iota}(P)/\rightsquigarrow$. *The simplicial structure on* $\mathcal{N}_{\mathsf{pOpe}_\iota}(P)$ *descends to* $\zeta_{\mathsf{pOpe}_\iota}(P)$, *and morphisms in* pOpe_ι *induce well-defined maps between quotients. Thus,* $\zeta_{\mathsf{pOpe}_\iota}$ *is a functor.*

Definition 4.13. *Let* $\zeta = \mathrm{Lan}_{\text{よ}}\zeta_{\mathsf{pOpe}_\iota} : \widehat{\mathsf{pOpe}_\iota} \to \mathsf{sSet}$ *be the left Kan extension of* $\zeta_{\mathsf{pOpe}_\iota}$ *along the Yoneda embedding* $\text{よ} : \mathsf{pOpe}_\iota \to \widehat{\mathsf{pOpe}_\iota}$.

Let $\mathfrak{Y} = \mathrm{Lan}_{\text{よ}}\mathcal{N}_{\mathsf{pOpe}_\iota} : \widehat{\mathsf{pOpe}_\iota} \to \mathsf{sSet}$ *be the left Kan extension of* $\mathcal{N}_{\mathsf{pOpe}_\iota}$ *along the Yoneda embedding* $\text{よ} : \mathsf{pOpe}_\iota \to \widehat{\mathsf{pOpe}_\iota}$.

Remark 4.14. The quotient map

$$\pi : \mathcal{N}_{\mathsf{pOpe}_\iota} \twoheadrightarrow \zeta_{\mathsf{pOpe}_\iota}$$

is a natural transformation of functors $\mathsf{pOpe}_\iota \to \mathit{sSet}$. This quotient map can be thought of as an analog of the natural transformation

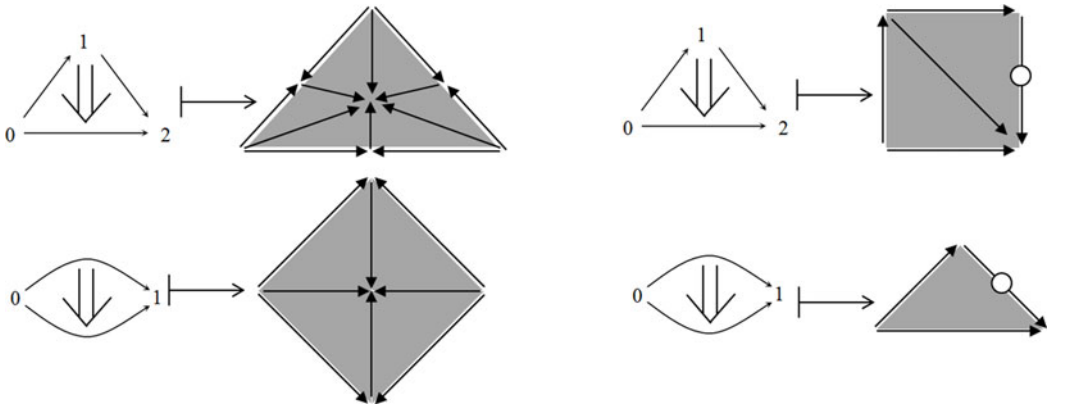
$$Sd \twoheadrightarrow Id_{\mathsf{sSet}}$$

of functors $\Delta \to \mathsf{sSet}$.

There is an obvious natural transformation $\pi : \mathfrak{Y} \twoheadrightarrow \zeta$ induced by $\pi : \mathcal{N}_{\mathsf{pOpe}_\iota} \twoheadrightarrow \zeta_{\mathsf{pOpe}_\iota}$.

Example 4.15. *In the picture below, we illustrate both functors* $\mathcal{N}_{\mathsf{pOpe}_\iota}$ *and* $\zeta_{\mathsf{pOpe}_\iota}$. *Degenerate edges of simplicial sets are marked with white dots.*

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Observation 4.16. *The functor ζ defined above is a left adjoint (in particular, it preserves colimits), being the left Kan extension along the Yoneda embedding.*

Theorem 4.17. $\zeta : \widehat{\mathsf{pOpe}_\iota} \to \mathsf{sSet}$ *preserves cellular monomorphisms.*

Proof. **Step 1: Reduction to generators.**

By Observation 4.16 it suffices to check this for generators, that is monomorphisms of the form $\partial_P : \partial 面_P \to 面_P$.

Step 2: Reduction to Set.

In sSet every monomorphism is cellular, so it is enough to show that $\zeta(\partial_P)$ is a monomorphism. Furthermore, since monomorphisms in sSet are pointwise, we only need to show that $\zeta(\partial_P)_n$ is a monomorphism in **Set** for every $n \in \mathbb{N}$.

Step 3: Exploitation of the confluence property.

Let us fix a pair $[x_0, \ldots, x_n], [y_0, \ldots, y_n] \in \zeta(\partial 面_P)$ of simplices, which are mapped to the same simplex in $\zeta(面_P)$. We can assume that $(x_0, \ldots, x_n)$ is in a normal form (when considered as an element of $面_{\langle x_n \rangle}$ and similarly for $(y_0, \ldots, y_n)$.

Then $\mathfrak{Y}(\partial_P)(x_0, \ldots, x_n)$ and $\mathfrak{Y}(\partial_P)(y_0, \ldots, y_n)$ have a common reduction. On the other hand, if a chain $(x_0, \ldots, x_n)$ is reduced in $\langle x_n \rangle$, then it must be also reduced in P, hence $(x_0, \ldots, x_n) = (y_0, \ldots, y_n)$. □

Example 4.18. *It is not true that ζ preserves arbitrary monomorphisms.*

Let G be the triangle and P the unique 1-dimensional globular opetope. Let $\sigma_1, \sigma_2 : P \to G$ be two distinct codegeneracy maps. Consider the following pushout:

$$\begin{array}{ccc} 面_G \amalg 面_G & \xrightarrow{(\sigma_1,\sigma_2)} & 面_P \\ \downarrow & & \downarrow \\ 面_G & \longrightarrow & X \end{array}$$

One can check that X is an opetopic set with a nondegenerate edge, satisfying $\zeta(X) \cong \{\bullet\}$. In particular, $\zeta(面_{\to} \hookrightarrow X)$ is not a monomorphism. Necessarily, X is not an EZ-presheaf.

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Theorem 4.19. *For $P \in \mathsf{pOpe}_t$, the following diagram is a pushout:*

$$\begin{array}{ccc} \zeta(\langle\gamma^2(\alpha_P)\rangle) * \{\bullet\} & \hookrightarrow & \zeta(\delta P) * \{\bullet\} \\ \downarrow & & \downarrow \\ \zeta(\langle\gamma^2\alpha_P\rangle) & \hookrightarrow & \zeta(P) \end{array}$$

where the left map is given by the formula

$$\big([x_0, \ldots, x_j] * (\bullet)\big) \mapsto [x_0, \ldots, x_j, \gamma^2(\alpha_P)]$$

and the right map is specified as

$$\big([x_0, \ldots, x_j] * (\bullet)\big) \mapsto [x_0, \ldots, x_j, \alpha_P]$$

Proof. Verification that formulas given in the statement yield well-defined morphisms of simplicial sets is straightforward.

Colimits in presheaves are computed pointwise, so it suffices to verify the pushout property in **Set** after evaluating on a representable. Given a cone:

$$\begin{array}{ccc} \big(\zeta(\langle\gamma^2\alpha_P\rangle) * \{\bullet\}\big)_n & \hookrightarrow & \big(\zeta(\delta P) * \{\bullet\}\big)_n \\ \downarrow & & \downarrow \\ \zeta(\langle\gamma^2\alpha_P\rangle)_n & \longrightarrow & X \end{array}$$

we construct a unique $f : \zeta(P)_n \to X$. For a reduced chain $(x_0, \ldots, x_n)$:

Case 1: $[(x_0, \ldots, x_n)] \in \zeta(\langle\gamma^2\alpha_P\rangle)$**.**

If $[(x_0, \ldots, x_n)] \in \zeta(\langle\gamma^2\alpha_P\rangle)$, then $f(x_0, \ldots, x_n)$ is forced to agree with the inclusion map $\zeta(\langle\gamma^2\alpha_P\rangle) \hookrightarrow X$.

Furthermore, suppose there exists an element $[(y_0, \ldots, y_k) * (\bullet, \ldots, \bullet)]$ in $\zeta(\delta P) * \{\bullet\}$ (where $(y_0, \ldots, y_k)$ is reduced in y_k) that maps to $[(x_0, \ldots, x_n)] \in \zeta(P)$. By confluence, $(y_0, \ldots, y_k) = (x_0, \ldots, x_k)$. Consequently, $[(y_0, \ldots, y_k) * (\bullet, \ldots, \bullet)]$ lies in $\zeta(\langle\gamma^2\alpha_P\rangle) * \{\bullet\}$, thus verifying that f defines a map of cones.

Case 2: $[(x_0, \ldots, x_n)] \in \zeta(\delta P) \setminus \zeta(\langle\gamma^2\alpha_P\rangle)$**.**

If $[(x_0, \ldots, x_n)] \in \zeta(\delta P) \setminus \zeta(\langle\gamma^2\alpha_P\rangle)$, then $f(x_0, \ldots, x_n)$ is forced to agree with the composite inclusion $\zeta(\delta P) \hookrightarrow \zeta(\delta P) * \{\bullet\} \hookrightarrow X$.

Assume there exists an element $[(y_0, \ldots, y_k) * (\bullet, \ldots, \bullet)]$ in $\zeta(\delta P) * \{\bullet\} \setminus \zeta(\delta P)$ (again with $(y_0, \ldots, y_k)$ reduced in y_k) that maps to $[(x_0, \ldots, x_n)] \in \zeta(P)$. By confluence, $(y_0, \ldots, y_k) = (x_0, \ldots, x_k)$, and all subsequent terms $x_{k+1}, \ldots, x_n$ must equal $\gamma(\alpha_P)$ or P. However, since $[(x_0, \ldots, x_n)] \in \zeta(\delta P) \setminus \zeta(\langle\gamma^2\alpha_P\rangle)$, the term x_{k+1} cannot be reduced to $\gamma^2(\alpha_P)$, which yields a contradiction with $x_{k+1} \in |\delta P|$.

Case 3: $[(x_0, \ldots, x_n)] \in \zeta(P) \setminus \zeta(\delta P)$**.**

Chain $(x_0, \ldots, x_n)$ contains a face $x_i \in \delta P \setminus \gamma^2(\alpha_P)$, and the smallest subsequent term x_j (with $i < j$) equals $\gamma(\alpha_P)$ or P. We define $f(x_0, \ldots, x_n)$ to coincide with the value of the inclusion $\zeta(\delta P) * \{\bullet\} \hookrightarrow X$ applied to $(x_0, \ldots, x_{j-1}, \bullet, \ldots, \bullet)$. An argument analogous to the previous case demonstrates that there is exactly one element in $\zeta(\delta P) * \{\bullet\}$ mapping to $(x_0, \ldots, x_n)$, ensuring uniqueness. □

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4.2 Ideals in opetopes

In this section, we address the issue of showing that pOpe_ι is a tidy Reedy category. Our results rely heavily on the combinatorial properties of positive opetopic cardinals, ι-contractions, and ideals. Therefore, we cite numerous important results from Zawadowski (2007a), where Zawadowski introduced positive opetopes.

The core of this chapter consists of computational tools for explicitly constructing connected colimits in the category $(\mathsf{pOpe}_\iota)_-$. However, we use these tools only to demonstrate the existence of binary pushouts in that category.

Definition 4.20 (Zawadowski (2007a)). *Let P and Q be positive opetopic cardinals, $f : P^* \to Q^*$ an ω-functor between ω-categories generated by them. The **kernel** of f is the set of faces of P sent by f to identities on cells of Q^* of lower dimension.* $\ker(f)$ *denotes the kernel of f.*

Remark 4.21. As shown by Zawadowski, this definition is equivalent to Definition 2.7, which follows from Lemma 4.23 below. Consequently, there is no need to work with ω-categories, allowing the content of this chapter to remain purely combinatorial. Furthermore, all notions introduced in Definition 2.7 – originally stated for positive opetopes – naturally extend to positive opetopic cardinals and are also equivalent to those in Definition 4.20.

Definition 4.22 (Zawadowski (2007a)). *We say that a set $\mathfrak{C}$ of faces of $P_{>0}$ is an **ideal** in P iff for any $b \in P_{>1}$:*

1_i. *if $\gamma(b) \in \mathfrak{C}$, then $b \in \mathfrak{C}$;*
2_i. *if $\delta(b) \subseteq \mathfrak{C}$, then $b \in \mathfrak{C}$;*
3_i. *if $b \in \mathfrak{C}$, then $|\delta(b) \setminus \mathfrak{C}| = |\gamma(b) \setminus \mathfrak{C}|$.*

Lemma 4.23 (Zawadowski (2007a), Lemma 10.4, Theorem 10.8). *Let $f : P^* \to Q^*$ be an ω-functor between ω-categories generated by positive opetopic cardinals. Then* $\ker(f)$ *is an ideal. Conversely, any ideal is a kernel of an ω-functor corresponding to the ι-contraction epimorphism.*

Definition 4.24 (Zawadowski (2007a)). *A face $u \in P_{>0}$ is **unary** if $\delta(u)$ contains one element.*

Definition 4.25 (Zawadowski (2007a)). *Let $U(P)$ be the set of unary faces in P, $\mathfrak{C} \subseteq P$ an ideal in P, and $\mathfrak{C} \neq \emptyset$. The face $u \in P$ is called **safe** for P if $u \in U(P) \setminus \gamma(P) \setminus \delta(U(P))$, that is u is a unary face in P that is not a codomain of any other face in P and it is not in the domain of a unary face in P.*

Lemma 4.26. *Let P be a positive opetopic cardinal, $a \in P$ be a face which is a codomain of some other face, that is, there exists $b \in P$ such that $a = \gamma(b)$. Then there exists $c \in P$ such that $a = \gamma(c)$ and $c \in P \setminus \gamma(P)$.*

Proof. Follows directly from the pencil-linearity applied to a and the first globularity axiom (if $\gamma(d) = b$, then the last face e in the poset $(\delta(b), <^-)$ satisfies $\gamma(e) = \gamma(b)$). □

The following Lemma says that we can always divide any opetopic cardinal by its safe face.

Lemma 4.27 (Zawadowski (2007a), Lemma 10.5). *Let P be an opetopic cardinal, and u a safe face for P. Then we can divide P by u, that is we have a quotient ω-functor $q_u : P^* \to P^*/u$ whose kernel is $\{u\}$.*

Lemma 4.28 (Zawadowski (2007a), Lemma 10.7). *Let $\mathfrak{C}$ be a non-empty ideal in P. There is always a face $u \in \mathfrak{C}$ safe for P.*

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Theorem 4.29 (Zawadowski (2007a), Theorem 10.8). *If $\mathfrak{C} \subseteq P_{\geq 1}$ is an ideal, then there is a ι-epi map $\pi_{\mathfrak{C}} : P \to P/\mathfrak{C}$ such that it has $\mathfrak{C}$ as its kernel. Moreover, $\pi_{\mathfrak{C}}$ is a universal map with this property, that is whenever there is an ω-functor $h : P^* \to Q^*$ such that $\mathfrak{C} \subseteq \ker(f)$, then there is a unique map $f' : (P/\mathfrak{C})^* \to Q$ such that $f = f' \circ \pi_{\mathfrak{C}}$.*

Definition 4.30 (Zawadowski (2007a)). *The relation $<_l^-$ (denoted also simply by $<_l$) on the set P_k for $0 \leq l < k$ is defined as follows:*

$$\alpha <_l \beta \iff \gamma^{(l)}\alpha <^- \gamma^{(l)}\beta$$

Theorem 4.31 (Zawadowski (2007a), Corollary 5.11, Corollary 5.12). *Let $\alpha, \beta \in P_k, \alpha \neq \beta$, then α and β are comparable in exactly one of the orders $<_0, \ldots, <_k, <^+$. The union of all those relations (denoted simply by $<$) is a linear order on P_k.*

Notation 4.32. *We denote the operation of taking the successor by succ (the linear order in which this is going to take place will be clear from the context).*

As we will see, the collection of linear orders $(P_0, <), (P_1, <), \ldots, (P_n, <)$ induces a canonical filtration of P.

Construction 4.33. *We begin by defining $\mathcal{F}_{-1} = \emptyset$ and $\mathcal{F}_0 := \{x_0\}$, where x_0 is the initial element of P_0.*

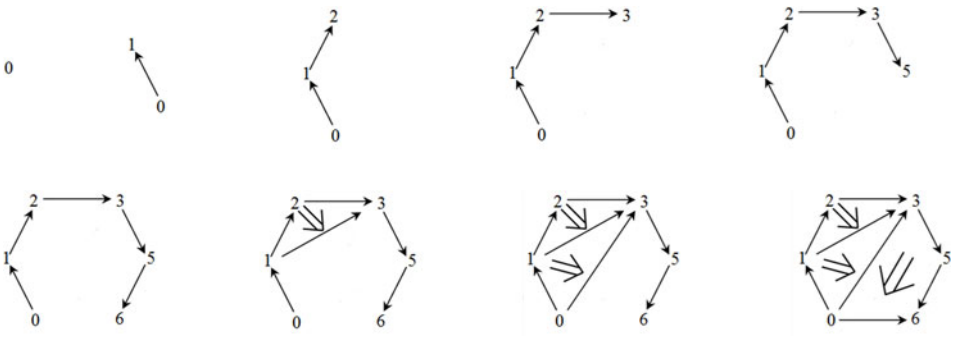
Assume inductively that the pair $\mathcal{F}_n, \alpha_n$ has been constructed, where $\mathcal{F}_n \subset \mathcal{P}(P)$ and $x \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ is a cell of minimal dimension. We then construct $\mathcal{F}_{n+1}$ as follows:

1. *If $P_{\dim(x)} \subset \mathcal{F}_n$, define $\mathcal{F}_{n+1} := \mathcal{F}_n \cup \{\alpha, z\}$, where:*
   - *z is the initial element of $(P_{\dim(x)+1} \setminus \mathcal{F}_n, <)$,*
   - *α is the $<^+$-smallest cell of P satisfying $\gamma(\alpha) = z$ (guaranteed by the pencil-linearity axiom).*

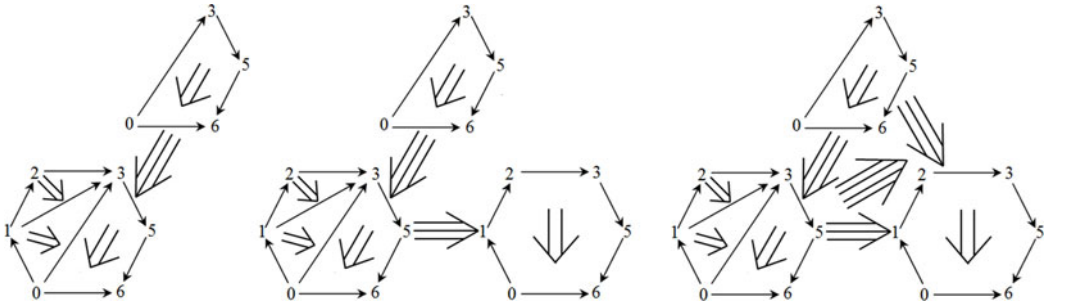
   Note that $\delta(\alpha) \subset \mathcal{F}_n$, because all elements of $\delta(\alpha)$ are $<^+$-smaller (and hence $<$-smaller) than z, so they must already belong to $\mathcal{F}_n$.
2. *If $P_{\dim(x)} \not\subset \mathcal{F}_n$, let z be the initial element of $(P_{\dim(x)} \setminus \mathcal{F}_n, <)$. There exists at least one cell α connecting z to $\mathcal{F}_n$ (i.e., satisfying $\delta(\alpha) \subset \mathcal{F}_n$ and $\gamma(\alpha) = z$). Among all such cells, select the $<^+$-minimal α and define $\mathcal{F}_{n+1} := \mathcal{F}_n \cup \{\alpha, z\}$.*

This procedure terminates after $\#|P|$ steps, with $\mathcal{F}_{\#|P|} = P$. It is straightforward to see that each $\mathcal{F}_i$ for $i > 0$ forms an opetopic cardinal. Note that with this construction the following holds: for every $y \in P$ and every index i, if $\gamma(y) \in \mathcal{F}_i$, then $y \in \mathcal{F}_i$.

Definition 4.34. *The filtration constructed above is called* ***the domain filtration****.*



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Example 4.35. *In the picture above, we show the sequence of consecutive terms in the domain filtration of a certain opetope P. Note that, in the left term of the last line, the new square-shaped cell is the lower one, not the upper one, as one might assume by simply observing the picture.*

Key properties (either already proven or obvious) of the domain filtration are summarized in the following theorem:

Theorem 4.36. *In the domain filtration, the first term is equal to the singleton of the initial vertex of P.*

Every step $\mathcal{F}_{i+1}$ of this filtration is produced from the previous one by adding exactly two new cells $\alpha, \gamma\alpha$ to $\mathcal{F}_i$, where α has the property that $\delta\alpha \subseteq \mathcal{F}_i$.

For any $\kappa \in P$, iff $\kappa \in \mathcal{F}_i$, then $\kappa \in \mathcal{F}_i \setminus \gamma\mathcal{F}_i \iff \kappa \in \mathcal{F}_{i+1} \setminus \gamma\mathcal{F}_{i+1}$.

Definition 4.37. *Following Zawadowski (2017), we note that the notion of an ι-map extends to arbitrary positive hypergraphs, not just positive opetopes or positive opetopic cardinals.* ***The category of positive hypergraphs with ι-maps*** *is denoted pHg_ι.*

Using the inclusion functor $\mathsf{pOpe}_\iota \hookrightarrow \mathsf{pHg}_\iota$, we obtain a functor

$$\mathcal{H}_\iota : \mathsf{pHg}_\iota \to \widehat{\mathsf{pOpe}_\iota}$$

defined by

$$H \mapsto \left[P \mapsto \mathsf{pHg}_\iota(P, H)\right].$$

This functor $\mathcal{H}_\iota$ induces a bijection

$$\mathsf{pHg}_\iota(P, H) \cong \widehat{\mathsf{pOpe}_\iota}(P, H)$$

when P is a positive opetope and H is a ***positive opetopic hypergraph*** *(i.e., for each $x \in |H|$, the hypergraph $\langle x \rangle$ is a positive opetope).*

Applying the functor $\mathcal{H}_\iota$ to the domain filtration of an opetopic cardinal, we can transfer this construction to the category $\widehat{\mathsf{pOpe}_\iota}$.

Lemma 4.38. *Let P be a positive opetopic hypergraph. Let X be the quotient of $\coprod_{c \in |P|} \daleth_{\langle c \rangle}$ induced by the inclusions $\daleth_{\langle c_1 \rangle} \hookrightarrow \daleth_{\langle c_2 \rangle}$ for every pair $c_1 \preceq c_2$. Then $X \cong \mathcal{H}_\iota(P)$. In particular, $\mathcal{H}_\iota(P) \in \widehat{\mathsf{pOpe}_{\iota EZ}}$.*

Proof. Immediate consequence of the Yoneda lemma. □

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Corollary 4.39. *Let P be a positive opetopic cardinal. There exists a filtration $\{U_k\}_{k=1,\ldots,m}$ of $\mathcal{H}_\iota(P)$ in which the first term is equal to the initial vertex of P and each term of the filtration is obtained from the previous one by the following pushout*

$$\begin{array}{ccc} \mathcal{H}_\iota(\delta Q) & \hookrightarrow & U_k \\ \downarrow & & \downarrow \\ \mathcal{H}_\iota(Q) & \hookrightarrow & U_{k+1} \end{array}$$

where $Q = \langle c \rangle$ for some $c \in |P|$.

Proof. Apply the functor $\mathcal{H}_\iota$ to the domain filtration of P. □

Theorem 4.40. *Let $f : P \to Q$ be a ι-epi, that is a contraction morphism between opetopes. Then there exists an opetope P' and a pair of distinct morphisms $\sigma_1, \sigma_2 : P' \to P$ such that $f \circ \sigma_1 = f \circ \sigma_2$.*

In other words, pOpe_ι *is degeneracy-forking.*

Proof. Using Lemmas 4.27 and 4.28, it suffices to show this for the contraction morphism $f_u : P \to P''$ with $\ker(f_u) = \{u\}$, where u is a safe face of P contained in the kernel of f.

Case 1: u is not the greatest element of P.

The opetope P' is constructed from P by replacing u with two cells u', u'' of the same shape as u, subject to the relations:

$$\delta(u') = \delta(u), \quad \gamma(u') = \delta(u''), \quad \gamma(u'') = \gamma(u).$$

Furthermore, if an element $a \in P$ satisfies $\delta(a) = \{u\} \sqcup D$ for some D, then in P', a satisfies $\delta(a) = \{u', u''\} \sqcup D$.

One may verify directly that P' satisfies all axioms of a dendritic face complex. Additionally, both u' and u'' are safe faces of P', with the corresponding contractions yielding σ_1 and σ_2.

Case 2: u is the greatest element of P.

We construct P' by defining:

$$\delta(\alpha_{P'}) = \{u', u''\} \quad \text{and} \quad \gamma(\alpha_{P'}) = u''',$$

where u', u'', u''' are cells isomorphic to u, with the relation $\gamma(u') = \delta(u'')$. These conditions uniquely determine P' as a dendritic face complex, and u' and u'' are safe faces of P'.

Quotienting P' by either u' or u'' results in a unary opetope where the image of $\alpha_{P'}$ becomes a safe face.

Finally, define the contractions:

$$\sigma_1 := P \twoheadrightarrow (P/u')/[\alpha_{P'}], \quad \sigma_2 := P \twoheadrightarrow (P/u'')/[\alpha_{P'}].$$ □

Definition 4.41. *Let P be a positive opetopic cardinal and $\mathfrak{C} \subset \mathcal{P}(P)$. We say that $\mathfrak{C}$ is a* ***pre-ideal in*** *P iff whenever $b \in P_{>0} \setminus \gamma(P)$*

1_{pi}. if $\gamma(b) \in \mathfrak{C}$, then $b \in \mathfrak{C}$ and $\delta(b) \subseteq \mathfrak{C}$;
2_{pi}. if $b \in \mathfrak{C}$, then $|\delta(b) \setminus \mathfrak{C}| \leq 1$.

Remark 4.42. We can think about pre-ideal as a set of faces that is weakly backward and upward closed.

It is not true that any set of faces of an opetopic cardinal P is contained in the least ideal in P. If a set $X = \{m_P\}$ contains just the main face m_P of a non-globular 2-opetope P, then there are at

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least two minimal ideals containing X, they must contain all but one face in $\delta(m_P)$. However, if the set of faces X is a pre-ideal then there is a unique ideal $\mathfrak{C}$ containing X. This is shown below.

Definition 4.43. *We define two operations on the set of cells of an opetopic cardinal P:*

$$Cl, Cl^{\infty} : \mathcal{P}(P) \to \mathcal{P}(P)$$

*called **immediate closure** and **closure**, respectively. We put*

$$Cl(\mathfrak{C}) = \mathfrak{C} \cup \{\alpha, \gamma(\alpha) | \alpha \in P_{>0} \setminus \gamma(P),\ \delta(\alpha) \subseteq \mathfrak{C}\}$$

Let $Cl^{\infty}(\mathfrak{C})$ denote the least closure of the set $\mathfrak{C}$ under the (monotone) operation Cl, that is, $Cl^{\infty}(\mathfrak{C}) = \cup_{n\in\mathbb{N}} Cl^n(\mathfrak{C})$.

*If $Cl(\mathfrak{C}) \subseteq \mathfrak{C}$ and $\mathfrak{C}$ is a pre-ideal, then we call $\mathfrak{C}$ **closed pre-ideal**.*

Lemma 4.44. *Any union of pre-ideals in opetopic cardinal P is a pre-ideal.*

Proof. Straightforward. □

Lemma 4.45. *Let $\mathfrak{C}$ be a pre-ideal in opetope P. Assume $\beta \in P \setminus \gamma(P)$ a face in P such that $\delta(\beta) \subseteq \mathfrak{C}$ but $\gamma(\beta) \notin \mathfrak{C}$. Then the set $\mathfrak{C}_1 = \mathfrak{C} \cup \{\beta, \gamma(\beta)\}$ is a pre-ideal in P.*

Proof. The only face b in P such that $\gamma(b) \in \mathfrak{C}_1 \setminus I$ is β. For $b = \beta$ and $\mathfrak{C}_1$ the condition 1_{pi} clearly holds. Thus the condition 1_{pi} holds for any face in P and $\mathfrak{C}_1$.

The face β is also the only face in $\mathfrak{C}_1 \setminus \gamma(P)$ that does not belong to $\mathfrak{C}$. As the condition 1_{pi} holds for $b = \beta$ it holds for all the faces in $P \setminus \gamma(P)$.

Thus $\mathfrak{C}_1$ is indeed a pre-ideal. □

By iteration of the above Lemma, we get

Corollary 4.46. *Let $\mathfrak{C}$ be a pre-ideal in opetopic cardinal P. Then $Cl(\mathfrak{C})$ is again a pre-ideal in P containing $\mathfrak{C}$.*

The main fact of this section in the following

Theorem 4.47. *Let $\mathfrak{C}$ be a closed pre-ideal in an opetopic cardinal P. Then $\mathfrak{C}$ is an ideal.*

Before proving the theorem, we state the following:

Corollary 4.48. *Let $\mathfrak{C}$ be a closed pre-ideal in an opetopic cardinal P. Then $Cl^{\infty}(\mathfrak{C})$ is the least ideal containing $\mathfrak{C}$.*

Proof. Any ideal in an opetope P must be closed under the operation Cl. Since the operation Cl is monotone, by 4.44 and 4.46, the sum $Cl^{\infty}(\mathfrak{C}) = \bigcup_{n\in\mathbb{N}} Cl^n(\mathfrak{C})$ is a closed pre-ideal. Thus, by 4.47, $Cl^{\infty}(\mathfrak{C})$ is the least ideal containing $\mathfrak{C}$, as required. □

Proof of Theorem 4.47. We proceed by induction on the cardinality of $|P|$, where P is an arbitrary positive opetopic cardinal.

Base case ($|P| = 1$ and $|P| = 3$):
Straightforward verification since P is either a point or an edge.

Induction step:
Let $\mathcal{F} \subset P$ be the second-to-last term in the domain filtration of P (see Theorem 4.36), and let $P \setminus \mathcal{F} = \{\alpha, \gamma\alpha\}$.

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By the inductive hypothesis and the last statement in Theorem 4.36, $\mathfrak{C} \cap \mathcal{F}$ is a closed pre-ideal in $\mathcal{F}$ and hence an ideal. Therefore, conditions 1_i, 2_i, and 3_i hold for all $b \in P \setminus \{\alpha, \gamma\alpha\}$. We now verify these conditions for the remaining cells α and $\gamma\alpha$.

Case 1: Verifying conditions for α

$\mathbf{1_i}(\alpha)$: Follows directly from $1_{pi}(\alpha)$, since $\alpha \in P_{\geq 0} \setminus \gamma P$.

$\mathbf{2_i}(\alpha)$: Immediate from the closedness of $\mathfrak{C}$.

$\mathbf{3_i}(\alpha)$: By $2_{pi}(\alpha)$, $|\delta(\alpha)| = 0$ or $|\delta(\alpha)| = 1$.

If $|\delta(\alpha)| = 0$, closedness implies $\gamma(\alpha) \in \mathfrak{C}$, reducing 3_i to the trivial equality $0 = 0$.

Conversely, if $\gamma(\alpha) \in \mathfrak{C}$, then $\delta(\alpha) \subseteq \mathfrak{C}$ by $1_{pi}(\alpha)$, so if $|\delta(\alpha)| = 1$, then $\gamma(\alpha) \notin \mathfrak{C}$, reducing 3_i to $1 = 1$.

Case 2: Verifying conditions for $\gamma\alpha$

$\mathbf{1_i}(\gamma\alpha)$: Observe that $\gamma^2\alpha$ is terminal in the poset $\left((\delta\alpha)_{n-2}, <^-\right)$ (where $n = \dim(\alpha)$). By 1_i and 3_i, if a cell $x \in \mathcal{F}$ lies in $\mathfrak{C}$, its entire lower set is contained in $\mathfrak{C}$ – by induction it is enough to show that for direct predecessors of x, that is for y such that there exists $z \in P$ with $y \in \delta z$ and $\gamma(z) = x$. Since $\gamma(z) \in \mathfrak{C}$, $\delta(z) \subset \mathfrak{C}$. When x is the final element of $\left((\delta\alpha)_{n-2}, <^-\right)$, its lower set is equal to the whole poset itself, in particular $\delta\gamma\alpha \subseteq \mathfrak{C}$.

$\mathbf{2_i}(\gamma\alpha)$: A dual argument applies. By a combination of 2_i and 3_i, if all predecessors of a given element are in $\mathfrak{C}$, then so is the element itself. Assume that $\delta\gamma\alpha \subseteq \mathfrak{C}$. From the globularity formula for $\delta\gamma(\alpha) = \delta\delta(\alpha) \setminus \gamma\delta(\alpha)$ follows that all minimal elements the poset of $(n-2)$-cells of $(\delta\alpha, <^-)$ are in $\mathfrak{C}$ (recall that minimal elements of $(Q, <^-)$ are exactly those which are not in $\gamma(Q)$), hence the whole poset itself, in particular the final $\gamma^2\alpha$.

$\mathbf{3_i}(\gamma\alpha)$: If $\gamma\alpha \in \mathfrak{C}$, then $\delta\alpha \subseteq \mathfrak{C}$ by $3_i(\alpha)$ (since $\alpha \in \mathfrak{C}$ by $1_i(\alpha)$). As $\mathfrak{C} \cap \delta\alpha$ is an ideal in $\delta\alpha$, applying 3_i to $(n-1)$-cells in $\delta\alpha$ (we just proved that all of them are in $\mathfrak{C}$) shows that for every $\kappa \in \left((\delta\alpha)_{n-2}, <^-\right)$ exactly one of the following two possibilities is true:

1. $\kappa \in \mathfrak{C}$ with all direct predecessors in $\mathfrak{C}$, or
2. $\kappa \notin \mathfrak{C}$ with exactly one direct predecessor not in $\mathfrak{C}$.

$\mathbf{2_i}(\gamma\alpha)$: Starting from the final element and labeling predecessors of each cell according to those two rules, we see that at most one of the minimal elements is not in $\mathfrak{C}$ (and this happens exactly when the final element $\gamma^2\alpha$ is not in $\mathfrak{C}$, so the equality 3_i reduces either to $0 = 0$ or $1 = 1$). □

Theorem 4.49. *Pushout of any pair of ι-epis between opetopes in* pOpe_ι *and* $(\mathsf{pOpe}_\iota)_-$ *exist and the embedding* $(\mathsf{pOpe}_\iota)_- \to \mathsf{pOpe}_\iota$ *preserves them.*

Proof. Let $\sigma_1 : P \to P_1$ and $\sigma_2 : P \to P_2$ be a pair of ι-epis between opetopes. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be their respective kernels. Then, by Lemma 4.44, the set $\mathfrak{C}_1 \cup \mathfrak{C}_2$ is a pre-ideal. Then, by Corollary 4.48, $\mathfrak{C} = Cl^\infty(\mathfrak{C}_1 \cup \mathfrak{C}_2)$ is the least ideal containing $\mathfrak{C}_1 \cup \mathfrak{C}_2$. Then the square

$$\begin{array}{ccc} P & \xrightarrow{\sigma_1} & P_1 = P/\mathfrak{C}_1 \\ \downarrow{\scriptstyle \sigma_2} & & \downarrow \\ P_2 = P/\mathfrak{C}_2 & \longrightarrow & P/\mathfrak{C} \end{array}$$

is a pushout both in pOpe_ι and $(\mathsf{pOpe}_\iota)_-$ by Theorem 4.29. □

Remark 4.50. The author conjectures that pOpe_ι is a residuation structure in the sense of Ara et al. (2025), 4.6.1., with σ_1/σ_2 given as $\pi_{\overline{\sigma_1(\ker(\sigma_2))}} \circ \sigma_1$.

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4.3 ζ preserves pushouts of codegeneracy maps

The functor $\zeta : \widehat{\mathsf{pOpe}_\iota} \to \mathsf{sSet}$ has a right adjoint $\rho : \mathsf{sSet} \to \widehat{\mathsf{pOpe}_\iota}$, given by

$$X \mapsto \big(P \mapsto \rho(X)_P = \mathsf{sSet}(\zeta(P), X)\big),$$

since it is the left Kan extension along the Yoneda embedding.

However, it is far from clear whether $\zeta \,\big|\, \widehat{\mathsf{pOpe}_{\iota EZ}}$ has a right adjoint. To show that ρ can also be regarded as a right adjoint of $\zeta \,\big|\, \widehat{\mathsf{pOpe}_{\iota EZ}}$, we will use the criterion formulated in Theorem 3.39. The applicability of this theorem to the adjunction $\zeta \dashv \rho$ is precisely the content of Corollary 4.62, which we prove in the following. The recipe for the kernel of the pushout of two morphisms in $(\mathsf{pOpe}_\iota)_-$, described implicitly in the proofs of results of the previous section, is used extensively.

From now up to the end of this section, we assume that P is an opetopic cardinal, not necessarily an opetope, unless it is explicitly stated otherwise.

Definition 4.51. *If P is an opetope and two (P, n)-chains have the same image under the map $\mathcal{N}_{\mathsf{pOpe}_\iota}(P) \twoheadrightarrow \zeta(P)$, we call them ζ-**equivalent**.*

Definition 4.52. *Let $\mathfrak{C}$ be an ideal of P and $\pi_{\mathfrak{C}} : P \to P/\mathfrak{C}$ be a quotient map. We say that two (P, n)-chains $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ are $\mathfrak{C}$-**equivalent**, if for every $i \in \{0, \ldots, n\}$ $\pi_{\mathfrak{C}}(x_i) = \pi_{\mathfrak{C}}(y_i)$.*

Lemma 4.53. *If $\mathfrak{C}$ is an ideal of P and x, y are cells of P such that $\pi_{\mathfrak{C}}(x) = \pi_{\mathfrak{C}}(y)$, then there exists a sequence $a_0 = x \preceq u_1 \succeq a_1 \preceq u_2 \succeq a_2 \preceq \ldots \succeq a_m = y$ of cells of P such that for every k $\pi_{\mathfrak{C}}(a_k) = \pi_{\mathfrak{C}}(u_k) = \pi_{\mathfrak{C}}(a_{k-1})$.*

Proof. If $\pi_{\mathfrak{C}}(x) = \pi_{\mathfrak{C}}(\gamma x)$, we set $u_1 = x$, $a_1 = \gamma x$. Similarly, if $\pi_{\mathfrak{C}}(\gamma^2 x) = \pi_{\mathfrak{C}}(\gamma x)$, we set $u_2 = \gamma x$, $a_2 = \gamma^2 x$. Proceeding in this way, we build a sequence as in the statement of the lemma, but connecting x with $\gamma^k x$ for k such that $\gamma^k x \notin \mathfrak{C}$. By symmetry, we can connect y with $\gamma^l y$ such that $\gamma^l y \notin \mathfrak{C}$.

Note that in this case $\dim(\gamma^k x) = \dim(\gamma^l y)$, so $\gamma^k x, \gamma^l y$ can be a connected by a sequence of $\dim(\gamma^k x)$-cells in $\mathfrak{C}$ (Zawadowski (2017), Corollary 2.9). Concatenating all three sequences, we get the claim. □

Definition 4.54. *A pair of (P, n)-chains $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ is **elementarily equivalent** if there exists an index i such that $j \neq i \implies x_j = y_j$ and there is a cell z_i of P s. t. $x_i \preceq z_i \succeq y_i$, $\pi_{\mathfrak{C}}(x_i) = \pi_{\mathfrak{C}}(z_i) = \pi_{\mathfrak{C}}(y_i)$ and $z_i \preceq x_{i+1}$. In that case, we say that this pair is **elementarily equivalent** by z_i. The index i is called the **change index** of an elementary equivalence.*

Definition 4.55. *Let $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ be a $\mathfrak{C}$-equivalent pair of (P, n)-chains. Let k be the largest index such that $x_k \neq y_k$.*

*Then the **distance** between $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ is a sequence of $n+1$ natural numbers*

$$(m_0, \ldots, m_k, 0, \ldots, 0)$$

where m_k is the smallest natural number m which satisfies the statement of the Lemma 4.53 applied to $P := \langle x_{k+1} \rangle$, $x := x_k$, $y := y_k$ and m_i for $i < k$ is the smallest natural number m which satisfies the statement of the Lemma 4.53 applied to $P := \langle x_{k+1} \rangle, x := x_i$, $y := y_i$.

Observation 4.56. *Distances of $\mathfrak{C}$-equivalent pairs of (P, n)-chains are well-ordered by the colexicographic order.*

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Lemma 4.57. *If $\mathfrak{C}$ is an ideal and (P,n)-chains $(x_0,\dots,x_n),(y_0,\dots,y_n)$ are $\mathfrak{C}$-equivalent, then there exists a sequence $\vec{x}_0,\dots,\vec{x}_m$ of (P,n)-chains such that $\vec{x}_0=(x_0,\dots,x_n),\vec{x}_m=(y_0,\dots,y_n)$ and every pair of consecutive terms in this sequence is elementarily equivalent.*

Moreover, if i is the largest index such that $x_i\neq y_i$, we can pick this sequence in such a way that every term (and every elementary equivalence between two consecutive terms) is of the form $(v_0,\dots,v_i,x_{i+1},\dots,x_n)$.

Proof. We proceed by induction on the distance tuple $(m_0,\dots,m_n)$ between pairs of (P,n)-chains, by assuming that all pairs of $\mathfrak{C}$-equivalent (P,n)-chains with distance less than $(m_0,\dots,m_n)$ (in the lexicographical order) are elementarily equivalent.

Base case: If two chains $(x_0,\dots,x_n)$ and $(y_0,\dots,y_n)$ have distance $(1,0,\dots,0)$, they are elementarily equivalent by definition. This establishes the base case.

Inductive step: Let $(x_0,\dots,x_n)$ and $(y_0,\dots,y_n)$ be $\mathfrak{C}$-equivalent chains with distance $(m_0,\dots,m_n)$. Let i be the maximal index where $x_i\neq y_i$. By the distance definition, there exists a connecting sequence in x_{i+1}:

$$a^i_0=x_i\preceq u^i_1\succeq a^i_1\preceq u^i_2\succeq\cdots\preceq u^i_{m_i}\succeq a^i_{m_i}=y_i$$

where $\pi_{\mathfrak{C}}(a^i_k)=\pi_{\mathfrak{C}}(u^i_k)$ for all k, and $m_j=0$ for $j>i$.

Case 1: If $x_{i-1}\preceq a^i_1$, construct the modified chain:

$$(x_0,\dots,x_{i-1},a^i_1,x_{i+1},\dots,x_n).$$

This chain is elementarily equivalent to $(x_0,\dots,x_n)$ because $\pi_{\mathfrak{C}}(a^i_1)=\pi_{\mathfrak{C}}(u^i_1)$ and u^i_1 contains both x_i and a^i_1. The new pair:

$$\big((x_0,\dots,x_{i-1},a^i_1,\dots,x_n),\ (y_0,\dots,y_n)\big)$$

has a smaller distance. By the inductive hypothesis, they are elementarily equivalent.

Case 2: If $x_{i-1}\not\preceq a^i_1$, then $\pi_{\mathfrak{C}}(x_{i-1})$ coincides with $\pi_{\mathfrak{C}}(x'_{i-1})$ for some cell x'_{i-1} in a^i_1. This gives a connecting sequence in u^i_1:

$$a^{i-1}_0=x_{i-1}\preceq u^{i-1}_1\succeq a^{i-1}_1\preceq\cdots\preceq u^{i-1}_{m'}\succeq a^{i-1}_{m'}=x'_{i-1}$$

with $\pi_{\mathfrak{C}}(a^{i-1}_k)=\pi_{\mathfrak{C}}(u^{i-1}_k)$ for all k.

Subcase 2.1: If $x_{i-2}\preceq a^{i-1}_1$, we perform substitutions:

$$\begin{aligned}&(\dots,x_{i-2},x_{i-1},x_i,\dots)\simeq(\dots,x_{i-2},x_{i-1},u^i_1,\dots)\\&\simeq(\dots,x_{i-2},u^{i-1}_1,u^i_1,\dots)\simeq(\dots,x_{i-2},a^{i-1}_1,u^i_1,\dots)\\&\simeq(\dots,x_{i-2},u^{i-1}_1,u^i_1,\dots)\end{aligned}$$

The resulting chain $(\dots,x_{i-2},u^{i-1}_1,u^i_1,\dots)$ has a smaller distance to $(y_0,\dots,y_n)$, allowing us to apply the inductive hypothesis.

Subcase 2.2: If $x_{i-2}\not\preceq a^{i-1}_1$, repeat the procedure recursively. At each step, the index decreases by 1. When reaching index 0, replace x_0 with a^0_1 directly (no further predecessors exist), which reduces the distance.

This process terminates because each iteration either reduces the index i or decreases the distance. □

Definition 4.58. *Let $\mathfrak{C}$ be a pre-ideal of the opetope P. We say that two (P,n)-chains $(x_0,\dots,x_n)$ and $(y_0,\dots,y_n)$ are **primitively equivalent**, if there exists an index i such that $x_j=y_j$ for $j\neq i$, $y_i\in\mathfrak{C}$ and $\dim(x_i)+1=\dim(y_i)$, $\delta y_i\setminus\{x_i\}\subset\mathfrak{C}$. The index i is called the **change index** of a primitive equivalence.*

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Lemma 4.59. *If $\mathfrak{C}$ is an ideal, then every elementary equivalence*

$$(x_0, \ldots, x_{i-1}, x_i, x_{i+1} \ldots, x_n) \sim (x_0, \ldots, x_{i-1}, y_i, x_{i+1} \ldots, x_n)$$

can be represented as a composition of primitive-form equivalences, each of them having the same change index as the original elementary equivalence.

Proof. Assume that the pair of chains given in the statement is elementarily equivalent via z_i. There exists a sequence $x_i = c_1 \prec c_2 \prec \ldots \prec c_m = z_i$ such that for every j, $\dim(c_j) + 1 = \dim(c_{j+1})$ and $c_j \in \langle z_i \rangle$.

We note that the chains $(x_0, \ldots, x_{i-1}, c_j, x_{i+1}, \ldots, x_n)$ and $(x_0, \ldots, x_{i-1}, c_{j+1}, x_{i+1}, \ldots, x_n)$ are primitively equivalent. Thus, we produce a sequence of primitive-form equivalences joining

$$(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \quad \text{with} \quad (x_0, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n).$$

Using the same method, we connect

$$(x_0, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \quad \text{with} \quad (x_0, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n)$$

via a sequence of primitive-form equivalences. Combining these sequences, we establish the claim. □

Theorem 4.60. *Let $f : \zeta(P) \to X$ be a morphism of simplicial sets and let $\mathfrak{C}$ be a pre-ideal of P. Suppose that the following holds: for every pair of primitively equivalent (P, n)-chains, the corresponding pair of simplices in $\zeta(P)$ represented by those chains is identified by f. Then f also identifies every pair of simplices represented by $\overline{\mathfrak{C}}$-equivalent (P, n)-chains.*

Remark 4.61. The proof establishes a stronger technical statement: If P is an opetope, $\mathfrak{C}$ is a pre-ideal and $\vec{x} = (x_0, x_1, \ldots, x_n), \vec{y} = (y_0, y_1, \ldots, y_n)$ form a pair of $\overline{\mathfrak{C}}$-equivalent (P, n)-chains, then there exists a sequence $\vec{x} = \vec{a}_1, \ldots, \vec{a}_m = \vec{y}$ of (P, n)-chains in which every two consecutive terms are either

- $\mathfrak{C}$-primitevely equivalent, or
- ζ-equivalent

Moreover, the change index of each $\mathfrak{C}$-primitive equivalence is bounded by the last index where $\vec{x}, \vec{y}$ differ.

Proof. It suffices to show that simplices represented by $\overline{\mathfrak{C}}$-elementarily equivalent chains are identified. By Lemma 4.59, we need only consider $\overline{\mathfrak{C}}$-primitively equivalent chains.

From the definition of the immediate closure operator, $\overline{\mathfrak{C}}$ is obtained from $\mathfrak{C}$ through finitely many extensions: Whenever $\alpha \in (P \setminus \gamma P) \setminus \mathfrak{C}$ satisfies $\delta\alpha \subset \mathfrak{C}$, we set $\mathfrak{D} = \mathfrak{C} \cup \{\alpha, \gamma\alpha\}$, choosing α of minimal dimension at each stage.

We proceed by induction: Assume f identifies all $\mathfrak{C}$-equivalent chains. We must show it identifies $\mathfrak{D}$-equivalent ones. New primitive equivalences in $\mathfrak{D}$ must involve α or $\gamma\alpha$. By induction hypothesis, assume f identifies chains differing below index i; we prove the statement for index i.

Base case (Index 0): Consider chains

$$(x_0, x_1, \ldots, x_n) \quad \text{and} \quad (x_0', x_1, \ldots, x_n)$$

which are ζ-equivalent to

$$(\gamma^{(0)} x_0, x_1, \ldots, x_n) \quad \text{and} \quad (\gamma^{(0)} x_0', x_1, \ldots, x_n)$$

respectively. Since $\pi_{\overline{\mathfrak{C}}}(x_0) = \pi_{\overline{\mathfrak{C}}}(x_0')$, we have $\pi_{\overline{\mathfrak{C}}}(\gamma^{(0)} x_0) = \pi_{\overline{\mathfrak{C}}}(\gamma^{(0)} x_0')$. The 0-cells identified by $\pi_{\overline{\mathfrak{C}}}$ are connected through $\overline{\mathfrak{C}} \cap \mathcal{F}_l$, where $\mathcal{F}_l$ is the largest one-dimensional term in domain filtration of P. Crucially, $\overline{\mathfrak{C}} \cap \mathcal{F}_l = \mathfrak{C} \cap \mathcal{F}_l$ because immediate closure cannot add new cells to $\mathcal{F}_l$

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(all edges present here are $<^-$-minimal). Thus the chains connect via $\mathfrak{C}$-elementary equivalences, which establishes the base case.

Induction step (Index > 0):

Case 1 (Primitive equivalence through $\gamma\alpha$): Consider

$$(x_0, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \sim (x_0, \ldots, x_{j-1}, \gamma\alpha, x_{j+1}, \ldots, x_n)$$

where $x_j \in \delta\gamma\alpha$ (otherwise those chains are ζ-equivalent). Construct a chain

$$(z_0, \ldots, z_{j-1}, \gamma^2\alpha, x_{j+1}, \ldots, x_n)$$

$\overline{\mathfrak{C}}$-equivalent to $(x_0, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)$ by:

- Taking z_{j-1} as a face of $\gamma^2\alpha$ with $\pi(z_{j-1}) = \pi(x_{j-1})$
- Taking z_{j-2} as a face of z_{j-1} with $\pi(z_{j-2}) = \pi(x_{j-2})$
- Continuing this face selection process recursively

This equivalence occurs entirely within $\delta\alpha$ (i.e., it can be viewed as an equivalence of $(\delta\alpha, j)$-chains when we truncate the chains from the right). By minimality of α, $\overline{\mathfrak{C}} \cap \delta\alpha \subset \mathfrak{C}$, making this a $\mathfrak{C}$-equivalence.

Similarly, the chain through $\gamma\alpha$ connects to

$$(y_0, \ldots, y_{j-1}, \gamma\alpha, x_{j+1}, \ldots, x_n)$$

with $y_{j-1} \preceq \gamma^2\alpha$, via $\overline{\mathfrak{C}}$-equivalences in $\gamma^2\alpha$ (since $\gamma\alpha \in \overline{\mathfrak{C}}$ implies $\pi_{\overline{\mathfrak{C}}}(\gamma\alpha) = \pi_{\overline{\mathfrak{C}}}(\gamma^2\alpha)$). These equivalences are again expressible through $\mathfrak{C}$-primitive equivalences. By induction,

$$(y_0, \ldots, \gamma^2\alpha, \ldots, x_n) \quad \text{and} \quad (z_0, \ldots, \gamma^2\alpha, \ldots, x_n)$$

are identified, completing the argument through three connecting sequences.

Case 2 (Primitive equivalence through α): The reasoning mirrors Case 1 precisely. For

$$(x_0, \ldots, x_{j-1}, x_j, \ldots, x_n) \sim (x_0, \ldots, x_{j-1}, \alpha, \ldots, x_n)$$

with $x_j \in \delta\alpha$, construct through recursive face selections in $\delta\alpha$

$$(z_0, \ldots, z_{j-1}, \gamma^2\alpha, \ldots, x_n)$$

which is $\overline{\mathfrak{C}}$-equivalent to $(x_0, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)$.

The chain

$$(x_0, \ldots, x_{j-1}, \alpha, \ldots, x_n)$$

similarly connects to

$$(y_0, \ldots, y_{j-1}, \alpha, \ldots, x_n)$$

via $\mathfrak{C}$-equivalences in $\gamma^2\alpha$. Induction again bridges the final identification and concatenation of three sequences gives the desired result. □

Corollary 4.62. *Let*

$$\begin{array}{ccc} P & \xrightarrow{\sigma_1} & Q_1 \\ {\scriptstyle \sigma_2}\downarrow & & \downarrow{\scriptstyle \overline{\sigma_2}} \\ Q_2 & \xrightarrow[\overline{\sigma_1}]{} & R \end{array}$$

be a pushout of codegeneracy maps between opetopes.

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Then the square

$$\begin{array}{ccc} \zeta(\text{よ}_P) & \xrightarrow{\zeta(\sigma_1)} & \zeta(\text{よ}_{Q_1}) \\ \downarrow{\scriptstyle \zeta(\sigma_2)} & & \downarrow{\scriptstyle \zeta(\overline{\sigma_2})} \\ \zeta(\text{よ}_{Q_2}) & \xrightarrow{\zeta(\overline{\sigma_1})} & \zeta(\text{よ}_R) \end{array}$$

is also a pushout.

Proof. We will use Theorem 4.60 several times, taking $\mathfrak{C} = \ker(\sigma_1) \cup \ker(\sigma_2)$.

We need to show that the canonical map $\theta : \left(\zeta(\text{よ}_{Q_1}) \amalg_{\zeta(\text{よ}_P)} \zeta(\text{よ}_{Q_2}) \to \zeta(\text{よ}_R)\right)$ is an isomorphism.

Claim 1: $\zeta : \mathsf{pOpe}_\iota \to \mathsf{sSet}$ **preserves epimorphisms.**

Let $P \to Q$ be a ι-epimorphism. For every (Q, n)-chain, its $\zeta(\sigma_i)$-preimage is constructed pointwise starting from the rightmost coordinate, analogous to the construction of the chain $(z_0, \ldots, z_{j-1}, \gamma^2\alpha, x_{j+1}, \ldots, x_n)$ in the proof of Theorem 4.60.

Claim 2: θ **is an epimorphism.**

The canonical map $\mu : \zeta(\text{よ}_P) \to \zeta(\text{よ}_{Q_1}) \amalg_{\zeta(\text{よ}_P)} \zeta(\text{よ}_{Q_2})$ is an epimorphism, being a composition of epimorphism and a pushout of an epimorphism.

The map $\zeta(\text{よ}_{Q_1}) \amalg_{\zeta(\text{よ}_P)} \zeta(\text{よ}_{Q_2}) \to \zeta(\text{よ}_R)$ is an epimorphism by the cancellation law for epimorphisms, applied to:

$$\left(\zeta(\text{よ}_P) \to \zeta(\text{よ}_R)\right) = \left(\zeta(\text{よ}_{Q_1}) \amalg_{\zeta(\text{よ}_P)} \zeta(\text{よ}_{Q_2}) \to \zeta(\text{よ}_R)\right) \circ \left(\zeta(\text{よ}_P) \to \zeta(\text{よ}_{Q_1}) \amalg_{\zeta(\text{よ}_P)} \zeta(\text{よ}_{Q_2})\right)$$

Claim 3: μ **identifies simplices represented by** $\mathfrak{C}$ **-primitively equivalent chains.**

Note that by Theorem 4.60, the map $\zeta(\text{よ}_P) \to \zeta(\text{よ}_R)$ is precisely the quotient map induced by these identifications.

Consider a primitively equivalent pair:

$$(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \sim (x_0, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

Without loss of generality, assume $y_i \in \ker(\sigma_1)$. If $x_i = \gamma y_i$, the chains are ζ-equivalent and thus identified by ζ regardless of any considerations concerning ideals, so assume $x_i \in \delta y_i$. Furthermore, if $x_i \notin \ker(\sigma_1)$, the chains are identified by $\zeta(\sigma_1)$; hence we assume $x_i \in \ker(\sigma_1)$ (which excludes the case where x_i is a point).

We inductively show each such pair is identified by μ, with the base case $i = -1$ being vacuously true (as chains are equal by assumption).

Almost all cells of δy_i (meaning all except at most one) lie in $\ker(\sigma_1)$, and x_i is identified by σ_1 with some $x_i' \in \partial\gamma y_i$. If $\delta y_i \subset \ker(\sigma_1)$, then $\sigma_1(x_i) = \sigma_1(y_i)$, so the chains are identified. This also holds if $\delta y_i \not\subset \ker(\sigma_1)$: the unique face z satisfying $z \in \delta y_i \setminus \ker(\sigma_1)$ must lie in $\ker(\sigma_2)$, since $\delta y_i \setminus \{x_i\} \subset \mathfrak{C}$.

Let

$$\gamma(x_i) = u_i^0 \lhd^+ u_i^1 \lhd^+ \ldots \lhd^+ u_i^m = \gamma^2(y_i)$$

with v_j (for $j = 1, \ldots, m$) satisfying $u_i^{j-1} \in \delta(v_j)$, $\gamma(v_j) = u_i^j$, and $v_j \in \delta(y_i)$.

Since $x_i \in \ker(\sigma_1)$, there exists a chain

$$(u_0^0, \ldots, u_{i-1}^0, u_i^0, x_{i+1}, \ldots, x_n)$$

identified by $\zeta(\sigma_1)$ with

$$(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

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Assume inductively we have constructed

$$(u_0^s, \ldots, u_{i-1}^s, u_i^s, x_{i+1}, \ldots, x_n)$$

which is identified with

$$(u_0^{s-1}, \ldots, u_{i-1}^{s-1}, u_i^{s-1}, x_{i+1}, \ldots, x_n)$$

via $\zeta(\sigma_1)$ or $\zeta(\sigma_2)$ for all $s < k$ $(1 \leq s \leq m)$.

The next term

$$(u_0^{s+1}, \ldots, u_{i-1}^{s+1}, u_i^{s+1}, x_{i+1}, \ldots, x_n)$$

exists because v_{s+1} lies in either $\ker(\sigma_1)$ or $\ker(\sigma_2)$ (the latter when $v_{s+1} = z$).

Consequently, the pair

$$(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n), (u_0^m, \ldots, u_{i-1}^m, \gamma^2 y_i, x_{i+1}, \ldots, x_n)$$

is identified by μ and the latter chain is ζ-equivalent to

$$(u_0^m, \ldots, u_{i-1}^m, y_i, x_{i+1}, \ldots, x_n)$$

The chains

$$(u_0^m, \ldots, y_i, \ldots, x_n), (x_0, \ldots, y_i, \ldots, x_n)$$

are $\overline{\ker(\sigma_1) \cup \ker(\sigma_2)}$-equivalent, so there is a sequence of $\ker(\sigma_1)$- and $\ker(\sigma_2)$-primitive-form equivalences connecting them (Remark 4.61), with change indices $\leq i-1$, hence μ identifies them by induction.

Thus, $\mathfrak{C} = \ker(\sigma_1) \cup \ker(\sigma_2)$ identifies $\mathfrak{C}$-primitively equivalent chains, allowing us to apply Theorem 4.53.

Claim 4: θ is a monomorphism.

It remains to show that if simplices

$$[x_0, \ldots, x_n], [y_0, \ldots, y_n]$$

(where $[\cdot]$ denotes ζ-equivalence classes) are identified by $\zeta(\sigma) : \zeta(\text{⅄}_P) \to \zeta(\text{⅄}_R)$, then they are also identified by μ (note that every simplex in $\zeta(\text{⅄}_{Q_1}) \amalg_{\zeta(\text{⅄}_P)} \zeta(\text{⅄}_{Q_2})$ is an image of some simplex in $\zeta(\text{⅄}_P)$).

By Lemma 4.10, their images under σ share the same γ-reduction. We can assume that both chains are in reduced form, with common reduction $(z_0, \ldots, z_n)$. If $\sigma(x_i) = z_i = \sigma(y_i)$ for all i, the chains are $\mathfrak{C}$-equivalent and thus μ identifies them by Theorem 4.60.

Otherwise, let k be the minimal index where $\sigma(x_k) \neq z_k$ (by symmetry between $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$, we assume that this happens for x_k). Then $\sigma(x_{k-1}) \in \gamma^t \sigma(x_k)$ for maximal $t > 0$ (vacuously true if $k = 0$).

There exists an ascending sequence of cells $x_0', \ldots, x_{k-1}'$ with $\sigma(x_i) = \sigma(x_i')$ and $x_{k-1}' \in \gamma^t x_k$. Chain

$$\vec{x}_1 = (x_0', \ldots, x_{k-1}', \gamma^t x_k, x_{k+1} \ldots, x_n)$$

is ζ-equivalent to

$$(x_0', \ldots, x_{k-1}', x_k, x_{k+1} \ldots, x_n)$$

which is $\overline{\mathfrak{C}}$-equivalent with

$$(x_0, \ldots, x_n)$$

Given $\vec{x}_j$, applying the same operation we get $\vec{x}_{j+1}$, leading to a sequence of chains in which every two consecutive terms are either ζ-equivalent or $\overline{\mathfrak{C}}$-equivalent. The first term of that

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sequence is $(x_0, \ldots, x_n)$ and the last one $(\tilde{x_0}, \ldots, \tilde{x_n})$ satisfies $\sigma(\tilde{x_j}) = z_j$. We proceed similarly with the chain $(y_0, \ldots, y_n)$.

Chains $(\tilde{x_0}, \ldots, \tilde{x_n})$ and $(\tilde{y_0}, \ldots, \tilde{y_n})$ are identified by $\zeta(よ_P) \to \zeta(よ_{Q_1}) \coprod_{\zeta(よ_P)} \zeta(よ_{Q_2})$ by the Theorem 4.60, because they are $\mathfrak{C}$-equivalent. Furthermore, every two consecutive terms of sequences mentioned above are identified (because they are either $\mathfrak{C}$-equivalent or ζ-equivalent).

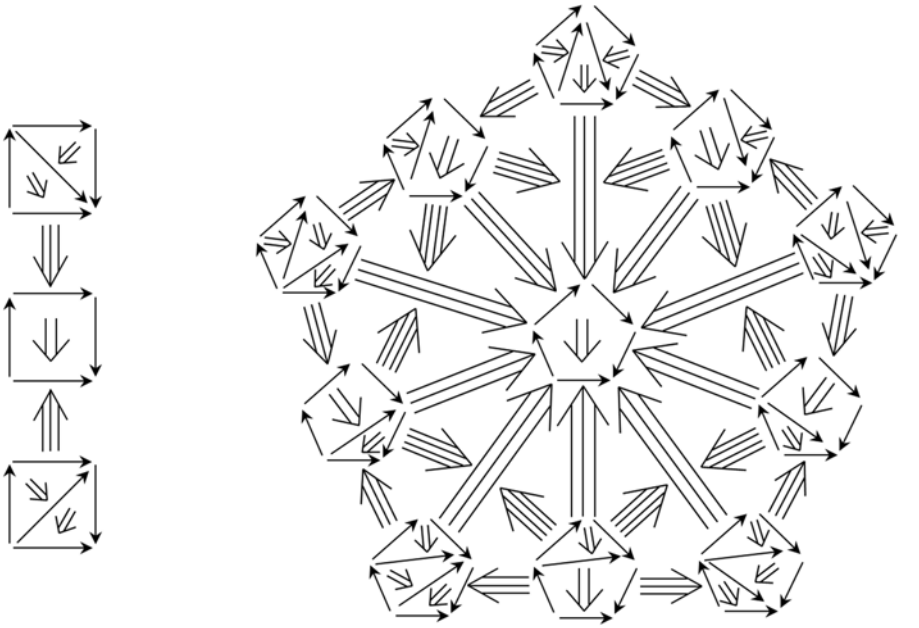
Claim 5: θ is an isomorphism.

Since sSet is a topos of presheaves, it is enough to prove that θ is both a monomorphism and an epimorphism, so this follows from Claims 2 and 4. □

4.4 Opetopic associahedron

Definition 4.63. *We define the* ***opetopic*** *n* ***-associahedron*** *as* $\rho(\Delta^n)$*, the image of the n-simplex under* ρ*.*

Example 4.64. *First three associahedra* $\rho(\Delta^0)$, $\rho(\Delta^1)$ *and* $\rho(\Delta^2)$ *are simply representables, that is* $よ_\bullet$, $よ_\to$ *and* $よ_\Delta$*. In the picture below we illustrated (nondegenerate cells of)* $\rho(\Delta^3)$ *and* $\rho(\Delta^4)$ *(note that some cells are repeated in order to allow us to incorporate all relations between them).*



Remark 4.65. Corollary 5.25 implies that opetopic associahedra are EZ-presheaves.

Lemma 4.66. *For every* $P \in \mathsf{pOpe}_t$*, the set* $\zeta(P)_0$ *is linearly ordered by the edges of* $\zeta(P)$*. Moreover, there exists a canonical isomorphism of linear orders* $P_0 \cong \zeta(P)_0$ *given by* $v \mapsto [(v)]$*, where both sides carry vertex orders induced by their connecting edges.*

Proof. The stated map is first a bijection of sets: normal forms of $(P, 0)$-chains demonstrate this. As ζ is a left adjoint, the partial order on $\zeta(P)$ contains the linear order on P_0 (i.e., the map is a preorder morphism). To conclude the equality of orders, we verify that for every edge $[(a, b)] \in \zeta(P)_1$, we have $\gamma^{(0)}a \leq \gamma^{(0)}b$. Here, $\gamma^{(0)}b$ is the last vertex of cell b, which by definition exceeds all other vertices of b, including $\gamma^{(0)}a$. □

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Corollary 4.67. *For every opetope P, the isomorphism*

$$\rho(\Delta^n)_P \cong \mathsf{Poset}(P_0, [n])$$

holds. In particular, $\rho(\Delta^n)$ contains a linear graph $Sp[n]$ with vertices canonically ordered.

Proof. By the Yoneda lemma and the $\zeta \dashv \rho$ adjunction:

$$\rho(\Delta^n)_P \cong \mathbf{Hom}(\text{よ}_P, \rho(\Delta^n)) \cong \mathbf{Hom}(\zeta(P), \Delta^n).$$

Since $\zeta(P)_0$ is linearly ordered, we further obtain:

$$\mathbf{Hom}(\zeta(P), \Delta^n) \cong \mathbf{Hom}(\zeta(P)_0, \Delta^n) \cong \mathsf{Poset}(\zeta(P)_0, [n]).$$

Applying Lemma 4.66 yields $\mathsf{Poset}(\zeta(P)_0, [n]) \cong \mathsf{Poset}(P_0, [n])$. □

In Zawadowski (2017), Zawadowski characterized the product $\text{よ}_P \times \text{よ}_{\to}$ using flags and p-flags (corresponding to codimension 0 and 1 cells, respectively). A straightforward adaptation of his method yields an inductive description of $\zeta(\Delta^n)$ as a subobject of $\zeta(\Delta^{n-1}) \times \text{よ}_{\to}$, which is essential for proving the contractibility of $\zeta(\Delta^n)$.

Definition 4.68. *Define a functor $A\colon \mathsf{sSet} \to \widehat{\mathsf{pOpe}_\iota}$ as the left Kan extension of $\rho|_{\mathsf{pOpe}_\iota}\colon \mathsf{pOpe}_\iota \to \mathsf{sSet}$ along the Yoneda embedding. Denote its right adjoint by S.*

Remark 4.69. By a standard result (Cisinski (2019), Proposition 3.1.13), A preserves monomorphisms.

Lemma 4.70. *The functor A takes values in $\widehat{\mathsf{pOpe}_{\iota EZ}}$.*

Proof. By Corollary 3.21, it suffices to show that A sends the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ to cellular monomorphisms. By the preceding remark and Lemma 3.17, this reduces to verifying that both $A(\partial\Delta^n)$ and $A(\Delta^n)$ are EZ-presheaves.

For $A(\Delta^n)$, this follows directly from its construction. For $A(\partial\Delta^n)$, observe that $\partial\Delta^n$ can be expressed as a pushout of representables along inclusions of common faces. Since A is cocontinuous and Lemma 3.34 ensures cellularity is preserved under pushouts, the claim follows. □

Lemma 4.71. *The functor A preserves finite products.*

Proof. It suffices to verify the claim for binary products of representables. Consider the sequence:

$$\big(A(\Delta^n) \times A(\Delta^m)\big)_P \cong \big(A(\Delta^n)\big)_P \times \big(A(\Delta^m)\big)_P \cong \mathsf{Poset}(P_0, [n]) \times \mathsf{Poset}(P_0, [m])$$

$$\cong \mathsf{Poset}(P_0, [n] \times [m])$$

This final expression motivates defining a functor $A'\colon \mathsf{Poset}_{\mathrm{fin}} \to \widehat{\mathsf{pOpe}_\iota}$ by:

$$A'(\mathcal{H}) := \big(P \mapsto \mathsf{Poset}(P_0, \mathcal{H})\big)$$

for finite posets $\mathcal{H}$. Each component functor $\big(A'(-)\big)_P$ preserves products because it is a Hom-functor in the first argument. As limits in $\widehat{\mathsf{pOpe}_\iota}$ are computed pointwise, A' preserves all products.

The chain of isomorphisms shows A and A' agree on posets $[n] \times [m]$, which correspond via the nerve functor to binary products $\Delta^n \times \Delta^m$ in sSet. Thus, we obtain:

$$A(\Delta^n \times \Delta^m) \cong A(\Delta^n) \times A(\Delta^m).$$

□

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Lemma 4.72. *If X is an EZ-presheaf and $\mathcal{G} \hookrightarrow \mathrm{Aut}(X)$ is a subgroup, then $X/\mathcal{G}$ remains an EZ-presheaf.*

Proof. Quotient under the action of automorphisms identifies nondegenerate cells only with nondegenerate cells since isomorphisms preserve and reflect nondegenerate cells.

Furthermore, if nondegenerate $a, b \in X_P$ satisfy $g(a) = b$ and $g(\sigma_1^* a) = \sigma_2^* b$ for some codegeneracy maps σ_1, σ_2 and $g \in \mathcal{G}$, then $\sigma_1 = \sigma_2$, so this quotient also cannot identify distinct degeneracies of a given cell. □

Corollary 4.73. *$A(\Delta^n) \cong \overset{n}{\underset{\rightarrow}{\curlywedge}} / \Sigma_n$, where the action of the symmetric group Σ_n is given by permutation of coordinates.*

Proof. In the category sSet, we have $(\Delta^1)^n / \Sigma_n \cong \Delta^n$. To see this, first observe that the vertices of $(\Delta^1)^n$ admit a canonical bijection

$$\big((\Delta^1)^n\big)_0 \ni v \longmapsto \phi_v \in \mathsf{Set}\big(\{1, \ldots, n\}, \{0, 1\}\big),$$

where $\phi_v(i)$ records the i-th coordinate of v. Two vertices $v, w \in \big((\Delta^1)^n\big)_0$ lie in the same Σ_n-orbit if and only if $\sum_{i=1}^n \phi_v(i) = \sum_{i=1}^n \phi_w(i)$.

Next, note that every nondegenerate cell of $(\Delta^1)^n$ is uniquely determined by its vertices. It follows that $(\Delta^1)^n / \Sigma_n$ is isomorphic to the largest subobject of $(\Delta^1)^n$ whose vertices are

$$(0, \ldots, 0, 0),\ (0, \ldots, 0, 1),\ (0, \ldots, 1, 1),\ \ldots,\ (0, 1, \ldots, 1),\ (1, \ldots, 1).$$

This subobject is evidently isomorphic to Δ^n, as it corresponds to the standard n-simplex with vertices ordered by increasing numbers of 1s.

Since A preserves the interval, finite products, and quotients, we conclude:

$$A(\Delta^n) \cong A\left((\Delta^1)^n / \Sigma_n\right) \cong A(\Delta^1)^n / \Sigma_n \cong \overset{n}{\underset{I}{\curlywedge}} / \Sigma_n.$$

□

Definition 4.74. *The **cylinder** $\mathrm{Cyl}(P)$ of an opetope P is a special hypergraph introduced by Zawadowski Zawadowski (2017). Before recalling the definition of $\mathrm{Cyl}(P)$, we introduce preliminary notions (here, we work with a fixed opetope P of dimension n).*

*A **flag** is a sequence $\vec{x} = [x_m, x_{m-1}, \ldots, x_0]$ of cells in P such that $x_i \in \partial x_{i+1}$ and $x_i \in P_i$ for all i. A flag is **maximal** if $m = n$.*

*For a flag $\vec{x} = [x_k, x_{k-1}, \ldots, x_0]$ that is neither $[x_n, \gamma(x_n), \ldots, \gamma^{(1)}(x_n), \gamma^{(0)}(x_n)]$ nor $[x_n, \gamma(x_n), \ldots, \gamma^{(1)}(x_n), \delta\gamma^{(1)}(x_n)]$, and where $k > 1$, the **low level** of $\vec{x}$ is defined as*

$$\mathrm{ll}(\vec{x}) = \max\{i < k - 1 \mid x_{i+1} \in \delta(x_{i+2})\}.$$

*A **punctured flag** (or **p-flag**) in P is a sequence $\vec{z} = \vec{x}(i)$, where $i = k - 1$ or $i = \mathrm{ll}(\vec{x})$ for some flag $\vec{x}$. If $i = k - 1$, $\vec{z}$ is a **high p-flag**; otherwise, it is a **low p-flag**.*

The faces of $\mathrm{Cyl}(P)$ are of three types:

1. ***Flat faces**: $\{-\} \times P \cup \{+\} \times P$ (denoted $-P$ and $+P$);*
2. ***Flags**: All flags of all faces of P;*
3. ***P-flags**: All p-flags of all faces of P.*

The dimension of a face $-p$ or $+p$ equals the dimension of p. The dimension of a flag or p-flag is the number of non-zero faces.

The complete description hypergraph $\mathrm{Cyl}(P)$ is long and technical. Here we include only those parts, which are critical for our reasoning.

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Lemma 4.75. *For a maximal flag $\vec{x}$ in* Cyl(P) *(i.e., a cell of maximal dimension), the set of vertices of $\vec{x}$ is given by:*

$$\langle \vec{x} \rangle_0 = \{-p \mid p \in P_0,\ p \leq^+ x_0\} \cup \{+p \mid p \in P_0,\ x_0 \leq^+ p\}.$$

Theorem 4.76. *The product $\daleth_P \times \daleth_{\rightarrow}$ has the following properties:*

- *it is an image of Cyl(P) under the functor $\mathsf{pHg}_\iota \hookrightarrow \widehat{\mathsf{pOpe}_\iota}$*
- *every nondegenerate cell of $\daleth_P \times \daleth_{\rightarrow}$ is contained in a nondegenerate cell of a dimension dim $P+1$*
- *nondegenerate cells of a maximal dimension are in a bijective correspondence with flags on P*
- *nondegenerate cells of a maximal dimension containing the vertex $(\gamma^{(0)}P, 0)$ are in a bijective correspondence with flags of the form $[\alpha_P, \ldots, \gamma^{(0)}P]$*
- *there is a linear order $<$ on the set of maximal flags, called **flag order**, such that every two consecutive flags in this order share exactly one face of codimension* 1. *Moreover,* Cyl(P) *is obtained from maximal flags exactly by those identifications.*

Definition 4.77. *The hypergraph-cone* Cone$_H(P)$ *over an opetope P is a subobject of* Cyl(P) *generated by the flags of the form $[\alpha_P, \ldots, \gamma^{(0)}P]$. The cone* Cone$(P)$ *over an opetope P is the image of* Cone$_H(P)$ *under the embedding $\mathsf{pHg}_\iota \hookrightarrow \widehat{\mathsf{pOpe}_\iota}$.*

Remark 4.78. There is a canonical inclusion $P \cong P \times \{0\} \overset{i_0}{\hookrightarrow}$ Cone(P).

If $\delta : Q \to P$ is a coface map s. t. $\gamma^{(0)}Q = \gamma^{(0)}P$, then there is an induced inclusion Cone(δ) : Cone$(Q) \hookrightarrow$ Cone(P). If $\gamma^{(0)}Q \neq \gamma^{(0)}P$, we define Cone$(\delta)$ to be the inclusion $Q \times \{0\} \hookrightarrow$ Cone(P)

Next, we extend the definition of a cone to a larger class of opetopic sets.

Definition 4.79. *Let $X \in \widehat{\mathsf{pOpe}_{\iota EZ}}$ be a free opetopic set, that is, an opetopic set with the property that for each nondegenerate cell α of X, the subobject of X generated by α is isomorphic with representable (counterpart of the notion of positive opetopic hypergraph on the level of presheaves). Furthermore, we assume that X has a distinguished vertex x, which is the last vertex in X with respect to the order induced by the edges of X.*

The cone Cone(X, x) *over X with respect to x is defined as*

$$\mathrm{colim}_{\alpha \in \mathcal{C}_{nd}(X)} \mathrm{Cone}(\alpha, x)$$

where $\mathcal{C}_{nd}(X)$ is a category with objects being nondegenerate cells of X and morphisms being inclusions of faces, and Cone(α, x) *is equal to* Cone(α) *if $\gamma^{(0)}\alpha = x$ and equal to α otherwise.*

Definition 4.80. *We define a sequence of opetopic sets $\{\widetilde{A}(n)\}_{n \in \mathbb{N}}$ inductively by the formula*

$$\widetilde{A}(0) = \{\bullet\},\ \widetilde{A}(n+1) = \mathrm{Cone}(\widetilde{A}(n), v_n)$$

where v_n is the unique vertex of $\widetilde{A}(n)$ which is not contained in the base of a cone, that is, $v_n \notin \mathcal{C}^0(\widetilde{A}(n-1))$.

Theorem 4.81. $\widetilde{A}(n) \cong \daleth_I^n / \Sigma_n$.

Proof. First we show that $\widetilde{A}(n) \cong V_n$, where V_n is the largest subobject of I^n with the set of vertices given by $\{(0, 0, \ldots, 0, 0), (0, 0, \ldots, 0, 1), (0, 0, \ldots, 1, 1), \ldots, (0, 1, \ldots, 1, 1), (1, 1, \ldots, 1, 1) =: v_n\}$

For $n = 0, 1$ it is obvious. For general n, we proceed by induction.

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Note that every (not necessarily nondegenerate) cell of I^n is uniquely determined by its shape and ordered list of vertices: it is obviously true for $\curlywedge_I^0$, $\curlywedge_I^1$, and the general case is shown by the induction using the canonical isomorphism $(\curlywedge_I^n)_P \cong (\curlywedge_I^{n-1})_P \times (\curlywedge_I)_P$.

By the inductive assumption and construction of a cone, it follows that $\widetilde{A}(n)$ contains vertices $(0,0,\ldots,0,0)$, $(0,\ldots,0,1,0)$, $(0,\ldots,1,1,0)$, $\ldots$, $(1,\ldots,1,1,0)$ (those are exactly the vertices from the base of a cone). An apex of a cone is equal to a vertex $(1,1,\ldots,1,1)$. Now from the Theorem 4.76 follows that $\widetilde{A}(n)$ contains all nondegenerate cells on those vertices: cells of the maximal dimension in $Cone(V_{n-1}, v_{n-1})$ are exactly those maximal cells of $V_{n-1} \times \curlywedge_I$ which contain both vertices $(v_{n-1}, 0)$ and $(v_{n-1}, 1)$, as we can see from the description of the set $\langle \vec{x} \rangle_0$ given in the Lemma 4.75.

To show that $V_n \cong \curlywedge_I^n / \Sigma_n$ we employ reasoning similar to the one given in the proof of the Corollary 4.73.

So the argument given in the proof of the Corollary 4.73 indeed applies also here. □

Corollary 4.82. $A(\Delta^n) \cong \mathrm{Cone}(A(\Delta^{n-1}), v_{n-1})$.

5. The Homotopy Theory of Opetopic Sets

5.1 Olschok-Cisinski theory for Eilenberg–Zilber presheaves

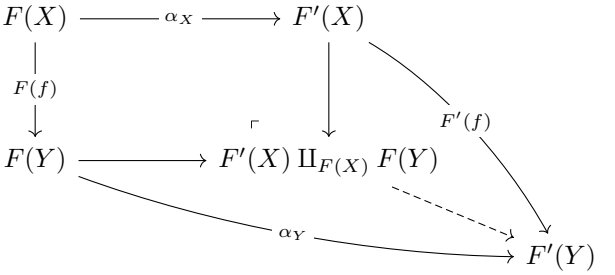
In this chapter, we briefly discuss the basic results of Cisinski's theory. Some parts of that theory were developed in the context of Grothendieck topoi; however, they are mostly used for categories of presheaves.

Olschok formulated a generalized version of that theory, which deals with locally presentable cartesian closed categories.

Our modification of his theory doesn't assume cartesian closedness of the underlying category.

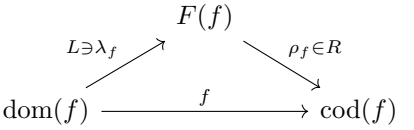
We begin by recalling several notions from Olschok's work.

Definition 3.1. Let $\mathcal{A}$ be a category with pushouts. Given a natural transformation $\alpha : F \dot{\to} F' : \mathcal{X} \to \mathcal{A}$ and a map $f : X \to Y$ let $f \star \alpha$ be the connecting map in the diagram below:



For a class J of maps, we write $J \star \alpha$ for $\{f \star \alpha \mid f \in J\}$.

Definition 5.1. *We say that a weak factorization system (L, R) is **cofibrantly generated** if $L = \mathrm{l}(\mathrm{r}(S))$ for some subset $S \subset L$. It is a functorial, if there exists a functor $F : \mathcal{C}^{\to} \to \mathcal{C}$ together with a pair of natural transformations $\lambda : \mathrm{dom} \to F$, $\rho : F \to \mathrm{cod}$ such that $\lambda_f \in L$ and $\rho_f \in R$ for every $f \in \mathcal{C}^{\to}$, and $f = \rho_f \circ \lambda_f$.*

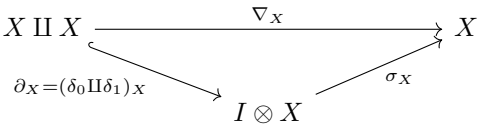


*A weak factorization system is called **cofibrant** if, for every object X, the morphism $\emptyset \to X$ belongs to L.*

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Observation 5.2. *For every weak factorization system on the category $\mathcal{C}$, there exists a trivial model structure on $\mathcal{C}$, with* $\mathrm{W} := \mathrm{Mor}(\mathcal{C})$, $\mathrm{Cof} = L$, $\mathrm{Fib} = R$.

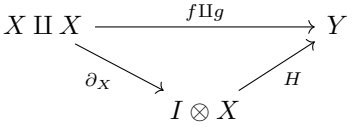
Definition 5.3. *For the model category $\mathcal{C}$, a* ***functorial cylinder*** *is a functor $I \otimes - : \mathcal{C} \to \mathcal{C}$, endowed with a pair of natural transformations*



in which the first one is a cofibration and the second one is a weak equivalence.

If $\mathcal{C}$ is just a category endowed with a weak factorization system, a functorial cylinder is understood as a functorial cylinder with respect to the trivial model structure.

Definition 5.4. *Given a functorial cylinder $I \otimes -$, two maps $f, g : X \to Y$ are called homotopic, if the map $f \amalg g : X \amalg X \to Y$ factors through $\partial_X : X \amalg X \to I \otimes X$*



The map $H : I \otimes X \to Y$ is called a homotopy between f and g.

Remark 5.5. The relation of being homotopic on the set $Hom(X, Y)$ is reflexive and symmetric, but not transitive in general. We denote its transitive closure by $\simeq$.

Definition 5.6. *Given a cofibrantly generated weak factorization system (L, R) with generating set M, a functorial cylinder $I \otimes -$ and the set $S \subset \mathrm{l}(\mathrm{r}(L))$ we define sets:*

$$\Lambda_I^0(S, L) = S \cup \{I \otimes X \cup \{\varepsilon\} \otimes Y \to I \otimes Y \mid X \to Y \in M,\ \varepsilon = 0, 1\}. \tag{1}$$

For $n \geq 0$, we put

$$\Lambda_I^{n+1}(S, L) = \{I \otimes X \cup \partial I \otimes Y \to I \otimes Y \mid X \to Y \in \Lambda_I^n(S, L)\}, \tag{2}$$

and finally:

$$\Lambda_I(S, L) = \bigcup_{n \geq 0} \Lambda_I^n(S, L). \tag{3}$$

Morphisms belonging to the class $\mathrm{l}(\mathrm{r}(\Lambda_I(S, L)))$ are called ***anodyne extensions****.*

Definition 5.7. *Morphisms having the right lifting property with respect to the class of anodyne extensions are called* ***naive fibrations****. If an object $X \in \mathcal{C}$ has the property that $X \to \{\bullet\}$ is a naive fibration, we say that X is* ***fibrant****.*

Definition 5.8. *A morphism $f : X \to Y$ is called a* ***weak equivalence*** *it the induced map*

$$\big(\mathrm{Hom}(Y, W)/\simeq\big) \to \big(\mathrm{Hom}(X, W)/\simeq\big)$$

is bijective for every fibrant object W.

We denote the class of weak equivalences by $W(I \otimes -, S, M)$.

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Definition 5.9. *Let (L, R) be a weak factorization system in a category $\mathcal{C}$. A functorial cylinder (I, ∂, σ) is **cartesian** if:*

- *the cylinder functor $I \otimes - : \mathcal{C} \to \mathcal{C}$ is a left adjoint.*
- *morphisms $I \otimes X \cup \{\varepsilon\} \otimes Y \to I \otimes Y$ and $I \otimes X \cup \partial I \otimes Y \to I \otimes Y$ are in L for $(X \to Y) \in L$ and $\varepsilon = 0, 1$.*

Theorem 5.10. *Let $\mathcal{K}$ be a locally presentable category and (L, R) a cofibrant weak factorization system generated by a set $M \subseteq L$. Let $(I \otimes -, \partial, \sigma)$ be a cartesian cylinder, and $S \subseteq L$ an arbitrary subset. Then, setting*

$$C := L, \quad W := W(I \otimes -, S, M), \quad F := \mathrm{r}(C \cap W)$$

gives a cofibrantly generated model structure (C, W, F) on $\mathcal{K}$. Moreover, $(I \otimes -, \partial, \sigma)$ is also a cylinder for this model structure.

Remark 5.11. In our work, we explain how, under certain additional assumptions, the assumption that $I \otimes -$ is left adjoint can be dropped.

Cisinski's theory, as described in Cisinski (2019), has become a standard tool for constructing cofibrantly generated model structures on categories of presheaves – and more generally, on Grothendieck toposes (see Cisinski (2006)). This framework was further generalized by Olschok Olschok (2011) to a broader class of locally presentable categories under certain conditions, including cartesian closedness.

Unfortunately, these tools cannot be directly applied in our setting, as the cartesian closedness of the category $\widehat{\mathsf{pOpe}_{\iota EZ}}$ remains an open question. Nevertheless, we were able to circumvent this issue by reproving several key lemmas from Olschok (2011) without relying on the assumption of cartesian closedness. Instead, we employ alternative assumptions that are more readily verifiable within the context of our work. This section is devoted to detailing these modifications.

Definition 5.12. *Let $\mathcal{A}$ be a tidy Reedy category.*

*We say that $\widehat{\mathcal{A}}$ is **category with proper point** if for every $X \in \mathrm{Ob}(\widehat{\mathcal{A}}) \setminus \{\emptyset\}$ there exists at least one morphism $\bullet \to X$ (which is necessarily a monomorphism), where $\bullet$ is the final object.*

In this section, we work with a fixed tidy Reedy category $\mathcal{A}$ such that $\widehat{\mathcal{A}}$ is a category with a proper point.

In the Theorem 3.16 in Olschok (2011), it is assumed that the cylinder functor is a left adjoint, but proof of the Theorem relies only on consequences of that fact, namely the Corollary 3.3 and the fact that cylinder preserves the initial object.

In the context of the category of EZ-presheaves, we don't know if it is true that the product-functor $\mathbb{1}_I \times (-)$ is a left adjoint (even though certainly it is when regarded as an endofunctor on the category of presheaves). Nevertheless, it preserves the initial object, and the appropriate analog of Corollary 3.3 (namely our Corollary 5.17) holds.

Definition 5.13. *Let J be a class of morphisms in the category $\widehat{\mathcal{A}}_{EZ}$.*

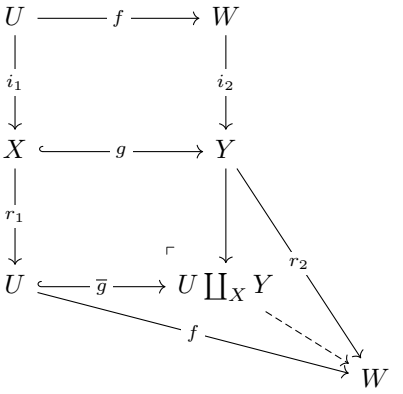
Then $\mathrm{l}_{\widehat{\mathcal{A}}}(J)$ ($\mathrm{r}_{\widehat{\mathcal{A}}}(J)$) denotes the class of all morphisms in $\widehat{\mathcal{A}}$ having the left lifting property (right lifting property, respectively) with respect to the class J, while $\mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(J)$ ($\mathrm{r}_{\widehat{\mathcal{A}}}(J)$) denotes the class of all morphisms in $\widehat{\mathcal{A}}_{EZ}$ having the left lifting property (the right lifting property, respectively) with respect to the class J.

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Lemma 5.14. *Let J be any class of cellular monomorphisms in $\widehat{\mathcal{A}}_{EZ}$. Let $\mathrm{l}_{\widehat{\mathcal{A}}}(\mathrm{r}_{\widehat{\mathcal{A}}}(J))$ be the saturated class generated by J in the category of all presheaves and $\mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J))$ be the saturated class generated by J in $\widehat{\mathcal{A}}_{EZ}$. Then $\mathrm{l}_{\widehat{\mathcal{A}}}(\mathrm{r}_{\widehat{\mathcal{A}}}(J)) \cap \mathrm{Mor}(\widehat{\mathcal{A}}_{EZ}) = \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J))$.*

Proof. By the small object argument, both classes are given by retracts of morphisms obtained by transfinite composition of pushouts of elements in J (in respective categories).

Suppose that $f \in \mathrm{l}_{\widehat{\mathcal{A}}}(\mathrm{r}_{\widehat{\mathcal{A}}}(J)) \cap \mathrm{Mor}(\widehat{\mathcal{A}}_{EZ}) \setminus \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J))$. This means that f is a retract of some g, which is a transfinite composition of pushouts of morphisms in J and the domain of g is not an EZ-presheaf.



Forming the pushout as on the diagram above, we see that f is also a retract of $\bar{g}$. The domain of $\bar{g}$ is an EZ-presheaf. Since pushouts commute with transfinite compositions, it follows that $\bar{g}$ is transfinite composition of pushouts of the same morphisms in J in terms of which g was expressed, but in the case of $\bar{g}$ pushouts are taken along the morphisms of EZ-presheaves, hence $\bar{g} \in \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J))$. Since this class is closed under retracts, we also have $f \in \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J))$, a contradiction. □

Lemma 5.15. *If a product $\prod_{t\in T} X_t$ of nonempty objects in $\widehat{\mathcal{A}}$, where $\mathcal{A}$ is a category with a proper point, is an EZ-presheaf, then every presheaf X_t is an EZ-presheaf.*

Proof. Suppose that X_{t_0} is not an EZ-presheaf for some t_0. For every $t \in T \setminus \{t_0\}$ we fix a vertex $x_t \in X_t$, that is, a subobject of X_t isomorphic with $\bullet$.

Next, we consider a subobject of the product given by $\prod_{t\in T\setminus\{t_0\}}\{x_t\} \times X_{t_0} \cong X_{t_0}$.

This subobject is a retract of $\prod_{t\in T} X_t$ by the projection map π_{t_0}. This gives a contradiction with the Corollary 3.22. □

Lemma 5.16. *Let ξ be any of the functorial inclusions $\{0\} \times (-) \hookrightarrow \text{ⱡ}_I \times (-)$, $\{1\} \times (-) \hookrightarrow \text{ⱡ}_I \times (-)$, $\partial\,\text{ⱡ}_I \times (-) \hookrightarrow \text{ⱡ}_I \times (-)$ and J be a class of cellular monomorphisms between EZ-presheaves over pOpe_ι. Then the following holds:*

$$\left(\mathrm{l}_{\widehat{\mathcal{A}}}(\mathrm{r}_{\widehat{\mathcal{A}}}(J)) \star \xi\right) \cap \mathrm{Mor}(\widehat{\mathcal{A}}_{EZ}) = \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J)) \star \xi$$

Proof. Let us fix a morphism $f : X \to Y$ from $\mathrm{l}_{\widehat{\mathcal{A}}}(\mathrm{r}_{\widehat{\mathcal{A}}}(J))$, where $Y \neq \emptyset$ (we leave it to the reader to handle this trivial case separately).

Claim 1: $Y \in \widehat{\mathcal{A}}_{EZ} \implies f \in \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J))$

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Since f is cellular, it is enough to show: that for a cellular monomorphism having an EZ-presheaf as a codomain implies having an EZ-presheaf as a domain (the reverse implication also holds, as we already showed).

Claim 1.1: $f \in \mathrm{l}(\mathrm{r}(\{\partial_P\}_{P\in\mathcal{A}})) \wedge Y \in \widehat{\mathcal{A}}_{EZ} \implies X \in \widehat{\mathcal{A}}_{EZ}$

By Observation 3.16 $\oint Y \setminus \oint X$ is well defined and a coproduct of connected categories (since it is a sum of connected components in Y), each of which has a terminal object. Since all connected components of $\oint Y$ have terminal objects (by virtue of Y being an EZ-presheaf), the same must hold for $\oint X$, showing that X is also an EZ-presheaf. The application of Lemma 5.14 finishes the proof of the claim.

Claim 2: $f \star \xi \in \mathrm{Mor}(\widehat{\mathcal{A}}_{EZ}) \implies Y \in \widehat{\mathcal{A}}_{EZ}$

Suppose that $f \star \xi$ is a morphism of EZ-presheaves, in particular $\mathrm{cod}(f \star \xi)$ is an EZ-presheaf. Lemma 5.15 and isomorphism $\mathrm{cod}(f) \times I \cong \mathrm{cod}(f \star \xi)$ imply that $\mathrm{cod}(f) = Y$ is indeed an EZ-presheaf, that is both the premise and the conclusion of the Claim 1 hold. □

Corollary 5.17 (Eilenberg–Zilber version of the Corollary 3.3 from Olschok (2011)). *Let J_1 and J_2 be two classes of cellular monomorphisms in $\widehat{\mathsf{pOpe}_{\iota EZ}}$. Then the following implication holds:*

$$J_1 \star \xi \subseteq J_2 \implies \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J_1)) \star \xi \subseteq \mathrm{l}_{\widehat{\mathcal{A}}_{EZ}}(\mathrm{r}_{\widehat{\mathcal{A}}_{EZ}}(J_2))$$

Proof. Inclusion $\mathrm{l}_{\widehat{\mathcal{A}}}(\mathrm{r}_{\widehat{\mathcal{A}}}(J_1)) \star \xi \subseteq \mathrm{l}_{\widehat{\mathcal{A}}}(\mathrm{r}_{\widehat{\mathcal{A}}}(J_2))$ follows from Olschok (2011), Corollary 3.3. Now it suffices to intersect both sides with $\mathrm{Mor}(\widehat{\mathcal{A}}_{EZ})$ and apply Lemma 5.14 and Lemma 5.16. □

Corollary 5.18 (Eilenberg–Zilber version of the Theorem 3.16 from Olschok (2011)). *Let $\mathcal{A}$ be a tidy Reedy category such that $\widehat{\mathcal{A}}_{EZ}$ is a category with a proper point. Let (L, R) be a cofibrant weak factorization system in $\widehat{\mathcal{A}}_{EZ}$ generated by a set $L \subseteq \{\partial_P\}_{P\in\mathcal{A}}$. Let I be a functorial cylinder and $S \subseteq L$ an arbitrary subset. Then, setting*

$$\mathrm{Cof} := L \quad \mathrm{W} := \mathcal{W}(I, S, \{\partial_P\}_{P\in\mathcal{A}}) \quad \mathrm{Fib} := \mathrm{r}(L \cap \mathcal{W})$$

gives a cofibrant model structure (Cof, W, Fib) *on $\widehat{\mathcal{A}}_{EZ}$, with $I \otimes X$ being a cylinder over X for every $X \in \widehat{\mathcal{A}}_{EZ}$.*

Definition 5.19. *The model structure obtained either from the original Cisinski–Olschok theory or from the modification described above is called a* ***generalized Cisinski model structure****.*

Remark 5.20. Proof of the Olschok theorem is based on the highly nontrivial theorem of Jeff Smith (and simultaneously generalizes the result of Cisinski, which has more elementary proof).

One can slightly modify the reasoning given by Cisinski to obtain the existence of the opetopic $(\infty, 0)$-model structure above without direct reliance on the Jeff Smith theorem.

In the following, we formulate a useful criterion for checking if a given functor with domain in a generalized Cisinski model structure is a left Quillen functor.

Lemma 5.21. *Let $\mathcal{C}$ be a category equipped with a generalized Cisinski model structure, generated by the cylinder functor $I \otimes - : \mathcal{C} \to \mathcal{C}$ and a set S of generating anodyne extensions.*

Let $\mathcal{D}$ be a model category and $F : \mathcal{C} \to \mathcal{D}$ a left adjoint such that the following conditions hold:

- *F preserves cofibrations,*
- *for every $c \in \mathcal{C}$, $F(I \otimes c)$ is a cylinder object for $F(c)$ in $\mathcal{D}$*
- *for every $f \in S$, $F(f)$ is an acyclic cofibration in $\mathcal{D}$*

Then F is a left Quillen functor.

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Proof. To prove F is left Quillen, it suffices to show F preserves acyclic cofibrations. Let $\mathsf{Cof}_{\mathcal{D}}$ and $\mathsf{W}_{\mathcal{D}}$ denote the cofibrations and weak equivalences in $\mathcal{D}$, respectively.

Step 1: Reduction to anodyne extensions

Let $f : K \to L$ be an acyclic cofibration in $\mathcal{C}$. Let $\kappa : Id_{\mathcal{C}} \to \mathfrak{P}$ be a functorial fibrant replacement obtained from the application small object argument to the set of generators of the class of anodyne extensions (in particular, it is a pointwise anodyne extension). Consider the following diagram:

$$\begin{array}{ccc} K & \xhookrightarrow{\kappa_K} & \mathfrak{P}(K) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathfrak{P}(f)} \\ L & \xhookrightarrow{\kappa_L} & \mathfrak{P}(L) \end{array}$$

Since $\mathfrak{P}(f)$ is an acyclic cofibration between fibrant objects, it is an anodyne extension. It follows that $F(\kappa_K), F(\kappa_L), F(\mathfrak{P}(f)) \in \mathsf{W}_{\mathcal{D}}$, the 2-out-of-3 property implies $F(f) \in \mathsf{W}_{\mathcal{D}}$. As F preserves cofibrations, $F(f) \in \mathsf{Cof}_{\mathcal{D}} \cap \mathsf{W}_{\mathcal{D}}$.

Step 2: Reduction to generating anodyne extensions.

Recall that the class of generating anodyne extension for the generalized Cisinski model structure is defined inductively through:

$$\begin{aligned} \Lambda^I_0(S,M) &= S \cup \{I \otimes K \cup \{\epsilon\} \otimes L \to I \otimes L \mid K \hookrightarrow L \in M,\ \epsilon \in \{0,1\}\} \\ \Lambda^I_{n+1}(S,M) &= \left\{I \otimes K \cup \partial I \otimes L \to I \otimes L \mid K \hookrightarrow L \in \Lambda^I_n(S,M)\right\} \\ \Lambda^I(S,M) &= \bigcup_{n \geq 0} \Lambda^I_n(S,M) \end{aligned}$$

Base Case ($n = 0$): Consider the following diagram:

$$\begin{array}{ccccc} \{\epsilon\} \otimes K & \hookrightarrow & \{\epsilon\} \otimes L & \xhookrightarrow{\sim} & I \otimes L \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & \nearrow & \\ I \otimes K & \hookrightarrow & I \otimes K \cup \{\epsilon\} \otimes L & & \end{array}$$

The left vertical arrow and the right upper arrow are acyclic cofibrations which are preserved by F by the assumption on I. Thus the right vertical arrow is also an acyclic cofibration preserved by F, being a pushout of one. Finally, the diagonal arrow on the right is a weak equivalence by 2-out-of-3.

Inductive step:

For $K \hookrightarrow L \in \Lambda^I_n(S,M)$ such that $F(K \hookrightarrow L) \in \mathrm{Cof}_{\mathcal{D}} \cap W_{\mathcal{D}}$:

$$\begin{array}{ccccc} \partial I \otimes K & \hookrightarrow & I \otimes K & \hookleftarrow & K \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \searrow \\ \partial I \otimes L & \hookrightarrow & I \otimes K \cup \partial I \otimes L & \hookrightarrow & I \otimes L \end{array} \qquad L$$

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The left vertical arrow is isomorphic with $K \amalg K \hookrightarrow L \amalg L$, hence it is in $\mathrm{Cof}_{\mathcal{D}} \cap W_{\mathcal{D}}$, hence so is its pushout, the second vertical arrow. Maps $K \hookrightarrow I \otimes K$ and $L \hookrightarrow I \otimes L$ are in $\mathrm{Cof}_{\mathcal{D}} \cap W_{\mathcal{D}}$ by the assumption on the cylinder. By 2-out-of-3 applied to the pentagon on the right, we see that $I \otimes K \cup \partial I \otimes L \to I \otimes L$ is in $\mathrm{Cof}_{\mathcal{D}} \cap W_{\mathcal{D}}$. □

5.2 Opetopic $(\infty, 0)$*-structure*

In this section, we apply the tools developed in the previous section to construct a particular model structure on the category $\widehat{\mathsf{pOpe}_{\iota EZ}}$, referred to as the $(\infty, 0)$-*model structure*. Moreover, we demonstrate that the functors introduced in the preceding chapter give rise to two Quillen adjunctions between this model structure and the classical Kan–Quillen model structure on sSet.

Definition 5.22. *Let P be a positive opetope. Let e be a cell of P of the codimension* 1. *We define e-**horn on** P $\Lambda^e P$ to be the biggest subobject of よ_P in the category of $\widehat{\mathsf{pOpe}_{\iota EZ}}$ which does not contain cells corresponding to morphisms $\langle e \rangle \hookrightarrow P$ and Id_P.*

Remark 5.23. Equivalently, $\Lambda^e P$ can be described as a subobject of よ_P (in either of categories $\widehat{\mathsf{pOpe}_{\iota}}$ or $\widehat{\mathsf{pOpe}_{\iota EZ}}$) generated by cells corresponding to morphisms $\langle c \rangle \hookrightarrow P$ for $c \in |P| \setminus \{e, \alpha_P\}$. In particular, for $e = \gamma \alpha_P$, we have $\Lambda^e P \cong \mathcal{H}_{\iota}(\delta P)$.

Definition 5.24. ***The opetopic*** $(\infty, 0)$***-structure*** *is a model structure on the category $\widehat{\mathsf{pOpe}_{\iota EZ}}$ obtained from the application of the Corollary 5.18 to the class*

$$S := \{\Lambda^e P \hookrightarrow \text{よ}_P\}_{P \in \mathsf{pOpe}_{\iota}, e \in P_{\dim(P)-1}}$$

and the functorial cylinder

$$I \otimes - = \text{よ}_{\to} \times -.$$

Theorem 5.25. *Functor $\rho : \mathsf{sSet} \to \widehat{\mathsf{pOpe}_{\iota}}$ takes values in a category $\widehat{\mathsf{pOpe}_{\iota EZ}}$. In particular, ζ and ρ restrict to a pair of adjoint functors between sSet and $\widehat{\mathsf{pOpe}_{\iota EZ}}$.*

Proof. Combine the Corollary 4.62 with the Theorem 3.39. □

Corollary 5.26. *For every $P \in \mathsf{pOpe}_{\iota}$, the simplicial set $\zeta(\text{よ}_P)$ is contractible in the Kan–Quillen model structure on sSet.*

Proof. We proceed by induction on the dimension of P.

Base cases (dimension 0 or 1):

For P being a point or an edge $\zeta(\text{よ}_P)$ is also a point or an edge, hence contractible in sSet.

Inductive step:

Consider the diagram from the Theorem 4.19. The top horizontal morphism is an acyclic cofibration, being a monomorphism between contractible objects; hence,a the bottom arrow is also an acyclic cofibration, being its pushout. By the inductive hypothesis, $\zeta(\gamma(\alpha_P))$ is contractible, as $\langle \gamma(\alpha_P) \rangle$ has lower dimension than P.

A weak equivalence from a contractible object ensures its codomain is contractible. Thus, $\zeta(\text{よ}_P)$, being the codomain of the bottom arrow, is contractible. □

Lemma 5.27. *If P is an opetopic cardinal, then δP is contractible.*

Proof. We will show that each term $\mathcal{F}_i$ in the domain filtration of P (see Theorem 4.36) with $i \geq 0$ is contractible, by induction on i and the cardinality of $|P|$.

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Base case:
$\mathcal{F}_0$ is clearly contractible, being simply a point.

Inductive step:
Assume that $\mathcal{F}_j(Q)$ is contractible for $\#|Q| < \#|P|$ and every $j \geq 0$.

$$\mathcal{F}_n(P) = \mathcal{F}_{n-1} \amalg_{\delta\alpha} \langle\alpha\rangle$$

If $\langle\alpha\rangle = \mathcal{F}_n(P)$, then Corollary 5.26 implies that $\mathcal{F}_n(P)$ is contractible.

Otherwise, $\#|\langle\alpha\rangle| < \#|P|$, hence $\delta\alpha$ is contractible by inductive hypothesis, being the second-to-last term in the domain filtration of $\langle\alpha\rangle$. Inclusion $\delta\alpha \hookrightarrow \langle\alpha\rangle$ is a cofibration, hence $\mathcal{F}_n(P)$ is contractible, being homotopy pushout of three contractible objects. □

Corollary 5.28. *For every horn $\Lambda^\alpha P$, the morphism $\zeta(\Lambda^\alpha P \hookrightarrow \curlywedge_P)$ is an acyclic cofibration of the Kan–Quillen model structure on* sSet.

Proof. By induction on the number of faces in δP.

Base case.

If δP has one face of maximal dimension (i.e., δP is representable), the claim follows because $\zeta(\Lambda^\alpha P \hookrightarrow P)$ is a morphism between contractible objects.

If δP has two faces of maximal dimension, denote the first face in δP by $\delta_f P$ and the last face in δP by $\delta_l P$. There are three cases:

- $\delta_f P$ is not in $\Lambda^\alpha P$. Then $\Lambda^\alpha P$ is isomorphic to $\langle\delta_l\alpha_P\rangle \coprod_{\langle\gamma^2(\alpha_P)\rangle} \langle\gamma(\alpha_P)\rangle$, hence contractible.
- γP is not in $\Lambda^\alpha P$. Then $\Lambda^\alpha P$ is isomorphic to $\delta_f P \coprod_{\gamma\delta_f P} \delta_l P$, hence contractible.
- $\delta_l P$ is not in $\Lambda^\alpha P$. Then $\Lambda^\alpha P$ is isomorphic to $\delta_f P \coprod_{\delta\delta_f P} \gamma\alpha_P$, hence contractible.

Inductive step.

Assume the claim holds for all P such that δP has at most n cells of maximal dimension. By the previous lemma, we may assume $\alpha \neq \gamma P$.

Choose a cell β of maximal dimension in δP, adjacent to the cells in $\delta P \setminus \Lambda^\alpha P$. Observe that $\Lambda^\alpha P \setminus \{\beta\}$ is a horn of an opetope P', obtained from P by merging α and β into a single cell κ.

Let Q be an opetope with domain spanned by α and β and codomain κ. Then $Q \coprod_\kappa P'$ is an inner horn of an opetope with codomain P. This horn is constructed from $\Lambda^\alpha P$ via two anodyne extensions, making it weakly equivalent to $\Lambda^\alpha P$. By the inductive assumption, $Q \coprod_\kappa P'$ is a binary horn and thus contractible. Since weak equivalences preserve contractibility, $\zeta(\Lambda^\alpha P \hookrightarrow P)$ is an acyclic cofibration. □

Lemma 5.29. *For every $X \in \widehat{\mathsf{pOpe}_{\iota EZ}}$, $\zeta(X \times \curlywedge_{\to})$ is a cylinder over $\zeta(X)$.*

Proof. Using induction by the skeleton, we reduce to the case of $X = \curlywedge_P$ for $P \in \mathsf{pOpe}_\iota$.

In Zawadowski (2017), the map $\curlywedge_P \times \{0\} \hookrightarrow \curlywedge_P \times \curlywedge_{\to}$ was explicitly described as a composition of pushouts of opetopic horn inclusions, thus the result follows from the previous Corollary. □

Theorem 5.30. *An adjunction $\zeta \dashv \rho$ is a Quillen adjunction between the Kan–Quillen structure on the category* sSet *and the $(\infty, 0)$-structure on the category $\widehat{\mathsf{pOpe}_{\iota EZ}}$.*

Proof. This follows the Theorem 4.17, the Corollary 5.28, and the Lemma 5.29 by using the criterion formulated in the Lemma 5.21. □

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With the aid of Corollary 4.82, we can prove the following:

Theorem 5.31. *$A(\Delta^n)$ is contractible in opetopic $(\infty, 0)$-structure.*

Proof. First, we fix an opetope P and show that $\mathrm{Cone}(P)$ is contractible, mirroring the reasoning given in Zawadowski (2017) Theorem 3.23 and preceding results.

Claim 1: $\mathrm{Cone}(P)$ **is contractible.**

$\mathrm{Cone}_H(P)$ is a subobject of $\mathrm{Cyl}(P)$ spanned by some initial (with respect to the flag order) segment of maximal flags. From Zawadowski (2017) Proposition 3.21, it follows that $\mathrm{Cone}_H(P)$ can be expressed in terms of the filtration

$$\langle x_0 \rangle = C^{\vec{x_0}} \hookrightarrow C^{\vec{x_1}} \hookrightarrow \ \ldots \hookrightarrow C^{\vec{x_l}} = \mathrm{Cone}(P)$$

where the first term is an opetope spanned by the initial flag of P and each term is obtained from the previous one by gluing the next (with respect to the flag order) maximal cell of $\mathrm{Cone}(P)$ to its predecessor along the unique common face of codimension 1.

When we apply to this filtration the functor $\mathsf{pHg}_\iota \hookrightarrow \widehat{\mathsf{pOpe}_\iota}$, we get a finite sequence in which the first term is contractible (being representable) and every other term is obtained from the previous one as a pushout of two inclusions between contractible cofibrant objects. From this, we see that every term in that sequence (in particular the last one) is contractible.

For more details, see proofs of Zawadowski (2017), Propositions 3.21, 3.22, and Theorem 3.22 (to simplify the comparison, we used analogous notation).

Claim 2: $A(\Delta^n)$ **is contractible (induction on** n**).**

Base case $(n = 0)$**:**

The claim is obviously true for $n = 0$, which establishes the inductive step.

Induction step:

Both

$$\mathrm{colim}_{\alpha \in \mathcal{C}_{nd}(A(\Delta^{n-1}))} \mathrm{Cone}(\alpha, \nu_{n-1}) \cong A(\Delta^n)$$

and

$$\mathrm{colim}_{\alpha \in \mathcal{C}_{nd}(A(\Delta^{n-1}))} \pitchfork_{\langle \alpha \rangle} \cong A(\Delta^{n-1})$$

represent the homotopy colimit of the same diagram (see Radulescu-Banu (2009), p. 101), and the second one is contractible by the inductive assumption, so the first one is also contractible. □

Corollary 5.32. *The adjunction $A \dashv S$ is a Quillen adjunction between the Kan–Quillen structure on the category sSet and opetopic $(\infty, 0)$-structure on $\widehat{\mathsf{pOpe}_{\iota EZ}}$*

Proof. By Remark 4.69, A preserves cofibrations.

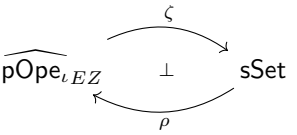
It suffices to show that A maps generating acyclic cofibrations $\Lambda^n_k \hookrightarrow \Delta^n$ to acyclic cofibrations, hence it is enough to show that both domain and codomain are mapped to contractible objects. For the codomain, this is precisely the corollary above, while for the domain, we use the standard description of Λ^n_k as a pushout of representables and cocontinuity of A. □

5.3 Quillen equivalence

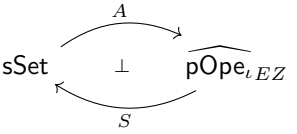
The goal of this section is to prove that opetopic $(\infty, 0)$-model structure is Quillen-equivalent with the Kan–Quillen model structure on sSet.

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Theorem 5.33. *Both adjunctions*



and



induce Quillen equivalences between the opetopic $(\infty, 0)$*-model structure on* $\widehat{\mathsf{pOpe}}_{\iota EZ}$ *and the Kan–Quillen model structure on* sSet.

Proof. To establish that the adjunctions (ζ, ρ) and (A, S) form Quillen equivalences, it suffices to demonstrate that both composite functors $\zeta \circ A$ and $A \circ \zeta$ induce equivalences on the respective homotopy categories Ho(sSet) and Ho($\widehat{\mathsf{pOpe}}_{\iota EZ}$). We analyze each composition systematically.

Claim 1: The composition $\zeta \circ A$ induces an equivalence on Ho(sSet)**.**

Step 1: Natural transformation construction.
By definition, the restrictions of A and ρ (the right adjoint to A) to representable objects in sSet (i.e., their compositions with the Yoneda embedding $\text{よ} : \Delta \to \mathsf{sSet}$) coincide. This identification allows us to reinterpret the counit transformation

$$\epsilon : \zeta \circ \rho|_{\Delta} \to \text{よ}_{\Delta}$$

as a natural transformation between restricted functors:

$$\epsilon' : \zeta \circ A|_{\Delta} \to \text{よ}_{\Delta}.$$

Since both ζ and A are left adjoints, they preserve colimits. We may therefore extend ε canonically along colimits over Δ to obtain a global natural transformation:

$$\mathcal{E} : \zeta \circ A \to \mathrm{Id}_{\mathsf{sSet}}.$$

Step 2: Proving $\mathcal{E}$ is a pointwise weak equivalence.
We verify that $\mathcal{E}_X$ is a weak equivalence for all $X \in \mathsf{sSet}$ via skeletal induction:

Base case – representables:
For $X = \Delta^n$ (the standard n-simplex), both $\zeta \circ A(\Delta^n)$ and $\mathrm{Id}_{\mathsf{sSet}}(\Delta^n) = \Delta^n$ are contractible simplicial sets by Theorem 5.31 and ζ being a left Quillen functor. The map $\mathcal{E}_{\Delta^n}$ between contractible objects is necessarily a weak equivalence.

Inductive step – arbitrary simplicial sets:
For general X, by Cisinski (2019) Corollaries 1.3.10, 2.3.19, and 2.3.29, the component $\mathcal{E}_X$ is shown to be a weak equivalence using induction by a skeleton.

Thus $\mathcal{E}$ is a natural weak equivalence, making $\zeta \circ A$ homotopy equivalent to the identity on sSet. This induces an equivalence $\mathrm{Ho}(\zeta \circ A) \simeq \mathrm{Id}_{\mathrm{Ho}(\mathsf{sSet})}$.

Claim 2: The composition $A \circ \zeta$ induces an equivalence on Ho($\widehat{\mathsf{pOpe}}_{\iota EZ}$)**.**

Step 1: Terminal object and reedy cofibrancy.

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Let $\bullet$ denote the terminal object in the category of *coopetopic diagrams* (functors $\mathsf{pOpe}_\iota \to \widehat{\mathsf{pOpe}_{\iota EZ}}$). Consider the unique morphisms:

$$\phi : A \circ \zeta|_{\mathsf{pOpe}_\iota} \to \bullet \quad \text{and} \quad \psi : \mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}|_{\mathsf{pOpe}_\iota} \to \bullet.$$

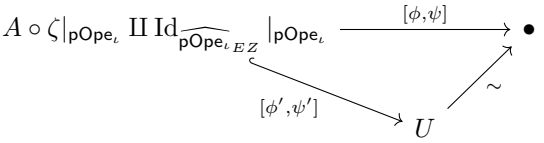
Equip this diagram category with the Reedy model structure. Since A, ζ, and $\mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}$ preserve cellular monomorphisms and are left adjoint, the diagrams $A \circ \zeta|_{\mathsf{pOpe}_\iota}$ and $\mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}|_{\mathsf{pOpe}_\iota}$ are Reedy cofibrant. Specifically, their latching maps are either $A \circ \zeta(\partial \curlywedge_P) \hookrightarrow A \circ \zeta(\curlywedge_P)$ or $\partial \curlywedge_P \hookrightarrow \curlywedge_P$. Hence, their coproduct $A \circ \zeta|_{\mathsf{pOpe}_\iota} \amalg \mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}|_{\mathsf{pOpe}_\iota}$ is also Reedy cofibrant.

Step 2: Weak equivalences to the terminal object.

All representable presheaves in $\widehat{\mathsf{pOpe}_{\iota EZ}}$ are cofibrant and contractible. Since A and ζ preserve cofibrant contractible objects (as left Quillen functors), both ϕ and ψ are weak equivalences between cofibrant objects.

Step 3: Factorization and cofibrant replacement.

Factor the universal morphism $[\phi, \psi] : A \circ \zeta|_{\mathsf{pOpe}_\iota} \amalg \mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}|_{\mathsf{pOpe}_\iota} \to \bullet$ as:



where $[\phi', \psi']$ is a cofibration and $U \to \bullet$ is an acyclic fibration. Since the domain is Reedy cofibrant, U inherits Reedy cofibrancy.

By the 2-out-of-3 property applied to the inclusions $i_{A \circ \zeta|_{\mathsf{pOpe}_\iota}}$ and $i_{\mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}|_{\mathsf{pOpe}_\iota}}$, the induced maps ϕ' and ψ' are weak equivalences between Reedy cofibrant diagrams.

Step 4: Left Kan extension and restricting to $\widehat{\mathsf{pOpe}_{\iota EZ}}$.

The left Kan extension $Lan_{\curlywedge} U$ along the Yoneda embedding $\curlywedge : \mathsf{pOpe}_\iota \to \widehat{\mathsf{pOpe}_{\iota EZ}}$ preserves cellular monomorphisms because U is Reedy cofibrant. Crucially, $Lan_{\curlywedge} U$ preserves EZ-presheaves, allowing us to regard it as an endofunctor on $\widehat{\mathsf{pOpe}_{\iota EZ}}$.

Furthermore, U being a cofibrant replacement of $\bullet$ ensures that each $Lan_{\curlywedge} U(\curlywedge_P)$ is contractible for $P \in \mathsf{pOpe}_\iota$.

Step 5: Connecting to the identity functor.

The natural transformations $Lan_{\curlywedge} \phi' : A \circ \zeta \to U$ and $Lan_{\curlywedge} \psi' : \mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}} \to U$ are pointwise weak equivalences (again by Cisinski (2019), Corollaries 1.3.10, 2.3.19, and 2.3.29). Thus $A \circ \zeta$ and $\mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}$ are connected through U by a zigzag of weak equivalences in the Reedy model structure, yielding:

$$\mathrm{Ho}(A \circ \zeta) \cong \mathrm{Ho}(\mathrm{Id}_{\widehat{\mathsf{pOpe}_{\iota EZ}}}) = \mathrm{Id}_{\mathrm{Ho}(\widehat{\mathsf{pOpe}_{\iota EZ}})}.$$

Therefore, $A \circ \zeta$ induces an equivalence on $\mathrm{Ho}(\widehat{\mathsf{pOpe}_{\iota EZ}})$. □

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